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Bifurcations of Positive and Negative Continua in Quasilinear Elliptic Eigenvalue Problems

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Abstract. The main result of this work is a Dancer-type bifurcation result for the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + h\big(x, u(x); \lambda\big) \quad \text{in} \quad \Omega\,; \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{P}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denotes the Dirichlet p-Laplacian on $W_0^{1,p}(\Omega)$, $1 < p < \infty$, and $\lambda \in \mathbb{R}$ is a spectral parameter. Let μ_1 denote the first (smallest) eigenvalue of $-\Delta_p$. Under some natural hypotheses on the perturbation function $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we show that the trivial solution $(0, \mu_1) \in E = W_0^{1,p}(\Omega) \times \mathbb{R}$ is a bifurcation point for problem (P) and, moreover, there are **two distinct continua**, $\mathcal{Z}_{\mu_1}^+$ and $\mathcal{Z}_{\mu_1}^-$, consisting of nontrivial solutions $(u, \lambda) \in E$ to problem (P) which bifurcate from the set of trivial solutions at the bifurcation point $(0, \mu_1)$. The continua $\mathcal{Z}_{\mu_1}^+$ and $\mathcal{Z}_{\mu_1}^-$ are *either* both unbounded in E, *or else* their intersection $\mathcal{Z}_{\mu_1}^+ \cap \mathcal{Z}_{\mu_1}^-$ contains also a point other than $(0, \mu_1)$. For the semilinear problem (P) (i.e., for $p = 2$) this is a classical result due to E. N. Dancer from 1974. We also provide an example of how the union $\mathcal{Z}_{\mu_1}^+ \cap \mathcal{Z}_{\mu_1}^-$ looks like (for $p > 2$) in an interesting particular case.

Our proofs are based on very precise, local asymptotic analysis for λ near μ_1 (for any $1 < p < \infty$) which is combined with standard topological degree arguments from global bifurcation theory used in Dancer's original work.

1. Introduction

This work is concerned with bifurcations of continua of "positive" and "negative" solutions to quasilinear elliptic problems of the following type:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + h\big(x, u(x); \lambda\big) \quad \text{in} \quad \Omega\,; \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.1}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), Δ_p stands for the Dirichlet p-Laplacian defined by $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$, λ ($\lambda \in \mathbb{R}$) serves as a bifurcation parameter, and $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $h(x, \,\cdot\,; \,\cdot\,)$ continuous for a.e. $x \in \Omega$. When considering bifurcations from a trivial solution, we naturally assume also $h(x, 0; \lambda) = 0$ and $h(x, u; \lambda)/|u|^{p-1} \to 0$ as $u \to 0$, pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$. A *trivial solution* of (1.1) is any pair $(0, \lambda) \in E \stackrel{\text{def}}{=} W_0^{1,p}(\Omega) \times \mathbb{R}$.

In analogy with classical results of Dancer [10] for the semilinear case $p = 2$, our main goal is to show the **existence of two distinct continua** of nontrivial (weak) solutions to problem (1.1), "positive" and "negative" ones, that bifurcate from the set of trivial solutions at the point $(0, \mu_1)$ in the **positive** and **negative** directions φ_1 and $-\varphi_1$, respectively (see Lemma 3.6). As usual, μ_1 denotes the first (smallest) eigenvalue of $-\Delta_p$ which is known to be simple with a positive eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$. Under a *continuum* in a Banach space we mean a closed connected set which contains at least two distinct points. Similarly to bifurcations from zero we treat also bifurcations from infinity under the condition $h(x, u; \lambda)/|u|^{p-1} \to 0$ as $|u| \to \infty$, pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

To be more specific about our present results, let us begin by considering the semilinear case $p = 2$ first: The classical global bifurcation result of Rabinowitz [31, Theorem 1.3] exhibits a continuum of nontrivial solutions to problem (1.1) which emanates from the set of trivial solutions at the bifurcation point $(0, \mu_1)$. Furthermore, Dancer's result [10, Theorem 2] guarantees the bifurcation of two continua of "positive" and "negative" solutions to problem (1.1) in the directions $\pm\varphi_1$. Indeed, in a sufficiently small neighborhood of $(0, \mu_1)$ these continua contain only solutions $(u, \lambda) \in E$ of problem (1.1) satisfying $u = \tau(\varphi_1 + v^\top)$ where $\tau \in \mathbb{R}$ and $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ as $\tau \to 0$. Hence, $u > 0$ in Ω ($u < 0$ in Ω, respectively) if and only if $\tau > 0$ ($\tau < 0$), provided $|\tau| > 0$ is small enough.

Now let us consider the quasilinear case $p \neq 2$. The analogue of Rabinowitz' result [31, Theorem 1.3] for problem (1.1) has been obtained in del Pino and Manásevich [30] with a continuum of nontrivial solutions bifurcating from the point $(0, \mu_1)$ and having the same properties as in the case $p = 2$. In the work reported here we obtain the corresponding analogue (Theorem 3.7 below) of Dancer's result [10, Theorem 2] for $1 < p < \infty$. We treat problems with a *more general* $(p-1)$-homogeneous part than just (1.1) treated in [30]. Similarly to a bifurcation from zero at $(0, \mu_1)$ sketched above, under a bifurcation from infinity at $(+\infty, \mu_1)$ ($(-\infty, \mu_1)$, respectively) we mean a continuum of solutions $(u, \lambda) \in E$ of problem (1.1) satisfying $u = t^{-1}(\varphi_1 + v^\top)$ where $0 \neq t \in \mathbb{R}$ and $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ as $t \to 0$. Again, $u > 0$ in Ω ($u < 0$ in Ω, respectively) if and only if $t > 0$ ($t < 0$), provided $|t| > 0$ is small enough.

In an analogy with the case $p = 2$, we use the fact that μ_1 is a simple eigenvalue of $-\Delta_p$ with a positive eigenfunction φ_1 in an essential way. Under a rather restrictive hypothesis, this extension of Dancer's result has already been stated in Drábek [14, Theorem 14.20, p. 191] without proof. His hypothesis [14, Eq. (14.43), p. 191] has been verified in Drábek et al. [16, Theorem 4.1] for the special case $h(x, u; \lambda) \equiv f(x)$ independent from u and λ, for bifurcations from infinity at $\lambda = \mu_1$. Extending a new asymptotic technique developed recently in Drábek et al. [16, Theorem 4.1] and Takáč [34, Section 5] and [35, Section 6], we are able to verify Drábek's hypothesis [14, Eq. (14.43), p. 191] and thus extend Dancer's result to a broader class of quasilinear elliptic operators of second order.

Last but not least, for the radially symmetric problem (1.1) in a ball $\Omega \subset \mathbb{R}^N$, a local bifurcation result of Crandall–Rabinowitz-type [9, Theorem 1.7, p. 325] has been obtained in García–Melián and Sabina de Lis [21, Theorem 2, p. 30]. We remark that Crandall–Rabinowitz' result guarantees only *local* existence of a (smooth) bifurcation *curve* of nontrivial solutions together with their *uniqueness*, whereas Dancer's result guarantees *global* existence of "positive" and "negative" bifurcation *continua* of nontrivial solutions *without* uniqueness. Since Crandall–Rabinowitz' result is concerned only with bifurcations from simple eigenvalues, one can clearly determine the "positive" and "negative" parts of the (smooth unique local) bifurcation curve of nontrivial solutions.

A direct consequence of our extension of Dancer's result is the following dichotomy for the simplified bifurcation problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + f(x) \quad \text{in} \quad \Omega\,; \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.2}$$

where $f \in L^\infty(\Omega)$, $f \not\equiv 0$ in Ω:

> There exist two continua $\mathcal{Z}^+_{\mu_1}$ and $\mathcal{Z}^-_{\mu_1}$ ($\subset E$) of solutions (u, λ) to problem (1.2) bifurcating from $(+\infty, \mu_1)$ and $(-\infty, \mu_1)$, respectively, such that *either*
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> (i) $\mathcal{Z}^+_{\mu_1} \cap \mathcal{Z}^-_{\mu_1} \neq \emptyset$, i.e., $\mathcal{Z}^+_{\mu_1} \cup \mathcal{Z}^-_{\mu_1}$ is a continuum connecting large positive with large negative solutions of types $(+\infty, \mu_1)$ and $(-\infty, \mu_1)$, respectively, *or else*
>
> (ii) the intersections of both $\mathcal{Z}^\pm_{\mu_1}$ with the set $\{(u, \lambda) \in E : |\lambda - \mu_1| > \delta\}$ are unbounded (in E) for every $\delta > 0$ small enough.

This extension also fills the gap left open in several results on global bifurcations from $(\pm\infty, \mu_1)$ for problem (1.1) obtained in Drábek et al. [16, Section 5] and in Drábek, Girg, and Takáč [15, Section 3] as well.

This work is organized as follows. For the sake of clarity of our presentation we always begin by treating the case of bifurcation from the trivial solution at $(0, \mu_1)$ in all details and then reduce our treatment of bifurcation from infinity at $(\pm\infty, \mu_1)$ to highlighting the necessary changes.

In the next section (Section 2) we introduce basic notations, state our hypotheses, and deduce a few simple consequences. Our main results are stated in Section 3: bifurcations from zero in Section 3.1 (Proposition 3.5 and Theorem 3.7) and bifurcations from infinity in Section 3.2 (Proposition 3.8 and Theorem 3.10). We begin Section 4 by showing the simplicity of the first eigenvalue μ_1 for the quasilinear eigenvalue problem (2.7) in Section 4.1 (Remark 4.1). In particular, we generalize also the well-known inequality of Díaz and Saa which they have established only for Δ_p; see Remark 4.4. In Section 4.2 we adapt some results from Arcoya and Gámez [6, Lemma 24, p. 1905] (proved there only for Δ_p and bifurcation from infinity) on the asymptotic analysis of local bifurcation from zero to our setting in problem (2.1); see Proposition 4.5. Analogous results for local bifurcations from infinity are established in Section 4.3 (Proposition 4.10). Section 5 contains the complete proofs of our main results, Proposition 3.5 and Theorem 3.7. In these proofs we employ the topological degree due to Browder and Petryshyn [7] and Skrypnik [33] which we describe in Section 5.1. The corresponding results for bifurcations from infinity, Proposition 3.8 and Theorem 3.10, respectively, are derived from Proposition 3.5 and Theorem 3.7 by applying the standard transformation $u \mapsto v = u/\|u\|^2_{W_0^{1,p}(\Omega)}$ for $u \in W_0^{1,p}(\Omega) \setminus \{0\}$; see Remark 3.9.

Section 6 features an interesting example of problem (1.1) with $\Omega = (0, \pi_p) \subset \mathbb{R}^1$ an interval and $2 < p < \infty$, cf. (6.1). The nonhomogeneous perturbation function $h(x, u; \lambda)$ is chosen in such a way that the continuum $\mathcal{Z}^+_{\mu_1} \cup \mathcal{Z}^-_{\mu_1}$ $(\subset E)$ oscillates through the hyperplane $\{(u, \lambda) \in E : \lambda = \mu_1\}$ in E while approaching the bifurcation point $(+\infty, \mu_1)$ or $(-\infty, \mu_1)$. In other words, $\lambda - \mu_1$ oscillates about zero as $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$. Rather involved asymptotic formulas from [16, Theorem 4.1] are required to handle these oscillations.

Finally, we collect some auxiliary results in Appendices A, B, and C. An a priori boundedness result in $L^\infty(\Omega)$ (due to Anane [4, Théorème A.1, p. 96]) is stated in Appendix A. Some useful consequences thereof (for bifurcations from zero and infinity) follow in Appendix B. Our treatment of Example 6.1 is based on a ramification of Erdélyi's asymptotic formula [17, Theorem on p. 52] which we establish in Appendix C.

2. Preliminaries

2.1. Notation

We set $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N} = \{1, 2, 3, \dots\}$. The closure, interior, and boundary of a set $S \subset \mathbb{R}^N$ are denoted by $\overline{S}$, $\mathrm{int}(S)$, and ∂S, respectively, and the characteristic function of S by $\chi_S : \mathbb{R}^N \to \{0, 1\}$. We write $|S|_N \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^N} \chi_S(x)\,\mathrm{d}x$ if S is also Lebesgue measurable. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \ge 1)$. Given an integer $k \ge 0$ and $0 \le \alpha \le 1$, we denote by $C^{k,\alpha}(\overline{\Omega})$ the Hölder space of all k-times continuously differentiable functions $u : \Omega \to \mathbb{R}$ whose all (classical) partial derivatives of order $\le k$ possess a continuous extension up to the boundary and are α-Hölder continuous on $\overline{\Omega}$. The norm $\|u\|_{C^{k,\alpha}(\overline{\Omega})}$ in $C^{k,\alpha}(\overline{\Omega})$ is defined in a natural way. As usual, we abbreviate $C^k(\overline{\Omega}) \equiv C^{k,0}(\overline{\Omega})$. The linear subspace of $C^k(\overline{\Omega})$ consisting of all C^k functions $u : \Omega \to \mathbb{R}$ with compact support is denoted by $C^k_{\mathrm{c}}(\Omega)$; we set $C^\infty_{\mathrm{c}}(\Omega) = \cap_{k=0}^\infty C^k_{\mathrm{c}}(\Omega)$. Given $1 \le p \le \infty$, we denote by $L^p(\Omega)$ the Lebesgue space of all (equivalence classes of) Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ with the standard norm. Finally, for an integer $k \ge 1$, we denote by $W^{k,p}(\Omega)$ the Sobolev space of all functions $u \in L^p(\Omega)$ whose all (distributional) partial derivatives of order $\leq k$ also belong to $L^p(\Omega)$. Again, the norm $\|u\|_{k,p} \equiv \|u\|_{W^{k,p}(\Omega)}$ in $W^{k,p}(\Omega)$ is defined in a natural way. The closure in $W^{k,p}(\Omega)$ of the set of all C^k functions $u : \Omega \to \mathbb{R}$ with compact support is denoted by $W_0^{k,p}(\Omega)$. We refer to Adams and Fournier [1] or Kufner, John, and Fučík [24] for details about these and other similar function spaces. All Banach and Hilbert spaces used in this article are real.

The Euclidean inner product in $\mathbb{R}^N$ is denoted by $\langle \cdot\, , \,\cdot \rangle$. We work with the standard inner product in $L^2(\Omega)$ defined by $\langle u, v\rangle_{L^2(\Omega)} \stackrel{\text{def}}{=} \int_\Omega uv\, dx$ for $u, v \in L^2(\Omega)$. The orthogonal complement in $L^2(\Omega)$ of a set $\mathcal{M} \subset L^2(\Omega)$ is denoted by $\mathcal{M}^{\perp, L^2}$,

$$\mathcal{M}^{\perp, L^2} \stackrel{\text{def}}{=} \left\{ u \in L^2(\Omega) : \langle u, v\rangle_{L^2(\Omega)} = 0 \quad \text{for all} \quad v \in \mathcal{M} \right\}.$$

The inner product $\langle \cdot\, , \,\cdot \rangle_{L^2(\Omega)}$ induces the canonical duality between the space of test functions $\mathcal{D}(\Omega) \equiv C_c^\infty(\Omega)$ and the space of distributions $\mathcal{D}'(\Omega)$. More generally, if X is a Banach space, $\mathcal{D}(\Omega) \subset X \subset \mathcal{D}'(\Omega)$, such that the embedding $\mathcal{D}(\Omega) \hookrightarrow X$ is dense and continuous, we denote by $\langle \cdot\, , \,\cdot \rangle_X$ the duality between X and its dual space X' induced by the canonical duality between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$. Since $\mathcal{D}(\Omega)$ is reflexive, also the embedding $X' \hookrightarrow \mathcal{D}'(\Omega)$ is dense and continuous. If no confusion may arise, we often leave out the index X in $\langle \cdot\, , \,\cdot \rangle_X$. In particular, the inner product $\langle \cdot\, , \,\cdot \rangle_{L^2(\Omega)}$ induces the duality $\langle \cdot\, , \,\cdot \rangle_{L^p(\Omega)}$ between the Lebesgue spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1 \leq p < \infty$ and $1 < p' \leq \infty$ with $1/p + 1/p' = 1$, and the duality $\langle \cdot\, , \,\cdot \rangle_{W_0^{1,p}(\Omega)}$ between the Sobolev space $W_0^{1,p}(\Omega)$ and its dual space $W^{-1,p'}(\Omega)$, as well. We use analogous notation also for the duality between the Cartesian products $[L^p(\Omega)]^N$ and $[L^{p'}(\Omega)]^N$.

2.2. Structural hypotheses

Let us consider the following more general version of problem (1.1), namely,

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x, \nabla u)\big) = \lambda\, B(x)\, |u|^{p-2} u + h\big(x, u(x); \lambda\big) \quad \text{in} \quad \Omega\,; \\ \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{2.1}$$

In the sequel we always assume that the domain Ω satisfies the following regularity hypothesis:

Hypothesis (Ω). If $N \geq 2$ then Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, and Ω satisfies also the interior sphere condition at every point of $\partial\Omega$. If $N = 1$ then Ω is a bounded open interval in $\mathbb{R}^1$.

It is clear that for $N \geq 2$, hypothesis $(\boldsymbol{\Omega})$ is satisfied if $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary.

We always assume that the function A of $(x, \xi) \in \Omega \times \mathbb{R}^N$ and its partial gradient $\partial_\xi A \equiv \left(\frac{\partial A}{\partial \xi_i}\right)_{i=1}^N$ with respect to $\xi \in \mathbb{R}^N$ satisfy the following structural hypothesis, upon the substitution $\mathbf{a}(x, \xi) \stackrel{\text{def}}{=} \frac{1}{p}\, \partial_\xi A(x, \xi)$ with $a_i = \frac{1}{p} \frac{\partial A}{\partial \xi_i}$:

Hypothesis (A). $A:\,\Omega\times\mathbb{R}^N\to\mathbb{R}_+$ verifies the *positive p-homogeneity* hypothesis

$$A(x,t\xi)=|t|^p\,A(x,\xi)\quad\text{for all}\quad t\in\mathbb{R}\tag{2.2}$$

and for all $(x,\xi)\in\Omega\times\mathbb{R}^N$. Furthermore, we assume that $A\in C^1(\Omega\times\mathbb{R}^N)$, and its partial gradient $\partial_\xi A:\,\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ satisfies $\frac{1}{p}\,\frac{\partial A}{\partial\xi_i}=a_i\in C^1(\Omega\times(\mathbb{R}^N\setminus\{0\}))$ for all $i=1,2,\dots,N$, together with the following *ellipticity* and *growth conditions*: There exist some constants $\gamma,\Gamma\in(0,\infty)$ such that

$$\sum_{i,j=1}^N \frac{\partial a_i}{\partial\xi_j}(x,\xi)\cdot\eta_i\eta_j\geq\gamma\cdot|\xi|^{p-2}\cdot|\eta|^2\,,\tag{2.3}$$

$$\sum_{i,j=1}^N \left|\frac{\partial a_i}{\partial\xi_j}(x,\xi)\right|\leq\Gamma\cdot|\xi|^{p-2}\,,\tag{2.4}$$

$$\sum_{i,j=1}^N \left|\frac{\partial a_i}{\partial x_j}(x,\xi)\right|\leq\Gamma\cdot|\xi|^{p-1}\,,\tag{2.5}$$

for all $x\in\Omega$, all $\xi\in\mathbb{R}^N\setminus\{0\}$, and all $\eta\in\mathbb{R}^N$.

It is evident that it suffices to require inequalities (2.3), (2.4), and (2.5) for $|\xi|=1$ only; the general case $\xi\in\mathbb{R}^N\setminus\{0\}$ follows from the positive p-homogeneity hypothesis (2.2).

Hypothesis (2.2) forces $A(x,\mathbf{0})=0$ and $\frac{\partial A}{\partial\xi_i}(x,\mathbf{0})=0$ for all $x\in\Omega$ and $i=1,2,\dots,N$. It follows that $A(x,\,\cdot\,)$ is strictly convex and satisfies

$$\frac{\gamma}{p-1}\,|\xi|^p\leq A(x,\xi)\leq\frac{\Gamma}{p-1}\,|\xi|^p\quad\text{for all}\quad\xi\in\mathbb{R}^N.\tag{2.6}$$

These inequalities are a direct consequence of Taylor's formula combined with (2.3) and (2.4), which yields

$$\frac{\gamma}{p-1}\,|\xi|^p\leq A(x,\xi)-A(x,\mathbf{0})-\big\langle\partial_\xi A(x,\mathbf{0}),\xi\big\rangle\leq\frac{\Gamma}{p-1}\,|\xi|^p$$

for all $(x,\xi)\in\Omega\times\mathbb{R}^N$.

The weight function B is assumed to satisfy

Hypothesis (B). $B:\,\Omega\to\mathbb{R}_+$ belongs to $L^\infty(\Omega)$ and does not vanish identically (almost everywhere) in Ω, i.e., $B\not\equiv 0$ in Ω.

Now consider the $(p-1)$-homogeneous nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x,\nabla u)\big)=\lambda\,B(x)\,|u|^{p-2}u\quad\text{in}\quad\Omega\,;\\ \qquad\qquad u=0\quad\text{on}\quad\partial\Omega\,,\end{cases}\tag{2.7}$$

with an eigenvalue $\lambda\in\mathbb{R}$ and an eigenfunction $u\in W_0^{1,p}(\Omega)\setminus\{0\}$.

Remark 2.1. The first (smallest) eigenvalue μ_1 for problem (2.7) is given by the Rayleigh quotient

$$\mu_1 = \inf \left\{ \int_\Omega A(x, \nabla u)\, \mathrm{d}x : \, u \in W_0^{1,p}(\Omega) \;\text{ with }\; \int_\Omega B(x)\, |u|^p \, \mathrm{d}x = 1 \right\}. \tag{2.8}$$

Since the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, the infimum above is attained and satisfies $0 < \mu_1 < \infty$. It is well-known that μ_1 is a simple eigenvalue for problem (2.7) with the associated eigenfunction φ_1 normalized by $\varphi_1 > 0$ in Ω and $\int_\Omega B(x)\varphi_1^p \mathrm{d}x = 1$; see Takáč, Tello and Ulm [37, Theorem 2.6, p. 80]. The special case of the positive Dirichlet p-Laplacian $\mathcal{A} = -\Delta_p$ is due to Anane [3, Théorème 1, p. 727] and in a more general domain Ω to Lindqvist [26, Theorem 1.3, p. 157]. Moreover, it is shown in Anane [3, Théorème 2, p. 727] or Anane and Tsouli [5, Prop. 2, p. 5] that μ_1 is an isolated eigenvalue of $\mathcal{A}$ and the next eigenvalue $\mu_2 > \mu_1$ has a variational characterization. An interested reader can easily derive this fact from the proof of the anti-maximum principle in Takáč [36, proof of Theorem 4.4, Eq. (4.13) on p. 408].

2.3. Hypotheses on the nonhomogeneous perturbation

Finally, we assume that h satisfies hypothesis $(\mathbf{H}_0)$ (for bifurcations from *zero*) or hypothesis $(\mathbf{H}_\infty)$ (for bifurcations from *infinity*) stated below:

Hypothesis $(\mathbf{H}_0)$. $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., $h(\,\cdot\,, u; \lambda) : \Omega \to \mathbb{R}$ is Lebesgue measurable for each pair $(u, \lambda) \in \mathbb{R}^2$ and $h(x, \,\cdot\,; \,\cdot\,) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$. Furthermore, we assume that there exists a constant $C \in (0, \infty)$ such that

$$\left| h(x, u; \lambda) \right| \leq C \, |u|^{p-1} \tag{2.9}$$

for all a.e. $x \in \Omega$ and all $(u, \lambda) \in \mathbb{R} \times \mathbb{R}$, and

$$h(x, u; \lambda)/|u|^{p-1} \to 0 \quad \text{as} \quad u \to 0 \tag{2.10}$$

uniformly for a.e. $x \in \Omega$ and uniformly in λ from bounded intervals in $\mathbb{R}$.

Hypothesis $(\mathbf{H}_0^n)$. We say that a sequence of functions $h_n : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, satisfy hypothesis $(\mathbf{H}_0^n)$ if functions h_n satisfy $(\mathbf{H}_0)$ for each $n \in \mathbb{N}$ and the bounds (2.9) and convergence in (2.10) are uniform in $n \in \mathbb{N}$.

Hypothesis $(\mathbf{H}_\infty)$. $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Furthermore, there exists a constant $C \in (0, \infty)$ such that

$$\left| h(x, u; \lambda) \right| \leq C\big(1 + |u|^{p-1}\big) \tag{2.11}$$

for all a.e. $x \in \Omega$ and all $(u, \lambda) \in \mathbb{R} \times \mathbb{R}$, and

$$h(x, u; \lambda)/|u|^{p-1} \to 0 \quad \text{as} \quad |u| \to \infty \tag{2.12}$$

uniformly for a.e. $x \in \Omega$ and in λ from bounded intervals in $\mathbb{R}$.

As we work with the zero Dirichlet boundary conditions, sometimes it will be necessary to assume inequality (2.9) also for bifurcations from infinity; this inequality is stronger than (2.11).

These hypotheses are satisfied by the following "canonical" examples of $h(x,u;\lambda) = g(x,u) + f(x)$ for $(x,u,\lambda) \in \Omega \times \mathbb{R} \times \mathbb{R}$:

- Condition (2.10) holds if $f \equiv 0$ in Ω and g takes either of the forms
$$g(x,u) = \begin{cases} c(x)\,|u|^{q(x)-2}u\,; \\ c(x)\,|u|^{q(x)-1}\,, \end{cases} \tag{2.13}$$
for $|u| \leq 1$, where $c, q \in L^\infty(\Omega)$ and q satisfies $\operatorname*{ess\,inf}_\Omega q > p$.
- Condition (2.12) holds if $f \in L^\infty(\Omega)$ and g takes either of the forms (2.13) for $|u| \geq 1$, where $c, q \in L^\infty(\Omega)$ and q satisfies $q(x) > 1$ for a.e. $x \in \Omega$ together with $\operatorname*{ess\,sup}_\Omega q < p$.

The following simple lemma is a very useful consequence of hypothesis $(\mathbf{H}_0)$ or $(\mathbf{H}_\infty)$ in our functional formulation of problem (2.1).

Lemma 2.2. *Let $u \in L^\infty(\Omega)$ be arbitrary, $u \not\equiv 0$ in Ω.*

(a) *If hypothesis $(\mathbf{H}_0)$ is satisfied, then*
$$h\big(x,u(x);\lambda\big)/\|u\|^{p-1}_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad \|u\|_{L^\infty(\Omega)} \to 0 \tag{2.14}$$
holds pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

(b) *If hypothesis $(\mathbf{H}_\infty)$ is satisfied, then*
$$h\big(x,u(x);\lambda\big)/\|u\|^{p-1}_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad \|u\|_{L^\infty(\Omega)} \to \infty \tag{2.15}$$
holds pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$.

The proof is given in the Appendix, Section B.1.

In order to obtain easy-to-verify a priori estimates for the "positive" and "negative" (nontrivial) branches of solutions to the bifurcation problem (2.1), we impose the following additional hypothesis $(\mathbf{H}_0')$ (for bifurcations from zero) or $(\mathbf{H}_\infty')$ (for bifurcations from infinity) on the perturbation function h where we assume that h is *independent* from λ:

Hypothesis $(\mathbf{H}_0')$. $h(x,u(x);\lambda) \equiv h(x,u(x))$ and there exist a constant $C_0 \in (0,\infty)$ and functions $f_{0+}, f_{0-} \in L^\infty(\Omega)$, $f_{0\pm} \not\equiv 0$ in Ω, and $g_0 : \mathbb{R} \to \mathbb{R}$, g_0 continuous with $g_0(\tau) \neq 0$ for $\tau \in \mathbb{R} \setminus \{0\}$, g_0 differentiable in $(-\delta,\delta) \setminus \{0\}$ for some $\delta > 0$, such that
$$\big|h(x,u)\big| \leq C_0 \left| g_0\left(\frac{u}{\varphi_1(x)}\right)\right| \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad u \in \mathbb{R}\,; \tag{2.16}$$
$$\frac{h(x,u)}{g_0\big(\frac{u}{\varphi_1(x)}\big)} \to f_{0\pm}(x) \quad \text{pointwise for a.e.} \quad x \in \Omega \quad \text{as} \quad u \to 0\pm\,; \tag{2.17}$$
$$\Gamma_0 \stackrel{\text{def}}{=} \sup_{0<|\tau|<\delta} \left| \frac{\tau\, g_0'(\tau)}{g_0(\tau)} \right| < \infty\,. \tag{2.18}$$

Hypothesis $(\mathbf{H}'_\infty)$. $h(x,u(x);\lambda) \equiv h(x,u(x))$ and there exist a constant $C_\infty \in (0,\infty)$ and functions $f_{\pm\infty} \in L^\infty(\Omega)$, $f_{\pm\infty} \not\equiv 0$ in Ω, and $g_\infty : \mathbb{R} \to \mathbb{R}$, g_∞ continuous with $g_\infty(\tau) \neq 0$ for $\tau \in \mathbb{R} \setminus \{0\}$, g_∞ differentiable in $(-\infty,-\delta) \cup (\delta,\infty)$ for some $\delta > 0$, such that

$$\big|h(x,u)\big| \le C_\infty \left|g_\infty\left(\frac{u}{\varphi_1(x)}\right)\right| \quad \text{for a.e.} \quad x \in \Omega \quad \text{and all} \quad u \in \mathbb{R}\,; \tag{2.19}$$

$$\frac{h(x,u)}{g_\infty\big(\frac{u}{\varphi_1(x)}\big)} \to f_{\pm\infty}(x) \quad \text{pointwise for a.e.} \quad x \in \Omega \quad \text{as} \quad u \to \pm\infty\,; \tag{2.20}$$

$$\Gamma_\infty \stackrel{\text{def}}{=} \sup_{|\tau|>\delta} \left|\frac{\tau\, g'_\infty(\tau)}{g_\infty(\tau)}\right| < \infty\,. \tag{2.21}$$

Some remarks on these hypotheses are in order.

Remark 2.3. Rewriting (2.17) as

$$\frac{h(x,u)}{|u|^{p-1}} \cdot \frac{\big|\frac{u}{\varphi_1(x)}\big|^{p-1}}{g_0\big(\frac{u}{\varphi_1(x)}\big)} \longrightarrow \frac{f_{0\pm}(x)}{\varphi_1(x)^{p-1}} \quad \text{for a.e.} \quad x \in \Omega \quad \text{as} \quad u \to 0\pm$$

we observe that (2.10) forces $g_0(\tau)/|\tau|^{p-1} \to 0$ as $\tau \to 0$. Similarly, combining (2.20) with (2.12) we arrive at $g_\infty(\tau)/|\tau|^{p-1} \to 0$ as $\tau \to \pm\infty$.

Remark 2.4. Condition (2.18) guarantees

$$\left|\frac{g_0\big(\tau(1+\theta)\big)}{g_0(\tau)} - 1\right| \le 4\,\Gamma_0\,|\theta| \tag{2.22}$$

for all $\tau,\theta \in \mathbb{R}$ such that $0 < |\tau| \le 1/2\delta$ and $|\theta| \le \theta_0 \stackrel{\text{def}}{=} \min\{1/2, \frac{1}{4\,\Gamma_0}\}$.

This can be seen as follows. Let $0 < |\tau| \le 1/2\delta$ and $|\theta| \le \theta_0$. From

$$g_0\big(\tau(1+\theta)\big) - g_0(\tau) = \tau\,\theta \int_0^1 g'_0\big(\tau(1+s\,\theta)\big)\,\mathrm{d}s$$

we obtain

$$\big|g_0\big(\tau(1+\theta)\big) - g_0(\tau)\big| \le |\tau|\,|\theta| \cdot \sup_{0\le s\le 1} \big|g'_0\big(\tau(1+s\,\theta)\big)\big|\,.$$

Now we apply (2.18) and $|\theta| \le 1/2$ to conclude that

$$\begin{aligned} \big|g_0\big(\tau(1+\theta)\big) - g_0(\tau)\big| &\le 2\,\Gamma_0\,|\theta| \cdot \sup_{0\le s\le 1} \big|g_0\big(\tau(1+s\theta)\big)\big| \\ &\le 2\,\Gamma_0\,|\theta| \cdot \left\{\sup_{0\le s\le 1} \big|g_0\big(\tau(1+s\theta)\big) - g_0(\tau)\big| + \big|g_0(\tau)\big|\right\}. \end{aligned} \tag{2.23}$$

Since $\theta \in \mathbb{R}$ is arbitrary with $|\theta| \le \theta_0$, (2.23) yields

$$\frac{1}{2} \cdot \sup_{0\le s\le 1} \big|g_0\big(\tau(1+s\theta)\big) - g_0(\tau)\big| \le 2\,\Gamma_0\,|\theta|\,\big|g_0(\tau)\big|$$

from which (2.22) follows.

Remark 2.5. Condition (2.21) guarantees

$$\left| \frac{g_\infty\big(\tau(1+\theta)\big)}{g_\infty(\tau)} - 1 \right| \leq 4\,\Gamma_\infty\,|\theta| \tag{2.24}$$

for all $\tau, \theta \in \mathbb{R}$ such that $|\tau| \geq 2\delta$ and $|\theta| \leq \theta_0 \stackrel{\text{def}}{=} \min\{1/2, \frac{1}{4\,\Gamma_\infty}\}$.

The proof of (2.24) is analogous to that of (2.22).

3. Main results

Set $E \stackrel{\text{def}}{=} W_0^{1,p}(\Omega) \times \mathbb{R}$. Under a *solution* of problem (2.1) (in the weak sense) we understand a pair $(u, \lambda) \in E$ that satisfies the integral identity

$$\int_\Omega \big\langle \mathbf{a}(x, \nabla u), \nabla \phi \big\rangle \,\mathrm{d}x = \lambda \int_\Omega B(x)\,|u|^{p-2} u\,\phi\,\mathrm{d}x + \int_\Omega h\big(x, u(x); \lambda\big)\,\phi\,\mathrm{d}x \tag{3.1}$$

for all $\phi \in W_0^{1,p}(\Omega)$. The last equation is equivalent to the operator equation

$$\mathcal{A}(u) = \lambda\,\mathcal{B}(u) + \mathcal{H}(u; \lambda) \tag{3.2}$$

with all terms valued in the dual space $X' = W^{-1,p'}(\Omega)$ of $X = W_0^{1,p}(\Omega)$ and the operators $\mathcal{A}, \mathcal{B}, \mathcal{H}(\,\cdot\,;\lambda) :\, X \to X'$ defined as follows, for all $u, \phi \in X$ and $\lambda \in \mathbb{R}$:

$$\big\langle \mathcal{A}(u), \phi \big\rangle_X = \int_\Omega \big\langle \mathbf{a}(x, \nabla u), \nabla \phi \big\rangle \,\mathrm{d}x\,; \tag{3.3}$$

$$\big\langle \mathcal{B}(u), \phi \big\rangle_X = \int_\Omega B(x)\,|u|^{p-2} u\,\phi\,\mathrm{d}x\,; \tag{3.4}$$

$$\big\langle \mathcal{H}(u; \lambda), \phi \big\rangle_X = \int_\Omega h(x, u; \lambda)\,\phi(x)\,\mathrm{d}x\,. \tag{3.5}$$

Owing to our conditions (2.2) through (2.5), the operator $\mathcal{A} :\, X \to X'$ is continuous, coercive, and strictly monotone. The operator $\tilde{\mathcal{B}} :\, X \to X'$ can be extended to a continuous operator $\mathcal{B} :\, L^p(\Omega) \to (L^p(\Omega))' = L^{p'}(\Omega)$ in a unique way. Consequently, $\mathcal{B}$ decomposed as

$$\mathcal{B} :\, X \hookrightarrow L^p(\Omega) \xrightarrow{\tilde{\mathcal{B}}} L^{p'}(\Omega) \hookrightarrow X'$$

is compact by Rellich's theorem. Finally, given $\lambda \in \mathbb{R}$, also the operator $\mathcal{H}(\,\cdot\,;\lambda) :$ $X \to X'$ can be extended to a continuous operator $\tilde{\mathcal{H}}(\,\cdot\,;\lambda) :\, L^p(\Omega) \to L^{p'}(\Omega)$ in a unique way. Again, $\mathcal{H} :\, X \times \mathbb{R} \to X'$ is compact by Rellich's theorem.

Furthermore, given any $F \in X'$, the mapping $u \mapsto \mathcal{A}(u) - F$ equals the Fréchet derivative of the energy functional

$$\mathcal{J}_0(u) \stackrel{\text{def}}{=} \frac{1}{p} \int_\Omega A(x, \nabla u)\,\mathrm{d}x - \langle F, u \rangle_X\,, \quad u \in X = W_0^{1,p}(\Omega)\,, \tag{3.6}$$

which is coercive and strictly convex on X. Thus, the equation $\mathcal{A}(u) = F$ has exactly one solution $u \in X$, i.e., $\mathcal{A}$ is invertible with the inverse mapping $\mathcal{A}^{-1}$: $X' \to X$ which is continuous because X is a uniformly convex space (cf. Takáč, Tello, and Ulm [37, Prop. 4.3, p. 87]). Consequently, the operator $\lambda\mathcal{B} + \mathcal{H}(\,\cdot\,;\lambda)$ on the right-hand side of equation (3.2) may be viewed as a compact perturbation of the invertible operator $\mathcal{A} :\ X \to X'$.

This setting will enable us to apply the Browder–Petryshyn degree theory (Browder and Petryshyn [7] or Skrypnik [33]) to the mapping

$$\Phi_\lambda(u) \equiv \Phi(u,\lambda) \stackrel{\text{def}}{=} \mathcal{A}(u) - \lambda\mathcal{B}(u) - \mathcal{H}(u;\lambda)\,, \quad (u,\lambda) \in E = X \times \mathbb{R}\,. \tag{3.7}$$

In terms of $\Phi(u,\lambda)$, equation (3.1) reads $\Phi(u,\lambda) = 0$. Now consider the $(p-1)$-homogeneous nonlinear eigenvalue problem (2.7), i.e., $\mathcal{A}(u) - \lambda\mathcal{B}(u) = 0$ for $(u,\lambda) \in E$; recall that $E = X \times \mathbb{R}$.

Definition 3.1. Let $\mu_0 \in \mathbb{R}$. We say that $(0,\mu_0) \in E$ is a *bifurcation point* (*from zero*) for problem (2.1) if there exists a sequence of pairs $\{(u_n,\lambda_n)\}_{n=1}^\infty \subset E$ such that equation (3.1) holds with $(u,\lambda) = (u_n,\lambda_n)$ for all $n = 1,2,\dots,$ and $(u_n,\lambda_n) \to (0,\mu_0)$ in E as $n \to \infty$.

Proposition 3.2. *Let* $(0,\mu_0) \in E$ *be a bifurcation point* (*from zero*) *for problem* (2.1). *Then* μ_0 *is an eigenvalue for the nonlinear eigenvalue problem* (2.7), *i.e.,* $\mathcal{A}(u) - \mu_0\mathcal{B}(u) = 0$ *with some* $u \in X \setminus \{0\}$.

The proof follows a standard pattern for this kind of result (the necessary condition for bifurcation via compactness). The reader is referred to the monograph by Fučík et al. [20, Proof of Theorem II.3.2, pp. 61–62] for details.

Definition 3.3. Let $\mu_0 \in \mathbb{R}$. We say that (∞,μ_0) is an (*asymptotic*) *bifurcation point from infinity* for problem (2.1) if there exists a sequence of pairs $\{(u_n,\lambda_n)\}_{n=1}^\infty \subset E$ such that equation (3.1) holds with $(u,\lambda) = (u_n,\lambda_n)$ for all $n = 1,2,\dots,$ and $(\|u_n\|_X,\lambda_n) \to (\infty,\mu_0)$ as $n \to \infty$.

For $u \in X$, $u \neq 0$, set $v = u/\|u\|_X^2$. Then (3.2) is equivalent to

$$\mathcal{A}(v) - \lambda\,\mathcal{B}(v) = \|v\|_X^{2(p-1)}\mathcal{H}\big(v/\|v\|_X^2;\lambda\big)\,,$$

and so the term

$$\mathcal{G}(v;\lambda) \stackrel{\text{def}}{=} \begin{cases} \|v\|_X^{2(p-1)}\mathcal{H}\big(v/\|v\|_X^2;\lambda\big) & \text{if} \quad v \neq 0\,; \\ 0 & \text{if} \quad v = 0\,, \end{cases}$$

for $\lambda \in \mathbb{R}$, represents a compact perturbation "of higher order" in the variable v in the equation

$$\mathcal{A}(v) - \lambda\,\mathcal{B}(v) = \mathcal{G}(v;\lambda)\,. \tag{3.8}$$

It follows immediately from this transformation that the pair (∞,μ_0) is a bifurcation point from infinity for (3.2) if and only if $(0,\mu_0)$ is a bifurcation point zero for (3.8). For $\mathcal{C} \subset X \times \mathbb{R}$ we define (the set) $\widetilde{\mathcal{C}}$ to be the closure in $X \times \mathbb{R}$ of the set of all pairs $(v,\mu) \in X \times \mathbb{R}$ such that $v \neq 0$ and $(v/\|v\|_X^2,\mu) \in \mathcal{C}$. Using this transformation, a necessary condition for bifurcations from infinity easily follows from Proposition 3.2.

Proposition 3.4. *Let (∞, μ_0) be a bifurcation point from infinity for problem* (2.1)*. Then μ_0 is an eigenvalue for the nonlinear eigenvalue problem* (2.7)*, i.e., $\mathcal{A}(u) - \mu_0 \mathcal{B}(u) = 0$ with some $u \in X \setminus \{0\}$.*

In our treatment below we will prove a number of bifurcation results only for bifurcations from zero and leave obvious adjustments for bifurcations from infinity to the interested reader. More precisely, the counterparts for bifurcations from infinity corresponding to bifurcations from zero are obtained by the standard transformation described in Definition 3.3 above. We will make necessary comments in case when this procedure is not straightforward.

3.1. Bifurcations from zero – main results

The closure of the set of all nontrivial solutions of problem (2.1) in $E = W_0^{1,p}(\Omega) \times \mathbb{R}$ will be denoted by $\mathcal{S}$, i.e.,

$$\mathcal{S} \stackrel{\text{def}}{=} \overline{\{(u,\lambda) \in E : \Phi(u,\lambda) = 0,\ u \neq 0\}}^{E} .$$

Our first result concerns global bifurcation of solutions $(u,\lambda) \in \mathcal{S}$ from zero at the point $(0, \mu_1) \in E$.

Proposition 3.5. *Let μ_1 be defined by formula* (2.8) *and assume that h satisfies hypothesis* ($\mathbf{H}_0$)*. Then the pair $(0, \mu_1)$ is a bifurcation point (from zero) for problem* (2.7)*. Moreover, there exists a maximal closed set $\mathcal{C} \subset \mathcal{S}$ (in the ordering by set inclusion), such that $\mathcal{C}$ is connected in E and has the following properties:*

(i) *there exists a sequence $\{(u_n, \lambda_n)\}_{n=1}^{\infty} \subset \mathcal{C}$ such that $(u_n, \lambda_n) \to (0, \mu_1)$ in E;*
(ii) *either $\mathcal{C}$ is unbounded, or else there exist another eigenvalue μ_0 of $\mathcal{A}(u) - \lambda \mathcal{B}(u) = 0$ such that $\mu_0 > \mu_1$ and another sequence $\{(u_n', \lambda_n')\}_{n=1}^{\infty} \subset \mathcal{C}$ satisfying $(u_n', \lambda_n') \to (0, \mu_0)$ in E.*

The proof of this result is postponed till Section 5.2.

Our main result below provides more details about the bifurcation from Proposition 3.5. In order to formulate and prove this result, it is convenient to introduce Dancer's notation [10]. Given any $\mu \in \mathbb{R}$ and $0 < s < \infty$, we consider an open neighborhood of $(0, \mu)$ in E defined by

$$E^s(\mu) \stackrel{\text{def}}{=} \left\{ (u,\lambda) \in E : \|u\|_{W_0^{1,p}(\Omega)} + |\lambda - \mu| < s \right\} .$$

Next, we define a functional $\ell \in W^{-1,p'}(\Omega)$ by

$$\ell(\phi) \stackrel{\text{def}}{=} \|\varphi_1\|_{L^2(\Omega)}^{-2} \int_\Omega \phi\, \varphi_1 \, \mathrm{d}x \quad \text{for all} \quad \phi \in W_0^{1,p}(\Omega) .$$

Thus, $\ell(\varphi_1) = 1$. Notice that, using the reflexivity of $W_0^{1,p}(\Omega)$, we may identify ℓ with the function $\|\varphi_1\|_{L^2(\Omega)}^{-2} \varphi_1$. Finally, for any $0 < \eta < \mu_1^{-1/p}$ we define

$$K_\eta \stackrel{\text{def}}{=} \left\{ (u,\lambda) \in E : |\ell(u)| > \eta \, \|u\|_{W_0^{1,p}(\Omega)} \right\}$$

and decompose it in a disjoint union $K_\eta = K_\eta^+ \cup K_\eta^-$ of the sets

$$K_\eta^\pm \overset{\text{def}}{=} \left\{ (u,\lambda) \in E : \pm\ell(u) > \eta\, \|u\|_{W_0^{1,p}(\Omega)} \right\}. \tag{3.9}$$

In particular, both K_η^ν ($\nu = \pm$) are convex cones, $K_\eta^- = -K_\eta^+$, and $\nu t\varphi_1 \in K_\eta^\nu$ for every number $t > 0$. The symbol $-\nu$ will denote the sign opposite to ν.

For a precise formulation of our main result, we also need the following lemma which is a variant of Rabinowitz' result [31, Lemma 1.24] adapted to the quasilinear problem (2.1) with λ in a neighborhood of the first eigenvalue μ_1. Here we introduce the (technical) constant

$$\eta_0 \overset{\text{def}}{=} \left(\frac{\gamma}{(p-1)\big[(\mu_1+1)\|B\|_{L^\infty(\Omega)} + C\big]|\Omega|_N} \right)^{1/p} \cdot \|\varphi_1\|_{L^\infty(\Omega)}^{-1} > 0\,, \tag{3.10}$$

where $\gamma > 0$ and $C > 0$ are the constants from inequalities (2.6) and (2.9), respectively.

Lemma 3.6. *For every $\eta \in (0,\eta_0)$ there exists a number S, $0 < S \leq 1$, such that*

$$\Big(\mathcal{S} \setminus \big\{(0,\mu_1)\big\}\Big) \cap \overline{E^S(\mu_1)} \subset K_\eta\,.$$

Moreover, if $(u,\lambda) \in (\mathcal{S} \setminus \{(0,\mu_1)\}) \cap \overline{E^S(\mu_1)}$ then $u = \tau(\varphi_1 + v^\top)$, where $\tau = \ell(u) \in \mathbb{R}$ and $v^\top \in C^{1,\beta'}(\overline{\Omega})$ $(0 < \beta' < \beta)$ satisfy $|\tau| > \eta\, \|u\|_{W_0^{1,p}(\Omega)}$ and $\ell(v^\top) = 0$ together with $|\lambda - \mu_1| \to 0$ and $\|v^\top\|_{C^{1,\beta'}(\overline{\Omega})} \to 0$ as $\tau \to 0$.

The proof of this result is postponed till Section 5.2.

Let $S > 0$ be the constant from Lemma 3.6. For $0 < \varepsilon \leq S$ and $\nu = \pm$ we define $\mathcal{D}_{\mu_1,\varepsilon}^\nu$ to be the component of $\overline{\{(0,\mu_1)\} \cup (\mathcal{S} \cap \overline{E_\varepsilon} \cap K_\eta^\nu)}$ containing $(0,\mu_1)$, and $\mathcal{Z}_{\mu_1,\varepsilon}^\nu$ to be the component of $\overline{\mathcal{Z}_{\mu_1} \setminus \mathcal{D}_{\mu_1,\varepsilon}^{-\nu}}$ containing $(0,\mu_1)$. Finally, we define $\mathcal{Z}_{\mu_1}^\nu$ to be the closure of $\bigcup_{0<\varepsilon\leq S} \mathcal{Z}_{\mu_1,\varepsilon}^\nu$ in X. Clearly, $\mathcal{Z}_{\mu_1}^\nu$ is connected. Thanks to the properties of S from Lemma 3.6, the definition of $\mathcal{Z}_{\mu_1}^\pm$ is independent from the choice of $\eta \in (0,\eta_0)$. Moreover, Lemma 3.6 guarantees that the union $\mathcal{Z}_{\mu_1} = \mathcal{Z}_{\mu_1}^+ \cup \mathcal{Z}_{\mu_1}^-$ coincides with the component of $\mathcal{S}$ containing the point $(0,\mu_1)$, by simple set-theoretical and topological arguments applied directly to the definition of $\mathcal{Z}_{\mu_1}^\nu$.

The following result is a close analogue of Dancer's result [10, Theorem 2, p. 1071] shown originally for abstract semilinear equations.

Theorem 3.7. *Either $\mathcal{Z}_{\mu_1}^+$ and $\mathcal{Z}_{\mu_1}^-$ are both unbounded, or else $\mathcal{Z}_{\mu_1}^+ \cap \mathcal{Z}_{\mu_1}^- \neq \{(0,\mu_1)\}$.*

Our proof of this result for the quasilinear boundary value problem (2.1) follows the same steps as does the proof for the semilinear case from [10]. Some asymptotic estimates are needed in this proof which, unlike in the semilinear case, are quite difficult to obtain due to the nonlinearity of the partial differential operator. These asymptotic estimates are our main contribution. Another difference consists in the fact that we use the Browder–Petryshyn degree instead of the more common Lerray–Schauder degree. The Browder–Petryshyn degree is better suited for a weak formulation of a quasilinear boundary value problem; see e.g., Drábek [14]. As we need the aforementioned asymptotic estimates, we postpone our proof of Theorem 3.7 until after the formulation and proof of these estimates.

3.2. Bifurcations from infinity – main results

In the preceding paragraph we considered $(0, \mu_1)$ as a bifurcation point for problem (3.8); cf., [14, Theorem 14.18]. Let us reformulate Proposition 3.5 and Theorem 3.7 in terms of bifurcation from infinity for problem (3.2) at (∞, μ_1). For simplicity, let us assume

$$h(\,\cdot\,,0;\lambda) \not\equiv 0 \quad \text{in} \quad \Omega \quad \text{for every} \quad \lambda \in \mathbb{R}\,. \tag{3.11}$$

This assumption will guarantee that there is no bifurcation from zero. Recall that, given a set $\mathcal{C} \subset X \times \mathbb{R}$, $\widetilde{\mathcal{C}}$ denotes the closure in $X \times \mathbb{R}$ of the set of all pairs $(v, \mu) \in X \times \mathbb{R}$ such that $v \neq 0$ and $(v/\|v\|_X^2, \mu) \in \mathcal{C}$.

Proposition 3.8. *Let h satisfy* $(\mathbf{H}_\infty)$ *and* (3.11). *Then the pair (∞, μ_1) is a bifurcation point from infinity for* (3.2). *Moreover, there exists a maximal* (*in the ordering by set inclusion*) *closed set $\mathcal{C} \subset X \times \mathbb{R}$, such that $\widetilde{\mathcal{C}}$ is connected in $X \times \mathbb{R}$ and the following properties hold:*

(i) *there exists a sequence $\{(u_n, \lambda_n)\}_{n=1}^\infty \subset \mathcal{C}$ such that $(\|u_n\|_X, \lambda_n) \to (\infty, \mu_1)$;*

(ii) *either $\mathcal{C}$ is unbounded in the λ-direction, or else there exists an eigenvalue μ_0 of the nonlinear eigenvalue problem* (2.7) *such that $\mu_0 > \mu_1$ and there is a sequence $\{(u_n, \lambda_n)\}_{n=1}^\infty \subset \mathcal{C}$ satisfying $(\|u_n\|_X, \lambda_n) \to (\infty, \mu_0)$.*

Remark 3.9. The assumption (3.11) implies that (3.2) cannot have a trivial solution $(u, \lambda) = (0, \lambda)$ in E and, therefore, $\mathcal{C}$ contains no sequence of pairs (u_k, λ_k) with $(u_k, \lambda_k) \to (0, \hat{\mu})$ in E for some $\hat{\mu} \in \mathbb{R}$. Hence, the statement of Proposition 3.8 follows directly from Proposition 3.5 using the transformation $u \mapsto v = u/\|u\|_X^2$.

Let S be as in Lemma 3.6. For $0 < \varepsilon \le S$ we define $\widetilde{\mathcal{D}}^{\pm}_{\mu_1,\varepsilon}$ as the component of $\overline{\{(0, \mu_1)\} \cup (\widetilde{\mathcal{S}} \cap \overline{E_\varepsilon} \cap K_\eta^{\pm})}$ containing $(0, \mu_1)$, and $\widetilde{\mathcal{Z}}^{\pm}_{\mu_1,\varepsilon}$ as the component of $\overline{\widetilde{\mathcal{Z}}_{\mu_1} \setminus \widetilde{\mathcal{D}}^{\mp}_{\mu_1,\varepsilon}}$ containing $(0, \mu_1)$. Finally, we define $\widetilde{\mathcal{Z}}^{\pm}_{\mu_1} \stackrel{\text{def}}{=} \bigcup_{0<\varepsilon\le S} \widetilde{\mathcal{Z}}^{\pm}_{\mu_1,\varepsilon}$. Recall that $\widetilde{\mathcal{S}}$ ($\widetilde{Z}^{\pm}_{\mu_1}$, respectively) denotes the closure in $E = X \times \mathbb{R}$ of the set of all pairs $(v, \lambda) \in E$ such that $v \neq 0$ and $(v/\|v\|_X^2, \mu) \in \mathcal{S}$ ($(v/\|v\|_X^2, \mu) \in Z^{\pm}_{\mu_1}$).

Theorem 3.10. *Let h satisfy* $(\mathbf{H}_\infty)$ *and* (3.11). *Then there is a pair of sets $\mathcal{Z}^+_{\mu_1}, \mathcal{Z}^-_{\mu_1} \subset \mathcal{S}$ such that $\widetilde{Z}^+_{\mu_1}$ and $\widetilde{Z}^-_{\mu_1}$ are both unbounded, or else $\widetilde{Z}^+_{\mu_1} \cap \widetilde{Z}^-_{\mu_1} \neq \{(0, \mu_1)\}$.*

4. Asymptotic estimates for λ near μ_1

Here we establish some local asymptotic estimates for λ near μ_1. We consider the energy functional

$$\mathcal{J}_\lambda(u) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega A(x, \nabla u)\,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega B(x)\,|u|^p\,\mathrm{d}x - \int_\Omega H(x, u)\,\mathrm{d}x \tag{4.1}$$

defined for $u \in W_0^{1,p}(\Omega)$ where

$$H(x,u) \stackrel{\text{def}}{=} \int_0^u h(x,t)\,\mathrm{d}t \quad \text{for} \quad x \in \Omega \quad \text{and} \quad u \in \mathbb{R}\,.$$

We assume that A and B satisfy hypotheses (**A**) and (**B**).

The critical points of $\mathcal{J}_\lambda$ coincide with the weak solutions of the boundary value problem (2.1) where $\mathbf{a}(x,\xi) = 1/p\,\partial_\xi A(x,\xi)$ and $h(x,u) = \partial_u H(x,u)$. Clearly, for $h \equiv 0$ in $\Omega \times \mathbb{R}$, the functional

$$\mathcal{J}_\lambda^{(0)}(u) = \frac{1}{p}\int_\Omega A(x,\nabla u)\,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega B(x)\,|u|^p\,\mathrm{d}x$$

is strictly convex on $W_0^{1,p}(\Omega)$ provided $\lambda \le 0$; if $\lambda > 0$ the convexity of $\mathcal{J}_\lambda^{(0)}$ is known to be lost, see Fleckinger et al. [18, Example 2, p. 148] for $1 < p < 2$ and del Pino, Elgueta and Manásevich [29, Eq. (5.26), p. 12] for $2 < p < \infty$, where such examples are constructed in an open interval $\Omega \subset \mathbb{R}^1$. However, if $u > 0$ almost everywhere in Ω, one may substitute $v = u^p$ and investigate the functional

$$\mathcal{K}_\lambda(v) \stackrel{\text{def}}{=} p\cdot\mathcal{J}_\lambda^{(0)}(v^{1/p}) = \int_\Omega A\big(x,\nabla(v^{1/p})\big)\,\mathrm{d}x - \lambda\int_\Omega B(x)\,v\,\mathrm{d}x \tag{4.2}$$

instead, which is defined on the set

$$\overset{\bullet}{V}_+ \stackrel{\text{def}}{=} \big\{v : \Omega \to (0,\infty) : v^{1/p} \in W_0^{1,p}(\Omega)\big\}\,.$$

The second summand in (4.2) being linear in the variable v, it suffices to focus on the convexity of the first one,

$$\mathcal{K}(v) \stackrel{\text{def}}{=} \int_\Omega A\big(x,\nabla(v^{1/p})\big)\,\mathrm{d}x\,, \quad v \in \overset{\bullet}{V}_+\,. \tag{4.3}$$

Notice that $\mathcal{K}$ is positively homogeneous, i.e.,

$$\mathcal{K}(tv) = t\cdot\mathcal{K}(v) \quad \text{for any number} \quad t > 0\,.$$

4.1. Convexity on the cone of positive functions

We would like to point out for future references that all results that are stated throughout this paragraph remain valid for any bounded domain $\Omega \subset \mathbb{R}^N$; the smoothness of its boundary $\partial\Omega$ required in hypothesis ($\boldsymbol{\Omega}$) is not necessary here.

It is shown in Takáč, Tello, and Ulm [37, Lemma 2.4, p. 79] that $\overset{\bullet}{V}_+$ is a convex cone, i.e., $v_0, v_1 \in \overset{\bullet}{V}_+ \Longrightarrow \alpha_0 v_0 + \alpha_1 v_1 \in \overset{\bullet}{V}_+$ for all $\alpha_0, \alpha_1 \in (0,\infty)$, and the functional $\mathcal{K} : \overset{\bullet}{V}_+ \to \mathbb{R}$ is *ray-strictly convex*, i.e., for all $v_0, v_1 \in \overset{\bullet}{V}_+$ and $\theta \in (0,1)$ we have

$$\mathcal{K}\big((1-\theta)v_0 + \theta v_1\big) \le (1-\theta)\,\mathcal{K}(v_0) + \theta\,\mathcal{K}(v_1)$$

where equality may hold only if v_0 and v_1 are colinear, i.e., $v_1 = \alpha v_0$ for some $\alpha \in (0,\infty)$. For the special case $A(x,\xi) = |\xi|^p$, $(x,\xi) \in \Omega \times \mathbb{R}^N$, this lemma is due to Díaz and Saa [12]. It has had a number of important consequences in the past; some of them are surveyed in Takáč [36, Sect. 3]. However, several of these important consequences can be obtained in a "parallel" way we adopt here, using supporting hyperplanes to the epigraph of $\mathcal{K}$. In particular, also this method yields the simplicity of the first (smallest) eigenvalue μ_1 for the Euler equation corresponding to the energy functional $\mathcal{J}_\lambda$ on $W_0^{1,p}(\Omega)$; recall that μ_1 is given by formula (2.8).

We begin with the notion of the *subdifferential* $\partial\mathcal{K}(v_0)$ of the functional $\mathcal{K} : \overset{\bullet}{V}_+ \to \mathbb{R}$ which we define as follows: Given $v_0 \in \overset{\bullet}{V}_+$, we introduce the weighted Sobolev space

$$\mathcal{D}(v_0) \overset{\text{def}}{=} \left\{ \phi : \Omega \to \mathbb{R} : \phi\, v_0^{-(p-1)/p} \in W_0^{1,p}(\Omega) \right\}$$

endowed with the natural norm

$$\|\phi\|_{\mathcal{D}(v_0)} \overset{\text{def}}{=} \left\| \phi\, v_0^{-(p-1)/p} \right\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega \left| \nabla \left(\phi\, v_0^{-(p-1)/p} \right) \right|^p \right)^{1/p} .$$

For $\phi \in \mathcal{D}(v_0)$ we set

$$\langle \partial\mathcal{K}(v_0), \phi \rangle \overset{\text{def}}{=} \int_\Omega \left\langle \mathbf{a}\big(x, \nabla(v_0^{1/p})\big), \nabla \left(\phi\, v_0^{-(p-1)/p} \right) \right\rangle \mathrm{d}x \,. \tag{4.4}$$

Combining our hypotheses on $\mathbf{a}(x,\xi) = 1/p\, \partial_\xi A(x,\xi)$ with the Hölder inequality, we conclude that the last integral is absolutely convergent and $\partial\mathcal{K}(v_0)$ is a bounded linear functional on $\mathcal{D}(v_0)$.

An easy calculation reveals that $v \in \mathcal{D}(v_0)$ whenever $v \in \overset{\bullet}{V}_+$ satisfies $v/v_0 \in L^\infty(\Omega)$.

Remark 4.1. It is easy to see that $v_0 \in \overset{\bullet}{V}_+ \cap C^0(\Omega)$ implies $\partial\mathcal{K}(v_0) \in \mathcal{D}'(\Omega)$. More precisely, $\partial\mathcal{K}(v_0)$ belongs to the dual space of the Fréchet space $C_c^1(\Omega)$ and thus to $\mathcal{D}'(\Omega)$. Employing formal integration by parts, we may write in $\mathcal{D}'(\Omega)$

$$\partial\mathcal{K}(v_0) = -\, v_0^{-(p-1)/p} \cdot \operatorname{div} \left(\mathbf{a}\big(x, \nabla(v_0^{1/p})\big) \right) . \tag{4.5}$$

In addition, the expression $\langle \partial\mathcal{K}(v_0), \phi \rangle$ gives the directional derivative of the functional $\mathcal{K}$ at v_0 in direction $\phi \in C_c^1(\Omega)$.

The conclusion of the lemma below is very close to being equivalent to the claim that $\mathcal{K} : \overset{\bullet}{V}_+ \to \mathbb{R}$ is ray-strictly convex. Its consequences are similar; see e.g., Picone's identity for the p-Laplacian used in Allegretto and Huang [2, Theorem 2.1, p. 821].

Lemma 4.2. *For any pair $v_0, v \in \overset{\bullet}{V}_+$ with $v \in \mathcal{D}(v_0)$ we have*

$$\mathcal{K}(v) \geq \langle \partial\mathcal{K}(v_0), v \rangle \,. \tag{4.6}$$

Equality holds if and only if the functions v_0 and v are colinear, i.e., $v = \alpha v_0$ for some $\alpha \in (0,\infty)$. In particular, we may take any pair $v_0, v \in \overset{\bullet}{V}_+$ satisfying $v/v_0 \in L^\infty(\Omega)$.

Proof. The function $A(x,\,\cdot\,)$ being strictly convex on $\mathbb{R}^N$, by the ellipticity condition (2.3) (hypothesis (**A**)), we observe that the inequality

$$A(x,\xi) - A(x,\xi_0) > \langle \partial_\xi A(x,\xi_0), \xi - \xi_0 \rangle$$

holds for all $\xi, \xi_0 \in \mathbb{R}^N$ with $\xi \neq \xi_0$. Using the identities $\mathbf{a}(x,\xi_0) = 1/p\, \partial_\xi A(x,\xi_0)$ and $A(x,\xi_0) = \langle \mathbf{a}(x,\xi_0), \xi_0 \rangle$, we can rewrite it equivalently as

$$A(x,\xi) > p \left\langle \mathbf{a}(x,\xi_0), \xi - \frac{p-1}{p}\,\xi_0 \right\rangle \quad \text{for} \quad \xi \neq \xi_0\,. \tag{4.7}$$

Now let $v, v_0 \in (0,\infty)$ and $\xi, \xi_0 \in \mathbb{R}^N$. Substituting the fractions ξ/v and ξ_0/v_0 for ξ and ξ_0, respectively, from (4.7) we derive

$$A\left(x, \frac{\xi}{v}\right) \geq p \left\langle \mathbf{a}\left(x, \frac{\xi_0}{v_0}\right), \frac{\xi}{v} - \frac{p-1}{p}\,\frac{\xi_0}{v_0} \right\rangle$$

where equality holds if and only if $\xi/v = \xi_0/v_0$. Finally, we apply the positive p-homogeneity hypothesis (2.2) to get

$$A\left(x, \frac{\xi}{p\, v^{(p-1)/p}}\right) \geq \frac{1}{v_0^{(p-1)/p}} \left\langle \mathbf{a}\left(x, \frac{\xi_0}{p\, v_0^{(p-1)/p}}\right), \xi - \frac{p-1}{p}\,\frac{v}{v_0}\,\xi_0 \right\rangle . \tag{4.8}$$

Equality holds if and only if $\xi/v = \xi_0/v_0$.

To conclude our proof, we use the identities $\nabla(v^{1/p}) = p^{-1} v^{-(p-1)/p} \nabla v$ and

$$\nabla\left(\frac{v}{v_0^{(p-1)/p}}\right) = \frac{1}{v_0^{(p-1)/p}} \left(\nabla v - \frac{p-1}{p}\,\frac{v}{v_0}\,\nabla v_0\right)$$

for $v_0, v \in \overset{\bullet}{V}_+$. If also $v \in \mathcal{D}(v_0)$, we may take $\xi = \nabla v$ and $\xi_0 = \nabla v_0$ a.e. in Ω, substitute them into inequality (4.8), and then integrate the result over Ω, thus arriving at

$$\int_\Omega A\big(x, \nabla(v^{1/p})\big)\, \mathrm{d}x \geq \int_\Omega \left\langle \mathbf{a}\left(x, \nabla(v_0^{1/p})\right), \nabla\left(v\, v_0^{-(p-1)/p}\right) \right\rangle \mathrm{d}x\,. \tag{4.9}$$

Equality holds if and only if $v^{-1}\nabla v = v_0^{-1}\nabla v_0$ a.e. in Ω. The latter equality is equivalent to $v/v_0 \equiv \text{const}$ in Ω. Clearly, (4.6) and (4.9) are the same.

The lemma is proved. $\square$

Remark 4.3. Lemma 4.2 implies immediately that the first eigenvalue μ_1 given by the Rayleigh quotient (2.8) must be simple; cf. Takáč, Tello, and Ulm [37, proof of Theorem 2.6, p. 81].

Indeed, since the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact by Rellich's theorem, the infimum in (2.8) is attained and satisfies $0 < \mu_1 < \infty$. Now write $u = u^+ - u^-$ where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, respectively, denote the positive and negative parts of a real-valued function $u \in W_0^{1,p}(\Omega)$. We have $u^\pm \in W_0^{1,p}(\Omega)$, see Gilbarg and Trudinger [22, Theorem 7.8, p. 153]. More precisely, also $\nabla u^+ = \nabla u$ almost everywhere in $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ and $\nabla u^+ = \mathbf{0}$ almost everywhere in $\Omega \setminus \Omega_+$. The corresponding result holds for u^-

and $-u$ as well. It follows from the proof of Theorem 2.6 (p. 81) in [37] that if a minimizer $u \in W_0^{1,p}(\Omega)$ for μ_1 in (2.8) changes sign in the set $\{x \in \Omega : \, B(x) > 0\}$, then also both functions $u^+ / \int_\Omega B(x)(u^+)^p \, dx$ and $u^- / \int_\Omega B(x)(u^-)^p \, dx$ are minimizers for μ_1. We apply the strong maximum principle (due to Tolksdorf [38, Prop. 3.2.1 and 3.2.2, p. 801] and Vázquez [40, Theorem 5, p. 200]) to conclude that any minimizer $u \in W_0^{1,p}(\Omega)$ for μ_1 is either almost everywhere positive or else almost everywhere negative in Ω; we may assume $u > 0$ a.e. in Ω. Hence, $\mathcal{K}(u^p) = \mu_1$. Denote by φ_1 any such minimizer for μ_1 (with $\varphi_1 > 0$ a.e. in Ω). From (4.5) we get $\partial\mathcal{K}(\varphi_1^p) = \mu_1 \, B(\,\cdot\,)$ in Ω. We conclude that equality must hold in $\mathcal{K}(u^p) \geq \langle \partial\mathcal{K}(\varphi_1^p), u^p \rangle$, which forces $u/\varphi_1 \equiv \mathrm{const}$ in Ω, by Lemma 4.2.

Furthermore, if the boundary $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, then $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,\alpha)$, by Proposition A.1. Finally, if Ω satisfies also an interior sphere condition at a point $x_0 \in \partial\Omega$, then $(\partial\varphi_1/\partial\nu)(x_0) < 0$, by the Hopf maximum principle (see [38, Prop. 3.2.1 and 3.2.2, p. 801] or [40, Theorem 5, p. 200]). We will need these facts throughout the rest of this work.

Remark 4.4. Lemma 4.2 implies also the *monotonicity* of the subdifferential $\partial\mathcal{K}$ in the following sense: If $u, v \in \overset{\bullet}{V}_+$ satisfy $u/v,\ v/u \in L^\infty(\Omega)$, then one has

$$\big\langle \partial\mathcal{K}(u) - \partial\mathcal{K}(v), u - v \big\rangle \geq 0 \tag{4.10}$$

where equality holds if and only if u and v are colinear. Indeed, using inequality (4.6) we get

$$\begin{aligned} \big\langle \partial\mathcal{K}(u) - \partial\mathcal{K}(v), u \big\rangle &= \mathcal{K}(u) - \big\langle \partial\mathcal{K}(v), u \big\rangle \geq 0\,, \\ \big\langle \partial\mathcal{K}(v) - \partial\mathcal{K}(u), v \big\rangle &= \mathcal{K}(v) - \big\langle \partial\mathcal{K}(u), v \big\rangle \geq 0\,. \end{aligned}$$

We obtain (4.10) by adding these two inequalities.

Notice that (4.10) is the well-known inequality of Díaz and Saa established in [12] for the special case $A(x,\xi) = |\xi|^p$, $(x,\xi) \in \Omega \times \mathbb{R}^N$: Let $u_0, u_1 \in W_0^{1,p}(\Omega)$ be such that $u_0 > 0$ and $u_1 > 0$ in Ω and both u_0/u_1 and u_1/u_0 are in $L^\infty(\Omega)$. Then we have

$$\int_\Omega \left(-\frac{\operatorname{div}\big(\mathbf{a}(x,\nabla u_0)\big)}{u_0^{p-1}} + \frac{\operatorname{div}\big(\mathbf{a}(x,\nabla u_1)\big)}{u_1^{p-1}} \right) (u_0^p - u_1^p)\, dx \geq 0 \tag{4.11}$$

where equality holds if and only if $v_1/v_0 \equiv \mathrm{const}$ in Ω.

4.2. Bifurcations from zero for λ near μ_1

Now, assuming hypothesis $(\mathbf{H}_0)$, let us consider problem (2.1) again. By Lemma B.2 in Appendix B.2, for any $\lambda \in \mathbb{R}$ sufficiently close to μ_1, every weak solution $u \in W_0^{1,p}(\Omega)$ of problem (2.1) with a sufficiently small norm $\|u\|_{W_0^{1,p}(\Omega)}$ takes the form

$$u = t(\varphi_1 + v^\top) \quad \text{where} \quad t \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad v^\top \in C^1(\overline{\Omega})\,, \tag{4.12}$$

and $v^\top$ satisfies $\langle v^\top, \varphi_1 \rangle = 0$ together with $|v^\top| \leq 1/2\varphi_1$ in Ω. Moreover, one has the asymptotic formulas $\lambda \to \mu_1$ and $\|v^\top\|_{C^1(\overline{\Omega})} \to 0$ as $t \to 0$ $(t \neq 0)$. In particular, we work only with such solutions $u \in C^1(\overline{\Omega})$ of problem (2.1) that are either positive or negative throughout Ω. More precisely, we have $t^{-1}u \geq 1/2\varphi_1$ in Ω.

We need a more accurate estimate on the difference $\lambda - \mu_1$ as $t \to 0$ which we derive from Lemma 4.2 next.

Proposition 4.5. *Let hypothesis* $(\mathbf{H}_0)$ *be satisfied. Then, for any* $t \in \mathbb{R} \setminus \{0\}$ *with* $|t|$ *small enough, we have*

$$\begin{aligned} -\big(|t|^{p-2}t\big)^{-1} & \left(\int_\Omega B\,(\varphi_1 + v^\top)^p \,\mathrm{d}x \right)^{-1} \int_\Omega h\big(x, t(\varphi_1 + v^\top);\lambda\big)\,(\varphi_1 + v^\top)\,\mathrm{d}x \\ &\leq \lambda - \mu_1 \\ &\leq -\big(|t|^{p-2}t\big)^{-1} \int_\Omega h\big(x, t(\varphi_1 + v^\top);\lambda\big) \left(1 + \frac{v^\top}{\varphi_1}\right)^{-(p-1)} \varphi_1 \,\mathrm{d}x\,, \end{aligned} \tag{4.13}$$

where $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ *as* $t \to 0$.

Proof. We treat the case $t > 0$ only. The case $t < 0$ is analogous; one has to replace u and $h(x,u;\lambda)$ by $-u$ and $-h(x,u;\lambda)$, respectively.

The lower bound on $\lambda - \mu_1$ is obtained as follows. The Euler equation for the minimizers in formula (2.8) for μ_1 combined with the normalization of φ_1 yield

$$\big\langle \partial\mathcal{K}(\varphi_1^p), \phi \big\rangle = \mu_1 \int_\Omega B\,\phi\,\mathrm{d}x \quad \text{for every} \quad \phi \in \mathcal{D}(\varphi_1)\,. \tag{4.14}$$

Multiplying equation (2.1) by u and integrating over Ω we get

$$\mathcal{K}(u^p) = \lambda \int_\Omega B\,u^p\,\mathrm{d}x + \int_\Omega h(x,u;\lambda)\,u\,\mathrm{d}x\,. \tag{4.15}$$

From inequality (4.6) we obtain $\mathcal{K}(u^p) \geq \langle \partial\mathcal{K}(\varphi_1^p), u^p \rangle$. Combining this inequality with (4.14) and (4.15) we arrive at

$$\lambda \int_\Omega B\,u^p\,\mathrm{d}x + \int_\Omega h(x,u;\lambda)\,u\,\mathrm{d}x \geq \mu_1 \int_\Omega B\,u^p\,\mathrm{d}x\,.$$

Finally, using (4.12), we get

$$\begin{aligned} (\lambda - \mu_1) & \int_\Omega B\,(\varphi_1 + v^\top)^p\,\mathrm{d}x \\ &\geq\; -\,t^{-(p-1)} \int_\Omega h\big(x, t(\varphi_1 + v^\top);\lambda\big)\,(\varphi_1 + v^\top)\,\mathrm{d}x \end{aligned} \tag{4.16}$$

for $t > 0$ small enough.

Now we estimate $\lambda - \mu_1$ from above. From equation (2.1) we deduce

$$\big\langle \partial\mathcal{K}(u^p), \phi \big\rangle = \lambda \int_\Omega B\,\phi\,\mathrm{d}x + \int_\Omega h(x,u;\lambda)\,u^{-(p-1)}\,\phi\,\mathrm{d}x \tag{4.17}$$

for every $\phi \in \mathcal{D}(\varphi_1)$. Formula (2.8) for μ_1 yields

$$\mathcal{K}(\varphi_1^p) = \mu_1 \int_\Omega B\,\varphi_1^p\,\mathrm{d}x = \mu_1\,. \tag{4.18}$$

From inequality (4.6) we obtain $\mathcal{K}(\varphi_1^p) \geq \langle \partial\mathcal{K}(u^p), \varphi_1^p \rangle$. We combine this inequality with (4.17) and (4.18) to get

$$(\lambda - \mu_1) \int_\Omega B\, \varphi_1^p \,\mathrm{d}x + \int_\Omega h(x,u;\lambda)\, u^{-(p-1)}\, \varphi_1^p \,\mathrm{d}x \leq 0\,.$$

Finally, using (4.12), we arrive at

$$\begin{aligned} \lambda - \mu_1 &= (\lambda - \mu_1) \int_\Omega B\, \varphi_1^p \,\mathrm{d}x \\ &\leq\; - t^{-(p-1)} \int_\Omega h\big(x, t(\varphi_1 + v^\top); \lambda\big) \left(1 + \frac{v^\top}{\varphi_1}\right)^{-(p-1)} \varphi_1 \,\mathrm{d}x \end{aligned} \tag{4.19}$$

for $t > 0$ small enough.

Hence, (4.13) is a combination of inequalities (4.16) and (4.19). $\square$

Our hypothesis $(\mathbf{H}_0)$ on $h(x,u;\lambda)$ implies the following asymptotic behavior of the integrals

$$\int_\Omega h\big(x, t(\varphi_1 + \phi); \lambda\big)\, (\varphi_1 + \phi) \,\mathrm{d}x \tag{4.20}$$

and

$$\int_\Omega h\big(x, t(\varphi_1 + \phi); \lambda\big) \left(1 + \frac{\phi}{\varphi_1}\right)^{-(p-1)} \varphi_1 \,\mathrm{d}x \tag{4.21}$$

from (4.13), for $|t| \to 0$, $t \neq 0$, and $\|\phi/\varphi_1\|_{L^\infty(\Omega)}$ small enough ($\phi \in C^1(\overline{\Omega})$). Given a number $0 < \eta \leq 1/2$, for each $t \in \mathbb{R}$ we define the expressions

$$\Theta_\eta^{(1)}(t) \stackrel{\text{def}}{=} \sup_{\substack{|\phi| \leq \eta\varphi_1 \\ |\lambda - \mu_1| \leq 1}} \left| \int_\Omega h\big(x, t(\varphi_1 + \phi); \lambda\big)\, (\varphi_1 + \phi) \,\mathrm{d}x \right| ; \tag{4.22}$$

$$\Theta_\eta^{(2)}(t) \stackrel{\text{def}}{=} \sup_{\substack{|\phi| \leq \eta\varphi_1 \\ |\lambda - \mu_1| \leq 1}} \left| \int_\Omega h\big(x, t(\varphi_1 + \phi); \lambda\big) \left(1 + \frac{\phi}{\varphi_1}\right)^{-(p-1)} \varphi_1 \,\mathrm{d}x \right| , \tag{4.23}$$

where both suprema are taken over all functions $\phi \in C^1(\overline{\Omega})$ that satisfy $|\phi| \leq \eta\varphi_1$ in Ω. Clearly, we have

$$\Theta_\eta^{(1)}(t) \leq (1 + \eta)\, \Theta_\eta(t) \leq \frac{3}{2}\, \Theta_\eta(t)\,; \tag{4.24}$$

$$\Theta_\eta^{(2)}(t) \leq (1 - \eta)^{-(p-1)}\, \Theta_\eta(t) \leq 2^{p-1}\, \Theta_\eta(t)\,, \tag{4.25}$$

where we have denoted

$$\Theta_\eta(t) \stackrel{\text{def}}{=} \int_\Omega \sup_{\substack{|s| \leq \eta \\ |\lambda - \mu_1| \leq 1}} \Big| h\big(x, t(1+s)\varphi_1; \lambda\big) \Big| \cdot \varphi_1 \,\mathrm{d}x\,. \tag{4.26}$$

Lemma 4.6. *Let hypothesis* $(\mathbf{H}_0)$ *be satisfied. Then, given any* $0 < \eta \leq 1/2$, *we have* $\Theta_\eta(t)/|t|^{p-1} \to 0$ *as* $|t| \to 0$. *In particular, also*

$$\Theta_\eta^{(1)}(t)/|t|^{p-1} \to 0 \quad \textit{and} \quad \Theta_\eta^{(2)}(t)/|t|^{p-1} \to 0 \quad \textit{as} \quad |t| \to 0\,.$$

Proof. We estimate

$$\Theta_\eta(t) \le \Theta_{1/2}(t) = \int_\Omega \sup_{\substack{|s|\le 1/2 \\ |\lambda-\mu_1|\le 1}} \left|h\big(x, t(1+s)\varphi_1; \lambda\big)\right| \cdot \varphi_1 \,\mathrm{d}x \tag{4.27}$$

where

$$\sup_{\substack{|s|\le 1/2 \\ |\lambda-\mu_1|\le 1}} \left|h\big(x, t(1+s)\varphi_1; \lambda\big)\right| \le C|t|^{p-1} \left(\frac{3}{2}\varphi_1\right)^{p-1} \quad \text{for a.e.} \quad x \in \Omega\,,$$

by (2.9). It follows that

$$|t|^{-(p-1)} \cdot \sup_{\substack{|s|\le 1/2 \\ |\lambda-\mu_1|\le 1}} \left|h\big(x, t(1+s)\varphi_1; \lambda\big)\right| \le C \left(\frac{3}{2}\varphi_1\right)^{p-1}$$

for a.e. $x \in \Omega$ and $0 < |t| \le 1$. Now we can apply the Lebesgue dominated convergence theorem to the integral in (4.27) to obtain $\Theta_\eta(t)/|t|^{p-1} \to 0$ as $|t| \to 0$, by (2.10).

The remaining two claims now follow from inequalities (4.24) and (4.25), respectively. $\square$

Given any function $u \in L^p(\Omega)$, for every $n = 1, 2, \dots$ we replace the reaction function $h(x,u)$ by the expression

$$\Big[h_n\big(u(\,\cdot\,);\lambda\big)\Big](x) \stackrel{\text{def}}{=} h\big(x, u(x);\lambda\big) + R_n(|t|)\varrho(|t|)\, B(x)\, \varphi_1(x)^{p-1}\,, \quad x \in \Omega\,, \tag{4.28}$$

where $t = \|\varphi_1\|^{-2}_{L^2(\Omega)} \int_\Omega u\,\varphi_1 \,\mathrm{d}x$, and $\varrho : \mathbb{R}_+ \to (0, +\infty)$ and $R_n : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions with the following properties:

(i) For every $0 < r \le 1$ we have

$$\varrho(r) > 2\left(\frac{1+\eta}{1-\eta}\right)^{\max\{1,p-1\}} \sup_{0<|t|\le r} \frac{\Theta_\eta(t)}{|t|^{p-1}} \tag{4.29}$$

and $\varrho(r) \to 0$ as $r \to 0$.

(ii) We require $R_n(r) = r^{p-1}$ if $0 \le r \le 1/(2n)$, $R_n(r)$ is monotone decreasing for $1/(2n) \le r \le 1/n$, and $R_n(r) = 0$ if $1/n \le r < \infty$.

The following lemma guarantees that the integrals in (4.13), with $[h_n(u;\lambda)](x)$ in place of $h(x,u;\lambda)$, are positive.

Lemma 4.7. *Assume that hypothesis* ($\mathbf{H}_0$) *is satisfied. Let* $0 < \eta \le 1/2$ *and* $n \in \mathbb{N}$ *be arbitrary. Then, for every* $u \in C^1(\overline{\Omega})$ *such that* $u = t(\varphi_1 + \phi)$*, where* $0 < |t| \le 1/(2n)$*,* $\int_\Omega \phi\,\varphi_1 \,\mathrm{d}x = 0$*, and* $\|\phi/\varphi_1\|_{L^\infty(\Omega)} \le \eta$*, we have*

$$\int_\Omega h_n\big(x, t(\varphi_1+\phi);\lambda\big)\,(\varphi_1+\phi)\,\mathrm{d}x \ge \frac{1}{2}(1-\eta)\,|t|^{p-1}\,\varrho(|t|) > 0 \tag{4.30}$$

and

$$\int_\Omega h_n\big(x, t(\varphi_1+\phi);\lambda\big)\left(1+\frac{\phi}{\varphi_1}\right)^{-(p-1)}\varphi_1\,\mathrm{d}x \\ \geq \frac{1}{2}(1+\eta)^{-(p-1)}\,|t|^{p-1}\,\varrho(|t|) > 0\,. \quad (4.31)$$

Proof. Recall that $0 < |t| \leq 1/(2n)$ implies $R_n(|t|) = |t|^{p-1}$. We combine (4.24) with (4.29) to get

$$\begin{aligned}\int_\Omega & h_n\big(x, t(\varphi_1+\phi);\lambda\big)\,(\varphi_1+\phi)\,\mathrm{d}x \\ &\geq -\Theta_\eta^{(1)}(t) + R_n(|t|)\,\varrho(|t|)\int_\Omega B\,\varphi_1^{p-1}\,(\varphi_1+\phi)\,\mathrm{d}x \\ &\geq -(1+\eta)\,\Theta_\eta(t) + (1-\eta)\,|t|^{p-1}\,\varrho(|t|) \geq \frac{1}{2}(1-\eta)\,|t|^{p-1}\,\varrho(|t|) > 0\,.\end{aligned}$$

Similarly, a combination of (4.25) with (4.29) yields

$$\begin{aligned}\int_\Omega & h_n\big(x, t(\varphi_1+\phi);\lambda\big)\left(1+\frac{\phi}{\varphi_1}\right)^{-(p-1)}\varphi_1\,\mathrm{d}x \\ &\geq -\Theta_\eta^{(2)}(t) + R_n(|t|)\,\varrho(|t|)\int_\Omega B\,\varphi_1^{p}\left(1+\frac{\phi}{\varphi_1}\right)^{-(p-1)}\mathrm{d}x \\ &\geq -(1-\eta)^{-(p-1)}\,\Theta_\eta(t) + (1+\eta)^{-(p-1)}\,|t|^{p-1}\,\varrho(|t|) \\ &\geq \frac{1}{2}(1+\eta)^{-(p-1)}\,|t|^{p-1}\,\varrho(|t|) > 0\,.\end{aligned}$$

The lemma is proved. □

We finish this section by inserting the integrals from Lemma 4.7 into Proposition 4.5. Thus, the boundary value problem they relate to reads

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x,\nabla u)\big) = \lambda\,B(x)\,|u|^{p-2}u + \big[h_n(u;\lambda)\big](x) \quad \text{in} \quad \Omega\,; \\ u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.32}$$

where $[h_n(u;\lambda)](x)$ has been defined in (4.28). In analogy with (4.12), we decompose a solution of (4.32) with $\|u\|_{L^\infty(\Omega)}$ sufficiently small as

$$u = t(\varphi_1 + v^\top) \quad \text{where} \quad t \in \mathbb{R}\setminus\{0\} \quad \text{and} \quad v^\top \in C^1(\overline{\Omega})\,, \tag{4.33}$$

and $v^\top$ satisfies $\langle v^\top, \varphi_1\rangle = 0$ together with $|v^\top| \leq 1/2\varphi_1$ in Ω. We insert estimates (4.30) and (4.31) into (4.13) in order to obtain the following asymptotic formulas for $\lambda - \mu_1$. Again, we work only with such solutions $u \in C^1(\overline{\Omega})$ of problem (4.32) that are either positive or negative throughout Ω.

Corollary 4.8. *Let hypothesis* $(\mathbf{H}_0)$ *be satisfied. Then, for any* $t \in \mathbb{R} \setminus \{0\}$ *with* $|t|$ *small enough, we have*

$$\lambda - \mu_1 \geq \frac{1}{2}(1+\eta)^{-p}(1-\eta)\,\varrho(-t) > 0 \quad \text{if} \quad t < 0\,; \tag{4.34}$$

$$\lambda - \mu_1 \leq -\frac{1}{2}(1+\eta)^{-(p-1)}\,\varrho(t) < 0 \quad \text{if} \quad t > 0 \tag{4.35}$$

together with $\|v_n^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ *as* $t \to 0$.

In the following lemma, $\Theta^{(i)}$ with $i = 1, 2$ are given by

$$\Theta^{(1)}(t) \stackrel{\text{def}}{=} \int_\Omega h\big(x, t(\varphi_1 + v^\top)\big)\,(\varphi_1 + v^\top)\,\mathrm{d}x \tag{4.36}$$

and

$$\Theta^{(2)}(t) \stackrel{\text{def}}{=} \int_\Omega h\big(x, t(\varphi_1 + v^\top)\big)\left(1 + \frac{v^\top}{\varphi_1}\right)^{-(p-1)} \varphi_1\,\mathrm{d}x\,, \tag{4.37}$$

respectively.

Lemma 4.9. *Let hypotheses* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_0')$ *be satisfied. Then we have*

$$\frac{\Theta^{(i)}(t)}{g_0(t)} \to \int_\Omega f_{0\pm}\varphi_1\,\mathrm{d}x \quad \text{as} \quad t \to 0\pm\,, \tag{4.38}$$

and, in particular,

$$\frac{\Theta^{(i)}(t)}{|t|^{p-1}} \to 0 \quad \text{as} \quad |t| \to 0\,, \quad |t| > 0\,. \tag{4.39}$$

Proof. For $t \in \mathbb{R} \setminus \{0\}$ we compute

$$\begin{aligned} \frac{\Theta^{(1)}(t)}{g_0(t)} &= \int_\Omega \frac{h\big(x, t(\varphi_1 + v^\top)\big)}{g_0(t)}(\varphi_1 + v^\top)\,\mathrm{d}x \\ &= \int_\Omega \frac{h\big(x, t(\varphi_1 + v^\top)\big)}{g_0\big(t(1 + \frac{v^\top}{\varphi_1})\big)} \cdot (\varphi_1 + v^\top) \cdot \frac{g_0\big(t(1 + \frac{v^\top}{\varphi_1})\big)}{g_0(t)}\,\mathrm{d}x \\ &\to \int_\Omega f_{0\pm}\varphi_1\,\mathrm{d}x \end{aligned} \tag{4.40}$$

by the Lebesgue dominated convergence theorem which makes use of (2.16) and (2.17) combined with Remark 2.4 and $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ as $|t| \to 0$. The case of $\Theta^{(2)}$ is analogous. □

4.3. Bifurcations from infinity for λ near μ_1

Given a Banach space X ($X = W_0^{1,p}(\Omega)$ in our case), we apply a standard method to transform bifurcations from infinity to bifurcations from zero using the transformation $\widetilde{\cdot} : u \mapsto \widetilde{u} \stackrel{\text{def}}{=} \|u\|_X^{-2} u$ which maps bijectively $\{u \in X : 1/r < \|u\|_X < \infty\}$ onto $\{\widetilde{u} \in X : 0 < \|\widetilde{u}\|_X < r\}$, for $0 < r < \infty$ small enough.

More precisely, in our case we take $r > 0$ small enough and thus obtain $u = \tau(\varphi_1 + v^\top)$ with $|\tau| \to \infty$ and $\|v^\top\|_{C^1(\overline{\Omega})} \to 0$ as $r \to 0$. This yields $\widetilde{u} = t(\varphi_1 + v^\top)$ with

$$t = \tau\|u\|_X^2 = \tau^{-1}\|\varphi_1 + v^\top\|_X^{-2} = \tau^{-1}\big(\|\varphi_1\|_X^{-2} + o(r)\big) \quad \text{as} \quad r \to 0\,.$$

This means that it suffices to replace the scalar $\tau \in \mathbb{R}\setminus\{0\}$ in $u = \tau(\varphi_1 + v^\top)$, $|\tau| \to \infty$ as $r \to 0$, by $t \in \mathbb{R}\setminus\{0\}$ from $\widetilde{u} = t(\varphi_1 + v^\top)$, $t \to 0$ as $r \to 0$. Consequently, we can easily reformulate all our results from the previous section (Section 4.2) as follows.

Proposition 4.10. *Let hypothesis* $(\mathbf{H}_\infty)$ *be satisfied. Then, for any* $\tau \in \mathbb{R}\setminus\{0\}$ *with* $|\tau|$ *large enough, we have*

$$\begin{aligned} -|\tau|^{p-2}\tau \left(\int_\Omega B\,(\varphi_1 + v^\top)^p\,\mathrm{d}x\right)^{-1} &\int_\Omega h\big(x, \tau(\varphi_1 + v^\top); \lambda\big)\,(\varphi_1 + v^\top)\,\mathrm{d}x \\ &\le \lambda - \mu_1 \qquad (4.41)\\ &\le -|\tau|^{p-2}\tau \int_\Omega h\big(x, \tau(\varphi_1 + v^\top); \lambda\big) \left(1 + \frac{v^\top}{\varphi_1}\right)^{-(p-1)} \varphi_1\,\mathrm{d}x\,, \end{aligned}$$

where $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ *as* $|\tau| \to \infty$.

Our hypothesis $(\mathbf{H}_\infty)$ on $h(x,u;\lambda)$ implies the following asymptotic behavior of the integrals (4.20) and (4.21) from (4.41), for $|\tau|$ large enough ($\tau \in \mathbb{R}$) and $\|\phi/\varphi_1\|_{L^\infty(\Omega)}$ small enough ($\phi \in C^1(\overline{\Omega})$).

Lemma 4.11. *Let hypothesis* $(\mathbf{H}_\infty)$ *be satisfied. Then, given any* $0 < \eta \le 1/2$, *we have* $\Theta_\eta(\tau)/|\tau|^{p-1} \to 0$ *as* $|\tau| \to \infty$. *In particular, also*

$$\Theta_\eta^{(1)}(\tau)/|\tau|^{p-1} \to 0 \quad \text{and} \quad \Theta_\eta^{(2)}(\tau)/|\tau|^{p-1} \to 0 \quad \text{as} \quad |\tau| \to \infty\,.$$

Given any function $u \in L^p(\Omega)$, for every $n = 1, 2, \dots$ next we replace the reaction function $h(x,u;\lambda)$ by the expression

$$\begin{aligned} \Big[h_n\big(u(\,\cdot\,);\lambda\big)\Big](x) &\stackrel{\text{def}}{=} h\big(x, u(x); \lambda\big) \\ &\quad + R_n(|\tau|)\,\varrho(|\tau|)\,B(x)\,\varphi_1(x)^{p-1}\,, \quad x \in \Omega\,, \qquad (4.42) \end{aligned}$$

where $\tau = \|\varphi_1\|_{L^2(\Omega)}^{-2} \int_\Omega u\,\varphi_1\,\mathrm{d}x$, and $\varrho : \mathbb{R}_+ \to (0,\infty)$ and $R_n : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions with the following properties:

(i) For every $1 \le r < \infty$ we have

$$\varrho(r) > 2\left(\frac{1+\eta}{1-\eta}\right)^{\max\{1,\,p-1\}} \sup_{|\tau| \ge r} \frac{\Theta_\eta(\tau)}{|\tau|^{p-1}} \qquad (4.43)$$

and $\varrho(r) \to 0$ as $r \to +\infty$.

(ii) We require $R_n(r) = 0$ if $0 \le r \le n$, $R_n(r)$ is monotone increasing for $n \le r \le 2n$, and $R_n(r) = r^{p-1}$ if $2n \le r < \infty$.

The following lemma guarantees that the integrals in (4.41), with $[h_n(u;\lambda)](x)$ in place of $h(x,u;\lambda)$, are positive.

Lemma 4.12. *Assume that hypothesis* $(\mathbf{H}_\infty)$ *is satisfied. Let* $0 < \eta \le 1/2$ *and* $n \in \mathbb{N}$ *be arbitrary. Then for every* $u \in C^1(\overline{\Omega})$ *such that* $u = \tau(\varphi_1 + \phi)$, *where* $|\tau| \ge 2n$, $\int_\Omega \phi\,\varphi_1\,\mathrm{d}x = 0$, *and* $\|\phi/\varphi_1\|_{L^\infty(\Omega)} \le \eta$, *we have*

$$\int_\Omega h_n\big(x, \tau(\varphi_1+\phi);\lambda\big)\,(\varphi_1+\phi)\,\mathrm{d}x \ge \frac{1}{2}(1-\eta)\,|\tau|^{p-1}\,\varrho\big(|\tau|\big) > 0 \tag{4.44}$$

and

$$\int_\Omega h_n\big(x, \tau(\varphi_1+\phi);\lambda\big)\left(1+\frac{\phi}{\varphi_1}\right)^{-(p-1)}\varphi_1\,\mathrm{d}x \ge \frac{1}{2}(1+\eta)^{-(p-1)}\,|\tau|^{p-1}\,\varrho(|\tau|) > 0\,. \tag{4.45}$$

We finish this section by inserting the integrals from Lemma 4.12 into Proposition 4.10. Thus, the boundary value problem they relate to reads

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x,\nabla u)\big) = \lambda\,B(x)\,|u|^{p-2}u + \big[h_n(u;\lambda)\big](x) \quad \text{in} \quad \Omega\,; \\ u = 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.46}$$

where $[h_n(u;\lambda)](x)$ has been defined in (4.42). In analogy with (4.12), we decompose a solution of (4.46) with $\|u\|_{L^\infty}(\Omega)$ sufficiently large as

$$u = \tau(\varphi_1 + v^\top) \quad \text{where} \quad \tau \in \mathbb{R}\setminus\{0\} \quad \text{and} \quad v^\top \in C^1(\overline{\Omega})\,, \tag{4.47}$$

and $v^\top$ satisfies $\langle v^\top, \varphi_1\rangle = 0$ together with $|v^\top| \le 1/2\varphi_1$ in Ω. We insert estimates (4.44) and (4.45) into (4.41) in order to obtain the following asymptotic formulas for $\lambda - \mu_1$. Again, we work only with such solutions $u \in C^1(\overline{\Omega})$ of problem (4.46) that are either positive or negative throughout Ω.

Corollary 4.13. *Let hypothesis* $(\mathbf{H}_\infty)$ *be satisfied. Then, for any* $\tau \in \mathbb{R}\setminus\{0\}$ *with* $|\tau|$ *large enough, we have*

$$\lambda - \mu_1 \ge \frac{1}{2}(1+\eta)^{-p}(1-\eta)\,\varrho(-\tau) > 0 \quad \text{if} \quad \tau < 0\,; \tag{4.48}$$

$$\lambda - \mu_1 \le -\frac{1}{2}(1+\eta)^{-(p-1)}\,\varrho(\tau) < 0 \quad \text{if} \quad \tau > 0\,, \tag{4.49}$$

together with $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ *as* $\tau \to \infty$.

In the following lemma, $\Theta^{(i)}$ with $i = 1, 2$ are given by (4.36) and (4.37), respectively.

Lemma 4.14. *Let hypotheses* $(\mathbf{H}_\infty)$ *and* $(\mathbf{H}'_\infty)$ *be satisfied. Then we have*

$$\frac{\Theta^{(i)}(\tau)}{g_\infty(\tau)} \to \int_\Omega f_{\pm\infty}\varphi_1\,\mathrm{d}x \quad \text{as} \quad \tau \to \pm\infty\,, \tag{4.50}$$

and, in particular,

$$\frac{\Theta^{(i)}(\tau)}{|\tau|^{p-1}} \to 0 \quad as \quad |\tau| \to \infty\,. \tag{4.51}$$

Proof. For $\tau \in \mathbb{R} \setminus \{0\}$ we compute

$$\begin{aligned} \frac{\Theta^{(1)}(\tau)}{g_\infty(\tau)} &= \int_\Omega \frac{h\big(x, \tau(\varphi_1 + v^\top)\big)}{g_\infty(\tau)} (\varphi_1 + v^\top)\, \mathrm{d}x \\ &= \int_\Omega \frac{h\big(x, \tau(\varphi_1 + v^\top)\big)}{g_\infty\big(\tau\big(1 + \frac{v^\top}{\varphi_1}\big)\big)} \cdot (\varphi_1 + v^\top) \cdot \frac{g_\infty\big(\tau\big(1 + \frac{v^\top}{\varphi_1}\big)\big)}{g_\infty(\tau)}\, \mathrm{d}x \\ &\to \int_\Omega f_\pm \varphi_1 \, \mathrm{d}x \end{aligned} \tag{4.52}$$

by the Lebesgue dominated convergence theorem which makes use of (2.19) and (2.20) combined with Remark 2.5 and $\|v^\top/\varphi_1\|_{L^\infty(\Omega)} \to 0$ as $|\tau| \to \infty$. □

5. Global bifurcation results

This section is divided into three paragraphs. The first one is devoted to preliminary results concerning Browder–Petryshyn and Skrypnik degree for perturbations of monotone operators. The definition and basic properties thereof can be found, e.g., in [42, Chapter 36, pp. 1002–1007]. Then we present proofs of global bifurcation results of Rabinowitz type, i.e., Propositions 3.5 and 3.8 in the second paragraph. Our main results, Theorems 3.7 and 3.10, which are global bifurcation results of Dancer's type, are proved in the third paragraph.

5.1. The Browder–Petryshyn and Skrypnik degree

Definition 5.1. Let us consider an operator $\mathcal{T} : X \to X'$ where X is a real separable reflexive Banach space. The operator $\mathcal{T}$ is said *to satisfy condition* $\alpha(X)$ if for an arbitrary sequence $\{u_n\}_{n=1}^\infty \subset X$ the relations

$$u_n \rightharpoonup u_0 \quad \text{weakly in } X \quad \text{and} \quad \limsup_{n\to\infty} \big\langle \mathcal{T}(u_n), u_n - u_0 \big\rangle_X \le 0 \tag{5.1}$$

imply $u_n \to u_0$ strongly in X.

It is proved in [14, Chapter 5, p. 188] that in the special case $\mathbf{a}(x, \mathbf{v}) \stackrel{\text{def}}{=} |\mathbf{v}|^{(p-2)}\mathbf{v}$ and $B(x) = 1$ for all $x \in \Omega$ and all $\mathbf{v} \in \mathbb{R}^N$ the operator $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{A} - \lambda\mathcal{B}$ satisfies condition $\alpha(X)$ from [33] (which is nothing else but condition (S_+) from [7]) and so its (Browder–Petryshyn) degree can be defined.

The following well-known inequalities,

($\mathrm{A_i}$) $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_X \ge \gamma(\|u\|_X + \|v\|_X)^{p-2}\|u - v\|_X^2$ for all $1 < p < 2$, $u, v \in X$, and some constant $\gamma > 0$;

($\mathrm{A_{ii}}$) $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_X \ge \gamma\|u - v\|_X^p$ for all $2 < p < \infty$, $u, v \in X$, and some constant $\gamma > 0$, cf. [8, Thm. 3; p. 736], play the crucial role in the verification of condition $\alpha(X)$ for $\mathcal{T} = \mathcal{A}$.

We prove only (A_i); the proof of (A_{ii}) is analogous and well-known. So let $1 < p < 2$. From (2.3) we deduce

$$\begin{aligned}\big[\mathbf{a}(x,\mathbf{u}) - \mathbf{a}(x,\mathbf{v})\big] \cdot (\mathbf{u}-\mathbf{v})\,\mathrm{d}x &= \int_\Omega \sum_{i,j=1}^N \left(\int_0^1 \frac{\partial a_i}{\partial \xi_j}\big(x,\mathbf{u} \right. \\ &\quad \left. + s(\mathbf{v}-\mathbf{u})\big)\,\mathrm{d}s\right)(u_i - v_i)(u_j - v_j)\,\mathrm{d}x \\ &\geq \gamma \int_\Omega \left(\int_0^1 \big|\mathbf{u} + s(\mathbf{v}-\mathbf{u})\big|^{p-2}\,\mathrm{d}s\right) |\mathbf{u}-\mathbf{v}|^2\,\mathrm{d}x \\ &\geq \gamma \int_\Omega \left(\max_{0\leq s\leq 1} \big|\mathbf{u} + s(\mathbf{v}-\mathbf{u})\big|\right)^{p-2} |\mathbf{u}-\mathbf{v}|^2\,\mathrm{d}x\,. \end{aligned} \tag{5.2}$$

Next, we use the classical Hölder inequality with the exponents $2/p$ and $\frac{2}{2-p}$ (note that $p/2 + \frac{2-p}{2} = 1$) to get the "reversed" Hölder inequality

$$\int_\Omega f(x)g(x)\,\mathrm{d}x \geq \left(\int_\Omega f(x)^{p/2}\,\mathrm{d}x\right)^{2/p} \left(\int_\Omega g(x)^{p/(p-2)}\,\mathrm{d}x\right)^{(p-2)/p} \tag{5.3}$$

for any measurable functions $f, g : \Omega \to \mathbb{R}$, $f \geq 0$ and $g > 0$ a.e. in Ω, such that $f\,g \in L^1(\Omega)$ and $1/g \in L^{p/(2-p)}(\Omega)$; hence, $f^{p/2} \in L^1(\Omega)$. Now take any vector-valued functions $\mathbf{u}, \mathbf{v} \in [L^p(\Omega)]^N$, $\|\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{v}\|_{L^p(\Omega)} > 0$. Inserting

$$f \stackrel{\text{def}}{=} |\mathbf{u}-\mathbf{v}|^2 \quad \text{and} \quad g \stackrel{\text{def}}{=} \left(\max_{0\leq s\leq 1} \big|\mathbf{u} + s(\mathbf{v}-\mathbf{u})\big|\right)^{p-2}$$

into (5.3) we arrive at

$$\begin{aligned}\int_\Omega |\mathbf{u}-\mathbf{v}|^2 \left(\max_{0\leq s\leq 1} \big|\mathbf{u} + s(\mathbf{v}-\mathbf{v})\big|\right)^{p-2} &\geq \left(\int_\Omega |\mathbf{u}-\mathbf{v}|^p\,\mathrm{d}x\right)^{2/p} \left(\int_\Omega \left(\max_{0\leq s\leq 1} \big|\mathbf{u} \right.\right. \\ &\quad \left.\left. + s(\mathbf{v}-\mathbf{u})\big|\right)^p\right)^{(p-2)/p} \\ &\geq \|\,\mathbf{u}-\mathbf{v}\,\|^2_{L^p(\Omega)} \left(\int_\Omega \big(|\mathbf{u}| + |\mathbf{v}|\big)^p\right)^{(p-2)/p} \\ &= \|\mathbf{u}-\mathbf{v}\|^2_{L^p(\Omega)} \big\||\mathbf{u}| + |\mathbf{v}|\big\|^{p-2}_{L^p(\Omega)} \\ &\geq \|\mathbf{u}-\mathbf{v}\|^2_{L^p(\Omega)} \big(\|\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{v}\|_{L^p(\Omega)}\big)^{p-2}. \end{aligned} \tag{5.4}$$

Combining inequalities (5.2) and (5.4) we conclude that

$$\int_\Omega \big[\mathbf{a}(x,\mathbf{u}) - \mathbf{a}(x,\mathbf{v})\big](\mathbf{u}-\mathbf{v})\,\mathbf{d}x \geq \gamma\big(\|\mathbf{u}\|_{L^p(\Omega)} + \|\mathbf{v}\|_{L^p(\Omega)}\big)^{p-2} \|\mathbf{u}-\mathbf{v}\|^2_{L^p(\Omega)}$$

which proves inequality (A_i).

By means of (A_i) and (A_{ii}), condition $\alpha(X)$ for $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{A}$ can be verified.

Proposition 5.2. *The operator $\mathcal{A} : X \to X'$ defined by identity* (3.3) *satisfies condition $\alpha(X)$.*

Proof. Assume that $\mathcal{A}$ and $\{u_n\}_{n=1}^{\infty} \subset X$ satisfy (5.1), i.e., $u_n \rightharpoonup u_0$ weakly in X and $\limsup_{n\to\infty} (\mathcal{A}(u_n), u_n - u_0)_X \le 0$. We combine the last inequality with $\lim_{n\to\infty} \langle \mathcal{A}(u_0), u_n - u_0\rangle_X = 0$ to obtain

$$\limsup_{n\to\infty} \big\langle \mathcal{A}(u_n) - \mathcal{A}(u_0), u_n - u_0 \big\rangle_X \le 0 \,. \tag{5.5}$$

Now we distinguish between the cases $1 < p < 2$ and $p \ge 2$.

Let $1 < p < 2$. From ($\mathrm{A_i}$) we find

$$\big\langle \mathcal{A}(u_n) - \mathcal{A}(u_0), u_n - u_0 \big\rangle_X \ge \gamma \big(\|u_n\|_X + \|u_0\|_X \big)^{p-2} \|u_n - u_0\|_X^2 \tag{5.6}$$

for $n = 1, 2, 3 \ldots$. Since every weakly convergent sequence in X is also bounded, we have $(\|u_n\|_X + \|u_0\|_X)^{p-2} \ge C > 0$ for some constant C. We apply (5.5) to (5.6) to get $\|u_n - u_0\|_X \to 0$ as $n \to \infty$ which verifies condition $\alpha(X)$.

For $p \ge 2$ we use inequality ($\mathrm{A_{ii}}$) in place of ($\mathrm{A_i}$) to get the same conclusion. □

A standard method for proving a discontinuity in the Browder–Petryshyn degree (cf. [14, Chapter 5, Thm. 14.18, p. 189] for the p-Laplacian) is based on the variational structure of the p-homogeneous part $\mathcal{A} - \lambda\mathcal{B}$ of the operator $\mathcal{A} - \lambda\mathcal{B} - \mathcal{H}$. For the sake of completeness we begin by proving the following result.

Proposition 5.3. *For all $r > 0$ and all $0 < \delta < \mu_2 - \mu_1$ we have*

$$\mathrm{Deg}\big[\mathcal{A} - (\mu_1 \pm \delta)\mathcal{B}; B_r(0), 0\big] = \mp 1 \,. \tag{5.7}$$

Proof. Given $R > 0$ fixed, we define $\psi : \mathbb{R}_+ \to \mathbb{R}$ as follows: $\psi(t) = 0$ for $0 \le t \le R$, $\psi(t) = \delta/R\,(t - R)^2$ for $R < t < 2R$, and $\psi(t) = 2\delta(t - 2R) + \delta R$ for $2R \le t < \infty$. Clearly, ψ is continuously differentiable, monotone increasing, and convex on $\mathbb{R}_+$, with $0 \le \psi'(t) \le 2\delta$ for every $t \in \mathbb{R}_+$.

Now consider the functional $F_\lambda : X \to \mathbb{R}$ defined by

$$F_\lambda(u) \stackrel{\mathrm{def}}{=} \frac{1}{p}\big\langle \mathcal{A}(u), u\big\rangle_X - \frac{\lambda}{p}\big\langle \mathcal{B}(u), u\big\rangle_X + \psi\left(\frac{1}{p}\big\langle \mathcal{B}(u), u\big\rangle_X\right), \quad u \in X \,.$$

Every critical point $u_0 \in X$ of F_λ is a solution of the operator equation

$$F'_\lambda(u) = \mathcal{A}(u) - \left[\lambda - \psi'\left(\frac{1}{p}\big\langle \mathcal{B}(u), u\big\rangle_X\right)\right]\mathcal{B}(u) = 0 \quad \text{in} \quad X' \,. \tag{5.8}$$

Of course, $u = 0 \in X$ is a solution. Assuming $-\infty < \lambda \le \mu_1 + \delta \,(< \mu_2)$ we have

$$(\lambda - 2\delta \le)\, \lambda - \psi'\left(\frac{1}{p}\big\langle \mathcal{B}(u), u\big\rangle_X\right) \le \lambda \le \mu_1 + \delta \,(< \mu_2) \,.$$

Therefore, if $u_0 \in X \setminus \{0\}$ is a nonzero solution of (5.8), we must have

$$\lambda - \psi'\left(\frac{1}{p}\big\langle \mathcal{B}(u_0), u_0\big\rangle_X\right) = \mu_1$$

and $u_0 = \alpha\varphi_1$ for some constant $\alpha \in \mathbb{R}\setminus\{0\}$, where α satisfies $\lambda - \psi'(|\alpha|^p/p) = \mu_1$. (Recall that $\langle \mathcal{B}(\varphi_1), \varphi_1\rangle_X = \int_\Omega B(x)\varphi_1^p \,\mathrm{d}x = 1$.) Since $\psi'(t) = 0$ for $0 \le t \le R$, $\psi'(t) = \frac{2\delta}{R}(t-R)$ for $R \le t \le 2R$, and $\psi'(t) = 2\delta$ for $2R \le t < \infty$, the last equation for $\alpha \in \mathbb{R}$ possesses a nonzero solution $\alpha = \pm\alpha_\lambda$ $(\alpha_\lambda > 0)$ if and only if $\lambda \in [\mu_1, \mu_1 + 2\delta]$. If $\lambda = \mu_1$, we may take any $\alpha \in \mathbb{R}$ with $0 < |\alpha|^p/p \le R$. If $\mu_1 < \lambda \le \mu_1 + \delta$, we determine $\alpha \in \mathbb{R}$ from

$$\psi'\big(|\alpha|^p/p\big) = \frac{2\delta}{R}\left(\frac{|\alpha|^p}{p} - R\right) = \lambda - \mu_1 \in (0,\delta]\,; \tag{5.9}$$

hence, $R < \frac{|\alpha|^p}{p} \le 3/2R\,(< 2R)$.

Let $\mu_1 - \delta \le \lambda < \mu_1$. The only critical point of F_λ is the zero function $u = 0 \in X$; it is the global minimizer for F_λ. We apply Skrypnik [33, Thm. 1.5.1, p. 42] to conclude that

$$\mathrm{Deg}\big[F_\lambda; B_r(0), 0\big] = 1 \quad \text{for all} \quad r > 0\,. \tag{5.10}$$

More precisely, this claim is proved in [33, Thm. 1.5.1, p. 42] for $r > 0$ small enough only. Arbitrary $r > 0$ is then allowed by the fact that $u = 0 \in X$ is the only critical point of F_λ.

Now let us consider the case $\mu_1 < \lambda \le \mu_1 + \delta$. The functional F_λ is coercive on X, owing to the following inequalities which hold for all $u \in X$ such that $1/p\langle \mathcal{B}(u), u\rangle_X \ge 2R$:

$$\begin{aligned}
F_\lambda(u) &= \frac{1}{p}\big\langle \mathcal{A}(u), u\big\rangle_X - \frac{\lambda}{p}\big\langle \mathcal{B}(u), u\big\rangle_X + 2\delta\left(\frac{1}{p}\big\langle \mathcal{B}(u), u\big\rangle_X - R\right)\\
&= \frac{1}{p}\big\langle \mathcal{A}(u), u\big\rangle_X - \frac{\lambda - 2\delta}{p}\big\langle \mathcal{B}(u), u\big\rangle_X - 2\delta R\\
&\ge \frac{1}{p}\big\langle \mathcal{A}(u), u\big\rangle_X - \frac{\mu_1 - \delta}{p}\big\langle \mathcal{B}(u), u\big\rangle_X - 2\delta\\
&\ge \frac{1}{p}\left(1 - \frac{\mu_1 - \delta}{\mu_1}\right)\big\langle \mathcal{A}(u), u\big\rangle_X - 2\delta R\\
&= \delta\left(\frac{1}{p\mu_1}\big\langle \mathcal{A}(u), u\big\rangle_X - 2R\right) \longrightarrow +\infty
\end{aligned}$$

as $\|u\|_X \to \infty$. Recall that $\alpha_\lambda \in (0,\infty)$ is uniquely determined by equation (5.9) and satisfies $R < \frac{\alpha_\lambda^p}{p} \le 3/2R\,(< 2R)$. It is easy to see that $F_\lambda(\pm\alpha_l\varphi_1) < 0 = F_\lambda(0)$. Since F_λ has no other critical points than $0 \in X$ and $\pm\alpha_\lambda\varphi_1$, both $\pm\alpha_\lambda\varphi_1$ must be the global minimizers for F_λ. By [33] again, we find

$$\mathrm{Deg}\big[F'_\lambda; B_\varrho(\pm\alpha_\lambda\varphi_1), 0\big] = 1 \quad \text{for every } \varrho > 0 \text{ small enough.} \tag{5.11}$$

We assume also $\varrho < 1/2\alpha_{\mu_1+\delta}\|\varphi_1\|_X$.

Set $r_\delta = 2\alpha_{\mu_1+\delta}\|\varphi_1\|_X$; hence, $\mu_1 < \lambda \le \mu_1 + \delta$ implies $\alpha_\lambda \le \alpha_{\mu_1+\delta}$ and therefore $r_\delta > \alpha_\lambda\|\varphi_1\|_X + \varrho$. Thus, by an argument with a homotopy connecting $F'_{\mu_1-\delta}$ with $F'_{\mu_1+\delta}$ we get

$$\mathrm{Deg}\big[F'_{\mu_1+\delta}; B_r(0), 0\big] = \mathrm{Deg}\big[F'_{\mu_1-\delta}; B_r(0), 0\big] = 1 \quad \text{for every} \quad r > r_\delta\,. \tag{5.12}$$

Now, since $u = 0 \in X$ is an isolated solution to (5.8), the degree $\mathrm{Deg}[F'_{\mu_1+\delta}; B_{\varrho'}(0), 0]$ is well defined for every $\varrho' > 0$ small enough; we assume $\varrho' < 1/2\alpha_{\mu_1+\delta}\|\varphi_1\|_X$. From the additivity property of the degree we deduce

$$\begin{aligned}
&\mathrm{Deg}\big[F'_{\mu_1+\delta}; B_\varrho(\alpha_{\mu_1+\delta}\varphi_1), 0\big] \\
&\qquad + \mathrm{Deg}\big[F'_{\mu_1+\delta}; B_\varrho(-\alpha_{\mu_1+\delta}\varphi_1), 0\big] \\
&\qquad + \mathrm{Deg}\big[F'_{\mu_1+\delta}; B_{\varrho'}(0), 0\big] \\
&\quad = \mathrm{Deg}\big[F'_{\mu_1+\delta}; B_r(0), 0\big] \Big(= \mathrm{Deg}\big[F'_{\mu_1-\delta}; B_r(0), 0\big] = 1\Big)
\end{aligned}$$

for every $r > r_\delta$. Finally, we apply (5.11) to get $\mathrm{Deg}[F'_{\mu_1+\delta}; B_{\varrho'}(0), 0] = -1$.

Since

$$\mathcal{A}(u) - (\mu_1+\delta)\mathcal{B}(u) = \mathcal{A}(u) - \left[(\mu_1+\delta) + \psi'\left(\frac{1}{p}\langle \mathcal{B}(u), u\rangle\right)\right]\mathcal{B}(u) = F'_{\mu_1+\delta}$$

holds whenever $\langle \mathcal{B}(u), u\rangle_X < R$, we find

$$\mathrm{Deg}\big[\mathcal{A} - (\mu_1+\delta)\mathcal{B}; B_{\varrho'}(0), 0\big] = \mathrm{Deg}\big[F'_{\mu_1+\delta}; B_{\varrho'}(0), 0\big] = -1$$

provided $\varrho' > 0$ from above is taken so small that also $\langle \mathcal{B}(u), u\rangle_X \le \frac{1}{\mu_1}\langle \mathcal{A}(u), u\rangle_X < R$ holds for every $u \in B_{\varrho'}(0) \subset X$. Since $u = 0 \in X$ is the only solution to the operator equation $\mathcal{A}(u) - (\mu_1+\delta)\mathcal{B}(u) = 0$ in X', we arrive at

$$\mathrm{Deg}\big[\mathcal{A} - (\mu_1+\delta)\mathcal{B}; B_r(0), 0\big] = -1 \quad \text{for every} \quad r > 0\,.$$

By analogous arguments, we infer from (5.10) that

$$\mathrm{Deg}\big[\mathcal{A} - (\mu_1-\delta)\mathcal{B}; B_r(0), 0\big] = 1 \quad \text{for every} \quad r > 0\,.$$

The proof is now complete. □

5.2. Proof of a Rabinowitz-type bifurcation theorem

As in the semilinear case, in order to prove Theorem 3.7, we begin with the proof of Proposition 3.5 (which is a Rabinowitz-type bifurcation theorem).

Proof of Proposition 3.5. We have $0 < \mu_1 < \mu_2$ by Remark 2.1. We begin the proof by showing that for every $0 < \delta < \mu_2 - \mu_1$ there exists $R > 0$ such that

$$\mathrm{Deg}\big[\Phi_{\mu_1\pm\delta}; B_r(0), 0\big] = \mp 1 \quad \text{whenever} \quad 0 < r < R\,. \tag{5.13}$$

By Proposition 3.2, for each $\lambda \in (-\infty, \mu_2)\setminus\{\mu_1\}$, $u = 0 \in X$ is an isolated solution of $\Phi_\lambda(u) = 0$. Thus one can find $R > 0$ small enough, such that $\mathrm{Deg}[\Phi_{\mu_1\pm\delta}; B_r(0), 0]$ remains constant with respect to $r \in (0, R)$. We will show later that there exists $R' \in (0, R)$ such that

$$\mathcal{A}(u) - (\mu_1\pm\delta)\mathcal{B}(u) - \alpha\mathcal{H}(u; \mu_1\pm\delta) \neq 0\,. \tag{5.14}$$

holds for all $u \in \partial B_{R'}(0)$ and $\alpha \in [0, 1]$. Therefore, the homotopy

$$\mathcal{A}(u) - (\mu_1\pm\delta)\mathcal{B}(u) - \alpha\mathcal{H}(u; \mu_1\pm\delta)\,, \quad \alpha \in [0, 1]\,,$$

connecting $\Phi_{\mu_1 \pm \delta}$ with $\mathcal{A} - (\mu_1 \pm \delta)\mathcal{B}$, is admissible with respect to $B_{R'}(0)$ and 0. Consequently, we have

$$\begin{aligned}\mathrm{Deg}\big[\Phi_{\mu_1 \pm \delta}; B_r(0), 0\big] &= \mathrm{Deg}\big[\Phi_{\mu_1 \pm \delta}; B_{R'}(0), 0\big] \\ &= \mathrm{Deg}\big[\mathcal{A} - (\mu_1 \pm \delta)\mathcal{B}; B_{R'}(0), 0\big] = \mp 1\,,\end{aligned}$$

by Lemma 5.3.

Using $\mathrm{Deg}[\Phi_{\mu_1 \pm \delta}; B_r(0), 0] = \mp 1$ we can proceed step by step as in the original proof of Rabinowitz [31, Theorem 1.3, pp. 490–491], cf. also Drábek [14, Theorem 14.9, pp. 178–183].

It remains to prove (5.14). Suppose the contrary, i.e., for all $R' \in (0, R)$ there exist $u \in \partial B_{R'}(0)$ and $\alpha \in [0, 1]$ such that

$$\mathcal{A}(u) - (\mu_1 \pm \delta)\mathcal{B}(u) - \alpha \mathcal{H}(u; \mu_1 \pm \delta) = 0\,. \tag{5.15}$$

Thus, we can find a sequence $\{r_n\}_{n=1}^{\infty} \subset (0, R)$, $r_n \to 0$, together with $\{u_n\}_{n=1}^{\infty} \subset X$, $u_n \in \partial B_{r_n}(0)$, and $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$, such that (5.15) holds with u_n and α_n in place of u and α, respectively. Since $u_n \in \partial B_{r_n}(0)$, we have $\|u_n\|_X \to 0$. Observe that the functions $\alpha_n\, h(\,\cdot\,,\,\cdot\,;\,\cdot\,) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy $(\mathbf{H}_0^n)$ because $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies $(\mathbf{H}_0)$ and $\alpha_n \in [0, 1]$. We apply Lemma B.2 with $\lambda_n = \mu_1 \pm \delta\, (< \mu_2)$ to equation (5.15) to conclude that $\lambda_n \to \mu_1$ as $n \to \infty$, which is absurd ($\delta > 0$). This concludes the proof. □

Now we continue by giving the proof of Lemma 3.6 which is another ingredient in the proof of Theorem 3.7.

Proof of Lemma 3.6. Suppose that for some $\eta \in (0, \eta_0)$ such a number $0 < S \leq 1$ does not exist. Then we can find a decreasing sequence $0 < S_n \leq 1$ with $S_n \searrow 0$ and another sequence $(u_n, \lambda_n) \in (\mathcal{S} \setminus \{(0, \mu_1)\}) \cap \overline{E^{S_n}(\mu_1)}$ such that $|\ell(u_n)| \leq \eta\, \|u_n\|_{W_0^{1,p}(\Omega)}$ for each $n = 1, 2, \ldots$. Notice that owing to $(u_n, \lambda_n) \neq (0, \mu_1)$ we must have $u_n \not\equiv 0$ in Ω for all $n \geq 1$ large enough, because μ_1 is an isolated eigenvalue of the $(p-1)$-homogeneous operator $\mathcal{A}$, as shown in Anane [3, Théorème 2, p. 727] or Anane and Tsouli [5, Prop. 2, p. 5]. Discarding a finite number of members of this sequence if necessary, we may assume $u_n \not\equiv 0$ in Ω for all $n \geq 1$.

Since $(u_n, \lambda_n) \in \mathcal{S}$ and $S_n \searrow 0$ as $n \to \infty$, we have also $\|u_n\|_{L^\infty(\Omega)} \to 0$ and $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to 0$, by Lemma B.2 (Appendix B). This lemma shows also that the normalized sequence $w_n \stackrel{\mathrm{def}}{=} u_n / \|u_n\|_{L^\infty(\Omega)}$, with $\|w_n\|_{L^\infty(\Omega)} = 1$, is the union of two disjoint subsequences $\{w_n'\}_{n=1}^{\infty}$ and $\{w_n''\}_{n=1}^{\infty}$, one of them possibly empty, such that, if nonempty, they satisfy $w_n' \to \varphi_1 / \|\varphi_1\|_{L^\infty(\Omega)}$ and/or $w_n'' \to -\varphi_1 / \|\varphi_1\|_{L^\infty(\Omega)}$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$. Consequently, we get

$$\big|\ell(w_n)\big| \to \ell\big(\varphi_1 / \|\varphi_1\|_{L^\infty(\Omega)}\big) = \|\varphi_1\|_{L^\infty(\Omega)}^{-1} \quad \text{as} \quad n \to \infty\,.$$

Furthermore, by our assumption we have

$$\big|\ell(w_n)\big| = \frac{|\ell(u_n)|}{\|u_n\|_{L^\infty(\Omega)}} \leq \eta\, \frac{\|u_n\|_{W_0^{1,p}(\Omega)}}{\|u_n\|_{L^\infty(\Omega)}} \quad \text{for each} \quad n = 1, 2, \ldots\,.$$

Combining the last two facts we arrive at

$$\|\varphi_1\|_{L^\infty(\Omega)}^{-1} \leq \eta \cdot \liminf_{n\to\infty} \frac{\|u_n\|_{W_0^{1,p}(\Omega)}}{\|u_n\|_{L^\infty(\Omega)}} . \tag{5.16}$$

To find an upper bound for the fraction above, we first take $(u,\lambda) = (u_n,\lambda_n)$ and $\phi = u_n$ in equation (2.1) which yields

$$\int_\Omega A(x,\nabla u_n)\,\mathrm{d}x = \lambda_n \int_\Omega B(x)\,|u_n|^p\,\mathrm{d}x + \int_\Omega h\big(x,u_n(x);\lambda_n\big)\,u_n(x)\,\mathrm{d}x .$$

Now we apply inequalities (2.6), $\lambda_n \leq \mu_1 + S_n \leq \mu_1 + 1$, and (2.9) to get

$$\begin{aligned}\frac{\gamma}{p-1}\int_\Omega |\nabla u_n|^p\,\mathrm{d}x &\leq \int_\Omega A(x,\nabla u_n)\,\mathrm{d}x \\ &\leq (\mu_1+1)\int_\Omega B(x)\,|u_n|^p\,\mathrm{d}x + C\int_\Omega |u_n|^p\,\mathrm{d}x \\ &\leq \big[(\mu_1+1)\|B\|_{L^\infty(\Omega)} + C\big]\int_\Omega |u_n|^p\,\mathrm{d}x \\ &\leq \big[(\mu_1+1)\|B\|_{L^\infty(\Omega)} + C\big]|\Omega|_N\,\|u_n\|_{L^\infty(\Omega)}^p .\end{aligned}$$

Consequently, we have $\|u_n\|_{W_0^{1,p}(\Omega)} \leq c_0\,\|u_n\|_{L^\infty(\Omega)}$ where

$$c_0 \stackrel{\text{def}}{=} \left(\frac{(p-1)\,\big[(\mu_1+1)\|B\|_{L^\infty(\Omega)} + C\big]\,|\Omega|_N}{\gamma}\right)^{1/p} > 0 .$$

Combining the last inequality with (5.16) we obtain $\|\varphi_1\|_{L^\infty(\Omega)}^{-1} \leq \eta\,c_0$ or, equivalently, $\eta \geq \eta_0$ with $\eta_0 > 0$ defined by (3.10). But this contradicts our choice of $\eta \in (0,\eta_0)$. The lemma is proved. □

Let us consider the sequence of boundary value problems

$$\begin{cases} -\operatorname{div}\big(\mathbf{a}(x,\nabla u)\big) = \lambda\,B(x)\,|u|^{p-2}u + h_n(x,u;\lambda) \quad \text{in} \quad \Omega ; \\ \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega , \end{cases} \tag{5.17}$$

where the sequence of functions $h_n : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies condition $(\mathbf{H}_0^n)$. With

$$\big\langle \mathcal{H}_n(u;\,\lambda),\,\phi\big\rangle_X = \int_\Omega h_n(x,u;\lambda)\,\varphi\,\mathrm{d}x \quad \text{satisfied for all} \quad u,\,\phi \in X ,$$

the operator formulation of (5.17) reads as follows,

$$\mathcal{A}(u) - \lambda\mathcal{B}(u) - \mathcal{H}_n(u;\lambda) = 0 \quad \text{in} \quad X' .$$

For each $n \in \mathbb{N}$, we also define

$$\mathcal{S}_n \stackrel{\text{def}}{=} \overline{\big\{(u,\lambda) \in E :\, \mathcal{A}(u) - \lambda\mathcal{B}(u) - \mathcal{H}_n(u;\lambda) = 0,\ u \neq 0\big\}}^{E} .$$

The following lemma is a version of Lemma 3.6 for (5.17) which is uniform in $n \in \mathbb{N}$.

Lemma 5.4. *Let $h_n : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy condition $(\mathbf{H}_0^n)$. For every $\eta \in (0, \eta_0)$ there exists a number S, $0 < S \leq 1$, such that*

$$\Big(\mathcal{S}_n \setminus \{(0,\mu_1)\}\Big) \cap \overline{E^S(\mu_1)} \subset K_\eta \qquad \text{for all} \quad n \in \mathbb{N}\,.$$

Moreover, if $(u,\lambda) \in (\mathcal{S}_n \setminus \{(0,\mu_1)\}) \cap \overline{E^S(\mu_1)}$ then $u = \tau(\varphi_1 + v^\top)$, where $\tau = \ell(u) \in \mathbb{R}$ and $v^\top \in C^{1,\beta'}(\overline{\Omega})$ $(0 < \beta' < \beta)$ satisfy $|\tau| > \eta \, \|u\|_{W_0^{1,p}(\Omega)}$ and $\ell(v^\top) = 0$ together with $|\lambda - \mu_1| \to 0$ and $\|v^\top\|_{C^{1,\beta'}(\overline{\Omega})} \to 0$ as $\tau \to 0$.

The proof of Lemma 5.4 is almost identical with the proof of Lemma 3.6. The only difference is that we consider $(u_m, \lambda_m) \in \mathcal{S}_{n_m}$ (here $\{\mathcal{S}_{n_m}\}_{m=1}^\infty \subset \{\mathcal{S}_n\}_{n=1}^\infty$). Then we continue literally as in the proof of Lemma 3.6 because all estimates are uniform with respect to $n \in \mathbb{N}$ due to $(\mathbf{H}_0^n)$.

5.3. Proof of a Dancer-type bifurcation theorem

As in the semilinear case in Dancer [10, Theorem 2, p. 1071], our proof of Theorem 3.7 is based on the following three lemmas.

Lemma 5.5. *Suppose $\delta_1, \delta_2 > 0$ are such that $0 < \delta_1 + \delta_2 < S$ and $\Phi_\lambda(u) \neq 0$ if $\|u\|_X = \delta_1$ and $|\lambda - \mu_1| \leq \delta_2$. If $0 < \sigma < \delta_2$ and $\beta > 0$ is sufficiently small, $\beta \equiv \beta(\sigma)$, then $\|u\|_X < \beta$ together with $\Phi_{\mu_1 \pm \sigma}(u) = 0$ imply $u = 0$ and, moreover,*

$$\mathrm{Deg}\big[\Phi_{\mu_1+\sigma}; W^\nu, 0\big] - \mathrm{Deg}\big[\Phi_{\mu_1-\sigma}; W^\nu, 0\big] = 1$$

where

$$W^\nu \stackrel{\text{def}}{=} \big\{u \in X : \, (u,\lambda) \in K_\eta^\nu \ \ \text{and} \ \ \beta < \|u\|_X < \delta_1\big\}\,, \qquad \nu = \pm\,.$$

Recall that η is arbitrary with $0 < \eta < \mu_1^{-1/p}$, and $K_\eta^\nu \subset X \times \mathbb{R}$ has been defined in (3.9). In the definition $W^\nu \subset X$ above we can use any $\lambda \in \mathbb{R}$.

Proof. We follow Dancer [10, Proof of Lemma 1, p. 1071]. We define

$$\mathcal{H}^-(u;\lambda) \stackrel{\text{def}}{=} \begin{cases} \mathcal{H}(u;\lambda) & \text{if} \quad \int_\Omega u\varphi_1 \,\mathrm{d}x \leq -\eta \|u\|_X\,; \\ \frac{\int_\Omega u\varphi_1 \,\mathrm{d}x}{\eta\|u\|_X}\, \mathcal{H}(u;\lambda) & \text{if} \quad -\eta\|u\|_X < \int_\Omega u\varphi_1 \,\mathrm{d}x \leq 0\,; \\ -\mathcal{H}(-u;\lambda) & \text{if} \quad \int_\Omega u\varphi_1 \,\mathrm{d}x > 0\,, \end{cases}$$

and

$$\Phi_\lambda^-(u) \stackrel{\text{def}}{=} \mathcal{A}(u) - \lambda\mathcal{B}(u) + \mathcal{H}^-(u;\lambda)\,. \tag{5.18}$$

The mapping $\Phi_\lambda^- : X \to X'$ is odd. Since the $(p-1)$-homogenous part of Φ_λ^- is the same as that of Φ_λ, also Φ_λ^- satisfies condition $\alpha(X)$.

By our hypothesis, the equation $\Phi_{\mu_1+\sigma}(u) = 0$ has no solution on ∂E_{δ_1}, ∂E_β, or in

$$E_{\delta_1} \setminus \big(W^+ \cup W^- \cup E_\beta\big)\,,$$

see Lemma 3.6. It follows that

$$\begin{aligned} \mathrm{Deg}\big[\Phi^-_{\mu_1+\sigma}; E_{\delta_1}, 0\big] &= \mathrm{Deg}\big[\Phi^-_{\mu_1+\sigma}; E_\beta, 0\big] + \mathrm{Deg}\big[\Phi^-_{\mu_1+\sigma}; W^-, 0\big] \\ &\quad + \mathrm{Deg}\big[\Phi^-_{\mu_1+\sigma}; W^+, 0\big] \end{aligned}$$

which is

$$\operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; W^-, 0\big] + \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; W^+, 0\big] = \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, E_{\delta_1}, 0\big] - \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, E_\beta, 0\big]\,.$$

By the oddness of $\Phi^-_{\mu_1+\sigma} : X \to X'$ and the definition of the Browder–Petryshyn degree, we find that

$$\operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, W^+, 0\big] = \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, W^-, 0\big]$$

and so

$$2\cdot \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, W^-, 0\big] = \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, E_{\delta_1}, 0\big] - \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}, E_\beta, 0\big]\,. \tag{5.19}$$

Analogously,

$$2\cdot \operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}, W^-, 0\big] = \operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}, E_{\delta_1}, 0\big] - \operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}, E_\beta, 0\big]\,. \tag{5.20}$$

As in the proof of Proposition 3.5 (in Section 5.2) one can show that

$$\operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}; E_\beta, 0\big] = -1 \quad\text{and}\quad \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; E_\beta, 0\big] = 1\,. \tag{5.21}$$

Subtracting (5.20) from (5.19) and using (5.21), we arrive at

$$\begin{aligned} 2\cdot\Big(\operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; W^-, 0\big] - \operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}; W^-, 0\big]\Big) &= \operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}; E_\beta, 0\big] \\ &\quad - \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; E_\beta, 0\big] \\ &= 1 - (-1) = 2\,. \end{aligned}$$

Since $\Phi^-_{\mu_1-\sigma}(u) = \Phi_{\mu_1-\sigma}(u)$ for all $u \in W^-$ and all $\lambda \in \mathbb{R}$, due to the definition of Φ^-_λ, we must have

$$\begin{aligned} \operatorname{Deg}\big[\Phi_{\mu_1+\sigma}; W^-, 0\big] - \operatorname{Deg}\big[\Phi_{\mu_1-\sigma}; W^-, 0\big] &= \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; W^-, 0\big] \\ &\quad - \operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}; W^-, 0\big] = 1\,. \end{aligned}$$

If $\|u\|_X = \delta_1$ and $|\lambda - \mu_1| \le \delta_2$, we have $\Phi_\lambda(u) \ne 0$ by our assumptions. Consequently, for $\sigma \in (0, \delta_2)$ the homotopy Φ^-_λ (where $\mu_1 - \sigma \le \lambda \le \mu_1 + \sigma$) is admissible on E_{δ_1} whence

$$\operatorname{Deg}\big[\Phi^-_{\mu_1-\sigma}; E_{\delta_1}, 0\big] = \operatorname{Deg}\big[\Phi^-_\lambda; E_{\delta_1}, 0\big] = \operatorname{Deg}\big[\Phi^-_{\mu_1+\sigma}; E_{\delta_1}, 0\big]$$

holds for all $\lambda \in [\mu_1 - \sigma, \mu_1 + \sigma]$. □

Remark 5.6. It is worthwhile to note that in proving a Dancer-type bifurcation result for an elliptic boundary value problem we have to consider an abstract functional differential equation; observe that the definition of $\mathcal{H}^-$ contains $\|u\|_X$ and $\int_\Omega u\varphi_1\,\mathrm{d}x$.

For $0 < \varepsilon < S$ we define $T^-_{\mu_1,\varepsilon}$ to be the component of $\mathcal{Z}_{\mu_1} \setminus (E^\varepsilon(\mu_1) \cap K^+_\eta)$ containing $(0, \mu_1)$.

Lemma 5.7. *If $0 < \varepsilon < S$, zero is an isolated solution of $\Phi_{\mu_1}(u) = 0$, and $T^-_{\mu_1,\varepsilon}$ is bounded in E, then*

$$\partial E^\varepsilon(\mu_1) \cap K^+_\eta \cap T^-_{\mu_1,\varepsilon} \ne \emptyset\,.$$

The proof of Lemma 5.7 would almost literally copy that of its semilinear counterpart in Dancer [10, Proof of Lemma 2, p. 1072], and therefore is omitted.

In Lemmas 5.5 and 5.7 we have assumed that zero is an isolated solution of $\Phi_{\mu_1}(u) = 0$. This is the case when, e.g., the function $h(x, u; \lambda)$ satisfies not only $(\mathbf{H}_0)$ but also $(\mathbf{H}_0')$ and, moreover, $\int_\Omega f_{0\pm}\varphi_1 \, dx \neq 0$. (Recall that functions $f_{0\pm}$ are defined in $(\mathbf{H}_0')$.) Under these assumptions it follows from the asymptotic estimates (4.13) in Proposition 4.5 that there exists $c > 0$ such that, for any sequence $\{(u_n, \lambda_n)\}_{n=1}^\infty \subset X \times \mathbb{R}$ of solutions to $\Phi_{\lambda_n}(u_n) = 0$ satisfying $\|u_n\|_X \to 0$ and $\lambda_n \to \mu_1$, one must have $\lambda_n \neq \mu_1$. Thus $\Phi_{\mu_1}(u) \neq 0$ provided $0 < \|u\|_X < c$, and so $u = 0$ is an isolated solution of $\Phi_{\mu_1}(u) = 0$.

In the following lemma we drop the assumption that zero is an isolated solution to $\Phi_{\mu_1}(u) = 0$. This is possible with help from an approximation scheme based on the results from Lemma 4.7.

Lemma 5.8. *The statement of* Lemma 5.7 *holds without the assumption that zero is an isolated solution of* $\Phi_{\mu_1}(u) = 0$.

Proof. We proceed as in the proof of [10, Lemma 3, p. 1072] by considering a sequence of boundary value problems $\Phi_\lambda^{(n)}(u) = 0$, where

$$\Phi_\lambda^{(n)}(u) \stackrel{\text{def}}{=} \mathcal{A}(u) - \lambda\mathcal{B}(u) - \mathcal{H}_n(u; \lambda), \tag{5.22}$$

with $\mathcal{H}_n(u; \lambda)$ being defined by (3.5) with $[h_n(u)](x)$ defined by (4.28) in place of $h(x, u; \lambda)$. Note that, by Lemma 4.7, the corresponding integrals with $[h_n(u)](x)$ defined by (4.28) in place of $h(x, u; \lambda)$ in (4.13) are positive for each $n \in \mathbb{N}$. Thus, for all $n \in \mathbb{N}$, zero is an isolated solution of $\Phi_{\mu_1}^{(n)}(u) = 0$ and, consequently, Lemma 5.7 applies.

By Lemma 5.4, let us first fix $\eta \in (0, \eta_0)$ and then choose S, $0 < S \leq 0$, such that

$$\mathcal{S}_n \setminus \{(0, \mu_1)\} \cap \overline{E^S(\mu_1)} \subset K_\eta \quad \text{for all} \quad n \in \mathbb{N}.$$

Now let $0 < \varepsilon < S$ and assume that $T_{\mu_1,\varepsilon}^-$ is bounded in E. Let T_n be a component of $\mathcal{S}_n \setminus (E^\varepsilon(\mu_1) \cap K_\varepsilon^+)$ containing $(0, \mu_1)$. Suppose that the conclusion of our lemma is false. This means that

$$\partial E^\varepsilon(\mu_1) \cap K_\eta^+ \cap T_{\mu_1,\varepsilon}^- = \emptyset.$$

Recall that, by the definition of $T_{\mu_1,\varepsilon}^-$, this set is connected and satisfies

$$E^\varepsilon(\mu_1) \cap K_\eta^+ \cap T_{\mu_1,\varepsilon}^- = \emptyset.$$

In addition, as $T_{\mu_1,\varepsilon}^-$ is assumed to be bounded, we can find $R > 0$ such that $T_{\mu_1,\varepsilon}^- \subset E^R(\mu_1)$.

We combine these facts with a classical topological result from Whyburn [41, Chap. I, Statement (9.3), p. 12], to conclude that

$$\left(\mathcal{S} \cap \overline{E^R(\mu_1)}\right) \setminus \left(E^\varepsilon(\mu_1) \cap K_\eta^+\right) = k_1 \cup k_2$$

where k_1, k_2 are compact sets in E, such that $k_1 \cap k_2 = \emptyset$, $T_{\mu_1,\varepsilon}^- \subset k_1$, and

$$\left(\mathcal{S} \cap \partial E^R(\mu_1)\right) \cup \left(\mathcal{S} \cap \partial E^\varepsilon(\mu_1) \cap K_\eta^+\right) \subset k_2.$$

Consequently, there exists a bounded open set U in E such that $k_1 \subset U$ and $k_2 \cap \overline{U} = \emptyset$. The properties of k_1 and k_2 entail

$$\begin{cases} (0, \mu_1) \in U\,, \quad \partial U \cap \mathcal{S} \subset E^{\varepsilon}(\mu_1) \cap K_{\eta}^{+}\,, \quad \text{and} \\ \partial E^{\varepsilon}(\mu_1) \cap K_{\eta}^{+} \cap U = \emptyset\,. \end{cases} \tag{5.23}$$

Recalling our definition of T_n as the component of $\mathcal{S}_n \setminus (E^{\varepsilon}(\mu_1) \cap K_{\varepsilon}^{+})$ containing $(0, \mu_1)$, we can apply Lemma 5.7 to the mapping $\Phi_{\lambda}^{(n)}$ in place of Φ_{λ}, thus obtaining

$$\partial E^{\varepsilon}(\mu_1) \cap K_{\eta}^{+} \cap T_n \neq \emptyset \quad \text{for each} \quad n \in \mathbb{N}\,. \tag{5.24}$$

Note that our asymptotic estimates (4.13), (4.30), and (4.31) guarantee the crucial assumption of Lemma 5.7, namely, that zero is an isolated solution of $\Phi_{\mu_1}^{(n)}(u) = 0$ for each $n \in \mathbb{N}$.

Combining the facts (5.23) and (5.24), we arrive at $\partial U \cap T_n \neq \emptyset$. So let us choose $(u_n, \lambda_n) \in \partial U \cap T_n$ for each $n \in \mathbb{N}$. Since U is bounded in E, we may pass to a subsequence $\{u_{n_k}, \lambda_{n_k}\}_{k=1}^{\infty}$ that converges weakly in E, that is, $u_{n_k} \rightharpoonup u^*$ weakly in $W_0^{1,p}(\Omega)$ and $\lambda_{n_k} \to \lambda^*$ in $\mathbb{R}$ as $k \to \infty$. Consequently, $u_{n_k} \to u^*$ strongly in $L^p(\Omega)$, by Rellich's theorem. Since $\Phi_{\lambda_{n_k}}^{(n_k)}(u_{n_k}) = 0$ for each $k \in \mathbb{N}$, with $\Phi_{\lambda_{n_k}}^{(n_k)}$ defined by (5.22), we conclude that

$$\mathcal{A}(u_{n_k}) \to \lambda^*\, \mathcal{B}(u^*) + \mathcal{H}(u^*; \lambda^*) \quad \text{in} \quad L^{p'}(\Omega) \quad \text{as} \quad k \to \infty\,.$$

This implies $u_{n_k} \to u^*$ strongly in $W_0^{1,p}(\Omega)$ as $k \to \infty$. (Note that $\mathcal{A}^{-1} : \ X' \to X$ is continuous due to ($\mathrm{A_i}$) or ($\mathrm{A_{ii}}$).) The embeddings $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ being compact, we have also $\mathcal{A}(u_{n_k}) \to \mathcal{A}(u^*)$ strongly in $W^{-1,p'}(\Omega)$. It follows that

$$\Phi_{\lambda^*}(u^*) \equiv \mathcal{A}(u^*) - \lambda^*\, \mathcal{B}(u^*) - \mathcal{H}(u^*; \lambda^*) = 0\,. \tag{5.25}$$

The boundary ∂U being closed, we conclude that $(u^*, \lambda^*) \in \partial U$. Moreover, by (5.25), we have also

$$(u^*, \lambda^*) \in \left(\mathcal{S} \cap \overline{E^R(\mu_1)}\right) \setminus \left(E^{\varepsilon}(\mu_1) \cap K_{\eta}^{+}\right) \subset k_2\,.$$

However, this contradicts $\partial U \cap k_2 = \emptyset$. □

Proof of Theorem 3.7. With Lemma 5.8 in hand, the proof of Theorem 3.7 follows the same pattern as in Dancer [10, Proof of Theorem 2, p. 1073]. Therefore, we omit the details. □

6. Parameter oscillations about μ_1

This section provides an example of oscillations of the parameter λ around μ_1 for solutions $(u, \lambda) \in E$, where $\|u\|_{W_0^{1,p}(\Omega)} \to \infty$ and $\lambda \to \mu_1$. We need some subtle regularity properties of φ_1 to perform some computations in the following example.

Unfortunately, these properties are known only for $p > 2$ in the one-dimensional case to which we restrict ourselves.

In the one-dimensional case it is convenient to work with $\Omega = (0, \pi_p)$, where

$$\pi_p \stackrel{\text{def}}{=} 2 \int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p \sin\frac{\pi}{p}}\,.$$

In the following example, we will use an asymptotic formula from [16] that takes a more readable form if the positive eigenfunction φ_1 is normalized by $\int_0^{\pi_p} \varphi_1^p\,\mathrm{d}x = 1$. For this asymptotic formula we also need the weighted Sobolev space $\mathcal{D}_{\varphi_1}$ defined to be the completion of $W_0^{1,p}(0,\pi_p)$ with respect to the norm $\|u\|_{\mathcal{D}_{\varphi_1}} \stackrel{\text{def}}{=} (\int_0^{\pi_p} |\varphi_1'(x)|^{p-2}|u'(x)|^2\mathrm{d}x)^{1/2}$. We have the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow C^\beta[0,\pi_p]$ where $\beta = \frac{1}{p-1} \in (0,1)$ (see [34, Lemma 4.5] or [35, Lemma 4.4]).

Example 6.1. Take $p > 2$, $0 \le \alpha < p-2$, and $f^\top \in L^\infty(0,\pi_p)$ with $\int_0^{\pi_p} f^\top \varphi_1 \mathrm{d}x = 0$ and $f^\top \not\equiv 0$. Let us consider the boundary value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' - \lambda |u|^{p-2}u + |u|^\alpha \sin(u) = f^\top + a\varphi_1 \quad \text{in} \quad (0,\pi_p) \\ u(0) = u(\pi_p) = 0\,. \end{cases} \tag{6.1}$$

Note that Theorem 3.10 applies to this problem. Below we will show that if either of the conditions $a = 0$ or $\alpha > 1/p' = 1 - (1/p)$ ($p' \stackrel{\text{def}}{=} p/(p-1)$) is satisfied, then there exist two continua $\mathcal{Z}_{\mu_1}^\pm \subset \mathcal{S}$ as in Theorem 3.10 and such that $\mathcal{Z}_{\mu_1}^+$ and $\mathcal{Z}_{\mu_1}^-$ exhibit the following additional “oscillation” phenomenon (with λ oscillating about μ_1). We write down this phenomenon for $\mathcal{Z}_{\mu_1}^+$ only; for $\mathcal{Z}_{\mu_1}^-$ it is analogous:

(OC) There exist a number $\delta > 0$ and two sequences $\{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty \subset \mathbb{R}$, $0 < \beta_n < \gamma_n < \beta_{n+1} < \gamma_{n+1}$ for all $n \in \mathbb{N}$, with $\beta_n, \gamma_n \to +\infty$ as $n \to \infty$, and such that for all $(u,\lambda) \in \mathcal{Z}_{\mu_1}^+ \cup \mathcal{Z}_{\mu_1}^-$ with $|\lambda - \mu_1| < \delta$ we have
  (i) $\int_0^{\pi_p} u\varphi_1\,\mathrm{d}x = \beta_n \implies \lambda > \mu_1$;
  (ii) $\int_0^{\pi_p} u\varphi_1\,\mathrm{d}x = \gamma_n \implies \lambda < \mu_1$.

As a consequence, the set $\mathcal{Z}_{\mu_1}^+$ being connected, for every $n \in \mathbb{N}$ large enough there exists $(u_n, \mu_1) \in \mathcal{Z}_{\mu_1}^+$ such that $\beta_n < \int_0^{\pi_p} u_n\varphi_1\,\mathrm{d}x < \gamma_n$; see Figure 1. Clearly, $\int_0^{\pi_p} u_n\varphi_1\,\mathrm{d}x \to +\infty$ as $n \to \infty$.

Notice that, in this example, $\int_0^{\pi_p} u_n\varphi_1\,\mathrm{d}x \to +\infty$ as $n \to \infty$ forces $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$. This means that Drábek’s hypothesis [14, Eq. (14.43), p. 191] for bifurcations from infinity at $(\pm\infty, \mu_1)$ is violated for the boundary value problem (6.1).

This example relates to results from Dancer [11] obtained for the semilinear case $p = 2$ and for $\alpha = 0$.

Proof of the statement (*OC*) *in Example* 6.1. Our proof is based on asymptotic estimate proved in [16] and *stationary phase argument*, see, e.g., [17].

Let $f = f^\top + a\varphi_1$ We take $t \in \mathbb{R} \setminus \{0\}$ with $|t|$ small enough. Multiplying equation (6.1) by t^α and denoting $w = t^{\alpha/(p-1)}u$, we arrive at

$$\begin{cases} -\big(|w'|^{p-2}w'\big)' - \lambda|w|^{p-2}w = t^\alpha\big(f^\top + a\varphi_1\big) \\ \qquad\qquad - |t|^{\alpha(p-\alpha-1)/(p-1)}|w|^\alpha \sin\big(t^{-\alpha/(p-1)}w\big)\,; \\ w(0) = w(\pi_p) = 0 \end{cases} \tag{6.2}$$

The right-hand side satisfies the assumptions of Theorem 4.1 from [16] and we thus obtain that large solutions $(u,\lambda) \in \mathcal{Z}^+_{\mu_1}$ satisfy $u = t^{-1}(\varphi_1 + v^\top)$ with $t \to 0+$ and

$$\begin{aligned} \lambda - \mu_1 = -t^{p-1-\alpha} \int_0^{\pi_p} \Big[t^\alpha a\varphi_1 + |\varphi_1 + v^\top|^\alpha \sin\big(t^{-1}(\varphi_1 + v^\top)\big)\Big]\,\varphi_1\,\mathrm{d}x \\ + (p-2)t^{2(p-1-\alpha)}\,\mathcal{Q}_0(V^\top, V^\top) + o\big(|t|^{2(p-1-\alpha)}\big) \end{aligned} \tag{6.3}$$

where $V^\top \in \mathcal{D}_{\varphi_1}$ is the limit $t^{-p+\alpha+1}v^\top \to V^\top$ in $\mathcal{D}_{\varphi_1}$ as $t \to 0+$. Thanks to the generalized Riemann–Lebesgue lemma [32, Prop. 2.1], we have

$$|\varphi_1 + v^\top|^\alpha \sin\big(t^{-1}(\varphi_1 + v^\top)\big) \overset{*}{\rightharpoonup} 0$$

weakly* in $L^\infty(0,1)$. Hence, the right hand-side of equation (6.2) converges weakly* in $L^\infty(0,1)$ to a function f^* given by $f^* = 0$ if $\alpha > 0$, and $f^* = f^\top$ if $\alpha = 0$ (in which is case $a = 0$ by our hypothesis). The limit function $V^\top \in \mathcal{D}_{\varphi_1}$ is the solution of the linearization of problem (6.2) at $(u,\lambda) = (\varphi_1, \mu_1)$, that is,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\left|\frac{\mathrm{d}\varphi_1}{\mathrm{d}x}\right|^{p-2}\frac{\mathrm{d}V^\top}{\mathrm{d}x}\right) - \mu_1\varphi_1^{p-2}V^\top = \frac{1}{p-1}\left(f^* - \varphi_1^{p-1}\int_0^{\pi_p}(f^*\varphi_1)\mathrm{d}x\right) \quad \text{in} \quad (0,1)\,; \\ V^\top(0) = 0, V^\top(1) = 0\,; \\ \int_0^1 V^\top\varphi_1\mathrm{d}x = 0\,; \end{aligned} \tag{6.4}$$

see [16, Thm. 4.1, pp. 445–446].

Owing to $0 \le \alpha < p-2$, we find that $t^{-1}v^\top \to 0$ in $\mathcal{D}_{\varphi_1}$ as $t \to 0+$. This fact will later allow us in Corollary C.2 to use a "stationary phase argument" to our problem and prove that

$$\begin{aligned} \int_0^{\pi_p} |\varphi_1 + v^\top|^\alpha \sin\big(t^{-1}(\varphi_1 + v^\top)\big)\varphi_1\mathrm{d}x = -K\big|\varphi_1(\pi_p/2)\big|^\alpha\varphi_1(\pi_p/2) \\ \cdot\ \sin\left(-\frac{\pi}{2p'} + t^{-1}\varphi_1(\pi_p/2)\right) \\ \cdot\ t^{-1/p'} + o\big(t^{-1/p'}\big) \end{aligned} \tag{6.5}$$

holds as $t \to 0+$. Here $1/p + 1/p' = 1$ and $K > 0$ is independent of t. Inserting (6.5) in the asymptotic estimate (6.3) and dividing it by $t^{p-1-\alpha+1/p'}$ we get

$$\begin{aligned}(\lambda - \mu_1)/t^{p-1-\alpha+1/p'} &= -\,K\,|\varphi_1(\pi_p/2)|^{\alpha}\,\varphi_1(\pi_p/2)\,\cdot\,\sin\left(-\frac{\pi}{2p'} + t^{-1}\varphi_1\big(\pi_p/2\big)\right) \\ &\quad + o(1) - t^{\alpha-1/p'}\,a\int_0^{\pi_p}\varphi_1^2\,\mathrm{d}x \\ &\quad + (p-2)t^{p-1-\alpha-1/p'}\,\mathcal{Q}_0(V^\top, V^\top) + o\left(|t|^{p-1-\alpha-1/p'}\right)\,.\end{aligned}$$

An easy calculation shows that

$$p - 1 - \alpha - 1/p' = \frac{(p-1)^2 - \alpha p}{p} > 0$$

thanks to $0 < \alpha < p - 2$. This means that

$$\begin{aligned}(\lambda - \mu_1)/t^{p-1-\alpha+1/p'} &= -K\big|\varphi_1(\pi_p/2)\big|^{\alpha}\varphi_1(\pi_p/2)\,\cdot\,\sin\left(-\frac{\pi}{2p'} + t^{-1}\varphi_1(\pi_p/2)\right) \\ &\quad - t^{\alpha-1/p'}a\int_0^{\pi_p}\varphi_1^2\,\mathrm{d}x + o(1)\end{aligned}$$

as $t \to 0+$. For $\alpha > 1/p'$ we find that

$$\begin{aligned}(\lambda - \mu_1)/t^{p-1-\alpha+1/p'} &= -K|\varphi_1(\pi_p/2)|^{\alpha}\varphi_1(\pi_p/2) \\ &\quad \cdot\,\sin\left(-\frac{\pi}{2p'} + t^{-1}\varphi_1(\pi_p/2)\right) + o(1)\end{aligned}$$

as $t \to 0+$. Then the desired numbers β_n and γ_n are, for instance,

$$\beta_n = \big(3/2 + 1/(2p') + 2n + 2n_0\big)\,\pi\int_0^{\pi_p}\varphi_1(x)^2\,\mathrm{d}x/\varphi_1(\pi_p/2)$$

and

$$\gamma_n = \big(1/2 + 1/(2p') + 2n + 2n_0\big)\,\pi\int_0^{\pi_p}\varphi_1(x)^2\,\mathrm{d}x/\varphi_1(\pi_p/2)$$

with $n_0 \in \mathbb{N}$ large enough. This completes the proof. □

Appendices

Appendix A. A priori regularity results

Here we state the main regularity result for a weak solution $u \in W_0^{1,p}(\Omega)$ of the Dirichlet boundary value problem

$$-\operatorname{div}\big(\mathbf{a}(x, \nabla u)\big) = h\big(x, u(x)\big) \quad \text{in} \quad \Omega\,; \quad u = 0 \quad \text{on} \quad \partial\Omega\,. \tag{A.1}$$

This a priori regularity is used throughout the entire article.

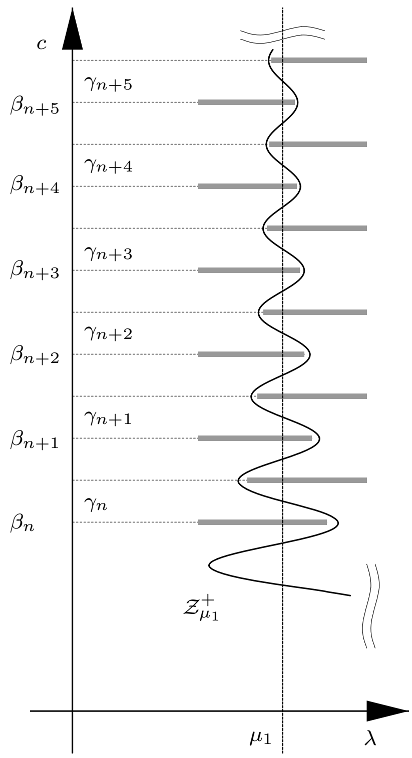

FIGURE 1. Bifurcations from infinity of solutions to (6.1): a sketch of the set $\mathcal{Z}^+_{\mu_1}$ for large positive solutions; here $c \stackrel{\text{def}}{=} \int_0^{\pi_p} u\varphi_1 \, dx$.

Proposition A.1. *Let* $1 < p < \infty$ *and let hypotheses* (**A**) *and* (**H**) *be satisfied. Assume that* $u \in W_0^{1,p}(\Omega)$ *is a weak solution of problem* (A.1)*. Then* $u \in C^{1,\beta}(\Omega)$ *where* $\beta \in (0,1)$ *is a constant independent from* u*. If, in addition,* $\partial\Omega$ *is a compact manifold of class* $C^{1,\alpha}$ *for some* $\alpha \in (0,1)$*, then* $\beta \in (0,\alpha)$ *can be chosen such that* $u \in C^{1,\beta}(\overline{\Omega})$*. Moreover,* β *is again independent from* u*, and* $\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C$ *where* $C > 0$ *is some constant depending solely upon* Ω*,* A*,* h*,* N*,* p*, and the norm* $\|u\|_{L^{p_0}(\Omega)}$ *with*

$$p_0 = \begin{cases} p^* = \frac{Np}{N-p} & \text{if} \quad p < N\,; \\ 2p & \text{if} \quad p \geq N\,. \end{cases}$$

Notice that, owing to the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p_0}(\Omega)$, we have also $\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C'$, where the constant C' depends solely upon Ω, A, h, N, p, and the norm $\|u\|_{W_0^{1,p}(\Omega)}$. Similarly, one obtains $\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C''$ as well, where the constant C'' depends solely upon Ω, A, h, N, p, and the norm $\|u\|_{L^\infty(\Omega)}$. These two consequences of Proposition A.1 will be used quite often in the sequel.

Proposition A.1 is, in fact, a combination of the following two lemmas, in which we keep our hypotheses and notation from the proposition:

Lemma A.2. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $g(\,\cdot\,, s) \in L^1_{\mathrm{loc}}(\Omega)$ for every $s \in \mathbb{R}$, and the following inequality holds with some constants $a > 0$ and $b \geq 0$:*

$$s \cdot g(x, s) \leq a|s|^p + b|s| \quad \textit{for all} \quad s \in \mathbb{R} \quad \textit{and a.e.} \quad x \in \Omega\,.$$

Assume that $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega \big\langle \mathbf{a}(x, \nabla u), \nabla\phi \big\rangle \,\mathrm{d}x = \int_\Omega g\big(x, u(x)\big)\,\phi \,\mathrm{d}x \quad \textit{for all} \quad \phi \in C_{\mathrm{c}}^\infty(\Omega)\,.$$

Then $u \in L^\infty(\Omega)$ and there exists a constant $c > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq c$, where c depends solely upon a, b, N, p, and $\|u\|_{L^{p_0}(\Omega)}$.

This is a special case of a more general result shown in Anane's thesis [4, Théorème A.1, p. 96]. Although his proof is carried out only for

$$\mathbf{a}(x, \xi) \equiv \frac{1}{p}\,\partial_\xi A(x, \xi) = |\xi|^{p-2}\xi\,, \quad (x, \xi) \in \Omega \times \mathbb{R}^N\,, \tag{A.2}$$

one can rewrite it directly for our more general case.

Lemma A.3. *Assume that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem* (A.1) *such that $u \in L^\infty(\Omega)$. Then $u \in C^{1,\beta}(\Omega)$ where $\beta \in (0, 1)$ is a constant independent from u. If, in addition, $\partial\Omega$ is a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, then $\beta \in (0, \alpha)$ can be chosen such that $u \in C^{1,\beta}(\overline{\Omega})$. Moreover, β is again independent from u, and $\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C'$ where $C' > 0$ is some constant depending solely upon Ω, A, h, N, p, and the norm $\|u\|_{L^\infty(\Omega)}$.*

The first statement of this lemma, interior regularity in $C^{1,\beta}(\Omega)$, was established independently by DiBenedetto [13, Theorem 2, p. 829] and Tolksdorf [39, Theorem 1, p. 127]. The second statement, regularity near the boundary, is due to Lieberman [25, Theorem 1, p. 1203]. The constant β depends solely upon α, N and p. We keep the meaning of the constants α and β throughout the entire article and denote by β' an arbitrary, but fixed number such that $0 < \beta' < \beta < \alpha < 1$. Last but not least, Lieberman's regularity results have been shown for the Neumann boundary conditions as well.

While Anane's proof of Lemma A.2 is based on the special form of $\mathbf{a}(x, \xi) \equiv 1/p\,\partial_\xi A(x, \xi)$ with the positively p-homogeneous potential $A(x,\,\cdot\,)$ satisfying also hypothesis (2.2), Lemma A.3 is valid with any vector field $\mathbf{a} \equiv (a_i)_{i=1}^N : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying

$$a_i \in C^0(\Omega \times \mathbb{R}^N) \cap C^1\Big(\Omega \times \big(\mathbb{R}^N \setminus \{0\}\big)\Big) \quad (i = 1, 2, \ldots, N)$$

together with the ellipticity and growth conditions (2.3), (2.4) and (2.5).

Appendix B. Bifurcations from zero or infinity

Let us consider a sequence of nontrivial solutions $\{(u_n, \lambda_n)\}_{n=1}^{\infty} \subset \mathcal{S}$ of problem (2.1), i.e., for each $n = 1, 2, \dots$ the integral identity

$$\int_\Omega \langle \mathbf{a}(x, \nabla u_n), \nabla \phi \rangle \, \mathrm{d}x = \lambda_n \int_\Omega B \, |u_n|^{p-2} u_n \, \phi \, \mathrm{d}x + \int_\Omega h(x, u_n; \lambda_n) \, \phi \, \mathrm{d}x \tag{B.1}$$

holds for all $\phi \in W_0^{1,p}(\Omega)$. We assume

$$-\infty < \lambda_n \le \mu_2 - \delta \quad (n = 1, 2, \dots) \tag{B.2}$$

where $\delta \in (0, \mu_2 - \mu_1)$ is a constant and μ_2 stands for the second eigenvalue of the quasilinear operator

$$\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) : u \mapsto -\operatorname{div}\Big(\mathbf{a}\big(\,\cdot\,, \nabla u(\,\cdot\,)\big)\Big) .$$

A variational characterization of μ_2 by a minimax formula is due to Anane [4, Remarques 2.2, pp. 15–16]. It is shown in Anane and Tsouli [5, Prop. 2, p. 5] that there is no eigenvalue in the open interval (μ_1, μ_2). Although the last two claims have been proved only for the case of the positive Dirichlet p-Laplacian $-\Delta_p$, that is, for

$$\mathbf{a}(x, \xi) \equiv \frac{1}{p} \, \partial_\xi A(x, \xi) = |\xi|^{p-2} \xi \, , \quad (x, \xi) \in \Omega \times \mathbb{R}^N \, ,$$

their proofs carry over directly to our more general case.

Now we need to distinguish between solutions (u_n, λ_n) with u_n having arbitrarily small or arbitrarily large norm (bifurcations from zero or infinity, respectively). More precisely, we will show that for this purpose any of the three norms $\|u_n\|_{W_0^{1,p}(\Omega)}$, $\|u_n\|_{L^\infty(\Omega)}$, or $\|u_n\|_{C^{1,\beta}(\overline{\Omega})}$ can be employed. Recall that $0 < \beta < \alpha < 1$ are constants from Proposition A.1. To verify this claim, we need to employ Lemma 2.2, the proof of which is given next.

B.1. Proof of Lemma 2.2

To verify (a), we first notice that $h(x, u(x); \lambda) = 0$ if $u(x) = 0$, and estimate

$$\frac{\big|h\big(x, u(x); \lambda\big)\big|}{\|u\|_{L^\infty(\Omega)}^{p-1}} = \frac{\big|h\big(x, u(x); \lambda\big)\big|}{\big|u(x)\big|^{p-1}} \left(\frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}} \right)^{p-1} \le \frac{\big|h\big(x, u(x); \lambda\big)\big|}{|u(x)|^{p-1}} \tag{B.3}$$

if $u(x) \neq 0$. From $\|u\|_{L^\infty(\Omega)} \to 0$ we get $u(x) \to 0$ uniformly for a.e. $x \in \Omega$. Finally, using (2.10) we arrive at (2.14).

To prove (b), let us take a sequence $\{u_n\}_{n=1}^{\infty} \subset L^\infty(\Omega)$ with $\|u_n\|_{L^\infty(\Omega)} \to \infty$ as $n \to \infty$. We split the domain $\Omega = A_n \cup B_n$ where

$$A_n = \Big\{ x \in \Omega : \big|u_n(x)\big| \le \|u_n\|_{L^\infty(\Omega)}^{\frac{1}{2}} \Big\} ,$$

$$B_n = \Big\{ x \in \Omega : \big|u_n(x)\big| > \|u_n\|_{L^\infty(\Omega)}^{\frac{1}{2}} \Big\} .$$

For $x \in A_n$ we infer from inequality (2.11) that

$$\frac{\bigl|h\bigl(x, u_n(x); \lambda\bigr)\bigr|}{\|u_n\|_{L^\infty(\Omega)}^{p-1}} = \frac{\bigl|h\bigl(x, u_n(x); \lambda\bigr)\bigr|}{1 + \bigl|u_n(x)\bigr|^{p-1}} \cdot \frac{1 + \bigl|u_n(x)\bigr|^{p-1}}{\|u_n\|_{L^\infty(\Omega)}^{p-1}} \leq C\, \frac{1 + \|u_n\|_{L^\infty(\Omega)}^{(p-1)/2}}{\|u_n\|_{L^\infty(\Omega)}^{p-1}}\,. \tag{B.4}$$

For $x \in B_n$ we infer from (B.3) that

$$\frac{\bigl|h\bigl(x, u_n(x); \lambda\bigr)\bigr|}{\|u_n\|_{L^\infty(\Omega)}^{p-1}} \leq \frac{\bigl|h\bigl(x, u_n(x); \lambda\bigr)\bigr|}{|u_n(x)|^{p-1}}\,. \tag{B.5}$$

Now let χ_{A_n} and χ_{B_n} denote the characteristic functions of the sets A_n and B_n, respectively. Combining inequalities (B.4) and (B.5) we arrive at

$$\frac{|h(x, u_n(x); \lambda)|}{\|u_n\|_{L^\infty(\Omega)}^{p-1}} \leq C\, \frac{1 + \|u_n\|_{L^\infty(\Omega)}^{(p-1)/2}}{\|u_n\|_{L^\infty(\Omega)}^{p-1}}\, \chi_{A_n}(x) + \frac{|h(x, u_n(x); \lambda)|}{|u_n(x)|^{p-1}}\, \chi_{B_n}(x) \tag{B.6}$$

for $x \in \Omega$. The first summand clearly tends to zero as $n \to \infty$, whereas the second one tends to zero pointwise for a.e. $x \in \Omega$ and uniformly for every $\lambda \in \mathbb{R}$, by (2.12). This proves (2.15). □

Lemma 2.2 has the following important corollary.

Corollary B.1. *Let* $1 \leq q < \infty$. *In both alternatives,* (a) *and* (b)*, of* Lemma 2.2 *we have*

$$\bigl\|h\bigl(\,\cdot\,, u(\,\cdot\,); \lambda\bigr)\bigr\|_{L^q(\Omega)} / \|u\|_{L^\infty(\Omega)}^{p-1} \to 0 \tag{B.7}$$

as $\|u\|_{L^\infty(\Omega)} \to 0$ *or* $\|u\|_{L^\infty(\Omega)} \to \infty$*, respectively, uniformly for every* $\lambda \in \mathbb{R}$.

Proof. In the situation of alt. (a), we can combine inequality (2.9) with the Lebesgue dominated convergence theorem to obtain (B.7) as $\|u\|_{L^\infty(\Omega)} \to 0$.

The same argument applies to alt. (b), of course, with (2.9) replaced by (2.11). □

B.2. A priori results – bifurcations from zero

Lemma B.2. *Let* $\{(u_n, \lambda_n)\}_{n=1}^\infty \subset \mathcal{S}$ *be as specified above. Then the following three statements are equivalent, as* $n \to \infty$:

(i) $\|u_n\|_{W_0^{1,p}(\Omega)} \to 0$;
(ii) $\|u_n\|_{L^\infty(\Omega)} \to 0$;
(iii) $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to 0$.

Moreover, in all three cases we have $\lambda_n \to \mu_1$ *and the sequence* $w_n \stackrel{\text{def}}{=} u_n / \|u_n\|_{L^\infty(\Omega)}$ *is the union of two disjoint subsequences* $\{w'_n\}_{n=1}^\infty$ *and* $\{w''_n\}_{n=1}^\infty$*, one of them possibly empty, such that, if nonempty, they satisfy* $w'_n \to \varphi_1 / \|\varphi_1\|_{L^\infty(\Omega)}$

and/or $w_n'' \to -\varphi_1/\|\varphi_1\|_{L^\infty(\Omega)}$ *in* $C^{1,\beta'}(\overline{\Omega})$ *as* $n \to \infty$. *Here,* $\beta' \in (0, \beta)$ *is arbitrary.*

Proof. Clearly, (iii) implies (i) and (ii). We will prove (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii).

(i) $\Longrightarrow$ (ii): The function $w_n \stackrel{\text{def}}{=} u_n/\|u_n\|_{W_0^{1,p}(\Omega)}$ satisfies $\|w_n\|_{W_0^{1,p}(\Omega)} = 1$ together with

$$\int_\Omega \big\langle \mathbf{a}(x, \nabla w_n), \nabla\phi \big\rangle \, \mathrm{d}x = \int_\Omega g_n\big(x, w_n(x)\big)\, \phi \, \mathrm{d}x \tag{B.8}$$

for all $\phi \in W_0^{1,p}(\Omega)$, by equation (B.1), where we have abbreviated

$$g_n(x, s) \stackrel{\text{def}}{=} \lambda_n\, B(x)\, |s|^{p-2} s + \frac{h\big(x, s\|u_n\|_{W_0^{1,p}(\Omega)}; \lambda_n\big)}{\|u_n\|_{W_0^{1,p}(\Omega)}^{p-1}}, \quad (x, s) \in \Omega \times \mathbb{R}\,.$$

Our hypotheses $0 \le B \in L^\infty(\Omega)$ and (2.9) guarantee

$$s \cdot g_n(x, s) \le a|s|^p + b|s| \quad \text{for all} \quad s \in \mathbb{R} \quad \text{and a.e.} \quad x \in \Omega\,,$$

where

$$a = (\mu_2 - \delta)\|B\|_{L^\infty(\Omega)} + C > 0 \quad \text{and} \quad b = 0\,.$$

We may apply Lemma A.2 to (B.8) to conclude that $w_n \in L^\infty(\Omega)$ and there exists a constant $c > 0$ such that $\|w_n\|_{L^\infty(\Omega)} \le c$, where c is independent from $n = 1, 2, \ldots$. Consequently, we have $\|u_n\|_{L^\infty(\Omega)} \le c\, \|u_n\|_{W_0^{1,p}(\Omega)}$ for $n = 1, 2, \ldots$, which proves (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii): This time we take $w_n \stackrel{\text{def}}{=} u_n/\|u_n\|_{L^\infty(\Omega)}$ which satisfies $\|w_n\|_{L^\infty(\Omega)} = 1$ together with

$$\int_\Omega \big\langle \mathbf{a}(x, \nabla w_n), \nabla\phi \big\rangle \, \mathrm{d}x = \lambda_n \int_\Omega B(x)\, |w_n|^{p-2} w_n\, \phi \, \mathrm{d}x + \int_\Omega \frac{h\big(x, w_n(x)\|u_n\|_{L^\infty(\Omega)}; \lambda_n\big)}{\|u_n\|_{L^\infty(\Omega)}^{p-1}}\, \phi \, \mathrm{d}x\,. \tag{B.9}$$

for all $\phi \in W_0^{1,p}(\Omega)$, by equation (B.1). We claim: $\liminf_{n\to\infty} \lambda_n \ge \mu_1$.

On the contrary, suppose that there is a subsequence of $\{(u_n, \lambda_n)\}_{n=1}^\infty$, denoted again in the same way, such that for some $\delta' \in (0, \mu_1)$ and for each $n = 1, 2, \ldots$ we have $\lambda_n \le \mu_1 - \delta'$. Taking $\phi = w_n$ in equation (B.9) we obtain

$$\int_\Omega A(x, \nabla w_n) \, \mathrm{d}x = \lambda_n \int_\Omega B(x)\, |w_n|^p \, \mathrm{d}x + \int_\Omega \frac{h\big(x, w_n(x)\|u_n\|_{L^\infty(\Omega)}; \lambda_n\big)}{\|u_n\|_{L^\infty(\Omega)}^{p-1}}\, w_n(x) \, \mathrm{d}x\,.$$

Now we use $\lambda_n \le \mu_1 - \delta'$ and the variational characterization of μ_1 from (2.8) to get

$$\frac{\delta'}{\mu_1} \int_\Omega A(x, \nabla w_n) \, \mathrm{d}x \le \int_\Omega \frac{\big|h\big(x, w_n(x)\|u_n\|_{L^\infty(\Omega)}; \lambda_n\big)\big|}{\|u_n\|_{L^\infty(\Omega)}^{p-1}}\, |w_n(x)| \, \mathrm{d}x\,. \tag{B.10}$$

Next, we apply Corollary B.1, alt. (a), and $\|w_n\|_{L^\infty(\Omega)} = 1$ to the right-hand side of equation (B.10) to conclude that $\int_\Omega A(x, \nabla w_n)\,\mathrm{d}x \to 0$ as $n \to \infty$. But this means $\|w_n\|_{W_0^{1,p}(\Omega)} \to 0$ as $n \to \infty$, by inequality (2.6). Finally, let us define $z_n \stackrel{\mathrm{def}}{=} w_n / \|w_n\|_{W_0^{1,p}(\Omega)}$. This function satisfies $\|z_n\|_{W_0^{1,p}(\Omega)} = 1$ together with

$$\int_\Omega \langle \mathbf{a}(x, \nabla z_n), \nabla\phi \rangle\,\mathrm{d}x = \lambda_n \int_\Omega B(x)\,|z_n|^{p-2} z_n\,\phi\,\mathrm{d}x + \int_\Omega \frac{h\big(x, \nu_n z_n(x); \lambda_n\big)}{\nu_n^{p-1}}\,\phi\,\mathrm{d}x$$

for all $\phi \in W_0^{1,p}(\Omega)$, by equation (B.9), where $\nu_n \stackrel{\mathrm{def}}{=} \|w_n\|_{W_0^{1,p}(\Omega)} \|u_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$. The same arguments we have used above in the proof of (i) $\Longrightarrow$ (ii) now reveal that also $\|w_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$, a contradiction with $\|w_n\|_{L^\infty(\Omega)} = 1$ for $n = 1, 2, \dots$. Therefore, our claim $\liminf_{n\to\infty} \lambda_n \geq \mu_1$ must be valid.

Taking $n \in \mathbb{N}$ large enough and using (B.2), we may assume $0 \leq \lambda_n \leq \mu_2 - \delta$ for every $n = 1, 2, \dots$. Thus, applying $0 \leq B \in L^\infty(\Omega)$ and (2.9) we observe that the function

$$f_n(x) \stackrel{\mathrm{def}}{=} \lambda_n\,B(x)\,|w_n|^{p-2} w_n + \frac{h\big(x, w_n(x)\|u_n\|_{L^\infty(\Omega)}; \lambda_n\big)}{\|u_n\|_{L^\infty(\Omega)}^{p-1}}\,, \quad x \in \Omega\,,$$

on the right-hand side of equation (B.9) is uniformly bounded a constant,

$$\big|f_n(x)\big| \leq M = (\mu_2 - \delta)\|B\|_{L^\infty(\Omega)} + C > 0\,, \quad x \in \Omega\,.$$

Now we may apply Lemma A.3 to (B.9) to conclude that $w_n \in C^{1,\beta}(\overline{\Omega})$ and there exists a constant $c' > 0$ such that $\|w_n\|_{C^{1,\beta}(\overline{\Omega})} \leq c'$, where c' is independent from $n = 1, 2, \dots$. Consequently, we have $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \leq c'\,\|u_n\|_{L^\infty(\Omega)}$ for $n = 1, 2, \dots$, which proves (ii) $\Longrightarrow$ (iii).

To complete the proof, we will derive $\lambda_n \to \mu_1$ from our proof of (ii) $\Longrightarrow$ (iii). Let us fix any $\beta' \in (0, \beta)$. The embedding $C^{1,\beta}(\overline{\Omega}) \hookrightarrow C^{1,\beta'}(\overline{\Omega})$ being compact by Arzelà–Ascoli's theorem, the sequence $\{w_n\}_{n=1}^\infty$ contains a subsequence that converges in $C^{1,\beta'}(\overline{\Omega})$ to some w; we denote it again by $w_n \to w$. Notice that $\|w\|_{L^\infty(\Omega)} = 1$. Extracting yet another convergent subsequence from $\{\lambda_n\}_{n=1}^\infty$ we may assume also $\lambda_n \to \lambda^*$. We let $n \to \infty$ in equation (B.9) and use Corollary B.1, alt. (a), to conclude that $w \in C^{1,\beta'}(\overline{\Omega})$ must satisfy

$$\int_\Omega Y \langle \mathbf{a}(x, \nabla w), \nabla\phi \rangle\,\mathrm{d}x = \lambda^* \int_\Omega B(x)\,|w|^{p-2} w\,\phi\,\mathrm{d}x \tag{B.11}$$

for all $\phi \in W_0^{1,p}(\Omega)$. Since $0 \leq \lambda^* \leq \mu_2 - \delta$ and μ_1 is the only eigenvalue of the operator $\mathcal{A}$ in the open interval $(-\infty, \mu_2)$, we must have $\lambda^* = \mu_1$. In addition, μ_1 being a simple eigenvalue, we have $w = \kappa\varphi_1$ in Ω where $\kappa \in \mathbb{R}$ satisfies $|\kappa| \cdot \|\varphi_1\|_{L^\infty(\Omega)} = 1$.

The sequence $\{\lambda_n\}_{n=1}^\infty \subset [0, \mu_2 - \delta]$ being bounded and the cluster point $\lambda^* = \mu_1$ unique, we conclude that $\lambda_n \to \mu_1$ $(n \to \infty)$ holds not only for a suitable subsequence of $\{\lambda_n\}_{n=1}^{\infty}$, but also for the entire original sequence as well. Finally, the original sequence $\{w_n\}_{n=1}^{\infty}$ can have at most two cluster points, $\pm\varphi_1/\|\varphi_1\|_{L^\infty(\Omega)}$.

The lemma is proved. □

B.3. A priori results – bifurcations from infinity

Lemma B.3. *Let $\{(u_n, \lambda_n)\}_{n=1}^{\infty} \subset \mathcal{S}$ be as specified above. Then the following two statements are equivalent, as $n \to \infty$:*

(i) $\|u_n\|_{W_0^{1,p}(\Omega)} \to \infty$;

(ii) $\|u_n\|_{L^\infty(\Omega)} \to \infty$.

Moreover, if

($\mathbf{B}'$) $0 < b_0 \le B(x) \le b_1 < \infty$ *for a.e.* $x \in \Omega$ (b_0, b_1 *– constants*),

then in both cases we have $\lambda_n \to \mu_1$ and the sequence $w_n \stackrel{\text{def}}{=} u_n/\|u_n\|_{L^\infty(\Omega)}$ is the union of two disjoint subsequences $\{w_n'\}_{n=1}^{\infty}$ and $\{w_n''\}_{n=1}^{\infty}$, one of them possibly empty, such that, if nonempty, they satisfy $w_n' \to \varphi_1/\|\varphi_1\|_{L^\infty(\Omega)}$ and/or $w_n'' \to -\varphi_1/\|\varphi_1\|_{L^\infty(\Omega)}$ in $C^{1,\beta'}(\overline{\Omega})$ as $n \to \infty$. Here, $\beta' \in (0,\beta)$ is arbitrary.

Finally, either of the statements (i) *and* (ii) *is equivalent to* (*as* $n \to \infty$)

(iii) $\|u_n\|_{C^{1,\beta}(\overline{\Omega})} \to \infty$

provided that, in addition to ($\mathbf{B}'$), *either of the following two conditions is satisfied:*

(I) $-\infty < \Lambda \le \lambda_n \le \mu_2 - \delta$ *for all* $n = 1, 2, \ldots$ (Λ, δ *– constants,* $\delta > 0$);

(II) *inequality* (2.9) *holds.*

Notice that (all or some of) ($\mathbf{B}$), (B.2), and (2.11), respectively, have been replaced by stronger hypotheses ($\mathbf{B}'$), (I), and (2.9).

Proof. It is obvious that either of (i) and (ii) implies (iii).

(ii) $\Longrightarrow$ (i) is proved by contradiction using similar arguments as in the proof of (i) $\Longrightarrow$ (ii) in Lemma B.2 above.

(i) $\Longrightarrow$ (ii) is proved by contradiction, as well. Taking $\phi = u_n$ in equation (B.1) we obtain

$$\int_\Omega A(x, \nabla u_n)\, \mathrm{d}x = \lambda_n \int_\Omega B(x)\, |u_n|^p\, \mathrm{d}x + \int_\Omega h\big(x, u_n(x); \lambda_n\big)\, u_n(x)\, \mathrm{d}x. \tag{B.12}$$

Now we apply inequalities (2.6), $\lambda_n \le \mu_2 - \delta$, and (2.11) to get

$$\frac{\gamma}{p-1} \int_\Omega |\nabla u_n|^p\, \mathrm{d}x \le \int_\Omega A(x, \nabla u_n)\, \mathrm{d}x \\ \le (\mu_2 - \delta) \int_\Omega B(x)\, |u_n|^p\, \mathrm{d}x + C \int_\Omega \big(|u_n|^p + |u_n|\big)\, \mathrm{d}x\,.$$

Consequently, if $\{u_n\}_{n=1}^{\infty}$ contains a subsequence bounded in $L^\infty(\Omega)$, then the same subsequence must be bounded also in $W_0^{1,p}(\Omega)$, thus contradicting (i).

To verify the remaining claims, we assume ($\mathbf{B}'$) throughout the rest of the proof.

Knowing (i) $\Longleftrightarrow$ (ii) already, let us assume (i). We begin by establishing

$$\lambda_n > -2C/b_0 \quad \text{for all} \quad n \geq n_0 \,, \tag{B.13}$$

where $C > 0$ is the constant from (2.11) and $n_0 \in \mathbb{N}$ is taken large enough. Contrary to (B.13), suppose that a subsequence of $\{\lambda_n\}_{n=1}^{\infty}$, denoted identically, satisfies $\lambda_n \leq -2C/b_0$ (< 0) for every $n \geq 1$. We apply inequalities (2.6), (2.11), and ($\mathbf{B}'$) to equation (B.12) to estimate

$$\begin{aligned}
\frac{\gamma}{p-1} \int_\Omega |\nabla u_n|^p \,\mathrm{d}x &\leq \int_\Omega A(x, \nabla u_n) \,\mathrm{d}x \\
&= \lambda_n \int_\Omega B(x)\, |u_n|^p \,\mathrm{d}x + \int_\Omega h\big(x, u_n(x); \lambda_n\big)\, u_n(x) \,\mathrm{d}x \\
&\leq \lambda_n\, b_0 \int_\Omega |u_n|^p \,\mathrm{d}x + C \int_\Omega \big(|u_n|^p + |u_n|\big) \,\mathrm{d}x \\
&\leq (\lambda_n b_0 + 2C) \int_\Omega |u_n|^p \,\mathrm{d}x + C\, |\Omega|_N \leq C\, |\Omega|_N \,.
\end{aligned}$$

This is a contradiction to (i), so (B.13) must be valid. The convergence $\lambda_n \to \mu_1$, together with all the remaining claims for $w_n \stackrel{\text{def}}{=} u_n/\|u_n\|_{L^\infty(\Omega)}$, is now derived in the same way as in the proof of Lemma B.2.

Finally, assume that (iii) and ($\mathbf{B}'$) hold together with (I) or (II). First, it turns out that condition (II) implies (I); this can be deduced from equation (B.12), using (II): If $\lambda_n \leq 0$ and $u_n \not\equiv 0$ in Ω, then in equation (B.12) we can estimate

$$\int_\Omega A(x, \nabla u_n) \,\mathrm{d}x - \lambda_n\, b_0 \int_\Omega |u_n|^p \,\mathrm{d}x \leq C \int_\Omega |u_n|^p \,\mathrm{d}x \,.$$

In particular, we must have $-\lambda_n\, b_0 \leq C$. Thus, condition (I) holds with $\Lambda = -C/b_0$. Assuming now condition (I), we may apply Lemma A.3 to equation (B.1) to conclude that if $\{u_n\}_{n=1}^{\infty}$ contains a subsequence bounded in $L^\infty(\Omega)$, then the same subsequence is bounded also in $C^{1,\beta}(\overline{\Omega})$, which contradicts (iii). Hence, (iii) $\Longrightarrow$ (ii). □

Appendix C. Stationary phase argument

We restrict ourselves to $2 < p < \infty$ throughout this section.

Lemma C.1 (A generalization of Erdélyi [17, Theorem on p. 52]). *Let $g : [0, \pi_p/2] \times \mathbb{R}_+ \to \mathbb{C}$ be continuous with $g(\,\cdot\,, \tau) \to \hat{g}$ uniformly on $[0, \pi_p/2]$ as $\tau \to +\infty$. (Hence, also $\hat{g} : [0, \pi_p/2] \to \mathbb{C}$ is continuous.) Assume that the functions $g(\,\cdot\,, \tau) : [0, \pi_p/2] \to \mathbb{C}$ are absolutely equicontinuous, that is,*

(AEC) *for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\int_M \mathrm{d}x < \delta \Longrightarrow \int_M \left| \frac{\partial g}{\partial x}(x, \tau) \right| \mathrm{d}x < \varepsilon$$

holds for every Lebesgue-measurable set $M \subset [0, \pi_p/2]$.

Finally, let $h : [0,\pi_p/2] \to \mathbb{R}$ *be continuously differentiable with*

$$h'(x) = (\pi_p/2 - x)^{\sigma-1} h_1(x)\,, \tag{C.1}$$

where $\sigma > 1$ *is a constant and* $h_1 \in C^1[0,\pi_p/2]$ *is a strictly positive function.*

Then there exists a constant $K \equiv K(\sigma, h_1(\pi_p/2)) > 0$*, independent from* τ*, such that*

$$\int_0^{\pi_p/2} g(x,\tau) \cdot e^{\mathrm{i}\tau h(x)} \mathrm{d}x = -K\hat{g}(\pi_p/2) \cdot e^{\mathrm{i}[-\frac{\pi}{2\sigma}+\tau h(\pi_p/2)]} \cdot \tau^{-1/\sigma} + o\big(\tau^{-1/\sigma}\big)$$

as $\tau \to +\infty$.

Proof. We proceed as in [17, proof of the theorem on p. 52]. We take $\lambda = \mu = 1$, $\alpha = 0$, $\beta = \pi_p/2$, $\varrho = 1$, and $\sigma > 1$. In [17] a stronger smoothness hypothesis is imposed on $g(\,\cdot\,,\tau)$, namely, that $g(\,\cdot\,,\tau) \in C^1[0,\pi_p]$ and $\left|\frac{\partial g}{\partial x}(x,\tau)\right| \le C \equiv$ const. $< +\infty$ for all $(x,\tau) \in [0,\pi_p/2] \times \mathbb{R}_+$. However, this hypothesis is essentially used only in an estimate contained in the second displayed formula from the bottom of page 55. Nevertheless, it is easy to see that this estimate holds also under the weaker hypothesis (AEC). Recall that we deal only with the special case $\lambda = \mu = 1$ in that estimate; hence, (AEC) is sufficient. The remaining parts of the proof are identical with [17]. □

Corollary C.2. *Let* $u = t^{-1}(\varphi_1 + v^\top)$ *be a solution of* (6.1) *with* $t > 0$ *small enough. Then the asymptotic formula*

$$\begin{aligned}\int_0^{\pi_p} |\varphi_1 + v^\top|^\alpha \sin\big(t^{-1}(\varphi_1 + v^\top)\big)\varphi_1 \mathrm{d}x = -K\big|\varphi_1(\pi_p/2)\big|^\alpha \varphi_1(\pi_p/2)&\\ \cdot\ \sin\left(-\frac{\pi}{2p'} + t^{-1}\varphi_1(\pi_p/2)\right)&\\ \cdot\ t^{-1/p'} + o\big(t^{-1/p'}\big)&\end{aligned}$$

holds as $t \to 0+$.

Proof. This claim is derived from Proposition C.1 as follows. One splits the integral as $\int_0^{\pi_p} \dots \mathrm{d}x = \int_0^{\pi_p/2} \dots \mathrm{d}x + \int_{\pi_p/2}^{\pi_p} \dots \mathrm{d}x$ and applies Proposition C.1 to both integrals on the right. We treat only the first integral in detail; the second one can be treated analogously. Observe that

$$\begin{aligned}&\int_0^{\pi_p/2} \sin\Big(t^{-1}\big(\varphi_1(x) + v^\top(x)\big)\Big)\big|\varphi_1(x) + v^\top(x)\big|^\alpha \varphi_1(x)\mathrm{d}x\\ &\qquad = \mathfrak{Im}\left[\int_0^{\pi_p/2} e^{\mathrm{i}t^{-1}\varphi_1(x)} e^{\mathrm{i}t^{-1}v^\top(x)}\big|\varphi_1(x) + v^\top(x)\big|^\alpha \varphi_1(x)\mathrm{d}x\right].\end{aligned}$$

We set $\tau = t^{-1}$, $g(x,\tau) = |\varphi_1(x) + v^\top(x)|^\alpha \cdot e^{\mathrm{i}\tau v^\top(x)} \cdot \varphi_1(x)$, and $h(x) = \varphi_1(x)$ for $x \in [0,\pi_p/2]$ and $\tau > 0$. Recall that $\varphi_1(x) > 0$ for every $x \in (0,\pi_p/2)$, by Remark 2.1. It is obvious that

$$g(x,\tau) \to \hat{g}(x) = \big|\varphi_1(x)\big|^\alpha \varphi_1(x) = \varphi_1(x)^{\alpha+1}$$

uniformly for $x \in [0, \pi_p/2]$ as $\tau \to +\infty$. This follows from $\tau^{p-1-\alpha} \cdot v^\top \to V^\top$ in $\mathcal{D}_{\varphi_1}$ as $\tau \to +\infty$ combined with $p-1-\alpha > 1$ and the embedding $\mathcal{D}_{\varphi_1} \hookrightarrow C^\beta[0, \pi_p]$ where $\beta = \frac{1}{p-1} \in (0,1)$ (see [34, Lemma 4.5] or [35, Lemma 4.4]). The first eigenfunction φ_1 of the p-Laplacian on the interval $(0, \pi_p)$ can be expressed by means of a special function $\sin_p$ and a constant π_p defined, e.g., in [27,28]; we set $\varphi_1(x) = \kappa \sin_p(x)$ where $\kappa = 1/\int_0^{\pi_p} (\sin_p(x))^p \,\mathrm{d}x$.

We set $\cos_p(x) \stackrel{\text{def}}{=} \sin_p'(x)$. As a consequence of formulas (30)–(32) on page 332 in [27], we obtain

$$\big|\cos_p(x)\big|^{p-2} \cos_p(x) = \Big(\frac{\pi_p}{2} - x\Big)(p-1)\big[1 - J(x)\big]$$

for $0 \le x \le \pi_p/2$, cf. estimate (33) on page 332 in [27], where we have introduced

$$J(x) \stackrel{\text{def}}{=} \begin{cases} \int_x^{\pi_p/2} |\sin_p(t)|^{p-2} \cos_p(t) \frac{t-x}{\pi_p/2-x} \,\mathrm{d}t & \text{if} \quad 0 \le x < \pi_p/2\,; \\ 0 & \text{if} \quad x = \pi_p/2\,. \end{cases}$$

Taking into account $\cos_p(x) > 0$ on $[0, \pi_p/2)$, we find $1 - J(x) > 0$ for $0 \le x \le \pi_p/2$ and thus

$$\cos_p(x) = \Big(\frac{\pi_p}{2} - x\Big)^{p'-1} (p-1)^{p'-1} \big[1 - J(x)\big]^{p'-1}\,.$$

This implies also $J(x) \to J(\pi_p/2) = 0$ as $x \to (\pi_p/2)-$. Hence, the function $h(x) = \sin_p x$ satisfies assumption (C.1), where $\sigma = p'$ and $h_1(x) = (p-1)^{p'-1} [1 - J(x)]^{p'-1}$ is continuously differentiable for $0 \le x \le \pi_p/2$. Indeed, substituting $\phi(t) \stackrel{\text{def}}{=} \frac{1}{p-1} |\sin_p(t)|^{p-2} \sin_p(t)$ for $0 \le t \le \pi_p$, we observe that both

$$J(x) = \int_x^{\pi_p/2} \phi'(t)\, \frac{t-x}{\pi_p/2 - x} \,\mathrm{d}t = \phi(\pi_p/2) - \frac{1}{\pi_p/2 - x} \int_x^{\pi_p/2} \phi(t) \,\mathrm{d}t \qquad \text{(C.2)}$$

and

$$\begin{aligned} J'(x) &= \frac{1}{(\pi_p/2 - x)^2} \int_x^{\pi_p/2} \phi'(t)(t-x) \,\mathrm{d}t - \frac{1}{\pi_p/2 - x} \int_x^{\pi_p/2} \phi'(t) \,\mathrm{d}t \\ &= \frac{1}{\pi_p/2 - x} J(x) - \frac{\phi(\pi_p/2) - \phi(x)}{\pi_p/2 - x} \end{aligned} \qquad \text{(C.3)}$$

are continuous functions of $x \in [0, \pi_p/2)$. Moreover, we compute

$$\lim_{x \to (\pi_p/2)-} J(x) = \phi(\pi_p/2) - \phi(\pi_p/2) = 0 = J(\pi_p/2)$$

and

$$\begin{aligned} \lim_{x \to (\pi_p/2)-} J'(x) &= - \lim_{x \to (\pi_p/2)-} \frac{J(\pi_p/2) - J(x)}{\pi_p/2 - x} - \lim_{x \to (\pi_p/2)-} \frac{\phi(\pi_p/2) - \phi(x)}{\pi_p/2 - x} \\ &= -J'(\pi_p/2) - \phi'(\pi_p/2) = -J'(\pi_p/2)\,, \end{aligned}$$

where $J'(\pi_p/2) = -\phi'(\pi_p/2)/2 = 0$, by the following calculation:

$$\begin{aligned}
J'(\pi_p/2) &= \lim_{x\to(\pi_p/2)-} \frac{J(\pi_p/2) - J(x)}{\pi_p/2 - x} \\
&= -\lim_{x\to(\pi_p/2)-} \frac{\phi(\pi_p/2) - \frac{1}{\pi_p/2-x}\int_x^{\pi_p/2}\phi(t)\mathrm{d}t}{\pi_p/2 - x} \\
&= -\lim_{x\to(\pi_p/2)-} \frac{1}{(\pi_p/2-x)^2}\left((\pi_p/2-x)\phi(\pi_p/2) - \int_x^{\pi_p/2}\phi(t)\mathrm{d}t\right) \\
&= -\lim_{x\to(\pi_p/2)-} \frac{1}{(\pi_p/2-x)^2}\int_x^{\pi_p/2}\big(\phi(\pi_p/2) - \phi(t)\big)\mathrm{d}t \\
&= -\lim_{x\to(\pi_p/2)-} \frac{1}{(\pi_p/2-x)^2}\int_x^{\pi_p/2}\int_t^{\pi_p/2}\phi'(\sigma)\mathrm{d}\sigma\mathrm{d}t \\
&= -\lim_{x\to(\pi_p/2)-} \frac{1}{(\pi_p/2-x)^2}\iint_{x\le t\le\sigma\le\pi_p/2}\phi'(\sigma)\mathrm{d}\sigma\mathrm{d}t = -\phi'(\pi_p/2)/2
\end{aligned}$$

as ϕ is continuous on $[0, \pi_p]$. This completes the proof. □

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