

KIDs are Non-Generic

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Abstract. We prove that the space-time developments of generic solutions of the vacuum constraint Einstein equations do not possess any global or local Killing vectors, when Cauchy data are prescribed on an asymptotically flat Cauchy surface, or on a compact Cauchy surface with mean curvature close to a constant, or for CMC asymptotically hyperbolic initial data sets. More generally, we show that non-existence of global symmetries implies, generically, non-existence of local ones. As part of the argument, we prove that generic metrics do not possess any local or global conformal Killing vectors.

1 Introduction

Let P be the linearization of the general relativistic constraints map, as defined by (8.1) below. Recall that a *Killing Initial Data* (KID) is a couple (N, Y) , defined on a spacelike hypersurface, where N is a function and Y is a vector field, such that $P^*(Y, N) = 0$, see (5.3)–(5.4) below. In vacuum space-times, with or without cosmological constant, KIDs are in one-to-one correspondence with Killing vectors in the associated space-time [13, 19].

A local Killing vector field is a solution X of the Killing equations defined on an open subset of a pseudo-Riemannian¹ manifold M ; local conformal Killing vector fields and local KIDs are defined in an analogous way.

When attempting to glue general relativistic initial data sets [12] one is faced with the need of proving the following:

Conjecture 1.1 *Generic general relativistic vacuum initial data sets have no local KIDs.*

The object of this paper is to establish such a fact under some supplementary conditions. For $\mathcal{U} \subset M$ let $\mathcal{K}(\mathcal{U})$ denote the set of KIDs on \mathcal{U} . We show, first, that non-existence of global KIDs implies, generically, non-existence of local ones:

Theorem 1.2 *Let $\Lambda \in \mathbb{R}$, and consider the collection of vacuum initial data sets with cosmological constant Λ on an n -dimensional manifold M with a $C^{k,\alpha}$ topology, $k \geq k_0(n)$, for some $k_0(n)$ ($k_0(3) = 6$)², $\alpha \in (0, 1)$. Let (K_0, g_0) in this*

¹ In our terminology a Riemannian metric is also pseudo-Riemannian.

²The function $k_0(n)$ is obtained, in dimension $n = 3$, by chasing the differentiability thresholds throughout the proof. We do not have an explicit estimate for $k_0(n)$ if $n > 3$, or for the function $\ell_0(n)$ appearing in Theorem 8.7 below, because some steps of the proof in those dimensions proceed via non-constructive arguments, see Section 7.

collection be such that

$$\mathcal{K}(M) = \{0\} . \tag{1.1}$$

1. Let $p \in M$ and consider the set

$$\mathcal{V}_p = \{\text{vacuum initial data such that } \mathcal{K}(\mathcal{U}) = \{0\}\}$$

for any neighborhood \mathcal{U} of p .

Then \mathcal{V}_p is open and dense in a neighborhood of (K_0, g_0) .

2. Define further:

$$\mathcal{V} = \{\text{vacuum initial data such that } \mathcal{K}(\mathcal{U}) = \{0\}\}$$

for any open subset \mathcal{U} of M .

Then \mathcal{V} is of second category in a neighborhood of (K_0, g_0) .

Identical results hold in the class of initial data with fixed constant $\text{tr}_g K$, as well as in the class of time symmetric initial data $K \equiv 0$.

(Recall that a set is of *second category* if it contains a countable intersection of open dense sets; in complete metric or Fréchet spaces such sets are dense.)

The $C^{k,\alpha}$ topology in Theorem 1.2, as well as in the remaining results below unless explicitly stated otherwise, can be understood as follows: one chooses some smooth complete Riemannian metric h on M , which is then used to calculate norms of tensors and their h -covariant derivatives. Other choices are possible, and this is discussed in more detail in Appendix A.

One expects that for generic initial data the no-global-KIDs condition (1.1) of Theorem 1.2 will be satisfied. Attempts to prove that require analytical tools which impose restrictions on the geometry. We concentrate therefore on three cases which seem to us to be the most important ones from the point of view of applications: compact manifolds without boundary, or asymptotically flat initial data sets, or conformally compactifiable initial data sets. Our next main result, when used in conjunction with Theorem 1.2, establishes Conjecture 1.1 in those cases:

Theorem 1.3 *Consider the following collections of vacuum initial data sets:*

1. $\Lambda = 0$ with an asymptotically flat region, or
2. $\text{tr}_g K = \Lambda = 0$ with an asymptotically flat region, or
3. $K = \Lambda = 0$ with an asymptotically flat region, or
4. with a conformally compactifiable region in which $\text{tr}_g K$ is constant, or

5. the trace of K is constant and the underlying manifold M is compact, with

$$(\operatorname{tr}_g K)^2 \geq \frac{2n}{(n-1)} \Lambda, \quad (1.2)$$

or

6. $K = 0$, M is compact, and the curvature scalar R satisfies $R = 2\Lambda \leq 0$,

with a $C^{k,\alpha} \times C^{k,\alpha}$ (weighted in the non-compact region) topology, with $k \geq k_0(n)$ for some $k_0(n)$ ($k \geq 6$ if $n = \dim M = 3$). For each such collection the subset of vacuum initial data sets without global KIDs is open and dense.

The weights in the asymptotic region should be chosen so that the metrics approach the Euclidean one as $r^{-\beta}$, for some $\beta \in (0, n-2]$. In the conformally compactifiable regions a topology as in [11, Theorem 6.7] with $0 \leq t < (n+1)/2$ should be used.

It would be of interest to have a version of points 4 and 5 without the CMC condition. Since the collection of initial data sets which have no global KIDs is open (see Proposition 4.2 below), for compact manifolds the proof of Theorem 1.3 also provides a large open collection of initial data sets which are close to CMC data and which have no global KIDs. However, the general case remains open. We think that the removal of the CMC condition in point 5 is the most notable problem left open by our paper.

Somewhat surprisingly, the above results require a considerable amount of non-trivial work. We first show that generic metrics have no local conformal Killing vectors, or local Killing vectors³. This is done by reducing the problem to a finite system of linear algebraic equations for the candidate vector, as well as a few of its derivatives, at a given point. While the argument is conceptually straightforward, there is some messy algebra involved when one wishes to show that those algebraic equations lead to the desired conclusion *for at least one metric*. This result is then used in the proof of Theorem 1.3. A similar argument is used for local KIDs, with an appropriately messier algebra. That would have settled the problem, if not for the fact that we want initial data satisfying the constraint equations. In order to take care of that we first use Taylor expansions to construct approximate solutions of the constraint equations near a point p . The gluing techniques of Corvino-Schoen [14] type, as extended in [10, 11], are then used to go from an approximate solution to a real one, establishing Theorem 1.2.

Some comments on the organization of this paper are in order. The heart of our analysis lies in Section 8, where we show how to perturb solutions of the vacuum constraint equations to solutions without KIDs, preserving the constraint

³The only related result known to us in the literature is in [15], where it is shown that on a compact boundaryless manifold the set of Riemannian metrics without nontrivial isometries is open and dense. The argument given there does not seem to be useful to get rid of local Killing vector fields, and makes essential use of the fact that M is compact without boundary. Moreover it is not clear how to adapt it to account for conformal Killing vectors, or for KIDs.

equations. This requires several preliminary results, such as a) perturbing initial data to get rid of KIDs, without necessarily satisfying any constraint equations, and b) perturbing metrics to get rid of conformal Killing vectors. The argument needed for a) is presented in Section 5 in dimension three, and in Section 7 in all dimensions. The advantage of the argument in Section 5 is that it gives an explicit differentiability threshold for the construction, in the physically important case $n = 3$, while the one in Section 7 leads to some uncontrollable, dimension dependent threshold. In Section 6 we show how to get rid of KIDs in the time-symmetric case, while remaining in the time-symmetric class. In Section 2 we construct functions that control existence, or lack thereof, of conformal Killing vector fields in dimension three. This result is the key for getting rid of KIDs on CMC initial data sets; it also sets the stage for the structure of the argument for KID-removal. As before, the higher-dimensional proof is carried out in Section 7, with some non-explicit differentiability threshold. In Section 3 we construct the corresponding functions for controlling Killing vectors. Here we obtain explicit differentiability thresholds in all dimensions. Perturbations removing conformal Killing vectors do of course remove Killing vectors as well, but the differentiability thresholds we obtain in the Killing case are explicit in all dimensions, and smaller than the corresponding conformal Killing threshold in dimension three. In Section 4 we show how the local perturbation arguments of the previous sections can be translated into category-type statements. This leads immediately to the question of topologies appropriate in our context, this is briefly discussed in Appendix A. Appendix B presents a monodromy-type argument for analytic overdetermined PDE systems, needed in the proofs of Section 7. All the results just described join forces in Section 9, where Theorems 1.2 and 1.3 are established.

2 Metrics without conformal Killing vectors near a point, $n=3$

We start with some preliminaries. Unless explicitly specified otherwise we assume in this section that dimension equals three. Recall that the Schouten tensor L_{ij} is given by

$$L_{ij} = R_{ij} - \frac{1}{4}g_{ij}R, \quad (2.1)$$

where g_{ij} is a pseudo-Riemannian metric and R_{ij} and R are respectively its Ricci and scalar curvature. Furthermore we define the Cotton tensor B_{ijk}

$$B_{ijk} = L_{i[j;k]}. \quad (2.2)$$

The tensor B_{ijk} has the following algebraic properties

$$B_{ijk} = B_{i[jk]}, \quad B^i{}_{ik} = 0, \quad B_{[ijk]} = 0, \quad (2.3)$$

which makes five degrees of freedom per space point. It also satisfies

$$B_{i[jk;l]} = 0. \quad (2.4)$$

Equivalently, we can take

$$H_{ij} = \epsilon^{kl} B_{ijkl} . \tag{2.5}$$

The tensor H_{ij} is symmetric, tracefree and divergence-free.

Suppose a metric has a conformal Killing vector X ,

$$D_i X_j + D_j X_i = \frac{2}{3} \varphi g_{ij} , \tag{2.6}$$

where φ here is the divergence of the vector field X . Then it has to be the case that

$$\mathcal{L}_X B_{ijk} = 0 . \tag{2.7}$$

The reason is that the map *Cotton* sending a metric to its Cotton tensor satisfies $\Phi^* Cotton[g] = Cotton[\Phi^*g]$ for any map Φ of M into itself. One now applies this relation to the case where Φ is a one-parameter family of diffeomorphisms generated by a conformal Killing vector X . Taking the derivative with respect to the parameter and using that the map *Cotton* is invariant under conformal rescaling of the metric one obtains (2.7). An equivalent form of (2.7) is the relation

$$\mathcal{L}_X H_{ij} = -\frac{1}{3} \varphi H_{ij} , \tag{2.8}$$

Taking cyclic permutations of the equation obtained by differentiating (2.6) one has

$$D_i F_{jk} = -R_{jki}{}^l X_l + \frac{2}{3} \varphi_{[j} g_{k]i} , \tag{2.9}$$

where we have defined $F_{ij} = D_{[i} X_{j]}$ and $\varphi_i = D_i \varphi$. The Lie derivative of the tensor $R_{ij} - Rg_{ij}/2(n-1)$ (here, for future reference, we work in general dimension n) equals

$$\mathcal{L}_X \left(R_{ij} - \frac{R}{2(n-1)} g_{ij} \right) = -\frac{(n-2)}{n} D_i D_j \varphi . \tag{2.10}$$

In dimension $n = 3$, to which we return now, this reads

$$D_i \varphi_j = -3 \mathcal{L}_X L_{ij} . \tag{2.11}$$

The identities (2.9)–(2.11), together with the relation

$$D_i X_j = F_{ij} + \varphi g_{ij}/3 , \tag{2.12}$$

imply that a conformal Killing vector, for which the quantities

$$(X, F_{ij}, \varphi, \varphi_i)$$

are all zero at the point p , has to vanish in a neighborhood of p . Using (2.9), (2.8) takes the form

$$X^k D_k H_{ij} + 2 F_{(i}{}^k H_{j)k} + \varphi H_{ij} = 0 . \tag{2.13}$$

Next we take a derivative of (2.13) with the result that

$$F_l^k D_k H_{ij} + \frac{4}{3} \varphi D_l H_{ij} + X^k D_l D_k H_{ij} + 2(D_l F_{(i}^k) H_{j)k} + 2F_{(i}^k D_{|l} H_{j)k} + \varphi_l H_{ij} = 0. \tag{2.14}$$

We are ready now to pass to the proof of the main result of this section:

Theorem 2.1 *Let (M, g) be a smooth three-dimensional pseudo-Riemannian¹ manifold.*

1. *There exists a non-trivial homogeneous polynomial*

$$Q(\cdot, \cdot, \cdot) : \mathbb{R}^6 \times \mathbb{R}^{3 \times 6} \times \mathbb{R}^{3 \times 3 \times 6} \rightarrow \mathbb{R}$$

such that if

$$Q(H, DH, D^2H)(p) \neq 0$$

(the \mathbb{R}^6 arises here because H is symmetric), then there exists a neighborhood \mathcal{O}_p of p on which there are no local conformal Killing vectors.

2. *Let Ω be a neighborhood of $p \in M$. For any $k \geq 5$ and $\epsilon > 0$ there exists a metric $g' \in C^\infty(M)$ such that*

$$\|g - g'\|_{C^k(\Omega)} < \epsilon,$$

with $g - g'$ supported in Ω , and such that $Q(H', DH', D^2H')(p)$ does not vanish.

Remark 2.2 A corresponding result in higher dimensions is proved in Theorem 7.4.

Remark 2.3 Recall that a polynomial in the curvature tensor and its derivatives is called *invariant* if it is independent of the frame used to evaluate its numerical value. Below we arbitrarily choose some orthonormal basis of $T_p M$ to define Q , and it is unlikely that the polynomial Q defined in our proof will be an invariant polynomial if the signature of the metric is Lorentzian; moreover, it is not clear how to modify Q to make it invariant while preserving the claimed properties. Note that one can view Q as a function on the frame bundle. In the Riemannian case we let \tilde{Q} be the integral of Q over those fibers with respect to the Haar measure, then \tilde{Q} is a non-trivial invariant polynomial with the properties as above.

We note that the polynomial constructed below provides a convenient tool to capture the fact that a certain geometrically defined matrix has rank larger than ten; the latter assertion provides an equivalent invariant statement, regardless of signature.

Proof. Before passing to the proof, some auxiliary results will be useful. Let the superscript “ \circ ” denote “value at the point p ”, e.g., $D_k \mathring{R}_{ij} := (D_k R_{ij})(p)$. In the calculations that follow we will assume that the metric has Riemannian signature. The remaining cases require trivial modifications, which we leave to the reader. We start with a Lemma:

Lemma 2.4 Consider a metric such that

$$\mathring{g}_{ij} = \delta_{ij}, \mathring{R}_{ij} = 0, D_k \mathring{R}_{ij} = 0. \quad (2.15)$$

Furthermore let the second derivatives of the curvature be such that

$$D_k \mathring{H}_{ij} = Ax_k y_{(i} z_j) + By_k x_{(i} z_j) + Cz_k x_{(i} y_j), \quad (2.16)$$

where (x, y, z) form an orthonormal basis of $T_p M$ and the three real numbers A, B, C are all non-zero. Then the set of algebraic equations for

$$w := (X_i, F_{ij} := D_{[i} X_{j]}, \varphi := D^k X_k, \varphi_i := D_i \varphi)(p)$$

obtained from the equations

$$[\mathcal{L}_X H_{ij} + \frac{1}{3} \varphi H_{ij}](p) = 0, \quad (2.17)$$

$$[\mathcal{L}_X D_k H_{ij} + 2C_{k(i}^m H_{j)m} + \frac{1}{3} D_k(\varphi H_{ij})](p) = 0, \quad (2.18)$$

$$[\mathcal{L}_X D_l D_k H_{ij} + C_{kl}^m D_m H_{ij} + 2(D_l C_{k(i}^m) H_{j)m} + 2C_{l(i}^m D_{|k|} H_{j)m} + 2C_{k(i}^m D_{|l|} H_{j)m} + \frac{1}{3} D_l D_k(\varphi H_{ij})](p) = 0, \quad (2.19)$$

with C_{jk}^i defined as

$$C_{jk}^i = \frac{1}{3} (2\varphi_{(j} \delta_{k)}^i - g_{jk} \varphi^i) \quad (2.20)$$

implies $w = 0$.

Remark 2.5 Equations (2.17)–(2.19) are necessarily satisfied by every conformal Killing vector field X : (2.17) is equivalent to (2.13), while Equations (2.18)–(2.19) are equivalent to the first and second covariant derivatives of (2.13).

Remark 2.6 It can be seen that $D_k \mathring{H}_{ij}$ in (2.16) satisfies the necessary algebraic requirements to arise from a metric (*i.e.*, being symmetric in (ij) and trace-free on all index pairs, compare (2.3)–(2.5)); this follows in any case from Proposition 2.7 below.

Proof. It immediately follows from Equations (2.15), (2.16) and (2.13) that $\mathring{X} = 0$.

Let a, b and c be defined as the following components of F in the basis (x, y, z) :

$$\mathring{F}_l^k = a(x_l y^k - y_l x^k) + b(z_l x^k - x_l z^k) + c(y_l z^k - z_l y^k). \quad (2.21)$$

Evaluating (2.14) at p , and using $\mathring{X} = 0$ we find that

$$\begin{aligned} 0 = & [b(A + C)z_l - a(A + B)y_l]y_{(i} z_j) + [a(A + B)x_l - c(B + C)z_l]x_{(i} z_j) \\ & + [c(B + C)y_l - b(A + C)x_l]x_{(i} y_j) + (aCz_l - bBy_l)x_i x_j + (cAx_l - aCz_l)y_i y_j \\ & + (bBy_l - cAx_l)z_i z_j + \frac{4}{3} \mathring{\varphi} (Ax_l y_{(j} z_i) + By_l x_{(j} z_i) + Cz_l x_{(j} y_i)). \end{aligned} \quad (2.22)$$

It follows by inspection that a, b, c and $\mathring{\varphi}$ have all to be zero. Differentiating (2.14) we find that

$$0 = (D_m \mathring{F}_{lk}) D^k \mathring{H}_{ij} + \frac{4}{3} \mathring{\varphi}_m D_l \mathring{H}_{ij} + 2(D_l \mathring{F}_{(i|k}) D_{m|} \mathring{H}_j)^k + 2(D_m \mathring{F}_{(i|k}) D_{l|} \mathring{H}_j)^k + \mathring{\varphi}_l D_m \mathring{H}_{ij}, \quad (2.23)$$

where we have used the vanishing of X and $D_i X^j$ at p . Next observe that, by Equations (2.15) and (2.9), there holds

$$D_i \mathring{F}_{jk} = \frac{2}{3} \mathring{\varphi}_{[j} g_{k]i}. \quad (2.24)$$

We now insert (2.24) into (2.23) to find that

$$0 = \frac{8}{3} \mathring{\varphi}_{(l} D_m) \mathring{H}_{ij} - \frac{1}{3} g_{ml} \mathring{\varphi}_k D^k \mathring{H}_{ij} + \frac{2}{3} \mathring{\varphi}_{(i} D_{|m|} \mathring{H}_j)l + \frac{2}{3} \mathring{\varphi}_{(i} D_{|l|} \mathring{H}_j)m - \frac{2}{3} \mathring{\varphi}_k g_{i(l} D_m) \mathring{H}_j^k - \frac{2}{3} \mathring{\varphi}_k g_{j(l} D_m) \mathring{H}_i^k. \quad (2.25)$$

Direct algebra using (2.16) shows that $\mathring{\varphi}_i$ vanishes, which is what had to be established. \square

Let us show now that

Proposition 2.7 *A metric satisfying (2.15)–(2.16) exists.*

Proof. We start with two elementary lemmata:

Lemma 2.8 *Suppose we are given, on a star-shaped domain Ω in $(\mathbb{R}^n, \delta_{ij})$, a tensor field B_{ijk} satisfying*

$$B_{ijk} = B_{i[jk]}, \quad (2.26)$$

$$B_{[ijk]} = 0, \quad (2.27)$$

$$B_{i[jk,l]} = 0. \quad (2.28)$$

Then there exists a tensor field $L_{ij} = L_{(ij)}$ such that

$$B_{ijk} = L_{i[j,k]}. \quad (2.29)$$

If B is a homogeneous polynomial of order p , then L can be chosen to be a homogeneous polynomial of order $p + 1$.

Proof. By Equations (2.26)–(2.28), there exists a tensor field M_{ij} , not necessarily symmetric in i and j , satisfying (2.29) with L_{ij} replaced by M_{ij} . From (2.27) it follows that there exists a covector field Λ_i with $M_{[ij]} = \Lambda_{[i,j]}$. Set $L_{ij} = M_{ij} - \Lambda_{i,j}$, then $L_{ij} = L_{(ij)}$ and satisfies (2.29) thus proving Lemma 2.8. The fact that solutions can be chosen as polynomials follows from the explicit formula for the primitive of a form used in the proof of the Poincaré Lemma. \square

We will also need the following variation of a result of Pirani [21]:

Lemma 2.9 *Let Ω be as in Lemma 2.8 and on it a tensor field R_{ijkl} having the symmetries of the Riemann tensor and obeying the differential identity*

$$R_{ij[kl,m]} = 0 . \tag{2.30}$$

Then there exists $h_{ij} = h_{(ij)}$ such that

$$R_{ijlm} = 2\partial_{[i}h_{j][l,m]} . \tag{2.31}$$

If moreover R_{ijkl} is a homogeneous polynomial in the manifestly flat coordinates ξ^i of order q , then h_{ij} can be chosen as a homogeneous polynomial of order $q + 2$.

Proof. This is proved by inspection of the proof in Pirani [21, pp. 279–280], using the fact that the proof there consists of the repeated use of the Poincaré Lemma. □

Returning to the proof of Proposition 2.7, let ξ be coordinates on Ω and define

$$B_{ijk} = \frac{1}{2}\epsilon^m{}_{jk}(\partial_n H_{im})\xi^n , \tag{2.32}$$

where the constants $\partial_n H_{im}$ are given by the right-hand-side of (2.16). The field B_{ijk} defined by (2.32) obviously satisfies (2.26), while (2.27)-(2.28) hold because (2.16) is trace-free in all indices. Now let L_{ij} be the homogenous quadratic polynomial guaranteed to exist by Lemma 2.8. As (2.16) is symmetric in i and j , the field B_{ijk} satisfies the second equation in (2.3). This implies

$$\partial^j L_{ij} = \partial_i L , \tag{2.33}$$

where $L = \delta^{ij}L_{ij}$. Consider the field S_{ijkl} defined by

$$S_{ijkl} = 2\delta_{k[i}L_{j]l} - 2\delta_{l[i}L_{j]k} , \tag{2.34}$$

it is a homogeneous quadratic polynomial in ξ which clearly has the symmetries of a Riemann tensor. Equation (2.33) implies that (2.30) holds, hence all the assumptions of Lemma 2.9 are fulfilled. Let h_{ij} be the fourth order homogeneous polynomial guaranteed to exist by Lemma 2.9, set

$$g_{ij} = \delta_{ij} + h_{ij} . \tag{2.35}$$

Since h vanishes to order three, both the Riemann tensor and its derivatives vanish at p , which justifies (2.15). Further, the Riemann tensor R_{ijkl} of g coincides with S_{ijkl} up to terms which give zero contribution at p in all the calculations relevant here, so that it is not difficult to show that g_{ij} satisfies (2.16), which proves Proposition 2.7. □

We can now pass to the

Proof of Theorem 2.1. Consider the linear map L which to

$$w = (X_i, F_{ij} := D_{[i}X_{j]}, \varphi := D^k X_k, \varphi_i := D_i\varphi)(p) \in \mathbb{R}^{10}$$

assigns

$$\begin{aligned} \mathbb{R}^{10} \ni w \rightarrow Lw := & \left(\mathcal{L}_X H_{ij} + \frac{1}{3}\varphi H_{ij}, \mathcal{L}_X D_k H_{ij} + 2C_{k(i}^m H_{j)m} + \frac{1}{3}D_k(\varphi H_{ij}), \right. \\ & \mathcal{L}_X D_l D_k H_{ij} + C_{kl}^m D_m H_{ij} + 2(D_l C_{k(i}^m H_{j)m} \\ & \left. + 2C_{l(i}^m D_{|k|} H_{j)m} + 2C_{k(i}^m D_{|l|} H_{j)m} + \frac{1}{3}D_l D_k(\varphi H_{ij}) \right)(p) \\ & \in \mathbb{R}^6 \otimes \mathbb{R}^{3 \times 6} \otimes \mathbb{R}^{3 \times 3 \times 6}. \end{aligned}$$

Here the Lie derivative is calculated using the usual formula for the Lie derivative of a tensor, and then the values of X and its derivatives as determined by w are inserted. Further, the second derivatives of φ are eliminated using (2.11). It follows from Lemma 2.4 and Proposition 2.7 that the set of metrics for which L is injective is not empty. Standard linear algebra implies that there exists a 10×10 matrix, say A , constructed by listing ten appropriately chosen rows of L , which has non-vanishing determinant when H arises from the metric of Proposition 2.7. Let Q be the sum of squares of determinants of all ten-by-ten submatrices of L , then $Q \geq (\det A)^2$ and therefore Q is not identically vanishing by construction. Clearly L is injective whenever Q is non-zero, which proves point 1.

To prove point 2, let g be an arbitrary metric, if $Q(p)$, evaluated for the metric g , does not vanish, then the result is true with $g' = g$. Otherwise, define

$$\begin{aligned} \mathcal{J}_5 := & \{ \text{the set of fifth jets of } g \text{ in normal coordinates at } p \\ & \text{as } g \text{ varies in the set of all Riemannian metrics} \}. \end{aligned} \tag{2.36}$$

This a linear space, an explicit parameterisation of which can be found in [23]. Let $e_i, i = 1, \dots, N$, be any basis of \mathcal{J}_5 , thus every $j \in \mathcal{J}_5$ can be written as

$$j = j^i e_i,$$

for some numbers $j^i \in \mathbb{R}$. By definition of \mathcal{J}_5 , for every $(j^i) \in \mathbb{R}^N$ there exists some Riemannian metric for which $j = j^i e_i$. Clearly the map $g \rightarrow (j^i)$ is continuous in a $C^\ell(\bar{\Omega})$, $\ell \geq 5$, topology on the set of metrics, and a small variation of j^i can be realized by a small variation of g . In a frame such that $g_{ij}(p) = \delta_{ij}$, the map that assigns to the fifth jets of g , at p , the values of the tensors H, DH , and DD^2H at p , is a polynomial on \mathcal{J}_5 .

We want to show that a small variation of g will make Q non-zero. Now, Q is a polynomial in the j^i 's. Let j_0^i be the values of the j^i 's corresponding to the metric g , and suppose that we have

$$\forall i_1, \dots, i_n \quad \frac{\partial^{i_1 + \dots + i_n} Q}{\partial^{i_1} j^1 \dots \partial^{i_n} j^N}(j_0^i) = 0.$$

Then the polynomial Q would identically vanish, contradicting its construction. Hence there exists at least one of the above partial derivatives which does not vanish, and therefore an appropriate, no matter how small, variation of g will lead to a non-vanishing value of Q at p . As the argument depends only upon the jets of g at p , the variation can be made supported in a ball containing p with radius as small as desired. \square

3 Metrics without Killing vectors near a point

Results on non-existence of Killing vectors follow of course immediately from those on non-existence of conformal Killing vectors, as established above. However, for Killing vectors in dimension three the differentiability threshold of Theorem 2.1 can be lowered to three. Further, for Killing vectors a simple proof can be given in all dimensions:

Theorem 3.1 *Let (M, g) be a n -dimensional pseudo-Riemannian manifold.*

1. *There exists a non-trivial homogeneous invariant polynomial $P_n[g] := (DR, \dots, D^{2n+1}R)$ of degree n , where R is the Ricci scalar, such that if*

$$P_n(DR, \dots, D^{2n+1}R)(p) \neq 0$$

at a point $p \in M$, then there exists a neighborhood \mathcal{O}_p of p such that there are no non-trivial Killing vectors on any open subset of \mathcal{O}_p . In dimension $n = 3$ there exists such a polynomial \hat{P}_3 which depends upon Ric and $D\text{Ric}$.

2. *Let Ω be a neighborhood of $p \in M$. For any $k \geq 2n + 1$ and $\epsilon > 0$ there exists a metric g' such that*

$$\|g - g'\|_{C^k(\bar{\Omega})} < \epsilon, \quad (3.1)$$

with $g - g'$ supported in Ω , and such that $P_n(DR', \dots, D^{2n+1}R')(p)$ does not vanish. In dimension three we can arrange for the non-vanishing of $\hat{P}_3(\text{Ric}', D\text{Ric}')(p)$ using a perturbation supported in Ω and satisfying (3.1) for each arbitrarily chosen $k \geq 3$.

Remark 3.2 The differentiability required above in dimension n is certainly not optimal, but it allows the simple proof below.

Remark 3.3 The polynomial P_n obtained here is completely useless from the point of view of Killing vectors in vacuum space-times, where the Ricci scalar vanishes. In this context it is of interest to have a statement as above with a polynomial depending only upon the Weyl tensor, and we prove existence of such polynomials in Theorem 7.4 below. Further, in Section 8 we will construct small perturbations of initial data which preserve the vacuum constraints.

Proof. If X is a Killing vector we have $\mathcal{L}_X(\Delta^k R) = 0$ for all k , where Δ^k denotes the k th power of the Laplace operator Δ . At p this gives the linear system of equations

$$A_{ij} X^i(p) = 0, \quad A_{ij} = D_i(\Delta^j R)(p), \quad j = 0, \dots, n - 1.$$

Let $P_n = \det(A_{ij})$. If $P_n(p)$ does not vanish, then $X(q) = 0$ for all q in the neighborhood of p defined as $\{q : P_n(q) \neq 0\}$, hence $X \equiv 0$. It is not too difficult to check, using Taylor expansions of the metric (point 2 of Proposition 5.4 below is useful here), that there exist metrics for which $P_n \neq 0$, and the result follows by a repetition of the arguments of the proof of Theorem 2.1.

In dimension 3 the number of the derivatives of the metric needed can be improved as follows: Let $G_{ij} = R_{ij} - g^{kl} R_{kl} g_{ij} / 2$, in the notation of Section 2 we assume that

$$\mathring{G}_{ij} = \lambda_1 x_i x_j + \lambda_2 y_i y_j + \lambda_3 z_i z_j, \tag{3.2}$$

with $(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \neq 0$. We set, as in (2.21),

$$\mathring{F}_{ij} = 2(ax_{[i} y_{j]} + bz_{[i} x_{j]} + cy_{[i} z_{j]}), \tag{3.3}$$

so that

$$\mathring{F}_{(i}{}^k \mathring{G}_{j)k} = b(\lambda_1 - \lambda_3)x_{(i} z_{j)} + c(\lambda_3 - \lambda_2)y_{(i} z_{j)} + a(\lambda_2 - \lambda_1)x_{(i} y_{j)}, \tag{3.4}$$

which has zero components on the diagonal. Finally we assume that

$$\mathring{G}_{ij;k} = \mu_1(x_i \delta_{jk} - 2x_k \delta_{ij}) + \mu_2(y_i \delta_{jk} - 2y_k \delta_{ij}) + \mu_3(z_i \delta_{jk} - 2z_k \delta_{ij}), \tag{3.5}$$

where $\mu_1 \mu_2 \mu_3 \neq 0$. We now set

$$\mathring{X} = \alpha x_i + \beta y_i + \gamma z_i. \tag{3.6}$$

Writing (3.5) in the form $\mathring{G}_{ij;k}^1 + \mathring{G}_{ij;k}^2 + \mathring{G}_{ij;k}^3$, we find

$$\mathring{G}_{ij;k}^1 \mathring{X}^k = \mu_1 \alpha (-2y_i y_j - 2z_i z_j - x_i x_j) + \text{off diagonal terms}, \tag{3.7}$$

$$\mathring{G}_{ij;k}^2 \mathring{X}^k = \mu_2 \beta (-2x_i x_j - 2z_i z_j - y_i y_j) + \text{off diagonal terms}, \tag{3.8}$$

$$\mathring{G}_{ij;k}^3 \mathring{X}^k = \mu_3 \gamma (-2x_i x_j - 2y_i y_j - z_i z_j) + \text{off diagonal terms}. \tag{3.9}$$

We first consider the relation $\mathcal{L}_X G_{ij} = 0$ with $i = j$. Then (3.4) gives no contribution, while from (3.7) we obtain a linear homogenous system for (α, β, γ) with coefficient matrix Δ given by

$$\Delta = \begin{pmatrix} \mu_1 & 2\mu_2 & 2\mu_3 \\ 2\mu_1 & \mu_2 & 2\mu_3 \\ 2\mu_1 & 2\mu_2 & \mu_3 \end{pmatrix}.$$

There holds $\det(\Delta) = 5\mu_1\mu_2\mu_3 \neq 0$. Thus, the equation $\mathcal{L}_X G_{ij} = 0$, satisfied by any Killing vector, leads to $\alpha = \beta = \gamma = 0$. The off-diagonal components of $\mathcal{L}_X G_{ij} = 0$ imply now, by (3.4), that $a = b = c = 0$. Since (3.2) is symmetric, and (3.5) satisfies the linearized Bianchi identities, the results in [23] show that there exists a metric $g_{ij} = \delta_{ij} + h_{ij}$, with $h_{ij} = O(\xi^2)$, satisfying (3.2) and (3.5). The proof is completed by the same argument as already given for general n . \square

4 Generic non-existence of local Killing, or conformal Killing, vector fields

In this section we only consider three-dimensional manifolds, the reader will easily formulate an equivalent statement and proof for local Killing vector fields in any dimension using Theorem 3.1, or for local conformal Killing vector fields using Theorem 7.4 below.

Theorem 4.1 *Let M be a three-dimensional manifold. Then*

1. *The set of pseudo-Riemannian metrics on M which have no local Killing vector fields is of second category in the C^3 topology.*
2. *The set of pseudo-Riemannian metrics on M which have no local conformal Killing vector fields is of second category in the C^5 topology.*

Proof. We start with the following:

Proposition 4.2 *Let Ω be a domain in M . Then:*

1. *The set of metrics on Ω which have no Killing vectors on Ω is open in a $C^k(\bar{\Omega})$ topology, $k \geq 2$.*
2. *The set of metrics on Ω which have no conformal Killing vectors on Ω is open in a $C^k(\bar{\Omega})$ topology, $k \geq 3$.*
3. *The set of initial data (g, K) on Ω which have no non-trivial KIDs on Ω is open in a $C^{k+1}(\bar{\Omega}) \oplus C^k(\bar{\Omega})$ topology, $k \geq 1$.*

Remark 4.3 The openness established here holds for any metrisable topology \mathcal{T}_k such that convergence in \mathcal{T}_k implies uniform convergence in C^k norm on compact sets, with $k \geq 2$ for Killing vectors, etc; see also Appendix A.

Proof. We will show that *existence* of Killing vectors, or conformal Killing vectors, or KIDs, is a closed property. We start with the slightly simpler case of conditionally compact Ω :

Lemma 4.4 *Proposition 4.2 holds if Ω has compact closure.*

Proof. 1. Let γ_i be a sequence of metrics with non-zero Killing vectors $X(i)$. Rescaling $X(i)$ we can assume that

$$\sup_{p \in \Omega} \gamma_i(X(i), X(i)) = 1. \tag{4.1}$$

We note that Killing vectors extend by continuity to $\overline{\Omega}$, we shall use the same symbol to denote that extension. Let $p_i \in \overline{\Omega}$ be such that the sup is attained, passing to a subsequence if necessary there exists p_* in $\overline{\Omega}$ such that $p_i \rightarrow p_*$. Now, Killing vectors satisfy the system of equations

$$D_i D_j X_k = R^\ell{}_{ijk} X_\ell, \tag{4.2}$$

which shows that second covariant derivatives of all the $X(i)$'s are uniformly bounded on $\overline{\Omega}$. Interpolation [16, Appendix] shows that the sequence $X(i)$ is uniformly bounded in C^2 . The existence of a subsequence converging in C^1 to a non-trivial Killing vector field follows from the Arzela-Ascoli theorem.

2. The argument is essentially identical, with the following modifications: we replace the normalization (4.1) by

$$\sup_{p \in \Omega} (|X(i)|_{\gamma_i} + |DX(i)|_{\gamma_i}) = 1. \tag{4.3}$$

Equation (4.2) is replaced by the set of equations (2.9)–(2.12). Those equations easily imply boundedness of the sequence $X(i)$ in C^3 , leading to a converging subsequence in C^2 .

3. Let (γ_i, K_i) be a sequence of metrics with non-zero KIDs $(Y(i), N(i))$. We use the normalization

$$\sup_{p \in \Omega} (|Y(i)|_{\gamma_i} + |DY(i)|_{\gamma_i} + |N(i)| + |DN(i)|_{\gamma_i}) = 1. \tag{4.4}$$

From (5.4) and (5.5) one obtains a uniform C^2 bound on $(Y(i), N(i))$, and one concludes as before. □

Returning to the proof of point 1 of Proposition 4.2, let Ω_j be an increasing sequence of conditionally compact domains such that $\Omega = \cup \Omega_j$. By Lemma 4.4 we have $\mathcal{K}(\Omega_j) \neq \{0\}$ for all j . The restriction map induces an injection $i_{i,j} : \mathcal{K}(\Omega_i) \rightarrow \mathcal{K}(\Omega_j)$, $i \geq j$, so that $1 \leq \dim i_{i,1}(\mathcal{K}(\Omega_i))$ for all i , with $i_{i+1,1}(\mathcal{K}(\Omega_{i+1})) \subset i_{i,1}(\mathcal{K}(\Omega_i)) \subset \mathcal{K}(\Omega_1)$. It follows that $F := \cap_i i_{i,1}(\mathcal{K}(\Omega_i)) \neq \{0\}$, and every element of F extends to a globally defined Killing vector field on Ω . □

Proof of Theorem 4.1. Let p_i , $i \in \mathbb{N}$ be a dense collection of points and let $B(p_i, 1/j)$, $j \geq N_i$, be a collection of coordinate balls with compact closure. Let $\mathcal{V}_{i,j}$ be the set of metrics such that $\mathcal{K}(B(p_i, 1/j)) = \{0\}$. By Proposition 4.2 the set $\mathcal{V}_{i,j}$ is open, and it is dense by Theorem 3.1. Then any metric in $\cap_{i,j} \mathcal{V}_{i,j}$ has no local Killing vectors. The argument for conformal Killing vector fields is identical, based on Theorem 2.1. □

5 Three-dimensional initial data sets without KIDs near a point

We now pass to the construction of initial-data sets without KIDS. Let $\mathcal{C}(K_{ij}, g_{ij}) := (J_i, \rho)$ be the constraints map,

$$\rho := R + K^2 - K_{ij}K^{ij} - 2\Lambda, \tag{5.1}$$

$$J_i := -2D^j(K_{ij} - Kg_{ij}), \tag{5.2}$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant. In this section, and *only* in this section, the symbol K denotes the trace of K_{ij} ; K stands for the full extrinsic curvature tensor elsewhere in this paper. Let P denote the linearization of \mathcal{C} , and let P^* be the formal adjoint of P . By definition, a KID (N, X^i) is a solution of the set of equations $P^*(N, X) = 0$; explicitly, in dimension n (*cf.*, *e.g.*, [10]),

$$D_{(i}X_{j)} = -NK_{ij}, \tag{5.3}$$

$$D_i D_j N = N(R_{ij} + K K_{ij} - 2K_{il}K_j{}^l) - \mathcal{L}_X K_{ij} + \frac{1}{(n-1)} \left(\frac{J_l X^l}{2} - (\rho + 2\Lambda) N \right) g_{ij}. \tag{5.4}$$

One checks that any KID (N, X^i) for which $X^i, F_{ij} = D_{[i}X_{j]}, N$ and $N_i := D_i N$ all vanish at p has to be zero in a neighborhood of p . This is proved in the usual way from (5.4) together with

$$D_\ell D_j X_i = -R_{jil}{}^k X_k - D_\ell(NK_{ij}) - D_j(NK_{\ell i}) + D_i(NK_{j\ell}). \tag{5.5}$$

(Equation (5.5) is obtained by considering cyclic permutations of first derivatives of (5.3).) Since $\mathcal{L}_X \text{Ric}(g) = \text{Ric}'(\mathcal{L}_X g)$, the usual formula for Ric' leads to

$$\begin{aligned} \mathcal{L}_X R_{ij} &= \Delta(NK_{ij}) + D_i D_j(NK) - 2D_{(i}D^l(NK_{j)l}) \\ &\quad - 2NR_{lijm}K^{lm} - 2NR_{(i}{}^l K_{j)l} \\ &= \Delta(NK_{ij}) + D_i D_j(NK) - 2D^l D_{(i}(NK_{j)l}). \end{aligned} \tag{5.6}$$

From now on we assume that $n = 3$. By taking the curl of (5.4) one also finds

$$\begin{aligned} R_{lij}{}^k D_k N &= -2\mathcal{L}_X D_{[l}K_{i]j} - 2C_{j[l}^m K_{i]m} \\ &\quad + 2D_{[l} \left[N(R_{ij]j} + K_{ij}K - 2K_{i]m}K_j{}^m) + \left(\frac{J_m X^m}{4} - \left(\frac{\rho}{2} + \Lambda \right) N \right) g_{ij} \right], \end{aligned} \tag{5.7}$$

where

$$C_{jk}^i = -g^{in} [D_j(NK_{kn}) + D_k(NK_{jn}) - D_n(NK_{jk})]. \tag{5.8}$$

We choose some $\alpha, \beta, \lambda_i, a_i \in \mathbb{R}$ and we consider initial data with the following properties at p :

$$\mathring{R}_{ij} = \frac{\beta}{3} \delta_{ij}, \quad \mathring{K}_{ij} = \frac{\alpha}{3} \delta_{ij}, \quad D_i \mathring{K}_{jl} = 0, \tag{5.9}$$

$$D_i \mathring{R}_{jk} = \lambda_x x_i y_{(j} z_{k)} + (\text{cyclic}), \quad (5.10)$$

where (cyclic) means cyclic permutations of (x, y, z) , and $\lambda_x \lambda_y \lambda_z \neq 0$. (This ansatz is general enough to lead to the required result, and simple enough so that the calculations are manageable. We will show shortly that such initial data exist.) We also assume that

$$D_i D_j \mathring{K}_{lm} = a_x x_i x_j y_{(l} z_{m)} + (\text{cyclic}), \quad (5.11)$$

with

$$\lambda_x - \lambda_y \neq \frac{a_x^2}{\lambda_x} - \frac{a_y^2}{\lambda_y} \quad (5.12)$$

and

$$\lambda_x + \lambda_y \neq 0, \quad \lambda_x + \lambda_z \neq 0, \quad \lambda_y + \lambda_z \neq 0. \quad (5.13)$$

For further reference we note that, in local coordinates ξ such that p corresponds to $\xi = 0$, (5.9)–(5.11) imply

$$R + K^2 - K_{ij} K^{ij} = \beta + \frac{2\alpha^2}{3} + O(\xi^2), \quad D^j (K_{ij} - K g_{ij}) = O(\xi^2). \quad (5.14)$$

In particular, if $\beta = 2\Lambda - 2\alpha^2/3$ then

$$\rho = R + K^2 - K_{ij} K^{ij} - 2\Lambda = O(\xi^2), \quad J_i = -2D^j (K_{ij} - K g_{ij}) = O(\xi^2). \quad (5.15)$$

Inserting (5.9) into (5.3) and (5.4) we find that

$$D_{(i} \mathring{X}_{j)} = -\frac{\alpha}{3} \mathring{N} \delta_{ij}, \quad (5.16)$$

$$\begin{aligned} D_i D_j \mathring{N} &= \left(\beta + \alpha^2 - \frac{3\rho}{2} - 3\Lambda \right) \frac{\mathring{N}}{3} \delta_{ij} \\ &= -\frac{\mathring{N}\beta}{6} \delta_{ij}, \end{aligned} \quad (5.17)$$

$$\Delta \mathring{N} = -\frac{\mathring{N}\beta}{2}. \quad (5.18)$$

Evaluating (5.6) at p , it follows that

$$\mathring{X}_m D^m \mathring{R}_{ij} = \mathring{N} \Delta \mathring{K}_{ij}, \quad (5.19)$$

and, from (5.7), that

$$\mathring{N} D_{[l} \mathring{R}_{i]j} = \mathring{X}_m D^m D_{[l} \mathring{K}_{i]j}. \quad (5.20)$$

From (5.19) we find, using the expansion $\mathring{X}_i = \alpha_x x^i + \alpha_y y^i + \alpha_z z^i$, that

$$\alpha_x \lambda_x = \mathring{N} a_x, \quad \alpha_y \lambda_y = \mathring{N} a_y, \quad \alpha_z \lambda_z = \mathring{N} a_z, \quad (5.21)$$

and from (5.20)

$$\dot{N}(\lambda_x - \lambda_y) = \alpha_x a_x - \alpha_y a_y, \quad \dot{N}(\lambda_y - \lambda_z) = \alpha_y a_y - \alpha_z a_z, \quad \dot{N}(\lambda_z - \lambda_x) = \alpha_z a_z - \alpha_x a_x. \tag{5.22}$$

Combining (5.22) with (5.21) and using (5.12), it follows that

$$\dot{N} = 0 = \alpha_x = \alpha_y = \alpha_z. \tag{5.23}$$

Using (5.23) in the first derivative of (5.4) and in (5.5), we infer that

$$D_l D_{(i} \dot{X}_{j)} = -\frac{\alpha}{3} \delta_{ij} D_l \dot{N}, \quad D_l D_i D_j \dot{N} = -\frac{\beta}{6} \delta_{ij} D_l \dot{N}. \tag{5.24}$$

We now take a derivative of (5.6) to obtain (recall that F_{ij} is the anti-symmetric part of $D_i X_j$)

$$\begin{aligned} \dot{F}_{km} D^m \dot{R}_{ij} + 2\dot{F}_{(i}{}^m D_{k|} \dot{R}_{j)m} = \\ (D_k \dot{N}) \Delta \dot{K}_{ij} + 2(D_l \dot{N}) D_k D^l \dot{K}_{ij} - 2(D_l \dot{N}) D_k D_{(i} \dot{K}_{j)}{}^l. \end{aligned} \tag{5.25}$$

Somewhat surprisingly, all terms involving α and β have dropped out. We have to compute the different terms entering (5.25). Writing \dot{F}_{ij} as

$$\dot{F}_{ij} = A_x y_{[i} z_{j]} + (\text{cyclic}), \tag{5.26}$$

we obtain

$$\dot{F}_{km} D^m \dot{R}_{ij} = \frac{1}{2} \lambda_x (A_y z_k - A_z y_k) y_{(i} z_{j)} + (\text{cyclic}), \tag{5.27}$$

$$\dot{F}_{im} D_k \dot{R}_j{}^m = \frac{1}{4} (\lambda_x x_k y_j + \lambda_y y_k x_j) (A_x y_i - A_y x_i) + (\text{cyclic}). \tag{5.28}$$

Also, decomposing $D_i \dot{N} = u_x x_i + u_y y_i + u_z z_i$, we have that

$$(D_k \dot{N}) \Delta \dot{K}_{ij} = (u_x x_k + u_y y_k + u_z z_k) (a_x y_{(i} z_{j)} + (\text{cyclic})). \tag{5.29}$$

and

$$2(D_l \dot{N}) (D_k D^l \dot{K}_{ij} - D_k D_{(i} \dot{K}_{j)}{}^l) = u_x (2a_x x_k - a_y y_k - a_z z_k) y_{(i} z_{j)} + (\text{cyclic}). \tag{5.30}$$

We now insert Equations (5.27)–(5.30) into (5.25). Contracting the resulting equation first with $x^k y^i z^j$ and cyclic permutations thereof, one sees that u_x, u_y, u_z have to vanish. Contracting, then, with terms of the form $x^k x^i y^j, x^k x^i z^j, y^k y^i x^j$, etc., we see that A_x, A_y, A_z are also zero, due to (5.13). Thus (N, X^i) is zero near p . We have thus proved:

Lemma 5.1 *Consider an initial data set (g_{ij}, K_{ij}) satisfying Eqs. (5.9)–(5.11) together with the conditions on the coefficients spelled out above. For any $\alpha, \beta, \Lambda \in \mathbb{R}$ the algebraic equations for $r = (X_i, F_{ij}, N, D_i N)$ obtained from (5.3)–(5.4) by taking derivatives up to order two imply the vanishing of $r(p)$. \square*

We also have the following KID-analogue of Proposition 2.7:

Proposition 5.2

1. A pair (g_{ij}, K_{ij}) satisfying (5.9)–(5.11) exists.
2. Further, one can choose $g_{ij} = \delta_{ij} + h_{ij}$ and K_{ij} so that, in local coordinates ξ , the tensor fields g_{ij} and K_{ij} satisfy the vacuum constraints up to terms which are of $O(|\xi|^2)$.

Proof. By Lemma 2.9 we can find h_{ij} of order $O(|\xi|^2)$, so that (5.9)–(5.10) are satisfied. For K_{ij} we choose

$$K_{ij} = \frac{\alpha}{3}\delta_{ij} + \frac{1}{2}(\mathring{K}_{ijklm} + \frac{\alpha}{3}\partial_l\partial_m\mathring{h}_{ij})\xi^l\xi^m, \tag{5.31}$$

where the second term on the right-hand side of (5.31) is given by the right-hand side of (5.11). One checks that (5.11) is valid. Point 2 follows from (5.15). \square

We are ready now to prove:

Theorem 5.3 *Let $\alpha, \beta \in \mathbb{R}$, $p \in M$, and consider the collection of all three-dimensional data sets (M, K_{ij}, g_{ij}) with $(K_{ij}, g_{ij}) \in C^k \times C^{k+1}$, $k \geq 3$, with the trace $K(p)$ of $K_{ij}(p)$ equal to α , and with $R(p) = \beta$.*

1. There exists a non-trivial homogeneous invariant polynomial $Q[K_{ij}, g_{ij}] := Q(R_{ij}, DR_{ij}, D^2R_{ij}, K_{ij}, DK_{ij}, D^2K_{ij}, D^3K_{ij})$ such that if

$$Q[K_{ij}, g_{ij}](p) \neq 0$$

at a point $p \in M$, then there exists a neighborhood \mathcal{O}_p of p for which there exist no non-trivial KIDS on any open subset of \mathcal{O}_p .

2. Let Ω be a domain in M with $p \in \Omega$. a) There exists a variation $(\delta K_{ij}, \delta g_{ij}) \in (C^\infty \times C^\infty)(\Omega)$, compactly supported in Ω , such that $Q[K_{ij} + \epsilon\delta K_{ij}, g_{ij} + \epsilon\delta g_{ij}](p) \neq 0$ for all ϵ small enough. b) The variation can be chosen so that it preserves the value of $R(p)$ and of $K(p)$. One can further arrange for the trace of $K_{ij} + \epsilon\delta K_{ij}$ to be equal to K throughout Ω when K is a constant.
3. If (K_{ij}, g_{ij}) is vacuum (with perhaps non-zero cosmological constant) with $(K_{ij}, g_{ij}) \in C^{k+\ell+1} \times C^{k+\ell+2}$, $\ell \geq 0$, then for any $p \in \Omega$ the variation of point 2 can be chosen to satisfy the linearized constraint equations up to error terms which are $o(r^\ell)$ in a $C^k(B(p_0, r))$ norm, for small r .

Proof. The proof of points 1 and 2 follows closely that of points 1 and 2 of Theorem 2.1. Given any constant α , the set \mathcal{J}_5 of (2.36) is replaced by the

$$\{\text{the set of fourth jets of } g_{ij} \text{ and of third jets of } K_{ij} \text{ at } p \text{ that one obtains as } g_{ij} \text{ varies in the set of all Riemannian metrics in normal coordinates near } p \text{ and as } K_{ij} \text{ varies in the set of all symmetric tensors with trace equal to a prescribed constant } \alpha\}. \tag{5.32}$$

The intermediate elements of the proof are provided by Lemma 5.1 and the first part of Proposition 5.2. The variations of g_{ij} and of the trace-free part of K_{ij} can be chosen to be polynomials multiplied by a smooth cut-off function, and are therefore smooth. One can then adjust the trace part of K_{ij} to achieve $K_\epsilon = \alpha$. We further note that the non-vanishing of some derivative of Q follows immediately from the fact that $Q(p)$ is a polynomial, when viewed as a function depending upon the jets of g_{ij} and K_{ij} in normal coordinates at p . Further details are left to the reader.

In order to prove point 3, for $r > 0$ it is useful to introduce the following set:

$$\mathscr{W}_{\ell+k} = \{ \text{jets at } p \text{ of order } (\ell + k + 1, \ell + k + 2) \text{ of } (K_{ij}, g_{ij}) \text{ such that} \\ \rho = o(|\xi|^{\ell+k}), J = o(|\xi|^{\ell+k}) \text{ in } B(0, r) \} .$$

Here ξ are supposed to be geodesic coordinates near p in the metric g_{ij} . Equivalently, if $(K_{ij}, g_{ij}) \in C^{k+\ell+1} \times C^{k+\ell+2}$ has jets in $\mathscr{W}_{\ell+k}$, then we have

$$D^\alpha \rho(p) = 0, D^\alpha J(p) = 0, \quad 0 \leq |\alpha| \leq \ell + k, \tag{5.33}$$

where the $\alpha = (i_1 \dots i_j)$'s are multi-indices, with $|(i_1 \dots i_j)| = i_1 + \dots + i_j$. Elements of $\mathscr{W}_{\ell+k}$ can be *uniquely* parameterized as follows: Taylor expanding g_{ij} and $P_{ij} := K_{ij} - Kg_{ij}$ in geodesic coordinates around p , one can write

$$g_{ij} = \delta_{ij} + \sum_{2 \leq |\alpha| \leq \ell+k+2} h_{ij\alpha} \xi^\alpha + O(|\xi|^{\ell+k+3}), \quad \text{with } h_{i(j_1 \dots j_p)} = 0, \tag{5.34}$$

$$P_{ij} = \mathring{P}_{ij} + \sum_{1 \leq |\alpha| \leq \ell+k+1} P_{ij\alpha} \xi^\alpha + O(|\xi|^{\ell+k+2}) \tag{5.35}$$

(see, e.g., [23] for a justification of the last condition in (5.34)). Then (5.33) can be solved by induction as follows: (5.33) with $|\alpha| = 0$ gives

$$\sum_{i,j} (h_{ijij} - h_{jjii}) = \sum_{i,j} \mathring{P}_{ij}^2 - (\sum_i \mathring{P}_{ii})^2 + 2\Lambda, \\ \sum_i P_{ijj} = 0.$$

For any given $\mathring{P}_{ij} \in \mathbb{R}^{m_0} := \mathbb{R}^6$ the first equation defines an affine subspace isomorphic to \mathbb{R}^{n_2} for some n_2 , in the vector space of second Taylor coefficients h_{ijkl} . The second equation defines a linear subspace isomorphic to \mathbb{R}^{m_1} in the space of P_{ijk} 's, for some m_1 . To understand (5.33) with $|\alpha| \geq 1$ we will need the following:

Proposition 5.4 *Let $k \in \mathbb{N}$, and suppose that $\dim M = n \geq 2$.*

1. *For every $J_i = J_{ij_1 \dots j_k} \xi^{j_1} \dots \xi^{j_k}$ and $p = p_{j_1 \dots j_{k+1}} \xi^{j_1} \dots \xi^{j_{k+1}}$ there exists $P_{ij} = P_{ijj_1 \dots j_{k+1}} \xi^{j_1} \dots \xi^{j_{k+1}}$, symmetric in i and j , such that*

$$\sum_i \partial_j P_{ij} = J_i, \quad \sum_i P_{ii} = p.$$

2. For every $f = f_{j_1 \dots j_k} \xi^{j_1} \dots \xi^{j_k}$ there exists $h_{ij} = h_{ijj_1 \dots j_{k+2}} \xi^{j_1} \dots \xi^{j_{k+2}}$, symmetric in i and j , with $h_{i(jj_1 \dots j_{k+2})} = 0$, such that

$$\sum_{i,j} (\partial_j \partial_i h_{ij} - \partial_i \partial_i h_{jj}) = f .$$

Proof. Consider a system of linear PDEs

$$Pu = I , \tag{5.36}$$

with constant coefficients, of order p , which can be written in the Cauchy-Kowalevskaja form with respect to a coordinate z . We claim that if I is a polynomial of order l , then there exists a solution of (5.36) which is a polynomial of order $l + p$. In order to see that, we note that (5.36) determines, at $z = 0$, the z -derivatives of u of order greater than or equal to p as polynomials in the remaining variables. So choosing zero Cauchy data on $\{z = 0\}$ one obtains a polynomial solution in z with polynomial coefficients, hence a polynomial. If P is homogeneous of order p , and if I is in addition homogenous of order l , then the above solution is a homogeneous polynomial of order $l + p$.

In order to prove point 1, we make the ansatz

$$P_{ij} = \partial_i W_j + \partial_j W_i + \frac{1}{n} \left(p - 2 \sum_{\ell} \partial_{\ell} W_{\ell} \right) \delta_{ij} ,$$

which leads to a homogeneous second order elliptic system for W , and the above argument applies.

In order to prove point 2, we first make the ansatz $h_{ij} = \frac{1}{n} h_{ll} \delta_{ij}$, solve the resulting Poisson equation in the class of homogeneous polynomials as described above, and introduce a metric $g_{ij} = \delta_{ij} + h_{ij}$. In geodesic coordinates y^i the metric g_{ij} will have an expansion with some new coefficients satisfying the symmetry condition in (5.34) [23]. One has $y^i = \xi^i + O(|\xi|^{k+3})$, which implies that the polynomial obtained from the y -Taylor coefficients of g_{ij} of order $k + 2$ provides the desired h_{ij} . \square

Proposition 5.4 shows that (5.33) can be used to inductively determine higher order Taylor coefficients $h_{ij\alpha}$ and $P_{ij\beta}$ in terms of lower order ones, as well as in terms of some free P -coefficients in $\mathbb{R}^{m_{|\beta|}}$, for some $m_{|\beta|} \in \mathbb{N}$, and some free h -coefficients in $\mathbb{R}^{n_{|\alpha|}}$, for some $n_{|\alpha|} \in \mathbb{N}$. It follows in particular that $\mathscr{W}_{\ell+k}$ is diffeomorphic to $\mathbb{R}^{N_{\ell+k}}$, for some $N_{\ell+k} \in \mathbb{N}$.

For solutions (K_{ij}, g_{ij}) of the constraint equations, the polynomial $Q[K_{ij}, g_{ij}](p)$ can be expressed as a polynomial of (K_{ij}, g_{ij}) -jets at p of order $(2, 3)$, call this polynomial \tilde{Q} . Since the $\mathscr{W}_{\ell+k}$'s are included in each other in the obvious way, \tilde{Q} can actually be viewed as a function defined on $\mathscr{W}_{\ell+k}$ which depends only on those coefficients which parameterise \mathscr{W}_1 . The pair (K_{ij}, g_{ij}) constructed in Proposition 5.2 has jets in \mathscr{W}_1 , which shows that \tilde{Q} is non-trivial on \mathscr{W}_1 . It then follows,

as in the proof of Theorem 2.1, that any jets in \mathscr{W}_1 can be ϵ -perturbed so that $\tilde{Q}(p)$ does not vanish on the perturbed jet, with the jets of the perturbation belonging to $\mathscr{W}_{\ell+k}$; by analyticity some of the derivatives of \tilde{Q} with respect to its arguments will not vanish at p .

It should be clear from (5.33) that the perturbed solution satisfies the properties described in the statement of point 3 of Theorem 5.3. \square

6 Riemannian metrics without static KIDs near a point

An interesting class of initial data is provided by the *time-symmetric* ones, $K \equiv 0$. In this case the KID equations (5.3)–(5.4) decouple, with X in (5.3) being simply a Killing vector field of g . It remains to analyse the equation for N ,

$$D_i D_j N = N R_{ij} + \Delta N g_{ij} . \tag{6.1}$$

A solution of (6.1) will be called a *static* KID, and the set of static KIDs on a set Ω will be denoted by $\mathcal{N}(\Omega)$. (The origin of the adjective “static” will be clarified shortly.) Since time-symmetric initial data are non-generic amongst all initial data, the results of the previous section do not say anything about non-existence of static KIDs, and separate treatment is required.

Taking the trace of (6.1) one obtains, in dimension n

$$\Delta N = -\frac{1}{n-1} N R , \tag{6.2}$$

so that (6.1) can be rewritten as

$$D_i D_j N = N \left(R_{ij} - \frac{1}{n-1} g_{ij} R \right) . \tag{6.3}$$

Calculating D^j of (6.3) and commuting derivatives one is led to (recall that the Einstein tensor is divergence-free)

$$N D_i R = 0 . \tag{6.4}$$

Since the zero-set of solutions of (6.1) has no interior except if $N \equiv 0$, we conclude that existence of non-trivial static KIDs implies that R is constant. It follows that a non-trivial solution of (6.3) does indeed correspond to initial data for a static solution of the vacuum Einstein equations with a cosmological constant. Further, one immediately obtains that generic C^2 metrics have no static KIDs: it suffices to vary the metric so that the scalar curvature is not constant.

From now on we assume $\dim M = 3$. In order to prepare the proof, that generic metrics with fixed constant value of scalar curvature have no static KIDs, we consider a metric g with Ricci tensor at p equal to

$$\overset{\circ}{R}_{ij} = A x_i x_j + B y_i y_j + C z_i z_j , \tag{6.5}$$

where we assume that $(A - B)(A - C)(B - C) \neq 0$, and we further suppose that

$$D_{[i}\mathring{R}_{j]l} = \alpha x_{[i}y_{j]}z_l + \beta z_{[i}x_{j]}y_l + \gamma y_{[i}z_{j]}x_l - \frac{1}{6}(\alpha + \beta + \gamma)\epsilon_{ijl} , \tag{6.6}$$

with $(\alpha, \beta, \gamma) \neq 0$. We also impose the condition that

$$D_i\mathring{R} = 0 . \tag{6.7}$$

Taking a curl of (6.1) we infer that

$$(2R_{j[l} - Rg_{j[l})D_{i]}N + g_{j[l}R_{i]k}D^kN = ND_{[l}R_{i]j} . \tag{6.8}$$

The left-hand side of (6.8), with

$$D_i\mathring{N} = ax_i + by_i + cz_i , \tag{6.9}$$

takes the form

$$\begin{aligned} &x_jx_{[l}[y_{i]}b(A - C) + z_{i]}c(A - B)] \\ &+ y_jy_{[l}[x_{i]}a(B - C) + z_{i]}c(B - A)] \\ &+ z_jz_{[l}[x_{i]}a(C - B) + y_{i]}b(C - A)] . \end{aligned} \tag{6.10}$$

Since no terms with this index structure occur in (6.6) we obtain that $D_i\mathring{N}$ vanishes, and using (6.8) allows us to finally conclude that

$$\mathring{N} = D_i\mathring{N} = 0 . \tag{6.11}$$

The arguments of proof of Proposition 2.7 apply and provide existence of a metric $g_{ij} = \delta_{ij} + h_{ij}$ satisfying (6.5) and (6.6). Clearly $A + B + C$ can be chosen so that \mathring{R} has any prescribed value. Now, we can multiply g_{ij} by $1 + \alpha$, where α is a homogeneous third order polynomial chosen so that $D_i\mathring{R}$ is zero. By conformal invariance this does not change the value of $\mathring{B}_{ij}k$, hence of (6.6) (compare (2.1)–(2.2)), and does not change the value of \mathring{R} either. A repetition of the remaining arguments of Section 5, with K_{ij} there set to zero, gives:

Theorem 6.1 *Let (M, g) be a Riemannian manifold with $g \in C^k$, $k \geq 2$. A necessary condition for a non-trivial $\mathcal{N}(\Omega)$ is that the scalar curvature of g be constant on Ω . Further, in dimension three, and for $k \geq 3$, the following hold:*

1. *There exists a non-trivial homogeneous invariant polynomial*

$$Q[g] := Q(\text{Ric}, D \text{Ric}) \quad \text{such that if } Q(\text{Ric}, D \text{Ric})(p) \neq 0$$

at a point $p \in M$, for a metric for which the gradient of the scalar curvature vanishes at p , then there exists a neighborhood \mathcal{O}_p of p for which there exist no non-trivial static KIDS on any open subset of \mathcal{O}_p .

2. *Let Ω be a domain in M , and let $p \in \Omega$. There exists a variation $\delta g \in C^\infty(\Omega)$, compactly supported in Ω , such that $Q[g + \epsilon\delta g](p) \neq 0$ for all ϵ small enough. If $g \in C^{k+\ell+2}$, $\ell \geq 0$, has constant scalar curvature, then the variation above can be chosen to have the same scalar curvature up to error terms which are $o(r^\ell)$ in a $C^k(B(p_0, r))$ norm, for small r .*

7 Results in general dimensions, with non-explicit orders of differentiability

The results obtained so far did require rather unpleasant, tedious, and lengthy calculations, and we will present here an argument which avoids those. The draw-back is that one does not obtain an explicit statement on the number of derivatives involved. However, non-genericity of KIDs is obtained in higher dimensions. Further, the proof below generalizes immediately, *e.g.*, to the Einstein-Maxwell equivalent of the KID equations, the details are left to the reader.

The starting point of the analysis in this section is the following result (recall that $n = \dim M$):

Lemma 7.1

1. For any $n \geq 2$ and for any signature there exists a real analytic compact pseudo-Riemannian manifold (M, g) without local Killing vectors.
2. For any $n \geq 3$ there exists a real analytic compact simply connected Riemannian manifold (M, g) without local conformal Killing vectors.
3. For any $n \geq 3$, $\Lambda \in \mathbb{R}$, $\tau \in \mathbb{R}$ there exists a real analytic vacuum initial data set (M, g, K) , with cosmological constant Λ , with $\text{tr}_g K = \tau$, and without local KIDs.
4. For any $n \geq 3$ and $\Lambda \in \mathbb{R}$, there exists a real analytic Riemannian or Lorentzian manifold (\mathcal{M}, g) , with $\dim \mathcal{M} = n + 1$, satisfying the vacuum Einstein equations with cosmological constant Λ and without local Killing vectors.

Remark 7.2 The main point of the Lemma is to construct *one single* example in each category listed. However, our argument makes it clear that there are actually lots of examples. For instance, the proof below shows that in point 1 for any analytic manifold M which is simply connected, compact, with $\dim M \geq 2$ one can find a g with the required properties.

Remark 7.3 If

$$\tau^2 \geq \frac{2n}{(n-1)} \Lambda, \quad (7.1)$$

then one can find an M as in point 3 which is compact (without boundary). The proof in the strict inequality case is given below. If the inequality in (1.2) is an equality, in the proof below one should instead choose (M, γ_0) to be any real analytic compact Riemannian manifold of positive Yamabe class. The monotone iteration scheme for solving the Lichnerowicz equation can then be handled by an argument in [17]. In that last reference only dimension three is considered, but the proof applies in any dimension.

Proof. 1: Let M be any simply connected, compact, analytic Riemannian manifold with $\dim M \geq 2$, let $p \in M$ and let g_0 be any smooth Riemannian metric on M

such that the polynomial $P_n = P_n[g]$ of Theorem 3.1 does not vanish at p . We need analytic approximations of g , for example for $0 \leq t < \epsilon$ we can let g_t be the family of metrics obtained by evolving g_0 using the Ricci flow, then the metrics g_t are indeed real analytic for $t > 0$. By continuity, reducing ϵ if necessary, we will have $P_n[g_t](p) \neq 0$, hence g_t will have no Killing vectors in a neighborhood of p . Now, a theorem⁴ of Nomizu [20] shows that on a simply connected analytic manifold every locally defined Killing vector extends to a globally defined one. This implies that for $0 < t < \epsilon$ the metrics g_t have no Killing vectors on any open subset of M .

2: For $n = 3$ this follows from Theorem 2.1. For any $n \geq 3$ one can argue as follows: Let M be any compact real analytic manifold of dimension not less than three. By [18] there exists on M a metric g with strictly negative Ricci curvature. It is well known that such metrics do not have non-trivial conformal Killing vectors, we recall the proof for completeness: from (2.6) with $2/3$ replaced by $2/n$ it follows that

$$D_i D_j X_k = -R_{jki}{}^l X_l + \frac{1}{n} (\varphi_i g_{jk} + \varphi_j g_{ik} - \varphi_k g_{ij}) , \tag{7.2}$$

hence

$$\Delta X_k = -R_k{}^l X_l - (1 - \frac{2}{n}) \varphi_k . \tag{7.3}$$

Multiplying by X^k and integrating over M one finds (recall that $\varphi = \operatorname{div} X$)

$$\int_M |DX|^2 - \operatorname{Ric}(X, X) + (1 - \frac{2}{n}) \varphi^2 = 0 ,$$

so that $X \equiv 0$ if $\operatorname{Ric} < 0$. Approximating g by real-analytic metrics g_t , $g_t \rightarrow g$ as $t \rightarrow 0$, one will have no conformal Killing vectors for g_t when t is small enough by Proposition 4.2. It then follows from Theorem B.1, Appendix B, that the g_t 's will have no local conformal Killing vector field either.

3 and 4: We start by noting that in dimension $n = 3$, an example of initial data as in point 3 can be obtained using vacuum Robinson-Trautman space-times with cosmological constant Λ (*cf.*, *e.g.*, [5]). Because of the parabolic character of the Robinson-Trautman equation, those metrics are always analytic away from the initial data surface. Further, if the initial metric h_0 on S^2 used in the Robinson-Trautman equation has no continuous global symmetries, then it follows from Proposition 4.2 that the evolved metrics h_t will not have any continuous global symmetries either, at least for t small enough. It is clear that the resulting four-dimensional metric 4g will then have no globally defined Killing vectors except the zero one. The non-existence of local Killing vectors follows then from Nomizu's theorem [20]. Finally, the initial data set of point 3 can be obtained as that induced by 4g on any hypersurface with $\operatorname{tr}_g K = \tau$ in \mathcal{M} ; such hypersurfaces can be obtained by solving a Dirichlet problem for the CMC equation on the boundary of a sufficiently small spacelike three-ball [4].

⁴We note that in [20] a Riemannian metric is assumed, but the proofs given there apply to any signature.

In any case, whatever $n \geq 3$ one can proceed as follows: consider, first, τ such that the inequality in (7.1) is strict, let (M, γ_0) be any real analytic compact Riemannian manifold of negative Yamabe class. Let L_0 be any non-zero, γ_0 -transverse and traceless tensor on M ; such tensors exist by [6]. For $t \in [0, \epsilon]$ let L_t be a family of analytic symmetric γ_0 -trace free tensors converging to L_0 . For example, L_t can be obtained from L_0 by heat flow using any analytic metric on M , and removing the γ_0 trace. Using the conformal⁵ method [17] with seed fields (γ_0, L_t) one obtains a family of real analytic vacuum CMC initial data sets (g_t, K_t) with cosmological constant Λ . Since γ_0 has no global conformal Killing vectors, g_t will have no global Killing vectors. Now, $\text{tr}_{g_t} K_t = \tau$ is a constant, which implies (see Remark 9.2 below) that any global KIDs for (g_t, K_t) are of the form $(N = 0, Y)$, where Y is a Killing vector of g_t , therefore none of the (g_t, K_t) 's has global KIDs. In the Lorentzian case we let $(\mathcal{M}, {}^{n+1}g_t)$ be the maximal globally hyperbolic vacuum development of (M, g_t, K_t) , then \mathcal{M} is diffeomorphic to $\mathbb{R} \times M$ (hence simply connected), and ${}^{n+1}g_t$ is analytic by [1]. In the Riemannian case we let \mathcal{M} be any simply connected and connected neighborhood \mathcal{U}_t of $M \times \{0\}$ in $M \times (-1, 1)$, chosen so that there exists a vacuum metric ${}^{n+1}g_t$ on \mathcal{U}_t with Cauchy data (g_t, K_t) on $M \times \{0\}$, obtained from the Cauchy-Kowalewska theorem. Suppose that there exists an open nonempty subset $\Omega_t \subset M$ such that $\mathcal{K}_t(\Omega_t) \neq \{0\}$, where \mathcal{K}_t denotes the set of KIDs with respect to (g_t, K_t) , then a standard argument [13]⁶, using the Cauchy-Kowalewska theorem, shows that there exists a non-trivial Killing vector X in a neighborhood of Ω_t in \mathcal{M} . By Nomizu's theorem [20] X extends to a globally defined Killing vector on \mathcal{M} , hence (M, g_t, K_t) has a globally defined KID, a contradiction. Thus there are no local KIDs on (M, g_t, K_t) , and $(\mathcal{M}, {}^{n+1}g_t)$ is a vacuum metric without local Killing vectors. This proves point 4, as well as Remark 7.3 in the case of a strict inequality there. To prove point 3 for the remaining values of τ one spans [4], within the Lorentzian solution \mathcal{M} just constructed, a CMC hypersurface of prescribed $\tau = \text{tr}_g K$ on the boundary of a small spacelike ball. The data induced on the resulting CMC hypersurface provide the desired initial data set. \square

We continue with the question of generic non-existence of KIDs, it should be clear that an identical argument applies to conformal Killing vectors (compare [7, Equation (1.15)]), or to Killing vectors. Let (g, K) be any vacuum analytic initial data on a simply connected manifold M which have no global KIDs. As explained above, it follows from a theorem of Nomizu [20], that such an initial data set will not have any local KIDs. Let $r(p) \in \mathbb{R}^M$ be as in Lemma 5.1, for some appropriate M , and for $\alpha \in \mathbb{N}^n$, let us write

$$D_\alpha r = P_\alpha r \tag{7.4}$$

⁵Since the inequality in (1.2) is strict, a small and a large constant provide barriers for the monotone iteration scheme.

⁶Compare the proof of [9, Theorem 2.1.1]; the Cauchy-Kowalewska theorem should be invoked for solvability of Eq. (2.1.5) there, or for uniqueness of solutions of Eq. (2.1.7) there.

for the linear system of equations obtained by calculating the $(k + 1)$ -st derivatives of r by differentiating (5.3)–(5.4) $|\alpha|$ times, and replacing the lower order derivatives that arise in the process by their values already calculated from the previous equations. Let us write $L_k r = 0$ for the system of equations that arise from first-order integrability conditions of the system (7.4) with $|\alpha| \leq k$. Choose some orthonormal frame, then L_k can be identified with a $N_k \times M$ matrix, for some N_k , with entries built out of the extrinsic curvature tensor K , of the Riemann tensor, and of their derivatives. Let Q_k denote the sum of squares of determinants of all $M \times M$ sub-matrices of L_k . Then the equation $L_k r = 0$ admits a non-trivial solution if and only if $Q_k = 0$. Suppose that there exists r_0 such that $L_k r_0 = 0$ for all k . One can then use (7.4) to calculate all the jets of r with initial value r_0 at p so that the Killing equations are satisfied to infinite order at p by a formal solution determined by those jets. To show convergence of the resulting Taylor series one can proceed as follows: let $x^i \in [-\epsilon, \epsilon]^n$ be local analytic coordinates around $p = \vec{0}$, we can solve the linear equation

$$\frac{\partial r}{\partial x^1} = P_1 r$$

along the path $[-\epsilon, \epsilon] \ni x^1 \rightarrow (x^1, 0, \dots, 0)$, with initial data r_0 at the origin, obtaining an analytic solution there. We can use the function so obtained as initial data for the equation

$$\frac{\partial r}{\partial x^2} = P_2 r$$

to obtain an analytic solution on $[-\epsilon, \epsilon]^2 \times \{0\} \times \dots \times \{0\}$. An inductive repetition of this procedure provides an analytic solution on $[-\epsilon, \epsilon]^n$ of the equation

$$\frac{\partial r}{\partial x^n} = P_n r,$$

such that the equation

$$\frac{\partial r}{\partial x^k} = P_k r \text{ holds on } [-\epsilon, \epsilon]^k \times \underbrace{\{0\} \times \dots \times \{0\}}_{n-k \text{ factors}}.$$

By choice of r_0 the analytic functions $L_k r$ have all derivatives vanishing at the origin, hence they vanish on $[-\epsilon, \epsilon]^n$. Standard arguments imply that the function r so obtained provides an analytic solution of the KID equations in a neighborhood of p . This gives a contradiction with the fact that (g, K) has no local KIDs near p . Therefore there exists k such that Q_k is non-zero for the initial data set under consideration. This Q_k provides the non-trivial polynomial needed in Theorem 5.3. When the metric involved is Riemannian we can integrate Q_k , viewed as a function on the frame bundle, over the rotation group to obtain an invariant polynomial.

We have therefore proved:

Theorem 7.4 *Theorem 6.1 remains valid in any dimension, with an invariant polynomial that depends upon some dimension-dependent number k of derivatives of g . Similarly Theorem 5.3 remains valid in any dimension, for some polynomial that depends upon $k + 1$ derivatives of g and k derivatives of K , for some dimension-dependent number k . In dimension $n \geq 4$ Theorem 3.1 remains valid with a polynomial which depends upon some dimension-dependent number k of derivatives of the Weyl tensor. Finally, Theorem 2.1 remains valid in any dimension $n \geq 3$ with a polynomial that depends upon some dimension-dependent number of derivatives of the Riemann tensor.*

Proof. The only statement which, at this stage, might require justification is the extension of Theorem 3.1: this result follows from point 4 of Lemma 7.1, as the polynomial obtained in that case by the proof above depends only upon the Weyl tensor. \square

8 From approximate linearized solutions to small vacuum perturbations

The perturbation results of the previous sections can be used to prove non-genericity of KIDs when no restrictions on ρ and J are imposed. They also apply if, *e.g.*, a *strict* dominant energy condition $\rho > |J|$ is imposed, for then a sufficiently small perturbation of the data will preserve that inequality. However, some more work is needed when vacuum initial data are considered, and this is the issue addressed in this section.

Let $\Omega \subset M$ be open and connected, and let $\mathcal{K}(\Omega)$ denote the set of KIDs defined on Ω ; each $\mathcal{K}(\Omega)$ is a finite-dimensional, possibly trivial, vector space. If $\Omega' \subset \Omega$ we have the natural map

$$i_{\Omega'} : \mathcal{K}(\Omega) \rightarrow \mathcal{K}(\Omega') ,$$

with $i_{\Omega'}(x)$ being defined as the restriction to Ω' of the KID $x \in \mathcal{K}(\Omega)$. A local KID vanishing on an open subset vanishes throughout the relevant connected component of its domain of definition, which shows that $i_{\Omega'}$ is injective.

We denote by $\underline{B}(p, r)$ the open geodesic ball of radius r , and for $a < b$ we set $\Gamma_p(a, b) := B(p, b) \setminus \underline{B}(p, a)$.

We will need the following result:

Proposition 8.1 *For every $p \in M$ and $\mathbb{R} \ni r > r_1 > 0$ there exists $0 < r_2 < r_1$ such that*

$$i_{\Gamma_p(r_2, r)} : \mathcal{K}(B(p, r)) \rightarrow \mathcal{K}(\Gamma_p(r_2, r))$$

is bijective.

The proof rests on the following lemma:

Lemma 8.2 *For every $p \in M$ and $r_1 > 0$ there exists $\sigma \in (0, 1)$ such that*

$$i_\sigma : \mathcal{K}(B(p, r_1)) \rightarrow \mathcal{K}(\Gamma_p(\sigma r_1, r_1))$$

is bijective. Here i_σ denotes $i_{\Gamma_p(\sigma r_1, r_1)}$.

Proof. As already pointed out, injectivity always holds. Suppose that surjectivity fails, then for every $\sigma \in (0, 1)$ there exists a KID $x_\sigma \in \mathcal{K}(\Gamma_p(\sigma r_1, r_1))$ such that $x_\sigma \notin i_\sigma(\mathcal{K}(B(p, r_1)))$. Choose any scalar product h on $\mathcal{K}(\Gamma_p(r_1/2, r_1))$. For $\sigma < 1/2$ without loss of generality we can assume that the restriction \hat{x}_σ of x_σ to $\Gamma_p(r_1/2, r_1)$ is h -orthogonal to the image of $i_{1/2}$, and that $h(\hat{x}_\sigma, \hat{x}_\sigma) = 1$. Since $\mathcal{K}(\Gamma_p(r_1/2, r_1))$ is finite-dimensional there exists a sequence $\sigma_i \rightarrow 0$ such that \hat{x}_{σ_i} converges to some \hat{x}_0 , with $h(\hat{x}_0, \hat{x}_0) = 1$. Further \hat{x}_0 is h -orthogonal to $i_{1/2}(\mathcal{K}(B(p, r_1)))$. It should be clear from (5.4)–(5.5) that for i such that $\sigma > \sigma_i$, the sequence of KIDs on $\Gamma_p(\sigma r_1, r_1)$ obtained by restricting x_{σ_i} to $\Gamma_p(\sigma r_1, r_1)$ converges, and defines a non-trivial KID which restricts to \hat{x}_0 on $\Gamma_p(r_1/2, r_1)$, with the limit being independent of σ in the obvious sense. This shows that there exists a KID x_0 defined on $B(p, r_1) \setminus \{p_1\}$ such that \hat{x}_0 is the restriction of x_0 to $\Gamma_p(r_1/2, r_1)$. But (5.4)–(5.5) further shows that x_0 can be extended to a KID defined on $B(p, r_1)$, still denoted by x_0 . It follows that $\hat{x}_0 = i_{1/2}(x_0)$, which contradicts orthogonality of \hat{x}_0 with the image of $i_{1/2}$. □

Proof of Proposition 8.1: Let $r_2 = \sigma r_1$, with σ given by Lemma 8.2. Every KID on $\Gamma_p(r_2, r)$ induces, by restriction, a KID on $\Gamma_p(r_2, r_1)$, therefore $\dim \mathcal{K}(\Gamma_p(r_2, r)) \leq \dim \mathcal{K}(\Gamma_p(r_2, r_1))$. By Lemma 8.2 we have $\dim \mathcal{K}(\Gamma_p(r_2, r_1)) = \dim \mathcal{K}(B(p, r))$. Again by restriction we have $\dim \mathcal{K}(B(p, r)) \leq \dim \mathcal{K}(\Gamma_p(r_2, r))$, whence the result. □

Corollary 8.3 *Suppose that $\mathcal{K}(B(p, r)) = \{0\}$. Then for any $\epsilon > 0$ there exists $\epsilon > r_1 > 0$ such that $\mathcal{K}(\Gamma_p(r_1, r)) = \{0\}$.* □

Recall that the *constraints map* has been defined by the formula:

$$\begin{pmatrix} J \\ \rho \end{pmatrix} (K, g) := \begin{pmatrix} 2(-\nabla^j K_{ij} + \nabla_i \operatorname{tr} K) \\ R(g) - |K|^2 + (\operatorname{tr} K)^2 - 2\Lambda \end{pmatrix}. \tag{8.1}$$

The following is one of the key steps of the proof:

Theorem 8.4 *For $\ell \in \mathbb{N}$, $\ell \geq 2$, $\alpha \in (0, 1)$, $p \in M$, $r, \eta > 0$, let the symbol P denote the linearization of the constraints operator (8.1) at*

$$(K, g) \in (C^{\ell+2, \alpha} \times C^{\ell+2, \alpha})(B(p, r)),$$

and let

$$x_\eta = (\delta K_\eta, \delta g_\eta) \in (C^{\ell+2, \alpha} \times C^{\ell+2, \alpha})(B(p, r))$$

be an “approximate solution” of the linearized constraint equations defined on $B(p, r)$, in the sense that:

$$\|Px_\eta\|_{(C^{\ell+1,\alpha} \times C^{\ell,\alpha})(B(p,r))} \leq \eta .$$

1. There exists a constant C such that if $i_{\Gamma_p(\sigma r, r)}$ is surjective for some $\sigma \in (0, 1/2]$, then there exists a solution $x \in (C^{\ell+2,\alpha} \times C^{\ell+2,\alpha})(B(p, r))$ of the linearized constraint equations supported in $B(p, r)$ such that

$$\|x - x_\eta\|_{(C^{\ell+2,\alpha} \times C^{\ell+2,\alpha})(B(p, \sigma r))} \leq C\eta . \tag{8.2}$$

x is smooth if (K, g) and x_η are.

2. For $\ell \geq 4$, for any (K_0, g_0) in $(C^{\ell+2,\alpha} \times C^{\ell+2,\alpha})(B(p, r))$, and for any r_0 such that $B(p, r_0)$ has smooth boundary, the constant C can be chosen independently of $\sigma \in (0, 1/2]$, (K, g) , and r satisfying $0 < r \leq r_0$, for all (K, g) sufficiently close in $(C^{\ell+2,\alpha} \times C^{\ell+2,\alpha})(B(p, r_0))$ to (K_0, g_0) .

Remark 8.5 The restriction $\sigma \leq 1/2$ is arbitrary, the argument applies with any $0 < \sigma \leq \sigma_0 \in (0, 1)$, with a constant in (8.2) depending perhaps upon σ_0 .

Proof. We use the definitions and notation of [11]. In particular if Ω is a domain with smooth boundary, then

$$\Lambda_{k,\alpha}^s = \mathring{H}_2^s \cap C_{k,\alpha}^s .$$

Roughly speaking, functions in that space behave as $o(x^s)$ near the boundary $\{x = 0\}$, with derivatives of order j , $0 \leq j \leq k$, being allowed to behave as $o(x^{s-j})$. In particular if $s > k + \alpha$ then functions in the space above are in $C^{k,\alpha}(\bar{\Omega})$. We will need the following result [11, Proposition 6.5]:

Proposition 8.6 *Suppose that $(K_0, g_0) \in (C^{k+2,\alpha} \times C^{k+2,\alpha})(M)$, $k \geq 2$, $\alpha \in (0, 1)$, and let $\Omega \subset M$ be a domain with smooth boundary and compact closure. For all $s \neq (n + 1)/2, (n + 3)/2$, the image of the linearization P , at (K_0, g_0) , of the constraints map, when defined on $(\Lambda_{k+2,\alpha}^{-s+1} \times \Lambda_{k+2,\alpha}^{-s+2})(\Omega)$, is*

$$\left\{ (J, \rho) \in \Lambda_{k+1,\alpha}^{-s} \times \Lambda_{k,\alpha}^{-s} \text{ such that } \langle (J, \rho), (Y, N) \rangle_{(L^2 \oplus L^2)(\Omega, d\mu_{g_0})} = 0 \right. \\ \left. \text{for all } (Y, N) \in H_1^{s-n} \times H_2^{s-n} \text{ satisfying } P^*(Y, N) = 0 \right\} .$$

Further $P^{-1}(0) \subset \Lambda_{k+2,\alpha}^{-s+1} \times \Lambda_{k+2,\alpha}^{-s+2}$ splits. □

The proof of point 1 of Theorem 8.4 will proceed in two steps:

STEP 1: We set $M := B(p, r)$, $k = \ell$, $(g_0, K_0) = (g, K)$, and we use Proposition 8.6 with $s = s_1$ for some $s_1 < -1$. Now, for such s the space $\mathcal{K}_0 \subset \mathcal{K}(B(p, r))$ above is the space of KIDs on $B(p, r)$ which vanish at $S(p, r) := \partial B(p, r)$ together with

their first derivatives; but Equations (5.4)–(5.5) imply that there are no such non-trivial KIDs. It follows that P is surjective, with the splitting property being equivalent to the fact that there exists a closed subspace $X \subset \Lambda_{k+2,\alpha}^{-s+1} \times \Lambda_{k+2,\alpha}^{-s+2}$ such that the restriction of P to X is an isomorphism. This shows that there exists $\hat{x}_\eta \in \Lambda_{k+2,\alpha}^{-s+1} \times \Lambda_{k+2,\alpha}^{-s+2}$ satisfying

$$\|\hat{x}_\eta\|_{\Lambda_{k+2,\alpha}^{-s+1} \times \Lambda_{k+2,\alpha}^{-s+2}} \leq C\eta,$$

and $P(\hat{x}_\eta) = -P(x_\eta) \Leftrightarrow P(x_\eta + \hat{x}_\eta) = 0$.

STEP 2: Now, because $s = s_1 < -1$, the correction term \hat{x}_η could be blowing-up near $S(p, r)$, while we want a solution which vanishes there to rather high order. To correct that, let φ be any smooth non-negative function which is identically one on $B(p, 5r/8)$, and vanishes on $\Gamma_p(3r/4, r)$, set

$$y_\eta = \varphi(x_\eta + \hat{x}_\eta).$$

Then $P(y_\eta)$ is supported in $\overline{\Gamma_p(5r/8, 3r/4)} \subset \Gamma(\sigma r, r)$. We now use Proposition 8.6 once again, with some $s = s_2 > \ell + 3$, to find $\tilde{x}_\eta \in (C^{\ell+2,\alpha} \times C^{\ell+2,\alpha})(\Gamma_p(\sigma r, r))$, which extends by zero both through $S(p, \sigma r)$ and through $S(p, r)$ in a $C^{\ell+2,\alpha} \times C^{\ell+2,\alpha}$ manner, such that

$$P(\tilde{x}_\eta + y_\eta) = 0 \iff P(\tilde{x}_\eta) = -Py_\eta =: z_\eta.$$

This will be possible if and only if z_η is orthogonal in $L^2(\Gamma_p(\sigma r, r))$ to $\mathcal{K}_0(\Gamma_p(\sigma r, r))$, where now $\mathcal{K}_0(\Gamma_p(\sigma r, r))$ coincides with the space of all KIDs on $\Gamma_p(\sigma r, r)$. Let, thus, $w = (Y, N) \in \mathcal{K}_0(\Gamma_p(\sigma r, r))$, by hypothesis there exists a KID \hat{w} defined on $B(p, r)$ such that w is the restriction to $\Gamma_p(\sigma r, r)$ of \hat{w} . We then have

$$\int_{\Gamma_p(\sigma r, r)} \langle w, Py_\eta \rangle = \int_{B(p, r)} \langle \hat{w}, Py_\eta \rangle = \int_{B(p, r)} \langle \hat{P}^* \hat{w}, y_\eta \rangle = 0.$$

Here the first and the second equalities are justified because Py_η is supported in $\overline{\Gamma_p(\sigma r, 3r/4)}$, while the last one follows because, by definition of a KID, $P^* \hat{w} = 0$. This provides the desired \tilde{x}_η . Setting $x_\eta = \varphi(x + \hat{x}_\eta) + \tilde{x}_\eta$, point 1 is proved.

To prove point 2, we first note that the value of σ does not affect the constant C , as that constant arises from step 1 of the proof of point 1: the perturbation \tilde{x}_η from step 2, which *could* depend upon σ , is supported away from $B(p, \sigma r)$. The result is proved now by the usual contradiction argument: Consider the map

$$\pi_{\mathcal{K}_0^{\perp g}} L_{\epsilon, x, x^{s-n/2}} : \mathcal{K}_0^{\perp g} \cap (\Lambda_{k+3,\alpha}^{-s} \times \Lambda_{k+4,\alpha}^{-s}) \longrightarrow \mathcal{K}_0^{\perp g} \cap (\Lambda_{k+1,\alpha}^{-s} \times \Lambda_{k,\alpha}^{-s}), \quad (8.3)$$

with $L_{\epsilon, x, x^{s-n/2}}$ being a regularized version, as in [11], of the map $L_{x, x^{s-n/2}}$ of [10, Section 5]. Equation (8.2) will fail to hold only if there exists a sequence of radii r_n and data (K_n, g_n) on $B(p, r_n)$ near $(K_0, g_0)|_{B(p, r_n)}$, with KIDs $(Y_n, N_n) \in \mathcal{K}_0^{\perp g}$ such that

$$\|(Y_n, N_n)\|_{\Lambda_{3,\alpha}^{-s} \times \Lambda_{4,\alpha}^{-s}} = 1 \text{ and } \|L_{\epsilon, x, x^{s-n/2}}(Y_n, N_n)\|_{\Lambda_{1,\alpha}^{-s} \times \Lambda_{0,\alpha}^{-s}} \leq 1/n.$$

Consider an extracted sequence, still denoted by r_n , converging to r_∞ . If $r_\infty > 0$, then $(K_0, g_0)|_{B(p, r_\infty)}$ would admit a KID vanishing, together with its first derivatives, at $S(p, r_\infty)$, a contradiction. On the other hand suppose that $r_\infty = 0$, introduce geodesic coordinates for the metrics (K_n, g_n) centred at p ; this might lead to a loss of two derivatives of the metric, so we increase the threshold on ℓ from two to four. Consider the sequence $(\tilde{K}_n, \tilde{g}_n)$ on $B(p, 1)$ obtained by scaling up the ball $B(p, r_n)$ to $B(p, 1)$. Then $(\tilde{K}_n, \tilde{g}_n)$ converges to $(0, \delta)$, where δ is the Euclidean metric on $B(p, 1)$. As before one obtains a contradiction because there are no KIDs vanishing, together with their first derivatives, on $S(p, 1)$ for $(K, g) = (0, \delta)$.

Smooth solutions can be obtained proceeding as above, but working instead with exponentially-weighted rather than power-weighted spaces. \square

The main result of this section is the following (see footnote 2):

Theorem 8.7 *Let M be a compact manifold with boundary, suppose that $\ell \geq \ell_0(n)$, $\alpha \in (0, 1)$ for some $\ell_0(n)$ ($\ell_0(3) = 6$), and let (M, K, g) be a $C^{\ell, \alpha} \times C^{\ell, \alpha}$ vacuum initial data set such that*

$$\mathcal{K}(M) = \{0\} .$$

For any $p \in M \setminus \partial M$ and for any $\epsilon > 0$ there exists $r > 0$ and an ϵ -small, in a $C^{\ell, \alpha} \times C^{\ell, \alpha}$ topology, vacuum perturbation (K_ϵ, g_ϵ) of (K, g) such that

$$\mathcal{K}(\mathcal{U}) = \{0\} \text{ for all } \mathcal{U} \text{ such that } \mathcal{U} \cap B(p, r) \neq \emptyset .$$

Further, (K_ϵ, g_ϵ) can be chosen to coincide with (K, g) in a neighborhood of ∂M .

Proof. For definiteness in the proof we will assume $n = 3$, for $n > 3$ in the argument below Theorem 5.3 should be replaced by its higher-dimensional generalization provided by Theorem 7.4. If the polynomial Q of point 1 of Theorem 5.3 does not vanish at p , we let $r > 0$ be small enough so that Q has no zeros on $B(p, r)$. Otherwise, let $\delta x := (\delta K, \delta g)$ be as in point 3 of Theorem 5.3 with $\ell = 1$ and $k = 3$. Let $\epsilon > 0$, as Q is a polynomial we have

$$Q[x + \epsilon \delta x](p) = \epsilon^j (Q^{(j)}[\delta x])(p) + O(\epsilon^{j+1}) ,$$

for some $j \geq 1$ such that $(Q^{(j)}[\delta x])(p) \neq 0$. By Proposition 8.1 for any $r > 0$ there exists $\sigma_r \in (0, 1)$ such that the conditions of Theorem 8.4 are satisfied. We then have

$$\|P\delta x\|_{(C^{3, \alpha} \times C^{2, \alpha})(B(p, r))} \leq \|P\delta x\|_{(C^4 \times C^3)(B(p, r))} \leq C_1 r ,$$

and Theorem 8.4 provides a solution $\delta \tilde{x}$ of the linearized constraint equations supported in $B(p, r)$ such that

$$\|\delta x - \delta \tilde{x}\|_{C^{4, \alpha}(B(p, \sigma_r r))} \leq CC_1 r .$$

Choosing r small enough so that $CC_1 r \leq \epsilon$ one obtains

$$Q[x + \epsilon \delta \tilde{x}](p) = \epsilon^j (Q^{(j)}[\delta x + O(\epsilon)])(p) + O(\epsilon^{j+1}) = \epsilon^j (Q^{(j)}[\delta x])(p) + O(\epsilon^{j+1}) .$$

Since $\delta\tilde{x}$ satisfies the linearized constraint equations and since $\mathcal{K}(M) = \{0\}$, it follows from [10, Theorem 5.6] together with the regularization technique from [11] that for ϵ small enough we can find $\delta\hat{x}(\epsilon)$, with $\|\delta\hat{x}(\epsilon)\|_{C^4(B(p,r))} \leq C_2\epsilon^2$, such that $x + \epsilon\delta\tilde{x} + \delta\hat{x}(\epsilon)$ satisfies the vacuum constraint equations. Choosing ϵ small enough so that $C_2\epsilon \leq \epsilon^{1/2}$ we then obtain

$$\begin{aligned} Q[x + \epsilon\delta\tilde{x} + \delta\hat{x}(\epsilon)](p) &= \epsilon^j(Q^{(j)}[\delta\tilde{x} + O(\epsilon^{1/2})])(p) + O(\epsilon^{j+1}) \\ &= \epsilon^j(Q^{(j)}[\delta x])(p) + O(\epsilon^{j+1/2}) \neq 0 \end{aligned}$$

for ϵ small enough. Replacing r by a smaller number if necessary so that $Q[x + \epsilon\delta\tilde{x} + \delta\hat{x}(\epsilon)]$ has no zeros on $B(p,r)$, the resulting data set has no KIDs in any subset of $B(p,r)$ by point 1 of Theorem 5.3.

The construction described so far leads to a perturbed initial data set which agrees with the starting one at ∂M to arbitrarily high order. Consider, next, a collar neighborhood $N_s = \{p \in M : d(p, \partial M) < s\}$ of ∂M . Arguments as in Lemma 8.2 show that $\mathcal{K}(M_s := M \setminus N_s) = \{0\}$ for s small enough. Applying the result already established to M_s one obtains a perturbation which vanishes on N_s . \square

An identical proof, based on the results in Section 6, gives:

Theorem 8.8 *Let (M, g) be a n -dimensional $C^{\ell, \alpha}$ compact Riemannian manifold with boundary, suppose that $\ell \geq \ell_0(n)$, $\alpha \in (0, 1)$ for some $\ell_0(n)$, and suppose that g has constant scalar curvature s . Assume that there are only trivial static KIDs,*

$$\mathcal{N}(M) = \{0\}.$$

For any $p \in M \setminus \partial M$ and for any $\epsilon > 0$ there exists $r > 0$ and an ϵ -small, in a $C^{\ell, \alpha}$ topology, perturbation g_ϵ of g with scalar curvature s such that

$$\mathcal{N}(\mathcal{U}) = \{0\} \text{ for all } \mathcal{U} \text{ such that } \mathcal{U} \cap B(p, r) \neq \emptyset.$$

Further, g_ϵ can be chosen to coincide with g in a neighborhood of ∂M .

9 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2: Let Q be the polynomial of Theorem 5.3, set

$$\hat{\mathcal{V}}_p = \{\text{vacuum initial data such that } Q[K, g](p) \neq 0\},$$

then $\hat{\mathcal{V}}_p$ is open and contained in \mathcal{V}_p . To show density, let $M_i \subset M$ be a sequence of relatively compact domains with smooth boundary such that $M = \cup_i M_i$. The argument of the proof of Lemma 8.2 shows that $\mathcal{K}(M_i) = \{0\}$ for i large enough. Point 1 follows then from Theorem 8.7 with M there equal to \overline{M}_i . The time-symmetric case is obtained similarly by Theorem 8.8. Point 2 is established by repeating the argument of the proof of Theorem 4.1. \square

Proof of Theorem 1.3: Openness follows from Proposition 4.2, it remains to establish density. We start by showing that for spatially compact CMC initial data KIDs are “purely spacelike”. Somewhat more generally, one has:

Proposition 9.1 *Consider a vacuum initial data set (M, g, K) with constant $\tau := \text{tr}_g K = 0$, suppose that (1.2) holds, and assume that (M, g) is geodesically complete (perhaps as a manifold with boundary). Let (N, Y) be a KID on M satisfying*

$$\lim_{r \rightarrow \infty} \sup_{q \in S_p(r)} |N(q)| = 0, \tag{9.1}$$

for some $p \in M$, where $S_p(r)$ is the boundary of the geodesic ball of radius r centred at p . If $N \not\equiv 0$, then K is pure trace, M is compact and (M, g) is Einstein.

Remark 9.2 A KID satisfying (9.1) will be called *asymptotically tangential*; a KID with $N \equiv 0$ will be called *tangential*. In the compact boundaryless case we have $S_p(r) = \emptyset$ for r large enough, so all KIDs are asymptotically tangential.

Proof. We note that if M has a boundary, then $S_p(r) \cap \partial M := \partial B_p(r) \cap \partial M \neq \emptyset$ for r large, so that (9.1) implies that N vanishes on ∂M . The KID equations imply

$$\Delta N = \left(|K|^2 - \frac{2\Lambda}{(n-1)} \right) N = \left(|\tilde{K}|^2 + \frac{(\text{tr}_g K)^2}{n} - \frac{2\Lambda}{(n-1)} \right) N,$$

where \tilde{K} is the trace-free part of K . Equation (1.2) and the maximum principle show that either $\tilde{K} \equiv 0$, with (1.2) being an equality, and $N = \text{const}$, or $N \equiv 0$. In the former case the KID equations further imply Ricci flatness of g . The case $N \not\equiv 0$ is compatible with (9.1) only if $S_p(r) = \emptyset$ for r sufficiently large, which is equivalent to compactness of M . \square

We note the following straightforward consequence of Theorem 2.1 and Proposition 4.2:

Proposition 9.3 *Consider the collection of Riemannian metrics on a three-dimensional manifold with a C^k (weighted, with arbitrary weights, in the non-compact case) topology, $k \geq 5$. The set of such Riemannian metrics which have no globally defined conformal Killing vectors is open and dense.*

Proof. Choose any relatively compact $\Omega \subset M$, then by Theorem 2.1 and Proposition 4.2 there exists an open and dense set of metrics which have no conformal Killing vector fields on Ω , then those metrics do not have globally defined conformal Killing vector fields either. \square

Our next result uses spaces $C_{\varphi, \psi}^{k, \alpha}$ defined in Appendix A. The weights φ and ψ in our next result have to be chosen in a way compatible with the conformal method in the asymptotically flat regions [8], similarly in the asymptotically hyperbolic regions [2], while $\varphi = \psi = 1$ in the compact case. The differentiability here are different, as compared to Theorem 1.3, because under the CMC restriction the conformal method can be used:

Corollary 9.4 *There exists $k_1(n)$, with $k_1(3) = 5$, such that for $k \geq k_1(n)$ and $\alpha \in (0, 1)$ the following holds: There exists a $C_{\varphi, \psi}^{k, \alpha} \times C_{\varphi, \psi}^{k-1, \alpha}$ -open and dense collection of vacuum CMC initial data sets (M, g, K) which are either*

1. *asymptotically flat with compact interior (then $\Lambda = 0$), or*
2. *asymptotically hyperbolic as in [2], or*
3. *defined on a compact M , with Λ satisfying (1.2),*

and which do not have any asymptotically tangential KIDs.

Proof. Let \mathcal{U} be the set of vacuum initial data $(g, K) \in C_{\varphi, \psi}^{k, \alpha} \times C_{\varphi, \psi}^{k-1, \alpha}$ such that g is Einstein. This class of initial data obviously forms a closed set with no interior. Let \mathcal{V}_1 be the complement of \mathcal{U} within the set of all vacuum $C_{\varphi, \psi}^{k, \alpha} \times C_{\varphi, \psi}^{k-1, \alpha}$ initial data, then \mathcal{V}_1 is open and dense. Choose any $p \in M$, and let \mathcal{V}_2 be the set of initial data in \mathcal{V}_1 such that K is not pure trace, and such that the polynomial $Q[H, DH, D^2H]$ of Theorem 2.1 does not vanish at p , then \mathcal{V}_2 is open. Consider any (g, K) which is not in \mathcal{V}_2 and which is not in \mathcal{U} , by Proposition 9.3 for any $\epsilon > 0$ there exists a metric $g'(\epsilon)$ which is in \mathcal{V}_2 (and therefore has no conformal Killing vectors) such that $\|g - g'(\epsilon)\|_{C_{\varphi, \psi}^5} \leq \epsilon$. Using K as the seed solution for the extrinsic curvature, the conformal method [2, 8, 17] allows one to solve for a nearby solution $(M, g(\epsilon), K(\epsilon))$ of the vacuum constraint equations. Let (N, Y) be a KID for $g(\epsilon)$. By Proposition 9.1 the KID (N, Y) is tangential, $N \equiv 0$, which implies that Y is a Killing vector field of $g(\epsilon)$. Now $g(\epsilon)$ is a conformal deformation of $g'(\epsilon)$, therefore Y is a conformal Killing vector field of $g'(\epsilon)$, hence $Y = 0$. It follows that \mathcal{V}_2 provides the desired open and dense set. \square

On compact boundaryless manifolds all KIDs are asymptotically tangential, and Theorem 1.3 is established in this case.

Consider, next, the asymptotically flat case, with an $r^{-\beta}$ weighted topology, $\beta \in (0, n - 2)$. Recall that we want to prove density of metrics without KIDs. For such β the result can be established as follows: consider the set of solutions of the constraint equations on $\mathbb{R}^3 \setminus B(0, R)$, which approach (g, K) at $S(0, R)$ exponentially fast as in [11, Theorem 6.6], and which are $r^{-\beta}$ -asymptotically flat. A straightforward generalization of [11, Corollary 6.3] applies to this space of initial data and shows that this collection forms a manifold. It follows that each linearized solution of the constraint equations constructed as at the beginning of the proof of Theorem 8.7 is tangent to a curve of solutions, which coincide with (g, K) away from the asymptotic region $\mathbb{R}^3 \setminus B(0, R)$. This establishes point 1 of Theorem 1.3. We note that the condition $\text{tr}_g K = 0$ is not necessarily preserved by the perturbation just constructed. However, it follows from the implicit function theorem, or from the results of Bartnik [3], that the deformed initial data set on $\mathbb{R}^3 \setminus B(0, R)$ can be deformed in the associated space-time to obtain a data set with vanishing mean extrinsic curvature, proving point 2. Point 3 is established as above using [11] in the $K \equiv 0$ setting.

In the conformally compactifiable case the argument is identical, based on [11, Theorem 6.7].

In the asymptotically flat case with $\beta = n - 2$ some more work is needed. For simplicity we consider only smooth initial data, but the construction works also in the finite differentiability case. The idea is to obtain solutions up to kernel using the techniques of [10, 14], and to show that one can correct for the kernel by changing the metric in the asymptotic region, the argument proceeds as follows. Let $\Gamma(R, 2R)$ be a coordinate annulus, with inner radius R and outer radius $2R$, contained in the asymptotically flat region, let $x = (K, g)$. Let $\delta x = (\delta K, \delta g)$ be a solution of the linearized constraint equations supported in $\Gamma(5R/4, 7R/4)$, constructed as at the beginning of the proof of Theorem 8.7, so that $x_\epsilon = x + \epsilon \delta x$ has no KIDs on $\Gamma(R, 2R)$ for all positive ϵ small enough. By construction x_ϵ fails to solve the constraint equations by $O(\epsilon^2)$. We use the terminology of [10, Sections 8.1 and 8.2]. Let $Q_0 = (m_0, \vec{p}_0, \vec{c}_0, \vec{J}_0)$ denote the Poincaré charges of $x_0 = x$, and for Q in a neighborhood of Q_0 let $y_Q = (K_Q, g_Q)$ be a reference family of metrics obtained on $\mathbb{R}^n \setminus B(R)$ as follows: by scaling, boosting, and space-translating (K, g) one is led to a family of initial data sets with mass m , ADM-momentum \vec{p} , and centre of mass \vec{c} covering a neighborhood of $(m_0, \vec{p}_0, \vec{c}_0)$. Choosing R large enough, a construction in [22] can be used to deform each of the solutions obtained so far to initial data sets with arbitrary angular momentum in a neighborhood of \vec{J}_0 .⁷ One can now glue x_ϵ with y_Q using the techniques described in detail in [10, 14] obtaining, for $\epsilon + |Q - Q_0|$ small enough, on $\Gamma(R, 2R)$ a “solution up-to-kernel” $z_{\epsilon, Q} = (K_{\epsilon, Q}, g_{\epsilon, Q})$ which smoothly extends across the inner sphere $B(0, R)$ to x , which smoothly extends across the exterior sphere $B(0, 2R)$ to y_Q , and which differs from x_ϵ by terms which are quadratic in ϵ and in $Q - Q_0$. Making ϵ and $|Q - Q_0|$ smaller if necessary, the arguments presented in Sections 8.1 and 8.2 of [10] show that one can find $Q(\epsilon)$ so that $z_{\epsilon, Q(\epsilon)}$ solves the constraints, providing the desired solution without global KIDs. \square

A Topologies

In this paper we prove both density and openness results, and there does not seem to be a topology which captures both features in an optimal way. The aim of this appendix is to discuss those issues in some detail.

As already pointed out in the introduction, a possible topology for which our results hold is the following: one chooses some smooth complete Riemannian metric h on M , which is then used to calculate norms of tensors and their h -covariant derivatives; we shall denote this topology by $\mathcal{T}^k(h)$. If M is compact, the resulting topology is h -independent, and all our results in the compact case hold with such topologies, for appropriate k 's. However, when M is not compact, there exist choices of h which will lead to different topologies; nevertheless, for each such

⁷The point of the current construction is to obtain a “reference family”, as defined in [10], near the initial data we started with. An alternative way is to first deform the initial data to data which are exactly Kerr outside of a compact set with large radius, and use the Kerr family as the reference family.

choice Theorems 1.2 holds. Further, all the results, *except the perturbations that remove global KIDs in an asymptotically flat or asymptotically hyperbolic region*, remain true if, *e.g.*, weighted $C_{\phi,\varphi}^{k,\alpha}$ topologies defined with respect to h are used, as defined in [10], with norm

$$\|u\|_{C_{\phi,\varphi}^{k,\alpha}(h)} = \sup_{x \in M} \sum_{i=0}^k \left(\|\varphi \phi^i \nabla^{(i)} u(x)\|_h + \sup_{0 \neq d_h(x,y) \leq \phi(x)/2} \varphi(x) \phi^{i+\alpha}(x) \frac{\|\nabla^{(i)} u(x) - \nabla^{(i)} u(y)\|_h}{d_h^\alpha(x,y)} \right) ,$$

with *any* weight functions ϕ and φ ; we shall denote such topologies by $\mathcal{T}_{\phi,\varphi}^{k,\alpha}(h)$. Finally, all openness and density results established in this paper, including statements involving the field equations, will hold with any choice of h and weight functions except for the following restriction: if (M, g, K) contains an asymptotically flat region, and one wishes to construct a perturbation that gets rid of a globally defined KID while preserving the field equations, then h should be chosen to be, *e.g.*, the Euclidean metric in the asymptotically flat region, with the weights $\phi = r$, $\varphi = r^{-\beta}$, for some $\beta \in (0, n - 2]$. Similarly, in the context of Corollary 9.4 and of point 4 of Theorem 1.3, the weights in the asymptotically hyperbolic region should be chosen in a way compatible with the asymptotic conditions in the conformally compactifiable region as in [2].

While the above topologies seem satisfactory for most purposes, the optimal topology for perturbations that get rid, *e.g.*, of Killing vectors, at a given point p , is that of convergence in the space of k th jets of the metric at p , with $k \geq k_0(n)$, for some $k_0(n)$ as described above, on the space of metrics which coincide with the starting metric g away from a compact neighborhood of p . However, this space is unnecessarily small for our openness results, which do not hold in such a weak topology in any case; see also Remark 4.3.

B “Local extends to global” in the simply connected analytic setting

In this appendix we wish to generalise Nomizu’s theorem [20] concerning Killing vectors to conformal Killing vectors and to KIDs. It should be clear that our argument applies to a large class of similar overdetermined systems with analytic coefficients, such as, *e.g.*, those considered in [7]. In particular the proof given here applies to Killing vector fields in arbitrary signature, and seems to be somewhat simpler than the original one.

Theorem B.1 *Let (M, g) be a simply connected analytic pseudo-Riemannian manifold.*

1. *Every locally defined conformal Killing vector extends to a globally defined one.*
2. *If, moreover, K is also analytic then every locally defined KID extends to a globally defined one.*

Proof. We give the proof for KIDs, the argument for conformal Killing vector fields is identical. Let r , P_α and L_k be as in Section 7.

We note the following:

Lemma B.2 *Consider a KID x defined on an open set Ω , let $\gamma : [0, 1] \rightarrow M$ be a differentiable path such that $\gamma : [0, 1) \rightarrow \Omega$, with $\gamma(1) \notin \gamma([0, 1))$. Then there exists a neighborhood \mathcal{U} of $\gamma([0, 1))$ and a KID \hat{x} defined on \mathcal{U} such that $x = \hat{x}$ on $\gamma([0, 1))$.*

Proof. Equation (7.4) shows that each covariant derivative $D_\alpha r$ of r satisfies along γ the linear equation

$$\frac{D(D_\alpha r \circ \gamma)}{ds} = \dot{\gamma}^\mu ((D_\mu D_\alpha r) \circ \gamma) = (P_{\mu\alpha} r) \circ \gamma,$$

with the multi-index $\mu\alpha$ in $P_{\mu\alpha}$ defined in the obvious way. It follows that each

$$F_\alpha(s) := (D_\alpha r \circ \gamma)(s)$$

extends by continuity to some values, denoted by $F_\alpha(1)$, such that

$$F_\alpha(1) = P_\alpha(\gamma(1))F(1),$$

where $F(1) = \lim_{s \rightarrow 1} r(\gamma(s))$. By continuity the integrability conditions $L_k = 0$ are satisfied by $F(1)$, and therefore, by the argument given after (7.4), there exists $\epsilon > 0$ and a solution of the KID equations defined on $B(\gamma(1), \epsilon)$ for some $\epsilon > 0$. We can cover $\gamma([0, 1))$ by a finite number of open balls $B_i := B(\gamma(s_i), r_i)$, $i = 1, \dots, N$, such that $s_1 = 0$, $s_N = 1$, $r_N \leq \epsilon$, with the balls pairwise disjoint except for the neighboring ones: $B_i \cap B_j = \emptyset$ if $|i - j| > 1$. It should be clear that the solution just constructed on $B(\gamma(1), \epsilon)$ coincides with that which exists already on the overlap with $B(\gamma(s_{N-1}), r_{N-1})$. The desired neighborhood is obtained by setting $\mathcal{U} = \cup_i B(\gamma(s_i), r_i)$. \square

Returning to the proof of Theorem B.1, let q be any point in Ω , let $p \in M$, and let $\gamma : [0, 1] \rightarrow M$ be any piecewise differentiable path without self-intersections with $\gamma(0) = q$, $\gamma(1) = p$. Let $I \subset [0, 1]$ be the set of numbers s such that there exists a neighborhood \mathcal{U}_s of $\gamma|_{[0, s]}$ and a KID x_s defined on \mathcal{U}_s such that $x_s = x$ near p . Then I is open by definition, it is closed by Lemma B.2, therefore $I = [0, 1]$.

We have thus shown:

Lemma B.3 *For any piecewise differentiable path $\gamma : [0, 1] \rightarrow M$ without self-intersections, with $\gamma(0) \in \Omega$, there exist a neighborhood \mathcal{U} of γ and a KID x_γ defined on \mathcal{U} , coinciding with x on $\mathcal{U} \cap \Omega$.* \square

Any γ as in Lemma B.3 allows us therefore to extend x to a neighborhood of p . It remains to show that this extension is γ -independent. Let thus γ and

$\hat{\gamma}$ be two differentiable paths from q to p without self-intersections, since M is simply connected there exist a homotopy of differentiable paths $\gamma_t : [0, 1] \rightarrow M$, $t \in [0, 1]$, with $\gamma_t(1) = p$, $\gamma_t(0) = q$, $\gamma_0 = \gamma$ and $\gamma_1 = \hat{\gamma}$. If any γ_t self-intersects at s_1 and s_2 , with $s_1 < s_2$, we replace it by a new path, still denoted by γ_t , obtained by staying at $\gamma_t(s_1)$ for $s \in [s_1, s_2]$; this procedure is repeated until all self-intersections of γ_t have been eliminated. Let $r(t)$ denote the value of r at p obtained from Lemma B.3 by following γ_t , then r is a continuous function of t . The set of t 's for which $r(t) = r(0)$ is closed by continuity of r , it is open by Lemma B.3, hence $r(0) = r(1)$, which establishes Theorem B.1. \square

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References

- [1] S. Alinhac and G. Métivier, Propagation de l'analyticité des solutions de systèmes hyperboliques non-linéaires, *Invent. Math.* **75**, 189–204 (1984).
- [2] L. Andersson and P.T. Chruściel, On asymptotic behavior of solutions of the constraint equations in general relativity with “hyperboloidal boundary conditions”, *Dissert. Math.* **355**, 1–100 (1996).
- [3] R. Bartnik, The existence of maximal hypersurfaces in asymptotically flat space-times, *Comm. Math. Phys.* **94**, 155–175 (1984).
- [4] ———, Regularity of variational maximal surfaces, *Acta Math.* **161**, 145–181 (1988).
- [5] J. Bičák and J. Podolský, The global structure of Robinson-Trautman radiative space-times with cosmological constant, *Phys. Rev. D* **55**, 1985–1993 (1996), gr-qc/9901018.
- [6] J.-P. Bourguignon, D.G. Ebin, and J.E. Marsden, Sur le noyau des opérateurs pseudo-différentiels à symbole surjectif et non injectif, *C. R. Acad. Sci. Paris Sér. A-B* **282**, Aii, A867–A870 (1976).
- [7] T. Branson, A. Čap, M. Eastwood, and R. Gover, *Prolongations of geometric overdetermined systems*, (2004), math.DG/0402100v2.
- [8] D. Christodoulou and N. Ó Murchadha, The boost problem in general relativity, *Comm. Math. Phys.* **80**, 271–300 (1980).
- [9] P.T. Chruściel, *On uniqueness in the large of solutions of Einstein equations (“Strong Cosmic Censorship”)*, Australian National University Press, Canberra, 1991.

- [10] P.T. Chruściel and E. Delay, On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications, *Mém. Soc. Math. de France* **94**, 1–103 (2003), gr-qc/0301073v2.
- [11] ———, Manifold structures for sets of solutions of the general relativistic constraint equations, *Jour. Geom Phys.* (2004), in press, gr-qc/0309001v2.
- [12] P.T. Chruściel, J. Isenberg, and D. Pollack, *Initial data engineering*, (2004), gr-qc/0403066.
- [13] B. Coll, On the evolution equations for Killing fields, *Jour. Math. Phys.* **18**, 1918–1922 (1977).
- [14] J. Corvino and R. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, gr-qc/0301071, 2003.
- [15] D.G. Ebin, *The manifold of Riemannian metrics*, Global Analysis, Proc. Sympos. Pure Math., vol. 15, 1970, pp. 11–40.
- [16] L. Hörmander, The boundary problems of physical geodesy, *Arch. Rat. Mech. Analysis* **62**, 1–52 (1976).
- [17] J. Isenberg, Constant mean curvature solutions of the Einstein constraint equations on closed manifolds, *Class. Quantum Grav.* **12**, 2249–2274 (1995).
- [18] J. Lohkamp, Metrics of negative Ricci curvature, *Ann. of Math. (2)* **140**, 655–683 (1994).
- [19] V. Moncrief, Space-time symmetries and linearization stability of the Einstein equations. II, *Jour. Math. Phys.* **17**, 1893–1902 (1976).
- [20] K. Nomizu, On local and global existence of Killing vector fields, *Ann. Math.* **72**, 105–120 (1960).
- [21] F.A.E. Pirani, *Introduction to gravitational radiation theory*, Lectures on general relativity, Brandeis, vol. 1, Prentice Hall, Englewood Cliffs, New Jersey, 1965.
- [22] R. Schoen, *in preparation*, (2003).
- [23] T.Y. Thomas, *The differential invariants of generalized spaces*, Cambridge University Press, 1934.

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