Ann. Henri Poincaré 5 (2004) 979 – 1012

1424-0637/04/050979-34
DOI 10.1007/s00023-004-0187-3

Annales Henri Poincaré

Negative Discrete Spectrum of Perturbed Multivortex Aharonov-Bohm Hamiltonians*

M. Melgaard, E.-M. Ouhabaz and G. Rozenblum

Abstract. The diamagnetic inequality is established for the Schrödinger operator $H_0^{(d)}$ in $L^2(\mathbb{R}^d)$, $d = 2, 3$, describing a particle moving in a magnetic field generated by finitely or infinitely many Aharonov-Bohm solenoids located at the points of a discrete set in $\mathbb{R}^2$, e.g., a lattice. This fact is used to prove the Lieb-Thirring inequality as well as CLR-type eigenvalue estimates for the perturbed Schrödinger operator $H_0^{(d)} - V$, using new Hardy type inequalities. Large coupling constant eigenvalue asymptotic formulas for the perturbed operators are also proved.

1 Introduction and main results

Consider a non-relativistic, spinless quantum particle in $\mathbb{R}^d$, $d = 2, 3$, interacting with a magnetic field B associated with finitely or infinitely many thin solenoids aligned along the x_3-axis which pass through the points λ of some discrete subset Λ of the x_1x_2 plane. The magnetic flux through each solenoid is a noninteger α_λ. If, moreover, the radii of the solenoids tend to zero, whilst the flux α_λ through each solenoid remains constant then one obtains a particle moving in $\mathbb{R}^d$ subject to a finite or an infinite sum of δ-type magnetic fields, the so-called Aharonov-Bohm fields or magnetic vortices, located at the points of Λ which may be interpreted as infinitely thin impurities within a superconductor. Setting $\Lambda^d = \Lambda \times \mathbb{R}^{d-2}$, the multiply-connected region $\mathbb{R}^d \setminus \Lambda^d$, in which the field B equals zero, represents the configuration space. In the case of a lattice (defined by $\lambda_{kl} = k\boldsymbol{\omega}_1 + l\boldsymbol{\omega}_2$, where $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ are vectors in $\mathbb{R}^2$ and (k, l) runs over the whole of $\mathbb{Z}^2$ or a subset of $\mathbb{Z}^2$) such a situation occurs experimentally in GaAs/AlGaAs heterostructures coated with a film of type-II superconductors [5, 11].

The vector potential $\boldsymbol{A}(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$ associated with B is chosen such that

$$A_1(x_1, x_2) = \operatorname{Im} \mathcal{A}(x_1, x_2), \quad \text{and} \quad A_2(x_1, x_2) = \operatorname{Re} \mathcal{A}(x_1, x_2), \tag{1.1}$$

where $\mathcal{A}(z) = \mathcal{A}(x_1, x_2)$, $z = x_1 + ix_2$, is a meromorphic function having simple poles at $\lambda \in \Lambda$ with residues α_λ; existence (and examples) of such functions $\mathcal{A}(z)$

*The main part of this work was carried out while the first author was an individual Marie Curie Post-Doc Fellow, supported by the European Union under grant no. HPMF-CT-2000-00973.

are discussed in Section 2. One easily verifies that

$$\partial_{x_1} A_2 - \partial_{x_2} A_1 = \sum_{\lambda \in \Lambda} \alpha_\lambda \delta(z - \lambda) = B$$

in the sense of distributions; as usual, it suffices to consider $\alpha_\lambda \in (0,1)$ due to gauge invariance.

The dynamics of a spinless particle moving in any of the above-mentioned configurations of Aharonov-Bohm (abbrev. A-B) solenoids in $\mathbb{R}^d$ is described by the Schrödinger operator

$$H_0^{(d)} = -(\nabla + i\boldsymbol{A})^2 \tag{1.2}$$

acting in $L^2(\mathbb{R}^d)$, where ∇ is the gradient on $\mathbb{R}^d$. Since the singularities of the A-B magnetic potential are very strong, the operator defined initially on functions with support away from the singularities is not essentially self-adjoint. In Section 2 we define the Friedrichs extension of $H_0^{(d)}$ by means of quadratic forms. In the case of a single A-B solenoid the corresponding standard A-B Schrödinger operator has been studied intensively in two dimensions and there is an ongoing discussion on the mathematical and physical reasonability of different self-adjoint extensions [31, 1, 10, 15, 35]. The Friedrichs extension considered herein corresponds to the model of solenoids being non-penetrable for electrons, and, moreover, with interaction preserving circular symmetry [1].

Within the theory of Schrödinger operators with magnetic fields $L(\boldsymbol{A}) = -(\nabla + i\boldsymbol{A})^2$ associated with a vector potential $\boldsymbol{A} = (A_1, \ldots, A_d)$ satisfying $A_j \in L^2_{loc}(\mathbb{R}^d)$, one of the fundamental facts is the diamagnetic inequality [2], viz., $|e^{-tL(\boldsymbol{A})}u| \le e^{-tL_0}|u|$ for all $t \ge 0$ and all $u \in L^2(\mathbb{R}^d)$; here L_0 denotes the negative Laplacian in $L^2(\mathbb{R}^d)$.

In Section 4 we show that this inequality is valid also for the Schrödinger operator $H_0^{(d)}$ in $L^2(\mathbb{R}^d)$ for *any* of the afore-mentioned A-B configurations.

Theorem 1.1 *The inequality*

$$|e^{-tH_0^{(d)}}u| \le e^{-tL_0}|u|$$

holds for all $t \ge 0$ and all $u \in L^2(\mathbb{R}^d)$

This result does not follow directly from the known diamagnetic inequality since the components (1.1) of the vector potential do not belong to $L^2_{loc}(\mathbb{R}^d)$; this latter condition is crucial in all existing proofs of the diamagnetic inequality for Schrödinger operators with magnetic fields.

Our proof of Theorem 1.1 uses a recent criterion (see Section 3) for the domination of semigroups due to Ouhabaz [25][1]. This criterion is a generalization (from operators to forms) of the Simon-Hess-Schrader-Uhlenbrock test for domination of semigroups [13].

[1]It might be possible to prove Theorem 1.1 from general results in [22] which, in their turn, are based on [25], but we prefer to give a direct proof.

As the first application of the diamagnetic inequality we establish the Lieb-Thirring inequality for the perturbed Schrödinger operator $H_0^{(d)} - V$ in Section 6. Here the electrostatic potential V is a nonnegative, measurable function on $\mathbb{R}^d$ belonging to an appropriate class, which guarantees that the form sum $H_0^{(d)} - V$ generates a semi-bounded, self-adjoint operator in $L^2(\mathbb{R}^d)$ with discrete spectrum below zero.

The classic Lieb-Thirring inequality [20] for a d-dimensional Schrödinger operator $L_0 - V$ in $L^2(\mathbb{R}^d)$, with $L_0 = -\Delta$ as above and $d \geq 1$, says that

$$\sum_j |\nu_j(L_0 - V)|^\gamma \leq b_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d}{2}}\, dx, \tag{1.3}$$

where $\nu_j(L_0 - V)$ denote the negative eigenvalues of $L_0 - V$, $\gamma > 0$ ($\gamma \geq 1/2$ for $d = 1$) and $V \in L^{\gamma+\frac{d}{2}}$. The constant $b_{\gamma,d}$ is expressible in terms of Γ-functions. The Lieb-Thirring inequality plays a crucial role in the problem of stability of matter (see, e.g., [21]), where the exact value of the constant is important (see [18], [14] for recent developments in obtaining sharp constants). One way of establishing (1.3) is to use the Cwikel-Lieb-Rozenblum (abbrev. CLR) estimate (see, e.g., [27]) which, in its original form, reads

$$N_-(L_0 - V) \leq C_d \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}}\, dx, \quad d \geq 3. \tag{1.4}$$

Here N_- denotes the number of negative eigenvalues of a self-adjoint operator, provided its negative spectrum is discrete. The single assumption, under which (1.4) is valid, is the finiteness of the integral on its right-hand side. In [33, p. 99-100] it is shown how one can obtain (1.3) provided (1.4) holds. This, however, does not produce the optimal constant in the Lieb-Thirring inequality.

The Lieb-Thirring inequality for d-dimensional Schrödinger operators with magnetic fields $L(\boldsymbol{A}) - V$, with $d \geq 3$ and $A_j \in L^2_{loc}(\mathbb{R}^d)$, takes the same form and can be obtained from the CLR-estimate for $L(\boldsymbol{A}) - V$ which is shown by means of the diamagnetic inequality (see, e.g., [33, p 168]).

In two dimensions there exist certain CLR-type estimates both for $L_0 - V$ [34, 6] and $L(\boldsymbol{A}) - V$ [29], provided $A_j \in L^2_{loc}(\mathbb{R}^2)$ for the latter operator. However, unlike in higher dimensions, these estimates, having a different form, do not produce Lieb-Thirring inequalities. Moreover, in our case, the components (1.1) of the magnetic potential do not belong to $L^2_{loc}(\mathbb{R}^2)$. Therefore the question on Lieb-Thirring inequalities for the perturbed Schrödinger operator $H_0^{(d)} - V$ was up to now open.

In the present paper we establish the following Lieb-Thirring (abbrev. LT) inequality for the perturbed Schrödinger operator $H_0^{(d)} - V$ in $L^2(\mathbb{R}^d)$, $d = 2, 3$, for any of the afore-mentioned configurations of A-B solenoids, with constants not depending on the configuration or the strength of magnetic fields.

Theorem 1.2 *Let ν_j denote the negative eigenvalues of $H_0^{(d)} - V$, $d = 2, 3$. If, moreover, $\gamma > 0$ and $V \in L^{\gamma+(d/2)}(\mathbb{R}^d)$ then*

$$LT_{\gamma,d} := \sum_j |\nu_j|^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d}{2}}\, dx,$$

where the constant $C_{\gamma,d}$ fulfills the following upper bounds for the most interesting values of γ:

$$C_{\gamma,2} \leq \begin{cases} 0.5300 & \text{for } \gamma = 1/2, \\ 0.3088 & \text{for } \gamma = 1, \\ 0.2275 & \text{for } \gamma = 3/2\ ; \end{cases}$$

and

$$C_{\gamma,3} \leq \begin{cases} 0.1542 & \text{for } \gamma = 1/2, \\ 0.0483 & \text{for } \gamma = 1, \\ 0.0270 & \text{for } \gamma = 3/2. \end{cases}$$

We note that the expression we obtain for the best constant in Theorem 1.2 is implicit; see (6.3).

Using Hardy-type inequalities enables one to further improve the LT estimates; see Section 7.

The diamagnetic inequality is one out of the two crucial ingredients in the proof of Theorem 1.2. The other is an abstract CLR estimate for generators of semigroups dominated by positive semigroups. To make the paper self-explanatory we formulate this rather recent result, obtained by Rozenblum and Solomyak, in Section 5.

An important application of eigenvalue estimates for Schrödinger operators is to deduce asymptotic formulas for the eigenvalues when the coupling constant q is present and it tends to infinity. The technology of getting the asymptotic formulas from the estimates is well-established nowadays (see, e.g., [27] and [7, 8]), and what is required from the estimates is that they have correct order in the coupling constant. For weakly singular magnetic fields such estimates were obtained by Lieb (see [33]) and Melgaard-Rozenblum [23] in dimensions $d \geq 3$, and by Rozenblum-Solomyak [29] in dimension $d = 2$ (see also [30]). In the case of a single A-B solenoid, the only existing estimate for the corresponding A-B Schrödinger operator $H_{AB}^{(2)} - qV$, by Balinsky, Evans and Lewis [4], deals with a rather special case of a radially symmetric potential (or with one majorized by a radially symmetric potential). There were no preceding results concerning eigenvalue estimates for many solenoids.

Based upon the diamagnetic inequality we establish CLR-type estimates (i.e., estimates having correct order in coupling constant q) for $H_0^{(d)} - qV$ for any of the A-B configurations mentioned above. To achieve this, we derive Hardy-type inequalities for each configuration, which allows us to carry over recent CLR-type estimates for the negative eigenvalues of two-dimensional Schrödinger operators, with a regularizing positive (Hardy) term added, to the operators $H_0^{(d)} - qV$. The Hardy-type inequalities are of interest by themselves and complement the recent result by Balinsky [3]. For finitely many A-B solenoids we prove the Hardy-type inequality by using a conformal mapping. This idea belongs to Balinsky but we use another, more explicit realization, which gives a better control over the weight function in the Hardy-type inequality. In the field of CLR-type estimates, sharp constants are unknown, and at present the known values of constants lie far above the expected ones. In applications to finding eigenvalue asymptotics, the values of these constants are of no importance, and thus we do not try to obtain the best values for our case either. Instead we demonstrate that the presence of the magnetic field and its particular configuration may improve, via Hardy inequalities, eigenvalue estimates, by compensating possible strong singularities or insufficient decay of the electric potential. Similar effect takes place for LT estimates as well.

After obtaining CLR-type estimates, to deduce the large coupling constant asymptotics for the eigenvalues of $H_0^{(d)} - qV$ is a standard job. Only a few remarks are needed.

The magnetic flux parameters α_λ are nonintegers throughout the paper. If α_λ are integers, the resulting operator is gauge equivalent to the negative Laplacian in $L^2(\mathbb{R}^d)$. This, however, does not reflect itself in the LT inequality but the eigenvalue estimates in Section 8 are no longer valid, as one can see, e.g., from the factor β^{-2} in formula (8.7).

2 The unperturbed Hamiltonian $H_0^{(d)}$

Choice of vector potential

As mentioned in the introduction, the vector potential $\boldsymbol{A}(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2), 0)$ associated with B is chosen such that

$$A_1(x_1, x_2) = \operatorname{Im} \mathcal{A}(x_1, x_2) \text{ and } A_2(x_1, x_2) = \operatorname{Re} \mathcal{A}(x_1, x_2), \tag{2.1}$$

where $\mathcal{A}(z) = \mathcal{A}(x_1, x_2)$, $z = x_1 + ix_2$, is a meromorphic function having (only) simple poles at $\lambda \in \Lambda$ with residues α_λ.

In the case where Λ is a finite set, say, $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, the function

$$\mathcal{A}(z) = \sum_{j=1}^{N} \frac{\alpha_{\lambda_j}}{z - \lambda_j}$$

has the desired properties. In the general case, where Λ is a discrete set with infinitely many points, without finite limit points, Mittag-Leffler's theorem guarantees the existence of a meromorphic function with the afore-mentioned properties, unique, up to an entire summand.

For an infinite regular lattice where all fluxes are equal to a noninteger α, we can construct such a function $\mathcal{A}(z)$ explicitly. Indeed let $\Phi(z)$ be an entire function such that its set of (only simple) zeros coincide with Λ. Then one can take $\mathcal{A}(z) = \alpha\Phi'(z)\Phi(z)^{-1}$. In particular, the Weierstrass function $\sigma(z)$ corresponding to the lattice can serve as $\Phi(z)$, and then $\Phi'(z)\Phi(z)^{-1}$ is the Weierstrass function $\zeta(z)$.

Magnetic quadratic forms

For $\boldsymbol{A} = (A_1, A_2)$ in (1.1) we observe that

$$A_1, A_2 \in L^\infty_{loc}(\mathbb{R}^d \setminus \Lambda^d).$$

Let

$$\Omega_n = \left(B(0,n) \times (-n,n)^{d-2}\right) \setminus \left(\cup_{\lambda\in\Lambda} B(\lambda, 1/n) \times \mathbb{R}^{d-2}\right), \ \ n \geq 2,$$

where $B(\lambda, r)$ denotes the disk with center λ and radius r. We define on $L^2(\Omega_n)$ (for each $n \geq 2$) the form

$$\mathfrak{h}_n^{(d)}[u,v] = \sum_{j=1}^{d} \int_{\Omega_n} \left(\frac{\partial u}{\partial x_j} + iA_j u\right) \overline{\left(\frac{\partial v}{\partial x_j} + iA_j v\right)} dx \tag{2.2}$$

on the domain $\mathcal{D}(\mathfrak{h}_n^{(d)}) = H_0^1(\Omega_n)$. The form is closed since $A_1, A_2 \in L^\infty(\Omega_n)$. The associated self-adjoint, nonnegative operators are denoted by $H_n^{(d)}$. These are operators in $L^2(\Omega_n)$. It is convenient to extend them to zero in $L^2(\mathbb{R}^d \setminus \Omega_n)$, thus getting operators in $L^2(\mathbb{R}^d)$; keeping the same notation for extended operators does not create misunderstanding.

Define, in addition, the (closed) form $\mathfrak{l}_n^{(d)}$ with the same form expression and domain as $\mathfrak{h}_n^{(d)}$ but with $A_1 = A_2 = 0$. The associated self-adjoint, nonnegative operators are denoted by $L_n^{(d)}$.

Define now the form $\mathfrak{h}^{(d)}$ by

$$\begin{aligned} \mathfrak{h}^{(d)}[u,v] &= \mathfrak{h}_n^{(d)}[u,v] \text{ if } u,v \in \mathcal{D}(\mathfrak{h}_n^{(d)}), \\ \mathcal{D}(\mathfrak{h}^{(d)}) &= \cup_n \mathcal{D}(\mathfrak{h}_n^{(d)}) = \cup_n H_0^1(\Omega_n). \end{aligned}$$

Lemma 2.1 *The form $\mathfrak{h}^{(d)}$ is closable.*

Proof. According to the definition, the form $\mathfrak{h}^{(d)}$ is closable if and only if any sequence $\{u_n\}$, $u_n \in \mathcal{D}(\mathfrak{h}^{(d)})$, for which

$$\lim_{n\to\infty} \|u_n\|_{L^2} = 0 \text{ and } \lim_{n,m\to\infty} \mathfrak{h}^{(d)}[u_n - u_m] = 0, \tag{2.3}$$

satisfies $\lim_{n\to\infty} \mathfrak{h}^{(d)}[u_n] = 0$. First observe that (2.3) implies

$$C := \sup_n \mathfrak{h}^{(d)}[u_n]^{1/2} < \infty. \tag{2.4}$$

Take $\epsilon > 0$ and choose n_0 such that

$$\mathfrak{h}^{(d)}[u_n - u_m] \leq \epsilon \text{ when } n, m \geq n_0. \tag{2.5}$$

Set, moreover, $K = \Omega_{n_0} \subset \mathbb{R}^d \setminus \Lambda^d$ such that $\operatorname{supp} u_{n_0} \subset K$. In view of (2.3),

$$\int_K |(\nabla + i\boldsymbol{A})(u_n - u_m)|^2 \, dx \leq \mathfrak{h}^{(d)}[u_n - u_m] \longrightarrow 0 \text{ as } n, m \to \infty, \tag{2.6}$$

$$\int_K |u_n|^2 \, dx \longrightarrow 0 \text{ as } n \to \infty, \tag{2.7}$$

and, since $\boldsymbol{A}$ is bounded on K,

$$\int_K |\boldsymbol{A} u_n|^2 \, dx \longrightarrow 0 \text{ as } n \to \infty. \tag{2.8}$$

Now,

$$\begin{aligned} &\left| \left(\int_K |\boldsymbol{A}(u_n - u_m)|^2 \, dx \right)^{1/2} - \left(\int_K |\nabla(u_n - u_m|^2 \, dx \right)^{1/2} \right| \\ &\qquad \leq \left(\int_K |(\nabla + i\boldsymbol{A})(u_n - u_m)|^2 \, dx \right)^{1/2}. \end{aligned} \tag{2.9}$$

According to (2.8), the first term on the left-hand side of the latter inequality tends to zero as $n, m \to \infty$ and, due to (2.6), the same holds for the right-hand side. Thus,

$$\int_K |u_n - u_m|^2 + |\nabla(u_n - u_m)|^2 \, dx \longrightarrow 0 \text{ as } n, m \to \infty.$$

Since the form of the classical Dirichlet Laplacian is closable it follows from the latter relation, in conjunction with (2.7) that

$$\int_K |\nabla u_n|^2 \, dx \to 0, \quad \int_K |u_n|^2 \, dx \to 0, \text{ as } n \to \infty. \tag{2.10}$$

Now,

$$\begin{aligned} \mathfrak{h}^{(d)}[u_n] &= \mathfrak{h}^{(d)}[u_n, u_n - u_{n_0}] + \mathfrak{h}^{(d)}[u_n, u_{n_0}] \\ &\leq \mathfrak{h}^{(d)}[u_n]^{1/2} \mathfrak{h}^{(d)}[u_n - u_{n_0}]^{1/2} + |\mathfrak{h}^{(d)}[u_n, u_{n_0}]|. \end{aligned} \tag{2.11}$$

It follows from (2.4) and (2.5) that

$$\mathfrak{h}^{(d)}[u_n]^{1/2} \mathfrak{h}^{(d)}[u_n - u_{n_0}]^{1/2} \leq C\epsilon^{1/2} \text{ when } n \geq n_0. \tag{2.12}$$

Since $\boldsymbol{A}$ is bounded on K we infer from (2.10) and (2.8) that

$$\mathfrak{h}^{(d)}[u_n, u_{n_0}] = \int_K (\nabla + i\boldsymbol{A}) u_n \overline{(\nabla + i\boldsymbol{A}) u_{n_0}} \, dx \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Substitution of (2.12)-(2.13) into (2.11) shows that $\lim_{n\to\infty} \mathfrak{h}^{(d)}[u_n] = 0$ as desired. $\square$

We denote the closure of $\mathfrak{h}^{(d)}$ by $\overline{\mathfrak{h}}^{(d)}$ and the associated semi-bounded (from below), self-adjoint operator by $H_0^{(d)}$. This is, in fact, the Friedrichs extension of the symmetric operator (1.2) defined initially on $C_0^\infty(\mathbb{R}^d \setminus \Lambda^d)$. The introduction of 'approximating' forms, however, is required for proving the diamagnetic inequality later. We define $\mathfrak{l}^{(d)}$ in a similar way, viz.

$$\begin{aligned} \mathfrak{l}^{(d)}[u,v] &= \mathfrak{l}_n^{(d)}[u,v] \text{ if } u,v \in \mathcal{D}(\mathfrak{l}_n^{(d)}), \\ \mathcal{D}(\mathfrak{l}^{(d)}) &= \cup_n \mathcal{D}(\mathfrak{l}_n^{(d)}) = \cup_{n\geq 2} H_0^1(\Omega_n). \end{aligned}$$

Then $\mathfrak{l}^{(d)}$ is closable. The closure $\overline{\mathfrak{l}}^{(d)}$ has domain $\mathcal{D}(\overline{\mathfrak{l}}^{(d)}) = H^1(\mathbb{R}^d)$. The associated nonnegative, self-adjoint operator is just the negative Laplacian in $L^2(\mathbb{R}^d)$; we suppress d and denote it by L_0.

3 Semigroup criterion

Throughout this section $\mathcal{H}$ denotes our Hilbert space $L^2(\mathbb{R}^d)$. For a given $u \in \mathcal{H}$ we denote by $\overline{u} := \mathrm{Re}\, u - i\mathrm{Im}\, u$ the conjugate function of u. By $|u|$ we denote the absolute value of u (i.e., the function $x \mapsto |u(x)| := \sqrt{u(x)\cdot\overline{u(x)}})$ and by $\operatorname{sign} u$ the function defined by

$$\operatorname{sign} u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Let $\mathfrak{s}$ be a sesquilinear form which satisfies

$$\mathcal{D}(\mathfrak{s}) \text{ is dense in } \mathcal{H}, \tag{3.1}$$

$$\mathrm{Re}\,\mathfrak{s}[u,u] \geq 0, \quad \forall u \in \mathcal{D}(\mathfrak{s}), \tag{3.2}$$

$$|\mathfrak{s}[u,v]| \leq C\|u\|_{\mathfrak{s}}\|v\|_{\mathfrak{s}}, \quad \forall u,v \in \mathcal{D}(\mathfrak{s}), \tag{3.3}$$

where C is a constant and $\|u\|_{\mathfrak{s}} = \sqrt{\mathrm{Re}\,\mathfrak{s}[u,u] + \|u\|^2}$, and, moreover,

$$\langle \mathcal{D}(\mathfrak{s}), \|\cdot\|_{\mathfrak{s}} \rangle \text{ is a complete space.} \tag{3.4}$$

Definition 3.1 Let $\mathcal{K}$ and $\mathcal{L}$ be two subspaces of $\mathcal{H}$. We shall say that $\mathcal{K}$ is an ideal of $\mathcal{L}$ if the following two assertions are fulfilled:
1) $u \in \mathcal{K}$ implies $|u| \in \mathcal{L}$.
2) If $u \in \mathcal{K}$ and $v \in \mathcal{L}$ such that $|v| \leq |u|$ then $v \cdot \operatorname{sign} u \in \mathcal{K}$.

Let $\mathfrak{s}$ and $\mathfrak{t}$ be two sesquilinear forms both of which satisfy (3.1)-(3.4). The semigroups associated to corresponding self-adjoint operators S, T will be denoted by e^{-tS} and e^{-tT}, respectively.

The following result was established by Ouhabaz [25, Theorem 3.3 and its Corollary].

Theorem 3.2 (Ouhabaz'96) *Assume that the semigroup e^{-tT} is positive. The following assertions are equivalent:*

1) $|e^{-tS}f| \le e^{-tT}|f|$ *for all* $t \ge 0$ *and all* $f \in \mathcal{H}$.
2) $\mathcal{D}(\mathfrak{s})$ *is an ideal of* $\mathcal{D}(\mathfrak{t})$ *and*

$$\operatorname{Re}\mathfrak{s}[u, |v|\operatorname{sign} u] \ge \mathfrak{t}[|u|, |v|] \tag{3.5}$$

for all $(u, v) \in \mathcal{D}(\mathfrak{s}) \times \mathcal{D}(\mathfrak{t})$ *such that* $|v| \le |u|$.

3) $\mathcal{D}(\mathfrak{s})$ *is an ideal of* $\mathcal{D}(\mathfrak{t})$ *and*

$$\operatorname{Re}\mathfrak{s}[u, v] \ge \mathfrak{t}[|u|, |v|] \tag{3.6}$$

for all $u, v \in \mathcal{D}(\mathfrak{s})$ *such that* $u \cdot \overline{v} \ge 0$.

The following lemma is useful when one wishes to apply the criteria in Theorem 3.2.

Lemma 3.3 *Let* $\Omega \subset \mathbb{R}^d$ *be an open set and* $u, v \in H^1(\Omega)$ *be functions satisfying* $u(x) \cdot \overline{v(x)} \ge 0$ *for a.e.* x *in* Ω. *Then*

1. $\operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \overline{v}\right) = |v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign}\overline{u}\right)$.
2. $|v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign}\overline{u}\right) = |u| \operatorname{Im}\left(\frac{\partial v}{\partial x_j} \cdot \operatorname{sign}\overline{u}\right)$.

Proof. Let $\chi_{\{u=0\}}$ denote the characteristic function of the set $\{\, x \mid u(x) = 0\}$. Since $(\partial u/\partial x_j) \cdot \chi_{\{u=0\}} = 0$, we have that

$$\frac{\partial u}{\partial x_j} \cdot \overline{v} = \frac{\partial u}{\partial x_j} \cdot \overline{v} \cdot \frac{v \cdot \overline{u}}{|u| \cdot |v|} \chi_{\{u \ne 0\}} \chi_{\{v \ne 0\}} = |v| \cdot \frac{\partial u}{\partial x_j} \cdot \frac{\overline{u}}{|u|} \cdot \chi_{\{u \ne 0\}}.$$

By taking the imaginary part on both sides of the latter equality, we obtain that

$$\operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \overline{v}\right) = |v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign}\overline{u}\right),$$

which verifies the first assertion. To prove the second assertion we start from

$$|v| \cdot u = |v| \cdot u \cdot \frac{v \cdot \overline{u}}{|u| \cdot |v|} \chi_{\{u \ne 0\}} \chi_{\{v \ne 0\}} = |u| \cdot v.$$

Hence,

$$\frac{\partial |v|}{\partial x_j} \cdot u + |v| \cdot \frac{\partial u}{\partial x_j} = \frac{\partial |u|}{\partial x_j} \cdot v + |u| \cdot \frac{\partial v}{\partial x_j}.$$

We multiply both sides by $\operatorname{sign}\overline{u} = (\overline{u}/|u|)\chi_{\{u \ne 0\}}$ and take the imaginary parts on both sides to obtain

$$|v| \operatorname{Im}\left(\frac{\partial u}{\partial x_j} \cdot \operatorname{sign}\overline{u}\right) = \operatorname{Im}\left(\frac{\partial v}{\partial x_j} \cdot \overline{u}\chi_{\{u \ne 0\}}\right) = \operatorname{Im}\left(\frac{\partial v}{\partial x_j} \cdot \overline{u}\right).$$

The latter in combination with the first assertion (with u substituted by v and vice-versa) shows the second assertion. □

4 Diamagnetic inequality for $H_0^{(d)}$

The usual diamagnetic inequality is established for vector potentials which belong to L^2_{loc} (see, e.g., [2]). In this section we establish the diamagnetic inequality for the Schrödinger operator $H_0^{(d)}$, i.e. when $A_j \notin L^2_{loc}$, $j = 1, 2$.

Denote by $e^{-tH_n^{(d)}}$ (resp. $e^{-tL_n^{(d)}}$) the semigroup associated with $H_n^{(d)}$ (resp. $L_n^{(d)}$) introduced in Section 2. For each n the diamagnetic inequality holds for these pairs of semigroups.

Proposition 4.1 *The inequality*

$$|e^{-tH_n^{(d)}} f| \leq e^{-tL_n^{(d)}} |f|$$

holds for all $t \geq 0$ and all $f \in L^2(\Omega_n)$ $(n \geq 2)$.

Proof. We give the proof for $d = 2$ and suppress the upper index in $\mathfrak{h}_n^{(2)}$. With a few obvious modifications the proof for $d = 3$ is the same. By the domination criterion in Theorem 3.2, assertion 3, it suffices to prove that

$$\mathrm{Re}\ \mathfrak{h}_n[u, v] \geq \mathfrak{l}_n[|u|, |v|] \tag{4.1}$$

for all $u, v \in \mathcal{D}(\mathfrak{h}_n) = H_0^1(\Omega_n)$ obeying $u \cdot \overline{v} \geq 0$.

Let $u, v \in H_0^1(\Omega_n)$ be such that $u \cdot \overline{v} \geq 0$. We have that

$$\begin{aligned}
I_1 \quad := \quad & \mathrm{Re} \int_{\Omega_n} \left\{ \frac{\partial u}{\partial x_1} \cdot \frac{\partial \overline{v}}{\partial x_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial \overline{v}}{\partial x_2} \right\} dx \\
= \quad & \int_{\Omega_n} \left\{ \mathrm{Re} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_1} \cdot \mathrm{sign}\, \overline{v} \right) \right. \\
& \left. + \mathrm{Re} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_2} \cdot \mathrm{sign}\, \overline{v} \right) \right\} dx \\
& + \int_{\Omega_n} \left\{ \mathrm{Im} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial v}{\partial x_1} \cdot \mathrm{sign}\, \overline{v} \right) \right. \\
& \left. + \mathrm{Im} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial v}{\partial x_2} \cdot \mathrm{sign}\, \overline{v} \right) \right\} dx \\
= \quad & \int_{\Omega_n} \left\{ \mathrm{Re} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_1} \cdot \mathrm{sign}\, \overline{v} \right) \right. \\
& + \mathrm{Re} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Re} \left(\frac{\partial v}{\partial x_2} \cdot \mathrm{sign}\, \overline{v} \right) \\
& + \mathrm{Im} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial u}{\partial x_1} \cdot \mathrm{sign}\, \overline{u} \right) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \\
& \left. + \mathrm{Im} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \mathrm{Im} \left(\frac{\partial u}{\partial x_2} \cdot \mathrm{sign}\, \overline{u} \right) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right\} dx,
\end{aligned}$$

where we applied Lemma 3.3, part 2, in the last equality. From [24, Lemma 4.1] we have that

$$\frac{\partial |u|}{\partial x_1} = \operatorname{Re}\left(\frac{\partial u}{\partial x_1} \operatorname{sign} \overline{u}\right), \quad \forall u \in H^1(\Omega_n) \supset H_0^1(\Omega_n).$$

Using this, we find that

$$\begin{aligned} I_1 = \int_{\Omega_n} \Bigg\{ & \frac{\partial |u|}{\partial x_1} \cdot \frac{\partial |v|}{\partial x_1} + \frac{\partial |u|}{\partial x_2} \cdot \frac{\partial |v|}{\partial x_2} + \left[\operatorname{Im}\left(\frac{\partial u}{\partial x_1} \operatorname{sign} \overline{u}\right)\right]^2 \frac{|v|}{|u|} \chi_{\{u\neq 0\}} \\ & + \left[\operatorname{Im}\left(\frac{\partial u}{\partial x_1} \operatorname{sign} \overline{u}\right)\right]^2 \frac{|v|}{|u|} \chi_{\{u\neq 0\}} \Bigg\} dx. \end{aligned} \tag{4.2}$$

Next, let $u, v \in H_0^1(\Omega_n)$ with $u \cdot \overline{v} \geq 0$. Using $\operatorname{Re} u \overline{\frac{\partial v}{\partial x_1}} = \operatorname{Re} \overline{u} \frac{\partial v}{\partial x_1}$ we have that

$$\begin{aligned} I_2 \quad := \quad & \operatorname{Re} \int_{\Omega_n} \left\{ -iA_1 \frac{\partial u}{\partial x_1} \overline{v} - iA_2 \frac{\partial u}{\partial x_2} \overline{v} + iA_1 u \overline{\frac{\partial v}{\partial x_1}} + iA_2 u \overline{\frac{\partial v}{\partial x_2}} \right\} dx \\ = \quad & \int_{\Omega_n} \Bigg\{ - \operatorname{Im}(-iA_1) \operatorname{Im}\left(\frac{\partial u}{\partial x_1} \overline{v}\right) - \operatorname{Im}(-iA_2) \operatorname{Im}\left(\frac{\partial u}{\partial x_2} \overline{v}\right) \\ & - \operatorname{Im}(iA_1) \operatorname{Im}\left(\overline{u} \frac{\partial v}{\partial x_1}\right) - \operatorname{Im}(iA_2) \operatorname{Im}\left(\overline{u} \frac{\partial v}{\partial x_2}\right) \Bigg\} dx. \end{aligned}$$

Using the first part of Lemma 3.3 we may rewrite I_2 as

$$\begin{aligned} I_2 = \int_{\Omega_n} \Bigg\{ & - \operatorname{Im}(-iA_1) \operatorname{Im}\left(\frac{\partial u}{\partial x_1} \operatorname{sign} \overline{u}\right) |v| - \operatorname{Im}(-iA_2) \\ & \times \operatorname{Im}\left(\frac{\partial u}{\partial x_2} \operatorname{sign} \overline{u}\right) |v| - \operatorname{Im}(iA_1) \operatorname{Im}\left(\frac{\partial v}{\partial x_1} \operatorname{sign} \overline{v}\right) |u| \\ & - \operatorname{Im}(iA_2) \operatorname{Im}\left(\frac{\partial v}{\partial x_2} \operatorname{sign} \overline{v} |u|\right) \Bigg\} dx. \end{aligned}$$

Next we apply the second part of Lemma 3.3 to the last two terms in I_2. It follows that

$$\begin{aligned} I_2 \quad = \quad & \int_{\Omega_n} \Bigg\{ (A_1 - A_1) \operatorname{Im}\left(\frac{\partial u}{\partial x_1} \operatorname{sign} \overline{u}\right) |v| \\ & + (A_2 - A_2) \operatorname{Im}\left(\frac{\partial u}{\partial x_2} \operatorname{sign} \overline{u}\right) |v| \Bigg\} dx = 0. \end{aligned} \tag{4.3}$$

For the last term in $\mathfrak{h}_n$, we have that

$$I_3 := \operatorname{Re} \int_{\Omega_n} (A_1^2 + A_2^2) u \cdot \overline{v} dx = \int_{\Omega_n} (A_1^2 + A_2^2) |u| \, |v| dx \tag{4.4}$$

for all $u, v \in H_0^1(\Omega_n)$ such that $u \cdot \overline{v} \geq 0$.

Since Re $\mathfrak{h}_n[u,v] = \sum_{j=1}^3 I_j$, we obtain from (4.2), (4.3), and (4.4) that

$$\begin{aligned}\operatorname{Re} \mathfrak{h}_n[u,v] = \int_{\Omega_n} \Bigg\{ & \frac{\partial|u|}{\partial x_1}\cdot\frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2}\cdot\frac{\partial|v|}{\partial x_2} + \left[\operatorname{Im}\left(\frac{\partial u}{\partial x_1}\operatorname{sign}\overline{u}\right)\right]^2 \\ & \times\frac{|v|}{|u|}\chi_{\{u\neq 0\}} + \left[\operatorname{Im}\left(\frac{\partial u}{\partial x_1}\operatorname{sign}\overline{u}\right)\right]^2 \frac{|v|}{|u|}\chi_{\{u\neq 0\}} + (A_1^2 + A_2^2)|u|\,|v| \Bigg\} dx.\end{aligned}$$

In this expression, the sum of the last three terms is nonnegative, so we infer that

$$\begin{aligned}\operatorname{Re} \mathfrak{h}_n[u,v] &\geq \int_{\Omega_n} \left\{ \frac{\partial|u|}{\partial x_1}\cdot\frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2}\cdot\frac{\partial|v|}{\partial x_2} + (A_1^2 + A_2^2)|u|\,|v| \right\} \\ &\geq \int_{\Omega_n} \left\{ \frac{\partial|u|}{\partial x_1}\cdot\frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2}\cdot\frac{\partial|v|}{\partial x_2} \right\} = \mathfrak{l}_n[|u|,|v|]\end{aligned}$$

for all $u, v \in H_0^1(\Omega_n)$ obeying $u \cdot \overline{v} \geq 0$. This verifies (4.1). $\square$

The semigroups associated with $H_0^{(d)}$ and L_0, introduced in Section 2, are denoted by $e^{-tH_0^{(d)}}$ and e^{-tL_0}, resp. By means of Proposition 4.1 we are ready to prove Theorem 1.1, i.e., the diamagnetic inequality for the operator $H_0^{(d)}$.

Proof of Theorem 1.1. Bear in mind that when $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are closed forms bounded from below then $\mathfrak{s}_1 \geq \mathfrak{s}_2$ means that $\mathcal{D}(\mathfrak{s}_1) \subset \mathcal{D}(\mathfrak{s}_2)$ and $\mathfrak{s}_1[u,u] \geq \mathfrak{s}_2[u,u]$ for $u \in \mathcal{D}(\mathfrak{s}_1)$. A sequence $\{\mathfrak{s}_n\}$ of closed forms bounded from below is nonincreasing if $\mathfrak{s}_n \geq \mathfrak{s}_{n+1}$ for all n.

The forms $\{\mathfrak{h}_n^{(d)}\}$ defined in (2.2) on the domains $\mathcal{D}(\mathfrak{h}_n^{(d)}) = H_0^1(\Omega_n)$ in $L^2(\Omega_n)$, $n \geq 2$, compose a nonincreasing sequence

$$\cdots \leq \mathfrak{h}_{n+1}^{(d)} \leq \mathfrak{h}_n^{(d)} \leq \mathfrak{h}_{n-1}^{(d)} \leq \ldots,$$

of closed, *non-densely defined* forms in $L^2(\mathbb{R}^d)$. The monotone convergence theorem for closed forms is also valid for non-densely defined forms [32, Theorem 4.1]. Hence, the latter theorem in conjunction with Lemma 2.1 yields that operators $H_n^{(d)}$ corresponding to forms $\mathfrak{h}_n^{(d)}$ (extended by zero outside Ω_n) converge in strong resolvent sense to ${H_0}^{(d)}$ or, equivalently,

$$e^{-tH_0^{(d)}} = \text{s}-\lim_{n\to\infty} e^{-tH_n^{(d)}}. \tag{4.5}$$

A similar argument yields

$$e^{-tL_0^{(d)}} = \text{s}-\lim_{n\to\infty} e^{-tL_n^{(d)}}. \tag{4.6}$$

Thus we can pass to the limit $n \to \infty$ in the diamagnetic inequality for operators $H_n^{(d)}, L_n^{(d)}$, Proposition 4.1 and therefore

$$\left| e^{-tH_0^{(d)}} f \right| = \lim_{n\to\infty} \left| e^{-tH_n^{(d)}} f \right| \le \lim_{n\to\infty} e^{-tL_n^{(d)}} |f| = e^{-tL_0} |f|$$

which proves the assertion. □

Remark 4.2 Our proof of the diamagnetic inequality also applies to the case where we have another metric, that is, the result holds also for operators of the type $(\nabla + i\boldsymbol{A})^* M(x)(\nabla + i\boldsymbol{A})$ where $M(x) = (m_{jk}(x))$ is a symmetric matrix with real-valued and bounded measurable coefficients (satisfying the classical ellipticity condition). The semigroup generated by this operator is dominated by the semigroup generated by the elliptic operator $\nabla^* M(x)\nabla$.

5 Abstract CLR eigenvalue estimates and semigroup domination

In this section we recall Rozenblum's and Solomyak's abstract CLR estimate for generators of positively dominated semigroup.

Let Ω be a space with σ-finite measure μ, $L^2 = L^2(\Omega, \mu)$. Let T be a nonnegative, self-adjoint operator in L^2, generating a positivity preserving semigroup $Q(t) = e^{-tT}$. We suppose also that $Q(t)$ is an integral operator with bounded kernel $Q(t; x, y)$ subject to

$$M_T(t) := \text{ess sup}_x Q(t; x, x), \;\; M_T(t) = O(t^{-\beta}) \text{ as } t \to 0 \text{ for some } \beta > 0. \tag{5.1}$$

We will write $T \in \mathcal{P}$ if T satisfies the afore-mentioned assumptions[2].

If $T \in \mathcal{P}$, the operator $T_\mu = T + \mu$ also belongs to $\mathcal{P}$. The corresponding semigroup is $Q_{T_\mu}(t) = e^{-\mu t} Q_T(t)$ and thus $M_{T_\mu}(t) = e^{-\mu t} M_T(t)$.

We say that the semigroup $P(t) = e^{-tS}$ is dominated by $Q(t)$ if the diamagnetic inequality holds, i.e., if any $u \in L^2$ satisfies

$$|P(t)u| \le Q(t)|u| \text{ a.e. on } \Omega. \tag{5.2}$$

In the latter case we write $S \in \mathcal{PD}(T)$.

Let now G be a nonnegative, continuous, convex function on $[0, \infty)$. To such a function we associate

$$g(\lambda) = \mathcal{L}(G)(\lambda) := \int_0^\infty z^{-1} G(z) e^{-z/\lambda}\, dz, \quad \lambda > 0, \tag{5.3}$$

provided the latter integral converges. In other words, $g(1/\lambda)$ is the Laplace transform of $z^{-1}G(z)$.

[2] Although the diagonal in $\Omega \times \Omega$ may be a set with measure zero in $\Omega \times \Omega$, the semigroup property defines $Q(t, \cdot, \cdot)$ as a function in $L^\infty(\Omega)$, see [28].

For a nonnegative, measurable function V such that the operator of multiplication by V is form-bounded with respect to T with a bound less than one, we associate the operators $T - V$, $S - V$ by means of quadratic forms. The number of negative eigenvalues (counting multiplicity) of $T - V$ is denoted by $N_-(T - V)$; if there is some essential spectrum below zero, we set $N_-(T - V) = \infty$.

Rozenblum and Solomyak [28, Theorem 2.4] have established the following abstract CLR estimate .

Theorem 5.1 *Let G, g and $T \in \mathcal{P}$ be as above and suppose that $\int_a^\infty M_T(t)\,dt < \infty$ for some $a > 0$. If $S \in \mathcal{PD}(T)$ then*

$$N_-(S - V) \le \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_\Omega M_T(t) G(tV(x))\,dx, \tag{5.4}$$

as long as the expression on the right-hand side is finite.

The assumption that V is form-bounded with respect to T with a bound smaller than one in conjunction with $S \in \mathcal{PD}(T)$ implies that V is form-bounded with respect to S with a bound less than one, thus $N_-(S - V)$ is well defined. In Section 6 we shall apply Theorem 5.1 to prove the LT inequality for $H_0^{(d)} - V$.

Rozenblum has also developed an abstract machinery which, in our situation, allows us to carry over *any*, sufficiently regular, bound for $N_-(T - qV)$ to $N_-(S - qV)$, as soon as the diamagnetic inequality (5.2) is valid for S, T [30, Theorem 4]. We customize it to our situation.

Theorem 5.2 *Assume that $T \in \mathcal{P}$, $S \in \mathcal{PD}(T)$ and $V \ge 0$ is a measurable function infinitesimally form-bounded with respect to T. Suppose that, for some $p > 0$,*

$$N_-(T - qV) \le Kq^p \tag{5.5}$$

for all $q > 0$ and some positive constant K. Then

$$N_-(S - qV) \le eC_pKq^p, \tag{5.6}$$

with a constant C_p which depends only on p.

6 Lieb-Thirring inequality for $H_0^{(d)} - V$

Having the diamagnetic inequality in Theorem 1.1 as well as the abstract CLR estimate in Theorem 5.1 at our disposal, we are ready to prove Theorem 1.2.

Before proceeding with the proof, observe that the assumption $V \in L^p(\mathbb{R}^d)$, $p > 1$ for $d = 2$ and $p \ge 3/2$ for $d = 3$, in Theorem 1.2 implies that V is infinitesimally L_0- form-bounded; Theorem 1.1 then implies that V is infinitesimally $H_0^{(d)}$-form-bounded. Thus, according to the KLMN Theorem, the form sum $H_0^{(d)} - V$ generates a lower semi-bounded, self-adjoint operator in $L^2(\mathbb{R}^d)$.

Proof of Theorem 1.2. As usual, it suffices to prove the estimate for $V \in L^1 \cap L^\infty$ and then approximate $V \in L^{\gamma+\frac{d}{2}}$ by such functions. It is well known that $L_0 \in \mathcal{P}$ and the kernel of its semigroup e^{-tL_0} on the diagonal is given by $Q(t;x,x) = (4\pi)^{-d/2}t^{-d/2}$. From Theorem 1.1 we have that $H_0^{(d)} \in \mathcal{PD}(L_0)$ and the kernel P of its semigroup obeys $P(t;x,x) \le (4\pi)^{-d/2}t^{-d/2}$.

Let $\mu > 0$ and define the auxiliary operators $S_\mu = H_0^{(d)} + \mu$ and $T_\mu = L_0 + \mu$. Now $L_0 \in \mathcal{P}$ and $H_0^{(d)} \in \mathcal{PD}(L_0)$ imply that $T_\mu \in \mathcal{P}$ and $S_\mu \in \mathcal{PD}(T_\mu)$. For the kernel $P_\mu(t;x,x) = e^{-\mu t}P(t;x,x)$ of the semigroup generated by S_μ we have therefore that $|P_\mu(t;x,x)| \le Q_\mu(t;x,x) = e^{-\mu t}Q(t;x,x) = (4\pi)^{-d/2}t^{-d/2}e^{-\mu t}$. Thus we may apply Theorem 5.1 which yields

$$N_-(S_\mu - V) \le \frac{1}{(4\pi)^{d/2}}\frac{1}{g(1)}\int_0^\infty \frac{dt}{t}\int_{\mathbb{R}^d} t^{-d/2}e^{-\mu t}G(tV(x))\,dx \tag{6.1}$$

for any nonnegative convex function $G(s)$ of subexponential growth, vanishing near zero (which ensures that the integral in (6.1) converges). We will not evaluate the integral in (6.1) as one might be inclined to do. Instead, for $\gamma > 0$, we recall that (see, e.g., [21])

$$\begin{aligned} LT_{\gamma,d} = \sum_j |\nu_j(H_0^{(d)} - V)|^\gamma &= -\int \mu^\gamma\,dN_\mu \\ &= \gamma\int_0^\infty \mu^{\gamma-1}N_-(S_\mu - V)\,d\mu. \end{aligned} \tag{6.2}$$

We substitute (6.1) into (6.2) and get that

$$LT_{\gamma,d} \le \frac{1}{(4\pi)^{d/2}}\frac{\gamma}{g(1)}\int_{\mathbb{R}^d} dx\int_0^\infty \mu^{\gamma-1}\,d\mu\int_0^\infty t^{-d/2}e^{-\mu t}G(tV(x))\,\frac{dt}{t}.$$

Making first the change of variables $s = V(x)t$ and then the change of variables $\tau = \mu/V(x)$ we obtain that

$$LT_{\gamma,d} \le \tilde{L}_{\gamma,d}\int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d}{2}}\,dx,$$

where

$$\tilde{L}_{\gamma,d} = \frac{1}{(4\pi)^{d/2}}\frac{\gamma}{g(1)}\int_0^\infty\int_0^\infty s^{-\frac{d}{2}-1}e^{-\tau s}G(s)\tau^{\gamma-1}\,ds\,d\tau.$$

Now, $\int_0^\infty \tau^{\gamma-1}e^{-\tau s}\,d\tau = s^{-\gamma}\Gamma(\gamma)$, where $\Gamma(\gamma)$ is the Gamma-function evaluated at γ. Choose $G(s) = (s-k)_+$ for some $k > 0$; this is Lieb's original choice. Then

$$\int_0^\infty s^{-\gamma}s^{-\frac{d}{2}-1}(s-k)_+\,ds = \frac{1}{\left(\gamma + \frac{d-2}{2}\right)\left(\gamma + \frac{d}{2}\right)k^{\gamma\frac{d}{2}}}.$$

Moreover,

$$g(1) = \int_1^\infty e^{-ks}s^{-2}\,ds \ge \frac{e^{-k}}{k} - \frac{2}{k}g(1),$$

i.e., $1/g(1) \le e^k(k+2)$. Thus

$$\tilde{L}_{\gamma,d} \le C_{\gamma,d} := \frac{\Gamma(\gamma)e^k(k+2)}{(4\pi)^{d/2}(\gamma+\frac{d-2}{2})(\gamma+\frac{d}{2})k^{\gamma+\frac{d-2}{2}}}. \tag{6.3}$$

The optimization problem for the expression in (6.3) does not admit an exact solution. For the three most interesting values of γ, namely $1, 1/2$ and $3/2$, one easily finds the numerical values of $C_{\gamma,d}$ given in the Theorem. □

Remark 6.1 In the case of a single A-B solenoid, A. Laptev pointed out to the authors that the LT inequality can be derived *without* using the diamagnetic inequality [19]. His argument goes as follows. When $\boldsymbol{A} = \alpha(-x_2/|x|^2, x_1/|x|^2)$ we may use the decomposition $L^2(\mathbb{R}^2) = L^2(\mathbb{R}^+, r\,dr) \otimes L^2(\mathbb{S}^1) = \oplus_{n\in\mathbb{Z}}\{L^2(\mathbb{R}^+, r\,dr)[e^{in\theta}/2\pi]\}$ ($[\cdot]$ denotes the linear span) to express the A-B Schrödinger operator as

$$H_{AB}^{(2)} = \oplus_{n\in\mathbb{Z}}\{H_n \otimes I_n\},$$

where H_n is the Friedrichs operator in $L^2(\mathbb{R}^+, r\,dr)$ associated with the quadratic form

$$\mathfrak{h}_n[u_n] = \int_0^\infty \left(|u_n'(r)|^2 + \frac{(n+\alpha)^2}{r^2}|u_n(r)|^2 \right) r\,dr.$$

Thus, with a slight abuse of notation, the quadratic form associated with H_{AB} is given by $\mathfrak{h}[u] = \sum_{n\in\mathbb{Z}} \mathfrak{h}_n[u_n]$. Taking $\alpha \in (0, 1/2)$, we note that $|n+\alpha|^2 \ge |1-\alpha|^2$ provided $n \ne 0$. As a consequence, we have that

$$\mathfrak{h}_n[u_n] \ge |1-\alpha|^2 \int_0^\infty \left(|u_n'(r)|^2 + \frac{n^2}{r^2}|u_n(r)|^2 \right) r\,dr = |1-\alpha|^2 \mathfrak{l}_n[u_n],$$

where $\mathfrak{l}[u] = \sum_{n\in\mathbb{Z}} \mathfrak{l}_n[u_n]$ is the quadratic form of the negative Laplacian in $L^2(\mathbb{R}^2)$. In conclusion, $\mathfrak{h}[u] \ge |1-\alpha|^2 \mathfrak{l}[u]$. The latter inequality immediately implies that the usual LT inequalities for $-\Delta - V$ carry over to the A-B Schrödinger operator $H_{AB}^{(2)} - V$ with a constant $L_{2,\gamma}/|1-\alpha|^2$, where $L_{2,\gamma}$ is the usual Lieb-Thirring constant. A similar reasoning was used in [4]. This argument, however, does not work for many A-B solenoids.

7 Hardy-type inequalities

In order to establish eigenvalue estimates in the two-dimensional case for various configurations of A-B solenoids (or magnetic vortices), we require certain Hardy-type inequalities which we will obtain in this section. Generally, a Hardy-type inequality is an estimate where the integral involving the gradient of the function majorizes the weighted integral of the square of the function itself.

The classical Hardy inequality

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx \le \text{ const. } \int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx, \quad u \in C_0^\infty(\mathbb{R}^d \setminus \{0\}),$$

does not hold for $d = 2$. It was discovered by Laptev and Weidl [17], however, that the presence of a magnetic field can improve this situation. In particular, if the gradient ∇ is replaced by the "magnetic gradient" $\nabla + i\boldsymbol{A}$, where $\boldsymbol{A}$ is the standard A-B vector potential (see (7.5) below), and the flux $\alpha = \frac{1}{2\pi}\int_{\mathbb{S}^1} \boldsymbol{A}\,dx$ is noninteger then ([17, Theorem 3])

$$\int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx \le \rho(\alpha)^{-2}\int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}), \tag{7.1}$$

where $\rho(\alpha) = \min_{k\in\mathbb{Z}} |k - \alpha|$.

We are going to find analogies of this fact for configurations of magnetic solenoids considered in Section 2. In what follows, we will freely interchange real and complex pictures in description of our magnetic object. Thus $x = (x_1, x_2)$, $z = x_1 + ix_2$, $\boldsymbol{A} = (A_1, A_2)$, $\mathcal{A} = (A_2 + iA_1)$, $dx = \frac{1}{2}dz d\overline{z}$ etc.

Finitely many solenoids

Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_J\}$ with $\lambda_j = (\lambda_{1,j}, \lambda_{2,j})$. For finitely many A-B solenoids located at the points of Λ the corresponding A-B vector potential is given by

$$\boldsymbol{A}(x) = \sum_{j=1}^{J} \frac{\alpha_j}{|x - \lambda_j|^2}(-x_2 + \lambda_{2,j}\ ,\ x_1 - \lambda_{1,j}) \tag{7.2}$$

for $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Lambda$ and α_j being the flux through the jth solenoid.

The aim is to establish the following Hardy-type inequality.

Proposition 7.1 *Suppose that $\alpha_j \notin \mathbb{Z}$, $j = 1, 2, \dots, J$, and that $\alpha_s := \sum_{j=1}^{J} \alpha_j \notin \mathbb{Z}$. Define*

$$W(x) = \min\{\rho(\alpha_j)^2, \rho(\alpha_s)^2\} \sum_{j=1}^{J} |x - \lambda_j|^{-2}. \tag{7.3}$$

Then there exists a constant C such that

$$\int_{\mathbb{R}^2} W(x)|u(x)|^2\,dx \le C \int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx \tag{7.4}$$

is valid for all $u \in C^\infty(\mathbb{R}^2 \setminus \Lambda)$ as long as the integral on the right-hand side is finite.

Note that the constant C above may depend on the configuration of the solenoids.

We will mostly use this and other similar inequalities for $u \in C_0^\infty(\mathbb{R}^2 \setminus \Lambda)$. Sometimes, however, we need the compact support condition dropped. Moreover, we want to stress that, unlike the nonmagnetic case, the magnetic Hardy inequalities hold without compactness of support; this simple observation was somehow overlooked in [17, 3].

We begin by showing a slightly modified version of [17, Theorem 3].

Lemma 7.2 *Assume that* $\alpha_0 \notin \mathbb{Z}$ *and let*

$$\boldsymbol{A}_0(x) = \frac{\alpha_0}{|x|^2}(-x_2, x_1) \tag{7.5}$$

Then

$$\rho(\alpha_0)^2 \int_{B(0,R)} \frac{|u(x)|^2}{|x|^2}\, dx \leq \int_{B(0,R)} |(\nabla + i\boldsymbol{A}_0)u(x)|^2\, dx \tag{7.6}$$

holds for any $u \in C^\infty(B(0,R) \setminus \{0\})$ *as long as the integral on the right-hand side is finite.*

Proof. In polar co-ordinates (r, θ), we have that

$$\nabla + i\boldsymbol{A}_0 = -\boldsymbol{e}_r(\partial/\partial_r) + (1/r)\boldsymbol{e}_\theta[(-\partial/\partial_\theta) + i\alpha_0]. \tag{7.7}$$

Therefore, for any function $f(r)e^{in\theta}$, $n \in \mathbb{Z}$, we have that

$$\begin{aligned}
&\int_{B(0,R)} |(\nabla + i\boldsymbol{A}_0)f(r)e^{in\theta}|^2\, r\, drd\theta \\
&\quad = \int_{B(0,R)} \left(|f_r^{'}|^2 + (1/r^2)|f(r)|^2(n+\alpha_0)^2\right) r\, drd\theta \\
&\quad \geq \int_{B(0,R)} \frac{1}{r^2}|f(r)|^2(n+\alpha_0)^2\, r\, drd\theta \\
&\quad \geq \rho(\alpha_0)^2 \int_{B(0,R)} \frac{|f(r)e^{in\theta}|^2}{r^2}\, r\, drd\theta.
\end{aligned}$$

This proves (7.6) for spherical functions and therefore for any $u \in C^\infty(B(0,R) \setminus \{0\})$ since the left-hand side and the right-hand side of (7.6) are both sums of contributions of spherical functions. $\square$

In a similar way we establish the following result.

Lemma 7.3 *Suppose that* $\alpha_s := \sum \alpha_j \notin \mathbb{Z}$. *Then, provided* $R > 0$ *is sufficiently large,* $\Omega_R = \{|x| > R\}$, *the inequality*

$$\int_{\Omega_R} |(\nabla + i\boldsymbol{A})u(x)|^2\, dx \geq \rho(\alpha_s)^2 \int_{\Omega_R} \frac{|u(x)|^2}{|x|^2}\, dx \tag{7.8}$$

holds for any $u \in C^\infty(\Omega_R)$ *as long as the integral on the left-hand side is finite.*

Proof. First we note that there exists a function φ such that $\boldsymbol{A}(x) - \boldsymbol{A}_s(x) = (\nabla\varphi)(x)$,

$$\boldsymbol{A}_s(x) = \frac{\alpha_s}{|x|^2}(-x_2, x_1), \tag{7.9}$$

for any $x \in \Omega_R$ provided $R > 0$ is large enough. Since the right-hand side of (7.8) is gauge invariant, it suffices to show (7.8) for the vector potential $\boldsymbol{A}_s$. Now we switch to polar co-ordinates and repeat the reasoning in Lemma 7.2. $\square$

Lemma 7.4 (Local Hardy inequality) *Let D be a bounded, simply-connected domain in $\mathbb{C}$ with smooth boundary and let $z_0 \in D$. Let $\mathcal{A}(z) = A_2(z)+iA_1(z)$, $z = x_1+ix_2$, be a (complex) magnetic vector potential such that $\mathcal{A}(z)$ is analytic in $D \setminus \{z_0\}$ and has a simple pole at z_0 with residue equal to μ_0, and let $\boldsymbol{A} = (A_1, A_2)$. Then, there exists a constant $C > 0$ such that for any $u \in C^\infty(D \setminus \{z_0\})$,*

$$\int_D \frac{|u(z)|^2}{|z-z_0|^2}\,dx \le \rho(\mu_0)^{-2} C \int_D |(\nabla + i\boldsymbol{A})u(z)|^2\,dx, \tag{7.10}$$

as long as the integral on the right-hand side is finite.

Proof. Let $w = y_1 + iy_2 = F(z), F : D \to B(0,1)$ be a conformal mapping of D onto the unit disk $B(0,1)$ so that z_0 is mapped to zero, and let $\tilde{u}(y) = \tilde{u}(w) = u(F^{-1}(w))$. Since D has a smooth boundary, F is smooth up to the boundary [26, p. 49], together with its inverse. For later purpose we note that there exists c such that

$$\frac{c}{|z-z_0|} \le \left|\frac{F_z'(z)}{F(z)}\right|. \tag{7.11}$$

Indeed, since F is smoothly invertible, F' is bounded away from 0. Therefore F'/F has the order of $1/F$ near z_0. Since F has a simple zero at z_0, it has the order of $|z-z_0|$, which verifies (7.11).

Let $\omega_{\boldsymbol{A}}$ denote the differential 1-form $A_1(z)dx_1 + A_2(z)dx_2$ and let $\boldsymbol{A}^F(w) = (A_1^F(w), A_2^F(w))$ be the transformed magnetic vector potential in $B(0,1)$ such that $F^*(\omega_{\boldsymbol{A}^F}) = \omega_{\boldsymbol{A}}$ (F^* denotes the pull-back), i.e.,

$$A_1^F(w)dy_1 + A_2^F(w)dy_2 = A_1(z)dx_1 + A_2(z)dx_2.$$

In particular, $\mathcal{A}^F$ has a simple pole at the origin with residue equal to μ_0. Since F is a conformal mapping it follows that

$$\int_D |(\nabla_x + i\boldsymbol{A})u(x)|^2\,dx = \int_{B(0,1)} |(\nabla_y + i\boldsymbol{A}^F)\tilde{u}(y)|^2\,dy \tag{7.12}$$

for any $u \in C^\infty(D)$.

Next we gauge away the regular part of $\mathcal{A}^F = A_2^F + iA_1^F$ (as we did in the proof of Lemma 7.3). From Lemma 7.2 we immediately get that

$$\int_{B(0,1)} \frac{|\tilde{u}(y)|^2}{|y|^2}\,dy \le \rho(\mu_0)^{-2} \int_{B(0,1)} |(\nabla_y + i\boldsymbol{A}_0)\tilde{u}(y)|^2\,dy, \tag{7.13}$$

where $\boldsymbol{A}_0$ is the pure A-B vector potential given in (7.5). Finally, we return to the domain D by making the inverse transform $F^{-1} : B(0,1) \to D$. Clearly,

$$\int_{B(0,1)} \frac{|\tilde{u}(w)|^2}{|w|^2}\,dy = \int_D |u(z)|^2 \left|\frac{F_z'(z)}{F(z)}\right|^2\,dx. \tag{7.14}$$

Using (7.11) in conjunction with (7.12) and (7.13) we arrive at (7.10). $\square$

We are ready to give the proof of Proposition 7.1

Proof of Proposition 7.1. Let $B(0,R)$ be a disk centered at the origin with a radius $R>0$ so large that all the points of Λ are in $B(0,R)$. Cover the disk $B(0,R)$ with simply connected domains Ω_j having smooth boundaries in such a way that Ω_j contains λ_j but no other point from Λ.[3] Let $\varkappa$ be the multiplicity of the covering of $B(0,R)$ and let Ω_R be the exterior of $B(0,R)$.

We clearly have that

$$\int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx$$
$$\geq (1+\varkappa)^{-1}\left(\int_{\Omega_R} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx + \sum_{j=1}^{J}\int_{\Omega_j} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx\right).$$

The first term on the right-hand side is estimated by the inequality in Lemma 7.3 and each of the terms in the sum on the right-hand side is estimated by the local Hardy inequality in Lemma 7.4. In this way, we obtain that

$$\int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2\,dx$$
$$\geq \rho(\alpha_s)^2\int_{\Omega_R}\frac{|u(x)|^2}{|x|^2}\,dx + (1+\varkappa)^{-1}\sum_{j=1}^{J} c_j\rho(\alpha_j)^2\int_{\Omega_j}\frac{|u(x)|^2}{|x-\lambda_j|^2}\,dx.$$

Since, inside Ω_j, $j>0$, we have $|x-\lambda_j|^{-2}\geq C\sum_{k=1}^{J}|x-\lambda_k|^{-2}$, and inside Ω_R, we have $|x|^{-2}\geq C\sum_{k=1}^{J}|x-\lambda_k|^{-2}$, this proves (7.4). □

Remark 7.5 Using a conformal mapping was inspired by A. Balinsky [3]. He has recently derived a Hardy-type inequality for an A-B Schrödinger operator on general punctured domains. His result, however, does not give sufficient control over the Hardy weight, in particular, it does not guarantee strict positivity of the weight everywhere. This does not fit our purpose and, consequently, we have derived a slightly modified Hardy-type inequality.

The inequality (7.4) has a shortcoming: if just one of the fluxes is very close to an integer, the weight on the left-hand side deteriorates. The following version of the Hardy inequality takes care of this situation: if the sum of fluxes is non-integer, we can exclude any solenoids we wish from the expression in (7.3).

[3] To construct such a covering, take, e.g., a direction not parallel to any of the straight lines passing through pairs of the points λ_j. Then one can draw straight lines parallel to this direction, which cut $B(0,R)$ into pieces, each of which contains only one of the points λ_j. Extending these pieces slightly to domains with smooth boundaries we obtain the desired covering, with multiplicity $\varkappa = 2$.

Proposition 7.6 *Suppose that $\alpha_j \notin \mathbb{Z}$, $j = 1, 2, \ldots, J$, and that $\alpha_s := \sum_{j=1}^{J} \alpha_j \notin \mathbb{Z}$. Let $\mathcal{J}_0$ be a subset in $\{1, \ldots, J\}$. Set*

$$W(x) = \min_{\{j \in \mathcal{J}_0\}} \{\rho(\alpha_j)^2, \rho(\alpha_s)^2\} \sum_{j \in \mathcal{J}_0} |x - \lambda_j|^{-2}, \tag{7.15}$$

if $\mathcal{J}_0$ is nonempty, and

$$W(x) = \rho(\alpha_s)^2 (1 + |x|^2)^{-1}. \tag{7.16}$$

otherwise. Then there exists a constant C such that (7.4) is satisfied for any $u \in C^\infty(\mathbb{R}^2 \setminus \Lambda)$.

Proof. We consider the case of empty $\mathcal{J}_0$ first. Let, as in the proof of Proposition 7.1, the ball $B(0, R)$ contain all points z_j. Let $\varphi, \psi \geq 0$ be smooth functions, $\varphi^2 + \psi^2 = 1$, $\varphi \in C_0^\infty(B(0, 3R))$, $1 - \varphi = 0$ outside $B(0, R)$, $|\nabla\varphi|, |\nabla\psi| < 2/R$. Then for any u,

$$\begin{aligned} J_{\boldsymbol{A}}(u) &:= \int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x)|^2 \, dx \\ &= J_{\boldsymbol{A}}(\varphi u) + J_{\boldsymbol{A}}(\psi u) - \int_{R<|x|<3R} (|\nabla\varphi|^2 + |\nabla\psi|^2)|u|^2 \, dx \\ &\geq J_{\boldsymbol{A}}(\varphi u) - 8R^{-2} \int_{R<|x|<3R} |u|^2 \, dx. \end{aligned} \tag{7.17}$$

Now we use the well-known fact (see, e.g., [33, page 2]) that (for any magnetic potential $\boldsymbol{A}$),

$$J_{\boldsymbol{A}}(v) \geq \int |\nabla|v||^2 dx. \tag{7.18}$$

Applying (7.18) to the function $v = \varphi u$ and substituting the result into (7.17), we obtain

$$J_{\boldsymbol{A}}(u) \geq \int |\nabla|\varphi u||^2 dx - 8R^{-2} \int_{R<|x|<2R} |u|^2 dx. \tag{7.19}$$

It follows from (7.6) that $u \in L^2_{loc}$ as soon as $J_{\boldsymbol{A}}(u)$ is finite. At the same time, (7.18)–(7.19) imply that $\nabla|\varphi u| \in L^2$. Thus $|\varphi u|$ belongs to the Sobolev space $H_0^1(B(0, 3R))$ and we can apply the Friedrichs inequality to the first integral in (7.19). The second term can be estimated from both sides by $\int_{R<|x|<3R} |x|^{-2}|u|^2 dx$. Thus we have

$$\begin{aligned} J_{\boldsymbol{A}}(u) &\geq C_1 \int |\varphi u|^2 \, dx - C_2 \int_{R<|x|<3R} |x|^{-2}|u|^2 \, dx \\ &\geq C_1 \int_{B_R} |u|^2 \, dx - C_2 \int_{\Omega_R} |x|^{-2}|u|^2 \, dx. \end{aligned}$$

For some $\epsilon > 0$, multiply the latter inequality by ϵ and add to (7.8), multiplied by $1-\epsilon$. We obtain

$$J_{\boldsymbol{A}}(u) \geq C_1\epsilon \int_{B(0,R)} |u|^2\,dx + (\rho(\alpha_s)^2(1-\epsilon) - C_2\epsilon)\int_{\Omega_R} |x|^{-2}|u|^2\,dx. \tag{7.20}$$

Choosing ϵ small enough (say, $\epsilon = \rho(\alpha_s)^2/(4C_2)$), we arrange that the constant before the last integral in (7.20) is greater than $C_3\rho(\alpha_s)^2$ and we obtain the required inequality. In the case of nonempty $\mathcal{J}_0$, we split $J_{\boldsymbol{A}}(u) = \frac{1}{2}J_{\boldsymbol{A}}(u) + \frac{1}{2}J_{\boldsymbol{A}}(u)$. For first term here we use the inequality we have just established. To estimate from below the second term, we act as in the proof of Proposition 7.1, i.e., consider the covering of the disk $B(0,R)$ by domains Ω_j but we write the local Hardy inequalities only for $j \in \mathcal{J}_0$, thus getting

$$\frac{1}{2}J_{\boldsymbol{A}}(u) \geq C\sum_{j\in\mathcal{J}_0} \rho(\alpha_j)^2 \int_{\Omega_j} |x-\lambda_j|^{-2}|u|^2dx.$$

Summing this with the estimate for the case of empty $\mathcal{J}_0$, we arrive at (7.4). □

Regular lattice of solenoids

For a regular lattice of A-B solenoids we establish the following Hardy-type inequality.

Proposition 7.7 *Let $\mathcal{A}(z) = \mathcal{A}(x_1+ix_2) = A_2 + iA_1$ be a magnetic potential such that $\mathcal{A}$ is analytical in $\mathbb{C}$ with exception of the points $z_{kl} = k\boldsymbol{\omega}_1 + l\boldsymbol{\omega}_2$, $k,l \in \mathbb{Z}$, and in these points $\mathcal{A}$ has simple poles with residue equal to some non-integer α. Then, for any $u \in C^\infty(\mathbb{C}\setminus \cup z_{jk})$,*

$$J_{\boldsymbol{A}}(u) = \int |(\nabla + i\boldsymbol{A})u|^2\,dx_1dx_2 \geq C\rho(\alpha)^2 \int |u|^2 W(z)\,dx_1dx_2,$$

where $C>0$, $\rho(\alpha) = \min_{k\in\mathbb{Z}}|k-\alpha|$ and $W(z)^{-1/2}$ is the distance from $z = x_1+ix_2$ to the nearest lattice point.

Proof. We consider first the case of a lattice Λ with $\boldsymbol{\omega}_1 = 1, \boldsymbol{\omega}_2 = i$. Write $J_{\boldsymbol{A}}(u) = 4\times\frac{1}{4}J_{\boldsymbol{A}}(u)$. Split the lattice Λ into four sublattices, $\Lambda = \Lambda_1\cup\Lambda_2\cup\Lambda_3\cup\Lambda_4$, where Λ_1 consists of the points $(2k,2l)$ and, $\Lambda_2 = \{(2k+1,2l)\}$, $\Lambda_3 = \{(2k,2l+1)\}$ and $\Lambda_4 = \{(2k+1,2l+1)\}$.

Around each point $z_{kl}\in\Lambda_j$, draw a disk D_{kl} with radius 0.8. Such a disk does not contain other points in the lattice. For this disk D_{kl} and any $u\in C_0^\infty(\mathbb{C}\setminus\cup z_{jk})$, we can apply the inequality (7.1),

$$\frac{1}{4}\int_{D_{kl}} |(\nabla + i\boldsymbol{A})u|^2\,dx_1dx_2 \geq \frac{1}{4}\rho(\alpha)^2\int_{D_{kl}} |u|^2|z-z_{kl}|^{-2}dx_1dx_2, \tag{7.21}$$

since, in the punctured disk $D_{kl} \setminus \{z_{kl}\}$, the vector potential $\boldsymbol{A}$ is gauge equivalent to the potential $\alpha/|z|$.

For j fixed ($j = 1, 2, 3, 4$) we now sum (7.21) over all $z_{kl} \in \Lambda_j$. The size of the disks is selected in such way that for j fixed, the corresponding disks are disjoint and therefore one can sum (7.21) termwise and get

$$\frac{1}{4} J_{\boldsymbol{A}}(u) \geq \frac{1}{4}\rho(\alpha)^2 \sum_{z_{kl}\in\Lambda_j} \int_{D_{kl}} |u|^2 |z - z_{kl}|^{-2}\, dx_1 dx_2 .$$

Next we sum the latter inequality over $j = 1, 2, 3, 4$, which yields

$$J_{\boldsymbol{A}}(u) \geq \frac{1}{4}\rho(\alpha)^2 \sum_{z_{kl}\in\Lambda} \int_{D_{kl}} |u|^2 |z - z_{kl}|^{-2}\, dx_1 dx_2 \tag{7.22}$$

Now we note that

$$|z - z_{kl}|^{-2} \geq CW(z) \text{ for } z \in D_{kl}$$

for some $C > 0$ and, moreover, the disks D_{kl} cover the plane. Therefore the expression in (7.22) majorizes $\rho(\alpha)^2 \int |u|^2 W(z)\, dx_1 dx_2$.

For an arbitrary lattice we perform the same reasoning, just with disks with radius 0.8 being replaced by equal ellipses of proper size, covering the plane, and with the local Hardy inequality in the ellipse used instead for the one in the disk. □

The weight function $W(z)$ in Proposition 7.7 is positive and separated from zero, $W(z) \geq W_0 > 0$. This implies, in particular, that the spectrum of the operator $H_0^{(2)}$ is separated from zero, i.e., the magnetic field produces a spectral gap. It is remarkable to compare this with the result of Geyler-Grishanov [12] who have shown that for another self-adjoint realization of the A-B operator corresponding to an infinite regular lattice of solenoids, the lowest point of the spectrum is zero and, moreover, an eigenvalue with infinite multiplicity.[4]

Hardy inequalities in dimension $d = 3$

Although in dimension three one does not need Hardy-type inequalities to establish CLR estimates (the latter is our main reason to study these inequalities), such three-dimensional versions are of certain interest.

In order to join all cases, we will denote, for a fixed configuration of vortices, by $\boldsymbol{W}(x_\perp)$, $x_\perp = (x_1, x_2)$, the weight which, according to Propositions 7.1, 7.6 or 7.7, enters into the Hardy inequality in $\mathbb{R}^2$, viz.

$$\int_{\mathbb{R}^2} |(\nabla + i\boldsymbol{A})u(x_\perp)|^2\, dx_1 dx_2 \geq \int_{\mathbb{R}^2} \boldsymbol{W}(x_\perp)|u(x_\perp)|\, dx_1 dx_2 . \tag{7.23}$$

[4] It is an interesting question, whether the lowest point of the spectrum of our operator is an eigenvalue.

Hence $\boldsymbol{W}(x)$ incorporates the weight W and coefficients depending on the configuration of solenoids and on fluxes α_s.

Proposition 7.8 *Under the assumptions of Propositions* 7.1*,* 7.6 *or* 7.7*, the following Hardy-type inequality holds:*

$$\int_{\mathbb{R}^3} |(\nabla + i\boldsymbol{A})u(x_\perp, x_3)|^2 \, d^3x \geq \int_{\mathbb{R}^3} \boldsymbol{W}(x_\perp)|u(x_\perp, x_3)| \, d^3x. \tag{7.24}$$

Of course, (7.24) is obtained from (7.23) by integration in x_3.

Improvement of LT estimates by means of Hardy inequalities

The LT-type inequality obtained in Section 6, does not show any dependence on the configuration of the magnetic fields. This is quite natural for the method of proving, since it was based upon comparing with the nonmagnetic case. By means of Hardy inequalities established above one can get certain improvements of the LT estimate. We explain, first of all, that one should not expect an improvement in the constant $C_{\gamma,d}$ in (1.3), at least for a finite system of A-B solenoids. In fact, the magnetic potential decays at infinity, and if the electric potential V is supported far away from the sources of the field, then the influence of the magnetic field on the eigenvalues must be negligible (one can give exact meaning to this statement). On the other hand, the Hardy term can compensate local singularities of V or insufficient decay and thus estimate $LT_{\gamma,d}$ even in the case when the right-hand side in (1.3) is infinite.

Proposition 7.9 *Let $\boldsymbol{A}$ be a magnetic potential in $\mathbb{R}^d$, $d = 2$ or $d = 3$, and let $\kappa \in (0, 1)$ be an arbitrary number. Then, for the sum $LT_{\gamma,d}$ of powers of the eigenvalues of the operator $H_0^d - V$, the following inequality holds:*

$$LT_{\gamma,d} \leq C_{\gamma,d}(1-\kappa)^{-\frac{d}{2}-\gamma} \int_{\mathbb{R}^d} (V(x) - \kappa \boldsymbol{W}(x_1, x_2))_+^{\frac{d}{2}+\gamma} \, dx, \tag{7.25}$$

where $\gamma > 0$ and $C_{\gamma,d}$ is the constant in (1.3)*.*

The proof follows immediately from the operator inequality

$$\begin{aligned} H_0^d - V &= (1-\kappa)H_0^d + \kappa H_0^d - V(x) \\ &\geq (1-\kappa)H_0^d + \kappa \boldsymbol{W}(x_1, x_2) - V(x) \\ &\geq (1-\kappa)(H_0^d - (1-\kappa)^{-1}(V(x) - \kappa \boldsymbol{W}(x_1, x_2))) \end{aligned}$$

and (1.3) applied to the electric potential $(V(x) - \kappa \boldsymbol{W}(x_1, x_2))_+$.

The inequality (7.25) includes certain situations which are worth being picked out, e.g., the case of an electric potential having singularities at the magnetic vortices, so that singularities are (partially) compensated by the magnetic field. In the case of the infinite lattice of solenoids, moreover, the Hardy weight $\boldsymbol{W}$ is separated from zero, $\boldsymbol{W} \geq c_0$, and thus an insufficiently slow decay of V, preventing the right-hand side of (1.3) from being finite, can be compensated; in view of (7.25),

$$LT_{\gamma,d} \leq C_{\gamma,d}(1-\kappa)^{-\frac{d}{2}-\gamma} \int_{\mathbb{R}^d} (V(x) - \kappa c_0)_+^{\frac{d}{2}+\gamma}\, dx. \tag{7.26}$$

Further versions of the LT-type inequalities, taking into account possible cancelling of the magnetic and electric field, can be obtained by averaging (7.25) in κ. Having chosen some nonnegative function $\zeta(\kappa)$ on $(0,1)$ which vanishes near one and has integral 1, we can multiply (7.25) by $\zeta(\kappa)$ and integrate. This produces new estimates, first for 'nice' V, but then, as usual, since the constants there do not depend on the potential, they extend by continuity to all V for which the quantity on the right-hand side is finite. We give here just one, simple example of this approach, where we do not care much about getting sharp constants but are interested in possible cancellation of singularities.

Take $\zeta(\kappa) = c_\sigma \kappa^\sigma$ for $\kappa < 1/2$ and $\zeta(\kappa) = 0$ otherwise; with $c_\sigma = (\sigma + 1)^{-1}2^{-\sigma-1}$, $\sigma > -1$. Then integrating of (7.25) with weight $\zeta(\kappa)$ gives

$$LT_{\gamma,d} \leq c_\sigma C_{\gamma,d} \int_0^{1/2} (1-\kappa)^{-\frac{d}{2}-\gamma}\kappa^\sigma \int_{\mathbb{R}^d} (V(x) - \kappa \boldsymbol{W}(x_1,x_2))_+^{\frac{d}{2}+\gamma}\, dx d\kappa. \tag{7.27}$$

The integral in (7.27) is estimated as follows. Change the order of integration and majorate $(1-\kappa)^{-\frac{d}{2}-\gamma}$ by $2^{\frac{d}{2}+\gamma}$. For the remaining integral in κ we have

$$\int_0^{1/2} \kappa^\sigma (V(x) - \kappa \boldsymbol{W}(x_1,x_2))_+^{\frac{d}{2}+\gamma}\, d\kappa \\ \leq \boldsymbol{W}(x_1,x_2)^{\frac{d}{2}+\gamma} \int_0^\infty \left(\frac{V(x)}{W(x_1,x_2)} - \kappa \right)_+^{\frac{d}{2}+\gamma} \kappa^\sigma\, d\kappa.$$

The latter integral is evaluated by the usual change of variables and equals $\boldsymbol{B}(\sigma + 1, \frac{d}{2}+\gamma+1)V(x)^{\frac{d}{2}+\gamma+\sigma+1}W(x_1,x_2)^{-\sigma-1}$; $\boldsymbol{B}(\cdot,\cdot)$ being the Beta function. Thus we come to the estimate

$$LT_{\gamma,d} \leq C_{\gamma,d} c_\sigma \boldsymbol{B}(\sigma+1, \frac{d}{2}+\gamma+1) \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}+\gamma+\sigma+1} W(x_1,x_2)^{-\sigma-1} dx. \tag{7.28}$$

In the inequality (7.28) one can choose the exponent σ depending on the particular potential V. The approach above has, in fact, a flavor of interpolation; similar results, based on Hardy inequalities, were obtained for $N_-(H_0 - V)$ in [9] (for the magnetic potential in L^2_{loc}, $d \geq 3$).

8 Eigenvalue estimates and large coupling constant asymptotics

In three dimensions the CLR estimate for $H_0^{(3)} - V, V \geq 0$, takes its standard form for any of the configurations of A-B solenoids considered in Section 2, viz.

$$N_-(H_0^{(3)} - V) \leq C_3 \int_{\mathbb{R}^3} V(x)^{\frac{3}{2}}\, dx. \tag{8.1}$$

This follows automatically from the non-magnetic inequality and domination, Theorem 5.2; the constant $C_3 > 0$ is absolute.

In the two-dimensional case there are a number of inequalities for the non-magnetic Schrödinger operator, fairly cumbersome, see [34, 6]. Due to domination, they carry over to the A-B Hamiltonian. However, the presence of the magnetic field improves the non-magnetic estimates considerably.

The aim of this section is *not* to obtain the most general nor the best possible bounds for the number of negative eigenvalues for the two-dimensional perturbed A-B Schrödinger operator; rather we demonstrate that *any* CLR-type eigenvalue estimate (existing or obtained in the future) for the (nonmagnetic) two-dimensional perturbed Schrödinger operator, *with a proper Hardy term added*, automatically produces a similar estimate for the A-B Schrödinger operator and, moreover, to show that further improvements, using the Hardy inequalities are possible.

A single solenoid

The (closed) quadratic form $\mathfrak{h}^{(2)}$ of the unperturbed A-B Schrödinger operator $H_0^{(2)}$ can be written as

$$\mathfrak{h}^{(2)}[u] = \frac{\mathfrak{h}^{(2)}[u]}{2} + \frac{\mathfrak{h}^{(2)}[u]}{2}. \tag{8.2}$$

Let $\beta = \rho(\alpha)$. To one of the two terms in (8.2), we apply the Hardy type inequality (7.1). This yields

$$\mathfrak{h}^{(2)}[u] \geq \frac{\mathfrak{h}^{(2)}[u]}{2} + \frac{\beta^2}{2} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\, dx. \tag{8.3}$$

Let $H_0^{(2)}(\beta^2 r^{-2})$, $r = |x|^{-2}$, denote the operator generated by the form on the right-hand side of (8.3). Since $H_0^{(2)}$ obeys the diamagnetic inequality, it follows from, e.g., the Trotter-Kato formula that $H_0^{(2)}(\beta^2 r^{-2})$ fulfills the diamagnetic inequality as well, in shorthand, $H_0^{(2)}(\beta^2 r^{-2}) \in \mathcal{PD}(\frac{L_0}{2} + \frac{\beta^2}{2} r^{-2})$. The latter fact in conjunction with Theorem 5.2 allows us to carry over all power in q bounds for the two-dimensional Schrödinger operator $L_0 + Cr^{-2} - qV$ to the A-B Schrödinger operator $H_0^{(2)} - qV$, with a coupling constant $q > 0$. In order to keep track of the influence of the value of β, we can write

$$\frac{L_0}{2} + \frac{\beta^2}{2} r^{-2} - qV \geq \frac{\beta^2}{2} L_0 + \frac{\beta^2}{2} r^{-2} - qV = \frac{\beta^2}{2}(L_0 + r^{-2} - 2\beta^{-2} qV).$$

Therefore, to estimate the number of eigenvalues for the operator with given β, we may use the existing estimates for the operator $L_0 + r^{-2}$ with potential $2\beta^{-2} qV$. Such estimates for the Schrödinger operator $L(r^{-2}, qV) := L_0 + r^{-2} - qV$ in $L^2(\mathbb{R}^2)$ have been studied in [34, 6, 16]. We present here only the results from [16], not the most general ones, but, probably, the most transparent.

Proposition 8.1 *Let $p > 1, V \geq 0$. Denote by $S_p(V)$ the expression*

$$\int_0^1 \left(\int_{\mathbb{S}^1} V(r\omega)^p d\omega \right)^{1/p} r\, dr \tag{8.4}$$

in polar co-ordinates (r, ω). Then for a constant C_p,

$$N_-(L(r^{-2}, qV)) \leq C_p q S_p(V)\,; \tag{8.5}$$

in particular, if V is radial, $V = V(r)$, then

$$N_-(L(r^{-2}, qV)) \leq Cq \int_{\mathbb{R}^2} V(x)\, dx. \tag{8.6}$$

Due to domination, this estimate is carried over to $H_0^{(2)} - qV$, with p chosen arbitrary:

Theorem 8.2 (A single solenoid) *For some C'_p, depending only on p,*

$$N_-(H_0^{(2)} - qV) \leq C'_p q \beta^{-2} S_p(V) \tag{8.7}$$

and, for a radial potential,

$$N_-(H_0^{(2)} - qV) \leq Cq\beta^{-2} \int_{\mathbb{R}^2} V(x)\, dx. \tag{8.8}$$

In a similar way, all estimates obtained in [6, 34] carry over to $H_0^{(2)} - qV$. Of course, the factor β^{-2} must arise in the estimates, as it was explained above.

Finitely many solenoids

Let $\mathfrak{h}^{(2)}$ be the (closed) quadratic form generating the unperturbed magnetic Schrödinger operator $H_0^{(2)}$ associated with finitely many A-B solenoids. Again, write (8.2) and apply the Hardy type inequality established in Proposition 7.1 to one of the two terms in (8.2). We get

$$\mathfrak{h}^{(2)}[u] \geq \frac{\mathfrak{h}^{(2)}[u]}{2} + \int_{\mathbb{R}^2} CW(x)|u(x)|^2\, dx, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus \Lambda). \tag{8.9}$$

where $W(x)$ is given in (7.3). Let $H_0^{(2)}(W(x))$ denote the operator generated by the form on the right-hand side of (8.9). As above, (8.9) together with Theorem 5.2 reduce the problem of estimating $N_-(H_0^{(2)} - qV)$ to the similar task for $N_-(L_0 + W(x) - qV)$.

Take some splitting of $\mathbb{R}^2$ into sets $\tilde{\Omega}_j$, $1 \le j \le J$, such that $\lambda_j \in \tilde{\Omega}_j$ and denote by V_j the function coinciding with V in $\tilde{\Omega}_j$ and vanishing outside (of course, at least one of these sets must be unbounded).

Then $L_0 + W - qV$ splits into the sum (in the sense of quadratic forms) of operators $J^{-1}(L_0 + W) - qV_j$, and from the Weyl inequality we infer that

$$N_-(L_0 + W(x) - qV) \le \sum_j^J N_-(J^{-1}L_0 + K|x - \lambda_j|^{-2} - qV_j), \tag{8.10}$$

where the constant K, depends on the positions of the solenoids λ_j and their fluxes α_j, according to the Hardy type inequality (7.4).

Separate terms in (8.10) have just been discussed above. For each j we define by $S_p(V_j)$ the expression of the form (8.4), with V replaced by V_j and integration performed in polar coordinates centered at λ_j. Proposition 8.1, applied to each V_j, leads to the following result.

Theorem 8.3 (Finitely many solenoids) *One has*

$$N_-(H_0^{(2)} - qV) \le Cq \sum_{1 \le j \le J} S_p(V_j), \tag{8.11}$$

where the constant C depends on the positions of the solenoids and their fluxes.

For fixed positions of solenoids, the constant C in (8.7) is determined by the flux which is the one closest to integers and it deteriorates if this flux approaches an integer. Using Proposition 7.6 one may exclude arbitrary solenoids from (8.7), and leaving only fluxes which are sufficiently far away from integers, one can get an improved estimate of the form (8.7), with summation performed only over $j \in \mathcal{J}_0$. Here the constant will depend only on the remaining fluxes $\alpha_j, j \in \mathcal{J}_0$ and the sum of all fluxes.

Regular lattice of solenoids

We consider the case with infinitely many solenoids located at the points of the lattice $\Lambda = \{\lambda_{kl} = (k, l) \in \mathbb{R}^2 \, : \, k, l \in \mathbb{Z}\}$. As it is typical for the two-dimensional case, one can here, as for the previous configurations, give different types of CLR estimates. We restrict ourselves to the two, most simple versions.

Theorem 8.4 *Let $V \in L^p(\mathbb{R}^2)$ locally, $p \ge 1$. Consider a partition of $\mathbb{R}^2$ into unit cubes Q_j. Then, for some constant C,*

$$N_-(H_0^{(2)} - qV) \le Cq \sum_j \|V\|_{L^p(Q_j)}, \tag{8.12}$$

where the norms involved are the L^p norms over the cubes.

Proof. We use the inequality, already mentioned, $W(x) \geq W_0 > 0$, where $W(x)$ is the weight function in Proposition 7.7. Thus, for the (nonmagnetic) two-dimensional Schrödinger operator $L_0 + W_0 - qV$, the estimate of the type (8.12) is contained, for example, in [6]. Then the Hardy inequality in Proposition 7.7 and the diamagnetic inequality, together with Theorem 5.2, imply that the same kind of estimate, with some other constant, holds for $H_0^{(2)} - qV$. □

Another estimate we give here is in flavor of Theorem 8.3.

Theorem 8.5 *Let $V(x)$ be split into the sum of nonnegative functions V_{kl}, viz.*

$$V(x) = \sum_{kl} V_{kl}.$$

Then, for some constant C_p,

$$N_-(H_0^{(2)} - qV) \leq C_p\, q \left(\sum_{kl} S_p(V_{kl})^{\frac{1}{2}} \right)^2, \tag{8.13}$$

as long as the quantity on the right-hand side is finite.

Proof. Similar to the reasoning in the previous proof, the diamagnetic inequality, the Hardy inequality, and Theorem 5.2 reduce our task to establishing (8.13) for the two-dimensional Schrödinger operator $L_0 + W - qV$.

Denote $S_p(V_{kl})$ by R_{kl} and suppose that the series $\sum R_{kl}^{1/2}$ converges to some number M. Set $\tau_{kl} = R_{kl}^{1/2} M^{-1}$, $\sum_{kl} \tau_{kl} = 1$. Then the series

$$\tilde{W}(z) = \sum_{kl} \tau_{kl} |x - \lambda_{kl}|^{-2}$$

converges for any $x \notin \Lambda$. Moreover, for some constant C, not depending on V, $\tilde{W}(x) \leq CW(x)$. Thus, due to the max-min principle, it suffices to prove the estimate (8.13) for the operator $L_0 + C\tilde{W}(z) - qV$. From Weyl's inequality it follows that

$$\begin{aligned} N_-(L_0 + C\tilde{W}(z) - qV) &= N_-\left(\sum_{kl} (\tau_{kl} L_0 + C\tau_{kl} |x - \lambda_{kl}|^{-2} - qV_{kl}) \right) \\ &\leq \sum_{kl} N_-(\tau_{kl} L_0 + C\tau_{kl} |x - \lambda_{kl}|^{-2} - qV_{kl}(|x - \lambda_{kl}|)) \\ &= \sum_{kl} N_-(L_0 + C|x - \lambda_{kl}|^{-2} - q\tau_{kl}^{-1} V_{kl}(|x - \lambda_{kl}|)). \end{aligned}$$

To each term in the latter sum we apply the estimate in (8.5), getting

$$N_-(L_0 + C\tilde{W}(z) - qV) \leq qC \sum_{kl} \tau_{kl}^{-1} S_p(V_{kl}) = qC \sum_{kl} \tau_{kl}^{-1} R_{kl} = qCM \sum_{kl} R_{kl}^{1/2},$$

which coincides with the expression in (8.13). □

Improving the estimates

Exactly as in the case of LT-type estimates, considered in Section 7, one can further improve CLR-type estimates by means of additional use of Hardy inequalities, which enables one to trace possible cancellations of singularities of the potential V and the Hardy weight. We present only one, simplest result in this direction, a substructure over Theorem 8.2; all other formulations can be obtained following this pattern, and the proofs just repeat the tricks used in Section 7.

Proposition 8.6 *Consider the case of a single solenoid, with radial potential $V(r)$. Then for any $\kappa \in (0,1)$*

$$N_-(H_0^{(2)} - qV) \leq Cq(1-\kappa)^{-1} S_p\left(\left(V - \frac{\kappa\beta^2}{2|x|^2}\right)_+\right). \tag{8.14}$$

Large coupling constant asymptotics for $H_0^{(d)} - qV$

The task of establishing large coupling constant asymptotics for Schrödinger-type operators is nowadays a routine matter as soon as correct estimates are obtained, see, e.g., [7, 27, 29], where this routine is described in details. Therefore, in the case of a singular magnetic field we just indicate those (minor) modifications one has to make.

Having the Schrödinger operator $H_0^{(d)}$ and a potential $0 \leq V \in L^1_{loc}$, we denote by $\Sigma(V)$ the quantity entering in the eigenvalue estimate for the particular configuration of A-B solenoids above. Thus, in dimension three, we set $\Sigma(V) = \int V(x)^{\frac{3}{2}} dx$. In dimension two, $\Sigma(V)$ is the right-hand side of (8.7) or (8.8) under the conditions of Theorem 8.2, and it equals the right-hand side of (8.11) under the conditions of Theorem 8.3. Finally, for the case of a lattice, $\Sigma(V)$ is the right-hand side of (8.12), respectively (8.13), under the conditions of Theorem 8.4, respectively Theorem 8.5.

Theorem 8.7 *For $d = 2,3$, let $H_0^{(d)}$ be the (multivortex) Aharonov-Bohm Schrödinger operator for any of the solenoid configurations in Section 2 and suppose $\Sigma(V)$ is finite. Then, for the negative eigenvalues of $H_{qV} = H_0^{(d)} - qV$, the following asymptotic formula holds:*

$$N_-(H_{qV}) \sim c_d q^{\frac{d}{2}} \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}} \text{ as } q \to \infty, \tag{8.15}$$

where c_d is the standard coefficient, $c_d = (2\pi)^{-d}\omega_d$ and ω_d is the volume, resp. area, of the unit ball, resp. disk, in $\mathbb{R}^d$.

Note that, similar to the nonmagnetic case, the asymptotic formula in dimension $d = 2$ requires some additional restrictions compared with just finiteness of the asymptotic coefficient in (8.15).

To prove the asymptotic formula, one has just to split, for given ϵ, the potential V into V_ϵ, supported away from the solenoids, and V'_ϵ, $\Sigma(V'_\epsilon) < \epsilon$. For the operator with potential V_ϵ one finds the asymptotics, for example, by means of Dirichlet-Neumann bracketing, and for the operator with potential V'_ϵ the results of this section give estimate with a small coefficient. The matter is completed by applying the asymptotic perturbation lemma from [7], see also the expositions in [27, 29].

Acknowledgment

M.M. thanks Professor J. Brasche for enlightening discussions and G.R. is grateful to Professor M. Solomyak for useful consultations. Moreover, the authors thank the Referee for constructive remarks and suggestions.

References

[1] R. Adami, A. Teta, On the Aharonov-Bohm Hamiltonian, *Lett. Math. Phys.* **43**, no. 1, 43–53 (1998).

[2] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic fields. I. General interactions. *Duke Math. J.* **45**, no. 4, 847–883 (1978).

[3] A.A. Balinsky, Hardy type inequality for Aharonov-Bohm magnetic potentials with multiple singularities, *Math. Res. Lett.* **10**, no. 2–3, 169–176 (2003).

[4] A.A. Balinsky, W.D. Evans, R.T. Lewis, On the number of negative eigenvalues of Schrödinger operators with an Aharonov-Bohm magnetic field, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457**, no. 2014, 2481–2489 (2001).

[5] S.J. Bending, K. von Klitzing, K. Ploog, Weak localization in a distribution of magnetic flux tubes *Phys. Rev. Lett.* **65**, 1060–1063 (1990).

[6] M.Sh. Birman, A. Laptev, M. Solomyak, The negative discrete spectrum of the operator $(-\Delta)^l - \alpha V$ in $L_2(\mathbb{R}^d)$ for $2l \geq d$, d even, *Ark. Mat.* **35**, 87–126 (1997).

[7] M.Sh. Birman, M. Solomyak, The principal term of the spectral asymptotics for "non-smooth" elliptic problems, *(Russian) Funkcional. Anal. i Priložen* **4**, no. 4, 1–13 (1970). Translated in *Functional Anal. Appl.* **4**, 265–275 (1970).

[8] M.Sh. Birman, M. Solomyak, Negative discrete spectrum of the Schrödinger operator with large coupling constant: a qualitative discussion; in *Order, disorder and chaos in quantum systems* (Dubna, 1989), pp. 3–16, Oper. Theory Adv. Appl., **46**, Birkhäuser, Basel, 1990.

[9] M.Sh. Birman, M. Solomyak, Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations, in: *Estimates and Asymptotics of Discrete Spectra*. Advances in Soviet Mathematics, **7**, Amer. Math. Soc., 1991.

[10] L. Dąbrowski, P. Šťovíček, Aharonov-Bohm effect with δ-type interaction, *J. Math. Phys.* **39**, no. 1, 47–62 (1998).

[11] A.K. Geim, V.I. Falko, S.V. Dubonos, I.V. Grigorieva, Single magnetic flux tube in a mesoscopic two-dimensional electron gas conductor, *Solid State Commun.* **82**, no. 10, 831–836 (1992).

[12] V.A. Geyler, E.N. Grishanov, Zero modes in a periodic system of Aharonov-Bohm solenoids, *JETP Letters* **75**, 354–356 (2002).

[13] H. Hess, R. Schrader, D.A. Uhlenbrock, Domination of semigroups and generalization of Kato's inequality, *Duke Math. J.* **44**, no. 4, 893–904 (1977).

[14] D. Hundertmark, A. Laptev, T. Weidl, New bounds on the Lieb-Thirring constants, *Invent. Math.* **140**, no. 3, 693–704 (2000).

[15] H.T. Ito, H. Tamura, Aharonov-Bohm effect in scattering by point-like magnetic fields at large separation, *Ann. Henri Poincaré* **2**, no. 2, 309–359 (2001).

[16] A. Laptev, Yu. Netrusov, On the negative eigenvalues of a class of Schrödinger operators; in *Differential operators and spectral theory*, pp. 173–186, Amer. Math. Soc. Transl. Ser. 2 **189**, Amer. Math. Soc., Providence, RI, 1999.

[17] A. Laptev, T. Weidl, Hardy inequalities for magnetic Dirichlet forms; in *Mathematical Results in Quantum Mechanics* (Prague, 1998), pp. 299–305, Oper. Theory Adv. Appl., **108**, Birkhäuser, Basel, 1999.

[18] A. Laptev, T. Weidl, Sharp Lieb-Thirring inequalities in high dimensions, *Acta Math.* **184**, 87–111 (2000).

[19] A. Laptev, private communication.

[20] E.H. Lieb, W. Thirring, Bound for the kinetic energy of fermions which proves the stability of matter, *Phys. Rev. Lett.* **35**, 687–689 (1975).

[21] E.H. Lieb, Kinetic energy bounds and their application to the stability of matter, in *Schrödinger operators*, pp. 371–382, Lecture Notes in Phys. **345**, Springer, Berlin, 1989.

[22] V. Liskevich, A. Manavi, Dominated semigroups with singular complex potentials, *J. Funct. Anal.* **151**, no. 2, 281–305 (1997).

[23] M. Melgaard, G. Rozenblum, Spectral estimates for magnetic operators, *Math. Scand.* **79**, no. 2, 237–254 (1996).

[24] E.-M. Ouhabaz, L^∞-contractivity of semigroups generated by sectorial forms *J. London Math. Soc.* (2) **46**, no. 3, 529–542 (1992).

[25] E.-M. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups, *Potential Anal.* **5**, no. 6, 611–625 (1996).

[26] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992.

[27] M. Reed, B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press, New York-London, 1978.

[28] G. Rozenblum, M. Solomyak, The Cwikel-Lieb-Rozenblum estimator for generators of positive semigroups and semigroups dominated by positive semigroups, *(Russian) Algebra i Analiz* **9**, no. 6, 214–236 (1997); translation in *St. Petersburg Math. J.* **9**, no. 6, 1195–1211 (1998).

[29] G. Rozenblum, M. Solomyak, On the number of negative eigenvalues for the two-dimensional magnetic Schrödinger operator, in *Differential operators and spectral theory*, pp. 205–217, Amer. Math. Soc. Transl. Ser. 2, **189**, Amer. Math. Soc., Providence, RI, 1999.

[30] G. Rozenblum, Domination of semigroups and estimates for eigenvalues, *(Russian) Algebra i Analiz* **12**, no. 5, 158–177 (2000); translation in *St. Petersburg Math. J.* **12**, no. 5, 831–845 (2001).

[31] S.N.M. Ruijsenaars, The Aharonov-Bohm effect and scattering theory, *Ann. Physics* **146**, no. 1, 1–34 (1983).

[32] B. Simon, A canonical decomposition for quadratic forms with applications to monotone convergence theorems, *J. Funct. Anal.* **28**, no. 3, 377–385 (1978).

[33] B. Simon, *Functional integration and quantum physics*, Academic Press, New York-London, 1979.

[34] M. Solomyak, Piecewise-polynomial approximation of functions from $H^l((0,1)^d)$, $2l = d$, and applications to the spectral theory of the Schrödinger operator, *Israel J. Math.* **86**, no. 1–3, 253–275 (1994).

[35] H. Tamura, Norm resolvent convergence to magnetic Schrödinger operators with point interactions, *Rev. Math. Phys.* **13**, no. 4, 465–511 (2001).

M. Melgaard
Department of Mathematics
Uppsala University
Polacksbacken
S-751 06 Uppsala, Sweden
email: melgaard@math.uu.se

E.-M. Ouhabaz
Laboratoire Bordelais d'Analyse et Géométrie
Université de Bordeaux 1
351, Cours de la Libération
F-33405 Talence cedex, France
email: ouhabaz@math.u-bordeaux.fr

G. Rozenblum
Department of Mathematics
Chalmers University of Technology
and University of Gothenburg
Eklandagatan 86
S-412 96 Gothenburg, Sweden
email: grigori@math.chalmers.se

Communicated by Bernard Helffer
submitted 02/12/03, accepted 12/03/04