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Many Body Physics: Unfinished Revolution

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Abstract. The study of many body physics has provided a scientific playground of surprise and continuing revolution over the past half century. The serendipitous discovery of new states and properties of matter, phenomena such as superfluidity, the Meissner, the Kondo and the fractional quantum hall effect, have driven the development of new conceptual frameworks for our understanding about collective behavior, the ramifications of which have spread far beyond the confines of terrestrial condensed matter physics – to cosmology, nuclear and particle physics. Here I shall selectively review some of the developments in this field, from the cold-war period, until the present day. I describe how, with the discovery of new classes of collective order, the unfolding puzzles of high temperature superconductivity and quantum criticality, the prospects for major conceptual discoveries remain as bright today as they were more than half a century ago.

1 Emergent Matter: a new Frontier

Since the time of the Greeks, scholars have pondered the principles that govern the universe on its tiniest and most vast scales. The icons that exemplify these frontiers are very well known – the swirling galaxy denoting the cosmos and the massive accelerators used to probe matter at successively smaller scales – from the atom down to the quark and beyond. These traditional frontiers of physics are concerned with reductionism: the notion that once we know the laws of nature that operate on the smallest possible scales, the mysteries of the universe will finally be revealed to us [1].

Over the last century and a half, in a period that stretches back to Darwin and Boltzmann – scientists have also become fascinated by another notion: the idea that to understand nature, one also needs to understand and study the principles that govern collective behavior of vast assemblies of matter. For a wide range of purposes, we already know the microscopic laws that govern matter on the tiniest scales. For example, a gold atom can be completely understood with the Schroedinger equation and the laws of quantum mechanics established more than seventy years ago. Yet, a gold atom is spherical and featureless – quite unlike the lustrous malleable and conducting metal which human society so prizes. To understand how crystalline assemblies of gold atoms acquire the properties of metallic gold, we need new principles – principles that describe the collective behavior of matter when humungous numbers of gold atoms congregate to form a metallic crystal. It is the search for these new principles that defines the frontiers

of many body physics in the realms of condensed matter physics and its closely related discipline of statistical mechanics.

In this informal article, I shall talk about the evolution of our ideas about the collective behavior of matter since the advent of quantum mechanics, hoping to give a sense of how often unexpected experimental discovery has seeded the growth of conceptually new ideas about collective matter. Given the brevity of the article, I must apologize for the necessarily selective nature of this discussion.

The past seventy years of development in the many body physics has seen a period of major unexpected conceptual and intellectual developments. Experimental discoveries of remarkable new phenomena, such as superconductivity, superfluidity, criticality, high temperature superconductivity, and the quantized Hall effect, have each prompted a renaissance in areas once thought to be closed to further fruitful intellectual study. Indeed, the history of the field is marked by the most wonderful and unexpected shifts in perspective and understanding that have involved a close linkage between experiment, new mathematics and new concepts.

I shall discuss three eras – the immediate aftermath of quantum mechanics – many body physics in the cold war – and the modern era of correlated matter physics. Over this period, the physicists view of the matter has evolved dramatically – witness the evolution in our view of "electricity" from the idea of the degenerate electron gas, to the concept of the Fermi liquid, to new kinds of electron fluid, such as a the Luttinger liquid or fractional quantum Hall state. Progress was not smooth and gradual, but often involved the agony, despair and controversy of the creative process. Even the notion that an electron is a fermion was controversial. Wolfgang Pauli – inventor of the the exclusion principle could not initially envisage that this principle would apply beyond the atom, to macroscopically vast assemblies of degenerate electrons: indeed, he initially preferred the idea that electrons were bosons. Pauli arrived at the realization that the electron fluid is a degenerate Fermi gas with great reluctance, and at the end of 1925 [2] gave way, writing in a short note to Schroedinger that

> *"With a heavy heart, I have decided that Fermi Dirac, not Einstein is the correct statistics, and I have decided to write a short note on paramagnetism."*

Wolfgang Pauli, letter to Schrödinger, November 1925 [2].

2   Unsolved riddles of the 1930s

The period of condensed matter physics between the two world-wards was characterized by a long list of unsolved mysteries in the area of magnetism and superconductivity [3]. Ferromagnetism had emerged as a shining triumph of the application of quantum mechanics to condensed matter. So rapid was the progress in this direction, that Néel and Landau quickly went on to generalize the idea, predicting the possibility of staggered magnetism, or antiferromagnetism in 1933 [4].
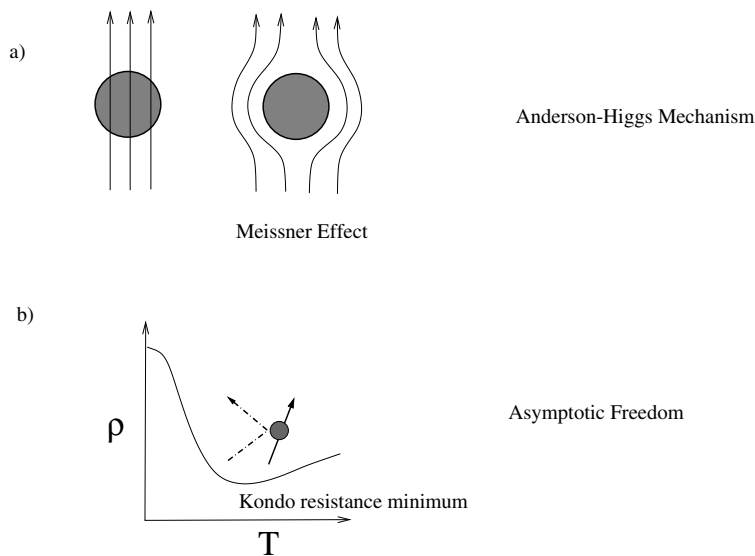
Figure 1: Two mysteries of the early era, whose ultimate resolution 30 years later linked them to profound new concepts about nature. (a) The Meissner effect, whose ultimate resolution led to an understanding of superconductivity and the discovery of the Anderson-Higgs mechanism, (b) The Kondo resistance minimum, which is linked to the physics of confinement.

In a situation with many parallels today, the experimental tools required to realize the predicted phenomenon, had to await two decades, for the development of neutron diffraction [5]. During this period, Landau became pessimistic and came to the conclusion that quantum fluctuations would most probably destroy antiferromagnetism, as they do in the antiferromagnetic 1D Bethe chain – encouraging one of his students, Pomeranchuk, to explore the idea that spin systems behave as neutral fluids of fermions [6].

By contrast, superconductivity remained unyielding to the efforts of the finest minds in quantum mechanics during the heady early days of quantum mechanics in the 1920s, a failure derived in part from a deadly early misconception about superconductivity [3]. It was not until 1933 that a missing elements in the puzzle came to light, with the Meissner and Ochensfeld discovery that superconductors are not perfect resistors, but perfect diamagnets. [7] It is this key discovery that led the London brothers [8] to link superconductivity to a concept of "rigidity" in the many body electron wavefunction, a notion that Landau and Ginzburg were to later incorporate in their order parameter treatment of superconductivity [9].

Another experimental mystery of the 1930's, was the observation of a mysterious "resistance minimum" in the temperature dependent resistance of copper, gold, silver and other metals [10]. It took 25 more years for the community to

link this pervasive phenomenon with tiny concentrations of atomic size magnetic impurities – and another 15 more years to solve the phenomenon – now known as the Kondo effect – using the concepts of renormalization and asymptotic freedom.

3 Many Body Physics in the Cold War

3.1 Physics without Feynman diagrams

Many body physics blossomed after the end of the second world war, and as the political walls between the east and west grew with the beginning of the cold war, a most wonderful period of scientific and conceptual development, with a frequent exchange of new ideas across the iron curtain, came into being. Surprisingly, the Feynman diagram did not really enter many body physics until the early 60s, yet without Feynman diagrams, the many body community made a sequence of astonishing advances in the 1950's [11].
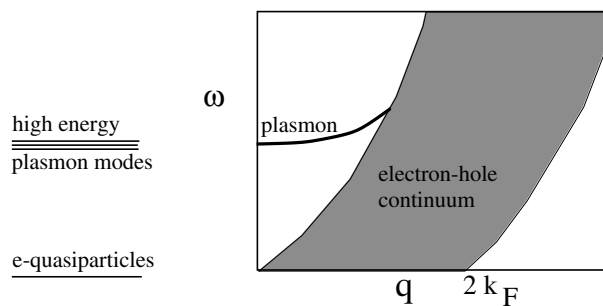

Figure 2: Illustrating the Pines-Bohm idea, that the physics of the electron fluid can be divided up into high energy collective "plasmon modes" and low energy electron quasiparticles.

The early 1950s saw the first appreciation by the community of the importance of collective modes. One of the great mysteries was the reason why the non-interacting Sommerfeld model of the electron fluid worked so well, despite interactions that are comparable to the kinetic energy of the electrons. In a landmark early paper, David Bohm and his graduate student, David Pines [12] realized that they could separate the strongly interacting gas via a unitary transformation into two well-separated sets of excitations – a high energy collective oscillations of the electron gas, called plasmons, and low energy electrons. The Pines-Bohm paper is a progenitor of the idea of renormalization: the idea that high energy modes of the system can be successively eliminated, to give rise to a renormalized picture of the residual low energy excitations.

Feynman diagrams entered many body physics in the late 1950s [11]. The first applications of the formalism of quantum field theory to many body physics to bulk electronic matter, made by Brueckner [13,] were closely followed by Goldstone and Hubbard's elegant re-derivations of the method using Feynman dia-

grams [14, 15]. A flurry of activity followed: Gell-Mann and Brueckner used the newly discovered "linked cluster theorem" to calculate the correlation energy of the high density electron gas [16], and Galitskii and Migdal [17, 18] in the USSR applied the methods to the spectrum of the interacting electron gas. Around the same time, Edwards [19] made the first applications of Feynman's methods to the problem of elastic scattering off disorder.

One of the great theoretical leaps of this early period was the invention of the concept of imaginary time. The earliest published discussion of this idea occurs in the papers of Matsubara [20]. Matsubara noted the remarkable similarity between the time evolution operator of quantum mechanics

$$U(t) = e^{-itH/\hbar} \tag{1}$$

and the Boltzmann density matrix

$$\rho(\beta) = e^{-\beta H} = U(-i\hbar\beta), \tag{2}$$

where $\beta = 1/(k_B T)$ and k_B is Boltzmann's constant. This parallel suggested that one could convert conventional quantum mechanics into finite temperature quantum statistical mechanics by using a time-evolution operator where real time is replaced by imaginary time,

$$t \to -i\tau\hbar. \tag{3}$$

Matsubara's ideas took a further leap into the realm of the practical, when Abrikosov, Gorkov and Dzyaloshinski (AGD) [21] who showed that the method was dramatically simplified by Fourier transforming the imaginary time electron Green function into the frequency domain. They noted for the first time that the antiperiodicity of the Green function $G(\tau + \beta) = -G(\tau)$ meant that the continuous frequencies of zero temperature physics are replaced by the discrete frequencies $\omega_n = (2n + 1)\pi T$, that we now call the "Matsubara frequency". In their paper, the finite temperature propagator

$$G(\omega_n, \vec{p}) = [i\omega_n + \mu - \epsilon(\vec{p})]^{-1} \tag{4}$$

for the electron makes its first appearance.

Another great conceptual leap of the early cold war, was Landau's development of the Quasiparticle excitation concept. Landau [22], stimulated by early measurements on liquid $He - 3$, realized that interacting fermi gases could be understood with the concept of "adiabaticity"- the notion that when interactions are turned on adiabatically, the original single-particle excitations of the Fermi liquid, evolve without changing their charge or spin quantum numbers, into "quasiparticle" excitations of the interacting system. Landau's Fermi liquid theory is the foundation for the modern "standard model" of the electron fluid.

3.2   Broken Symmetry

Two monumental achievements of the cold-war era deserve separate mention: the discovery of "broken symmetry" and the renormalization group. In 1937, Landau [23] formulated the concept of broken symmetry – proposing that phase transitions take place via the process of symmetry reduction, which he described in terms of his order parameter concept. In the early fifties, Onsager and Penrose [24], refined Landau's concept of broken symmetry to propose that a superfluidity could be understood as a state of matter in which the two-particle density matrix

$$\rho(r, r') = \langle \hat{\psi}^\dagger(r)\hat{\psi}(r') \rangle \tag{5}$$

can be factorized:

$$\rho(r, r') = \psi^*(r)\psi(r) + \text{small terms} \tag{6}$$

where

$$\psi(r) = \sqrt{\rho_s}e^{i\phi} = \langle N - 1|\hat{\psi}(r)|N\rangle. \tag{7}$$

is the order parameter of the superfluid, ρ_s is the superfluid density and ϕ the phase of the condensate, This concept of "off-diagonal long-range order" soon became generalized to fermi systems as part of the BCS theory of superconductivity [25, 26], where the off-diagonal order parameter

$$F(x - x') = \langle N - 2|\hat{\psi}_\downarrow(x)\hat{\psi}_\uparrow(x')|N\rangle, \tag{8}$$

defines the wavefunction of the Cooper pair.

One of the remarkable spin-offs of superconductivity, was that it led to an understanding of how a gauge boson can acquire a mass as a result of symmetry breaking. This idea was first discussed by Anderson in 1959 [27], and in more detail in 1964 [28, 29], but the concept evolved further and spread from Bell Laboratories to the particle physics community, ultimately re-appearing as the Higg's mechanism for spontaneous symmetry breaking in a Yang Mills theory. The Anderson-Higgs mechanism is a beautiful example of how the study of cryogenics led to a fundamentally new way of viewing the universe, providing a mechanism for the symmetry breaking between the electrical and weak forces in nature.

Another consequence of broken symmetry concept is the notion of "generalized rigidity" [30], a concept which has its origins in London's early model of superfluidity, according to which, if the phase of a boson or Cooper pair develops a rigidity, then it costs a phase bending energy

$$U(x) \sim \frac{1}{2}\rho_s(\nabla\phi(x))^2, \tag{9}$$

from which we derive that the "superflow" of particles is directly proportional to amount of phase bending, or the gradient of the phase

$$j_\text{S} = \rho_s\nabla\phi. \tag{10}$$

Anderson noted [30] that we can generalize this concept to a wide variety of broken symmetries, each with their own type of superflow (see table 1). Thus broken translation symmetry leads to the superflow of momentum, or sheer stress, broken spin symmetry leads to the superflow of spin or spin superflow. There are undoubtedly new classes of broken symmetry yet to be discovered.

Table 1. Order parameters, broken symmetry and rigidity

Name	Broken Symmetry	Rigidity/Supercurrent
Crystal	Translation Symmetry	Momentum superflow (Sheer stress)
Superfluid	Gauge symmetry	Matter superflow
Superconductivity	E.M. Gauge symmetry	Charge superflow
Antiferromagnetism	Spin rotation symmetry	Spin superflow (x-y magnets only)
?	Time Translation Symmetry	Energy superflow ?

3.3   Renormalization group

The theory of second order phase transitions was studied by Van der Waals in the 19th century, and thought to be a closed field. Two events – the experimental observation of critical exponents that did not fit the predictions of mean-field theory [31, 32], and the solution to the 2D Ising model [33], forced condensed matter physicists to revisit an area once thought to be closed. The revolution that ensued literally shook physics from end to end, furnishing us with a spectrum of new concepts and terms, such as

- scaling theory [34, 35, 36],

- universality – the idea that the essential long-distance physics is independent of all but a handful of short-distance details, such as the dimensionality of space and the symmetry of the order parameter.

- renormalization – the process by which short-distance, high energy physics is absorbed by adjusting the parameters inside the Lagrangian or Hamiltonian.

- fixed points – the limiting form of the Lagrangian or Hamiltonian as short-distance, high energy physics is removed

- running coupling constant,

- cross-over,

- upper critical dimensionality – the dimension above which mean-field theory is valid.

that appeared as part of the new "renormalization group" [37, 38, 39, 40]. The understanding of classical phase transitions required the remarkable fusion of universality, together with the new concepts of scaling, renormalization and the application of tools borrowed from quantum field theory. These developments are a main stay of modern theoretical physics, and their influence is felt far outside the realms of condensed matter.

One of the unexpected dividends of the renormalization group concept, in the realm of many body theory, was the solution of the Kondo effect: the condensed matter analog of quark confinement. By the late fifties, the resistance minima in copper, gold and silver alloys that had been observed since the 1930s [10], had been identified with magnetic impurities, but the mechanism for the minimum was still unknown. In the early 60's, Jun Kondo [41] was able to identify this resistance minimum, as a consequence of antiferromagnetic interactions between the local moments and the surrounding electron gas. The key ingredient in the Kondo model, is an antiferromagnetic interaction between a local moment and the conduction sea, denoted by

$$H_I = J\vec{\sigma}(0) \cdot \vec{S} \tag{11}$$

$\vec{S}$ is a spin $1/2$ and $\vec{\sigma}(0)$ is the spin density of the conduction electrons at the origin. Kondo [41] found that when he calculated the scattering rate of electrons off a magnetic moment to one order higher than Born approximation,

$$\frac{1}{\tau} \propto [J\rho + 2(J\rho)^2 \ln \frac{D}{T}]^2, \tag{12}$$

where ρ is the density of state of electrons in the conduction sea and D is the width of the electron band. As the temperature is lowered, the logarithmic term grows, and the scattering rate and resistivity ultimately rises, connecting the resistance minimum with the antiferromagnetic interaction between spins and their surroundings.

A deeper understanding of this logarithm required the renormalization group concept [42, 40, 43]. By systematically taking the effects of of high frequency virtual spin fluctuations into account, it became clear that the bare coupling J is replaced by a renormalized quantity

$$J\rho(\Lambda) = J\rho + 2(J\rho)^2 \ln \frac{D}{\Lambda} \tag{13}$$

that depends on the scale Λ of the cutoff, so that the scattering rate is merely given by $1/\tau \propto (\rho J(\Lambda))^2|_{\Lambda \sim T}$. The corresponding renormalization equation

$$\frac{\partial J\rho}{\partial \ln \Lambda} = \beta(J\rho) = -2(J\rho)^2 + O(J^3) \tag{14}$$

contains a "negative β function": the hallmark of a coupling which dies away at high energies (asymptotic freedom), but which grows at low energies, ultimately reaching a value of order unity when the characteristic cut-off is reduced to the scale of the so called "Kondo temperature" $T_K \sim De^{-1/J}$.

The "Kondo" effect is a manifestation of the phenomenon of "asymptotic freedom" that also governs quark physics. Like the quark, at high energies the local moments inside metals are asymptotically free, but at energies below the Kondo temperature, they interact so strongly with the surrounding electrons that they become screened or "confined" at low energies, ultimately forming a Landau Fermi liquid [43]. It is a remarkable that the latent physics of confinement, hiding within cryostats in the guise of the Kondo resistance minimum, remained a mystery for more than 40 years, pending purer materials, the concept of local moments and the discovery of the renormalization group.

3.4   The concept of Emergence

The end of the cold-war period in many body physics is marked by Anderson's statement of the concept of emergence. In a short paper, originally presented as part of a Regent's lecture entitled "More is different" at San Diego in the early seventies [44], Anderson defined the concept of emergence with the now famous quote

> "at each new level of complexity, entirely new properties appear, and the understanding of these behaviors requires research which I think is as fundamental in its nature as any other."

Anderson's quote underpins a modern attitude to condensed matter physics – the notion that the study of the collective principles that govern matter is a frontier unto itself, complimentary, yet separate to those of cosmology, particle physics and biology.

4   Condensed Matter Physics in the New Era

4.1   New States of Matter

Towards the end of the 1970's few condensed matter physicists had internalized the consequences of emergence. In the early eighties, condensed matter physicists were by enlarge, content with a comfortable notion that they had outlined the principle constraints on the behavior and possible ground-states of dense matter. Superconductivity was widely believed to be limited below about $25K$ [45].

The "vacuum" state of metallic behavior was firmly believed to be the Landau Fermi liquid, and no significant departures were envisaged outside the realm of one-dimensional conductors. Tiny amounts of magnetic impurities were known to be anathema to superconductivity. These principles were so entrenched in the community that the first observation [46] of heavy electron superconductivity in the magnetic metal UBe_{13} was mis-identified as an artifact, delaying acceptance of this phenomenon by another decade. By the end of the 80's all of these popularly held principles had been exploded by an unexpected sequence of discoveries, in the areas of heavy electron physics, the quantum Hall effect and the discovery of high temperature superconductivity.
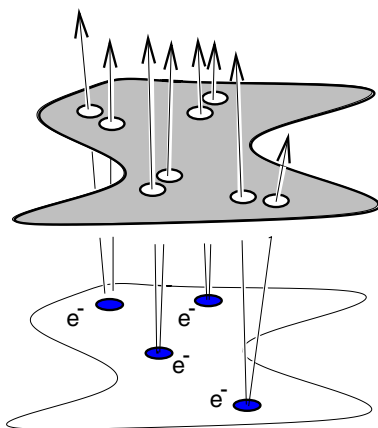

Figure 3: Illustrating the binding of two vortices to each electron, to form the $\nu = 1/3$ Laughlin ground-state.

4.1.1 Fractional Quantum Hall Effect

In the 1930's Landau had discussed the quantum mechanics of electron motion in a magnetic field [47], predicting the quantization of electron kinetic energy into discrete Landau levels

$$\frac{\hbar^2(k_x^2 + k_y^2)}{2m} \to \frac{\hbar eB}{m}(n + \frac{1}{2}), \qquad (n = 0,\ 1,\ 2\dots). \qquad (15)$$

Landau quantization had been confirmed in metals, where it produces oscillations in the field-dependent resistivity (Shubnikov de Haas oscillations) and magnetization (de Haas van Alphen oscillations), and the field was thought mature. In the seventies, advances in semiconductor technology and the availability of high magnetic fields, made it possible examine two dimensional electron fluids at high fields, when the spacing of the Landau levels is so large that the electrons drop into the lowest Landau level, so that their dynamics is entirely dominated by mutual

Coulomb interactions. Remarkably, the Hall constant of these electron fluids was found to be quantized with values

$$R_H = \frac{V}{I} = \frac{h}{\nu e^2}, \tag{16}$$

where at lower fields, $\nu = 1, 2, 3 \ldots$ is an integer, but at higher fields, ν acquires a fractionally quantized values $\nu = 1/3,\ 1/5,\ 1/7 \ldots$. Laughlin [48] showed that the fractional quantum Hall effect is produced by interactions, which stabilize a new type of electron fluid where the Landau level has fractionally filling factor $\nu = 1/(2M+1)$. In Laughlin's approach, the electron fluid is pierced by "vortices" which identify zero's in the electron wavefunction. Laughlin proposed that electrons bind to these vortices to avoid other electrons, and he incorporated this physics into his celebrated wavefunction by attaching each electron to an even number $2M$ of vortices.

$$\Psi(\{z_i\}) = \prod_{i>j}(z_i - z_j)^{2M+1} \exp\left[\sum_i |z_i|^2/4l_o^2\right] \tag{17}$$

where $l_o = \sqrt{\hbar/eB}$ is the magnetic length. The excitations in this state are gapped, with both fractional charge and fractional statistics: an entirely new electronic ground state. Moreover, the wavefunction is robust against the details of the Hamiltonian from which it is derived.

This break-through opened an entire field of investigation [49, 50] into the new world of highly correlated electron physics, bringing a whole range of new concepts and language, such as

- fractional statistics quasiparticles –

- composite fermions –

- Chern-Simons terms.

Equally importantly, the fractional quantum Hall effect made the community poignantly aware of the profound transformations that become possible in electronic matter when the strength of interactions becomes comparable to, or greater than the kinetic energy.

4.1.2 Heavy Electron Physics

The discovery of heavy electron materials in the late seventies [51, 52] forced condensed matter physicists to severely revise their understanding about how local moments interact with the electron fluid. In the late seventies, electron behavior in metals was neatly categorized into

1. "delocalized" behavior, where electrons form Bloch waves, and

2. "localized" behavior, where the electrons in question are bound near a particular atom in the material. Such unpaired spins form tiny atomic magnets

called "local moments" that tend to align at low temperatures and are extremely damaging to superconductivity.

Heavy fermion metals completely defy these norms, for they contain a dense array of magnetic moments, yet instead of magnetically ordering the moments develop a highly correlated paramagnetic ground-state with the conduction electrons. When this happens, the resistivity of the metal drops abruptly, form a highly correlated Landau Fermi liquid in which electron masses rise in excess of 100 times the bare electron mass [53].

Heavy electron physics is, in essence the direct descendant of the resistance minimum physics first observed in simple metals in the early 1930's. Our current understanding of heavy fermions is based on the notion, due to Doniach [54], that the "Kondo effect" seen for individual magnetic moments, survives inside the dense magnetic arrays of in heavy fermion compounds to produce the heavy fermion state. The heavy electrons that propagate in these materials are really the direct analogs of nucleons formed from confined quarks. Curiously, one of the most useful theoretical methods for describing these systems was borrowed from particle physics. Heavy electrons are formed in f-orbitals which are spin-orbit coupled with a large spin degeneracy $N = 2j + 1$. One of the most useful methods for developing a mean-field description of the heavy electron metal is the $1/N$ expansion, inspired by analogies with the $1/N$ expansion in the number of colors in Quantum Chromodynamics [55, 56, 57, 58].

4.1.3 High Temperature Superconductivity

The discovery of high temperature superconductivity, with transition temperatures that have spiraled way above the theoretically predicted maximum possible transition temperatures, to its current maximum of $165K$, stunned the physics community. These systems are formed by adding charge to an insulating state where electrons are localized in an antiferromagnetic array. Several aspects of these materials radically challenge our understanding of correlated electron systems, in particular:

- The close vicinity between insulating and superconducting behavior in the phase diagram, which suggests that the insulator and superconductor may derive from closely related ground-state wavefunctions [59, 60].

- The "strange metal" behavior of the optimally doped materials. Many properties of this state tell us that it is not a Landau Fermi liquid, such as the linear resistivity

$$\rho = \rho_o + AT \tag{18}$$

extending from the transition temperature, up to the melting temperature. This linear resistivity is known to originate in an electron-electron scattering rate $\Gamma(T) \sim k_B T$, that grows linearly with temperature, which has been

called a "marginal Fermi liquid" [61]. In conventional metals, the inelastic scattering rate grows quadratically with temperature. Despite 15 years of effort, the origin of the linearity of the scattering rate remains a mystery.

- The origin of the growth of a pseudogap in the electron spectrum for "under-doped" superconductors. This soft gap in the excitation spectrum signals the growth of correlations amongst the electrons prior to superconductivity, and some believe that it signals the formation of pairs, without coherence [62].

The radical simplicity of many of the properties of the cuprate superconductors leads many to believe that their ultimate solution will require a conceptually new description of the interacting electron fluid.
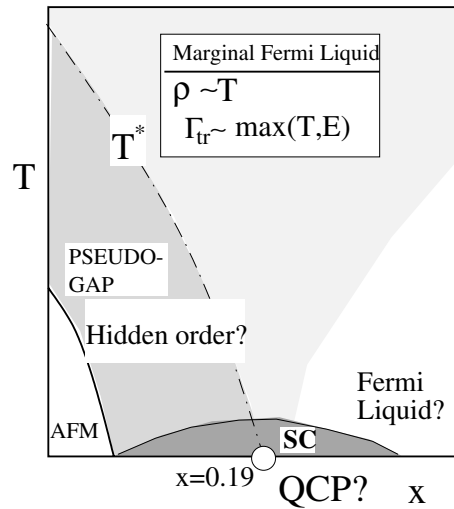

Figure 4: Schematic phase diagram for cuprate superconductors, where x is doping and T the temperature, showing the location of a possible quantum critical point.

The qualitative phase diagram is shown in Fig(4), showing three distinct regions – the over-doped region, the fan of "marginal Fermi liquid behavior" and the under-doped region. The theoretical study of this phase diagram has proven to be a huge engine for new ideas, such as

- Spin charge separation- the notion that the spin-charge coupled electron breaks up into independent collective charge and spin excitations, as in one dimensional fluids.

- Hidden order – the notion that the pseudo-gap is a consequence of the formation of an as-yet unidentified order parameter, such as orbital magnetism (d-density waves) [63, 64] or stripes [65].

- Quantum criticality – the notion that the strange-metal phase of the cuprates is a consequence of a "quantum critical point" around a critical doping of about $x_c \sim 0.2$ [66] In this scenario, the pseudo-gap is associated with the growth of "hidden order" and marginal Fermi liquid behavior is associated with the quantum fluctuations emanating from the quantum critical point.

- Pre-formed pairs – the idea that the under-doped pseudo-gap region of the phase diagram is a consequence of the formation of phase-incoherent pairs which form at the pseudo-gap temperature [62].

- Resonating Valence Bonds – the idea that superconductivity can be regarded as a fluid of spinless charged holes, moving in a background of singlet spin pairs. [59]

- New forms of gauge theory, including Z_2 [67], $SU(2)$ [68] and even super-symmetric gauge theories [69] that may describe the manifold of states that is highly constrained by the strong coulomb interactions between electrons in the doped Mott insulator.

Many of these ideas enjoy some particular realization in non-cuprate materials, and in this way, cuprate superconductivity has stimulated a huge growth of new concepts and ideas in many body physics.

4.2   Quantum Criticality

The concept of quantum criticality: the idea that a zero temperature phase transition will exhibit critical order parameter fluctuations in both space and time, was first introduced by John Hertz during the hey-days of interest in critical phenomena, but was regarded as an intellectual curiosity. [70] Discoveries over the past decade and a half have revealed the ability of zero-temperature quantum phase transitions to qualitatively transform the properties of a material at finite temperatures. For example, high temperature superconductivity is thought to be born from a new metallic state that develops at a certain critical doping in copper-perovskite materials. [66] Near a quantum phase transition, a material enters a weird state of "quantum criticality": a new state of matter where the wavefunction becomes a fluctuating entangled mixture of the ordered, and disordered state. The physics that governs this new quantum state of matter represents a major unsolved challenge to our understanding of correlated matter.

   A quantum critical point (QCP) is a singularity in the phase diagram: a point $x = x_c$ at zero-temperature where the characteristic energy scale $k_B T_o(x)$ of excitations above the ground-state goes to zero. (Fig. 1.). [71, 72, 73, 74, 75, 76] The QCP affects the broad wedge of phase diagram where $T > T_o(x)$. In this region of the material phase diagram, the critical quantum fluctuations are cut-off by thermal fluctuations after a correlation time given by the Heisenberg uncertainly

principle

$$\tau \sim \frac{\hbar}{k_B T}. \tag{19}$$

As a material is cooled towards a quantum critical point, the physics probes the critical quantum fluctuations on longer and longer time-scales. Although the "quantum critical" region of the phase diagram where $T > T_o(x)$ is not a strict phase, the absence of any scale to the excitations other than temperature itself qualitatively transforms the properties of the material in a fashion that we would normally associate with a new phase of matter.
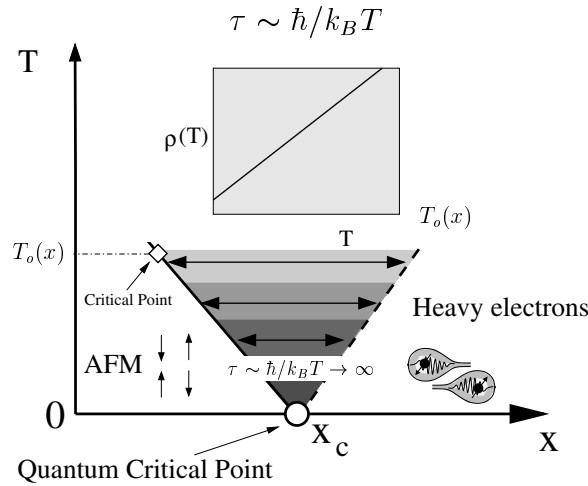
Figure 5: Quantum criticality in heavy electron systems. For $x < x_c$ spins become ordered for $T < T_o(x)$ forming an antiferromagnetic Fermi liquid ; for $x > x_c$, composite bound-states form between spins and electrons at $T < T_0(x)$ producing a heavy Fermi liquid. "Non-Fermi liquid behavior", in which the characteristic energy scale is temperature itself, and resistivity is quasi-linear, develops in the wedge shaped region between these two phases. The nature of the critical Lagrangian governing behavior at x_c is currently a mystery.

Quantum criticality has been extensively studied in heavy electron materials, in which the antiferromagnetic phase transition temperature can be tuned to zero by the application of a pressure, field or chemical doping. Close to quantum criticality, these materials exhibit a number of tantalizing similarities with the cuprate superconductors [76]:

- a predisposition to form anisotropic superconductors,

- the formation of a strange metal with quasi-linear resistivity in the critical region

- the appearance of temperature as the only scale in the electron excitation spectrum at criticality, reminiscent of "marginal Fermi liquid behavior"

Hertz proposed that quantum criticality could be understood by extending classical criticality to order parameter fluctuations in imaginary time, using a Landau Ginzburg functional that includes the effects of dissipation:

$$F = \int_0^{1/T} d\tau \int d^d x \left\{ |(\nabla + i\mathbf{Q}_o)\psi|^2 + \xi^{-2}|\psi|^2 + U|\psi|^4 \right\} + F_D \qquad (20)$$

where $\mathbf{Q}_o$ is the ordering vector of the antiferromagnet, ξ the correlation length which vanishes at the QCP and

$$F_D = \sum_{i\omega_n} \int \frac{d^3q}{(2\pi)^3} |\psi(q,\omega_n)|^2 \frac{|\omega_n|}{\Gamma_{\mathbf{q}}} \qquad (21)$$

is a linear damping rate derived from the density of particle-hole excitations in the Fermi sea. An important feature of this "ϕ^4" Lagrangian is that the momentum dependence enters with twice the power of the frequency dependence, the time dimension counts as $z = 2$ space dimensions, and the effective dimensionality of the theory is

$$D = d + z = d + 2, \qquad (22)$$

so that $D = 5$ for the three dimensional model, pushing it above its upper critical dimension.

In heavy electron materials, there is a growing sense that the Hertz approach can not explain the physics of quantum criticality. Many of the properties of the QCP, such as the appearance of non-trivial exponents in the quantum spin correlations, with T as the only energy scale, suggest that the underlying critical Lagrangian lies *beneath* its critical dimension. Also, all experiments indicate that the energy spectrum of the quasiparticles in the Landau Fermi liquid either side of the QCP, *telescopes to zero*, driving the masses of quasiparticle excitations to infinity, and pushing the characteristic Fermi temperature to zero at the quantum critical point. Yet the Hertz model predicts that most the electron quasiparticle masses should remain finite at an antiferromagnetic QCP.

This has led some to propose that unlike classical criticality, we can not use Landau Ginzburg theory as a starting point for an examination of the fluctuations: a *a new mean-field theory* must be found. One of the ideas of particular interest, is the idea of "local quantum criticality", whereby the quantum fluctuations of the spins become critical in time, but not space at a QCP. Another idea, is that at a heavy electron quantum critical point, the heavy electron quasiparticle disintegrates into separate spin and charge degrees of freedom. Both ideas require radically new kinds of mean-field theory, raising the prospect of a discovery of a wholly new class of critical phenomenon [76].

I should add that Chapline and Laughlin have suggested that quantum criticality may have cosmological implications, proposing that the event horizon of a black hole might be identified with a quantum critical interface where the characteristic scales of particle physics might, in complete analogy with condensed matter, telescope to zero [77].
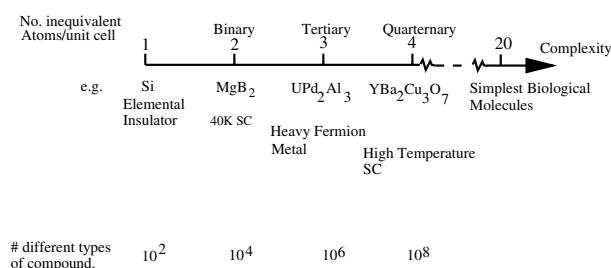
Figure 6: The "axis of complexity".

5   The nature of the Frontier

This article has tried to illustrate how condensed matter physics has had a central influence in the development of our ideas about collective matter, both in the lab, and on a cosmological scale. Many simple phenomena seen in the cryostat, illustrate fundamentally new principles of nature that recur throughout the cosmos. Thus, the discovery of superconductivity and the Meissner effect has contributed in a fundamental way to our understanding of broken symmetry and the Anderson Higg's mechanism. In a similar way, the observation of the resistance minimum in copper, provides an elementary example of the physics of confinement, and required an understanding of the principles of the renormalization group for its understanding. The interchange between the traditional frontiers – and the emergent frontier of condensed matter physics is as live today, as it has been over the past four decades – for example – insights into conformal field theory gained from the study of 2D phase transitions [78] currently play a major role in the description D-brane solitons [79] in superstring theory. In the future, newly discovered phenomena, such as quantum criticality are likely to have their cosmological counterparts as well.

One way of visualizing the frontier, is to consider that in the periodic table, there are about 100 elements. As we go out along the complexity axis(Fig. 6), from the elements to the binary, tertiary and quaternary compounds, the number of possible ordered crystals exponentiates by at least a factor of 100 at each stage, and with it grows the potential for discovery of fundamentally new states of matter. Only two years ago – a new high temperature superconductor MgB_2 was discovered amongst the binary compounds – and the vast phase space of quaternary compounds has barely been scratched by the materials physicist. This is a frontier of exponentiating possibilities, spanning from the simplest collective properties of the elements, out towards the most dramatic emergent phenomenon of all – that of life itself.

Curiously, this new frontier continues to preserve its links with technology and applications. During the past four decades, the size of semi-conductor memory has halved every 18 months, following Moore's law [80]. Extrapolating this unabated

trend into the future, sometime around 2020, the number of atoms required to store a single bit of information will reach unity, forcing technology into the realm of the quantum. Just as the first industrial revolution of the early 19th century was founded on the physical principles of thermodynamics, and the wireless and television revolutions of the 20th century were built largely upon the understanding of classical electromagnetism, we can expect that technology of this new century will depend on the new principles – of collective and quantum mechanical behavior that our field has begun, and continues to forge today.

Acknowledgments

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