

Addendum to: Capillary Floating and The Billiard Ball Problem

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Abstract. We compare the results of our earlier paper on the floating in neutral equilibrium at arbitrary orientation in the sense of Finn-Young with the literature on its counterpart in the sense of Archimedes. We add a few remarks of personal and social-historical character.

Humans were experimenting with floating long before they were able to come up with any understanding of this phenomenon. Current theory as it has developed over the past centuries is quite complicated, and various simplified models have been proposed. Each such model singles out some of the physical aspects of floating and disregards others. For instance, the “archimedean” floating accounts for buoyancy in a gravity field but disregards the fluid surface tension. On the other hand, Thomas Young [28]¹ based the theory on surface tension and concluded that the *contact angle* of the liquid surface with the body must be prescribed.² We refer the reader to the papers of Finn et al. [8–10] for elaborations. Here we are concerned only with the geometric aspects of floating models.

A case of particular interest appears when the floating solid is an infinite cylinder resting horizontally on the liquid. A three-dimensional floating model then reduces, by translational invariance, to a two-dimensional theory, expressed by suitable conditions on the cross-section of the cylinder, $\Omega \subset \mathbb{R}^2$, a bounded domain with a piecewise smooth boundary $\partial\Omega$. We identify the set of directions of oriented straight lines in \mathbb{R}^2 with the circle $\{\alpha : 0 \leq \alpha < 2\pi\}$. Finn introduced the *neutral equilibrium* model in which the fluid is horizontal and meets the body at a prescribed angle, say $\pi - \gamma$. See Fig. 1. This floating model takes into account the capillary forces and the liquid surface tension but disregards the gravity. We will simply refer to it as the *Finn-Young floating*. According to this model, Ω floats in every orientation at the contact angle $\pi - \gamma$ if and only if every directed chord making angle γ with $\partial\Omega$ at one end, makes the same angle with $\partial\Omega$ at the other end. In [8] Finn posed

Question 1. *What convex domains Ω (other than the round disc) satisfy this condition and for what $0 < \gamma < \pi/2$?*

Using the unpublished work [12], the author provided a fair amount of information on this in [14]: there is a dense countable set $\Gamma \subset (0, \pi/2)$, and for any $\gamma \in \Gamma$ there is a real analytic one-parameter family of distinct, smooth, strictly convex domains $\Omega_{\gamma,\tau}$ that float in every orientation at the contact angle $\pi - \gamma$. The set Γ is as follows:

$$\Gamma = \bigcup_{n=2}^{\infty} \{0 < \gamma < \pi/2 : \tan n\gamma = n \tan \gamma\}. \quad (1)$$

The domains $\Omega_{\gamma,\tau}$ are explicitly described via the Fourier coefficients of their radius of curvature functions [14]. Incidentally, these results also bear on the *billiard ball problem*, which yielded basic open questions in geometry and analysis [13].

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¹ Of the Rosetta stone fame.

² See also a pioneering study of capillary floating by Laplace [18].

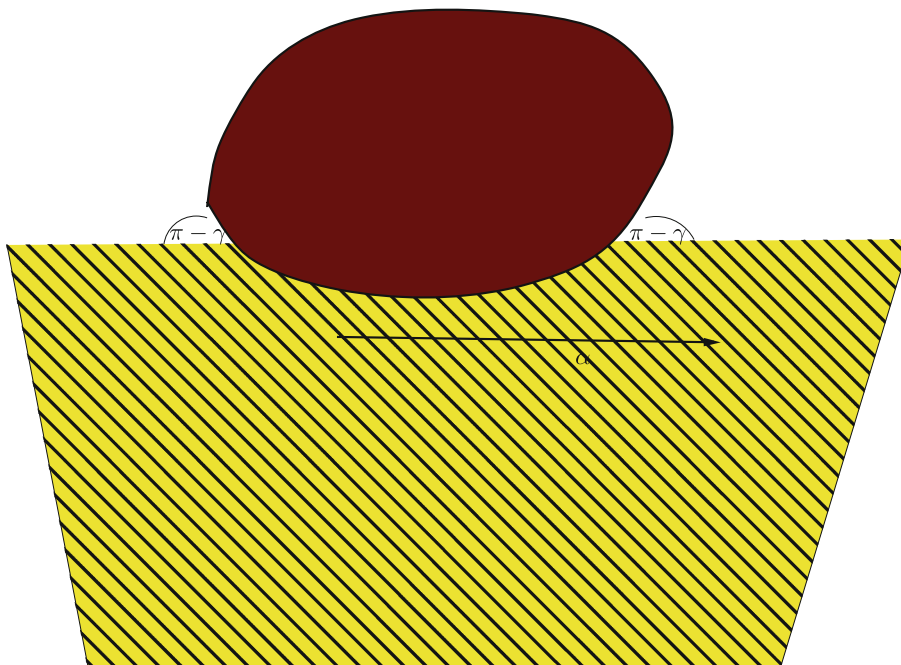


FIG. 1. Finn-Young floating at the orientation α with the contact angle $\pi - \gamma$

Remark 1 The work [20] contains an earlier surprising result on the Finn-Young floating. It says that every smooth, strictly convex plane domain Ω floats in neutral equilibrium at any contact angle in at least four distinct orientations. The bound is the best possible.

We will now turn to the archimedean floating. If the fluid surface is assumed flat, the round ball floats in every orientation and at every density less than the density of the liquid. Suppose that a bounded solid floats in every orientation at a fixed density. One of the problems of the famous “Scottish Café” [19], attributed to S. Ulam, asks if the solid is a round ball. The two-dimensional version of this problem, also attributed to Ulam [2], asks the same question for bounded plane domains.³ For reader’s convenience, we will now formulate the corresponding geometric problem.

Let $\Omega \subset \mathbb{R}^2$ be a bounded region with a piecewise smooth boundary $\partial\Omega$. The following sets are in natural one-to-one correspondences: Ordered pairs of points in $\partial\Omega$, arc segments in $\partial\Omega$, and directed chords in Ω with the endpoints on $\partial\Omega$. We will use the notation $||$ for the arc length; in particular, $|\partial\Omega|$ is the perimeter of Ω . Let $\Omega(A) \subset \mathbb{R}^2$ be the region enclosed between the arc segment A and the corresponding chord $C(A)$.

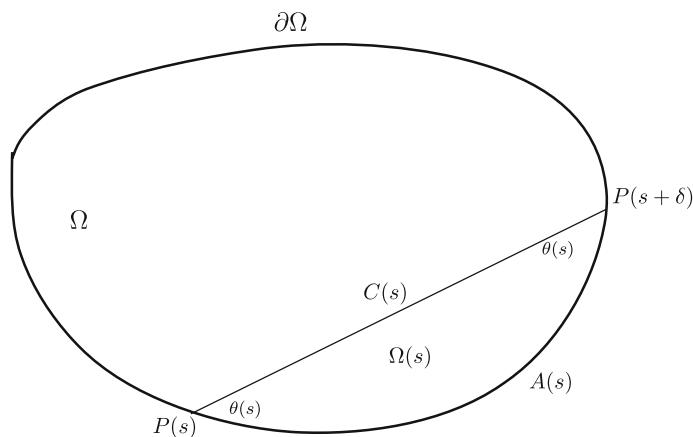
Definition 1. Let $0 < \delta \leq 1/2$. We say that Ω floats (in the archimedean sense) in equilibrium in every orientation at the density δ if for all arc segments $A \subset \partial\Omega$ satisfying $|A| = \delta|\partial\Omega|$ the areas $|\Omega(A)|$ are equal.

The archimedean counterpart of Question 1 is thus the following.

Question 2. What plane domains (not necessarily convex), other than the round disc, satisfy the condition of Definition 1, and what δ are possible?

Question 2 has been studied much more than Question 1, but despite the efforts of geometers that go back to the 1920s [2, 11, 21–23, 25, 26, 30], the results are less complete. We will now very briefly discuss

³ The two-dimensional version comes from the archimedean floating of infinite cylinders.

FIG. 2. Archimedean floating in every orientation at density δ

the literature on Question 2 using the term *floating domains* for plane regions satisfying the conditions of Definition 1. See Fig. 2 for the notation. We assume without loss of generality that $|\partial\Omega| = 1$, and parameterize $\partial\Omega$ by the arc length $0 \leq s < 1$. Let $P(s), P(s + \delta)$ be the endpoints of the moving arc segment $A(s)$. By elementary differential geometry, the condition $|\Omega(s)| = \text{const}$ holds if and only if $|C(s)| = \text{const}$ if and only if the angles between $C(s)$ and $\partial\Omega$ at the points $P(s), P(s + \delta)$ are equal.⁴ Let $\theta(s)$ be the angle at $P(s)$. Note that $\theta(\cdot) \neq \text{const}$, in general. Moreover, it follows from [17] that $\theta(\cdot) = \text{const} \neq \pi/2$ implies that Ω is a disc.

Question 2 has been fully answered for convex Ω and $\delta = 1/2$. In [2] Auerbach describes the curves $\partial\Omega$ satisfying this condition; he calls them *the Zindler curves* [30]. Auerbach characterizes these curves via the Fourier expansion of their radius of curvature functions. He emphasizes an analogy between the Zindler curves and the *curves of constant width*.

The work [22] contains a surprising construction of a class of nonconvex domains satisfying the conditions of Question 2 with $\delta = 1/2$. Let $P = A_1 \cdots A_{2n} \subset \mathbb{R}^2$ be a $2n$ -gon such that (i) the sides of P have the same length; (ii) there is $1 < k \leq n$ such that all diagonals $A_i A_{i+k}$ (with the indexing convention $j + 2n = j$) have the same length. Then the quadrilateral $Q_j, 1 \leq j \leq n$, built on any pair of ‘opposite’ sides of P is inscribed into a circle, say C_j . Let $\partial\Omega$ be the closed curve formed by the appropriate arcs in $C_j, 1 \leq j \leq n$. The domain $\Omega = \Omega(P)$ is obtained by replacing each side of P by an arc of the corresponding circle C_j . Then Ω is a floating domain with $\delta = 1/2$ [22]. Moreover, assumption (i) may be replaced by the weaker assumption (i’) that in each of n pairs of the ‘opposite’ sides of P both sides have the same length. The authors state in [22] that the class of polygons satisfying assumptions (i’), (ii) is quite large. A simple example is the $2n$ -gon obtained from the regular n -gon by adding as vertices the midpoints of its sides. Figure 3, taken from the Russian original of [22], shows the floating domain obtained by this construction from the equilateral triangle.⁵

As for Question 2 when $\delta \neq 1/2$, there is a controversy or, at least, confusion. According to Tabachnikov [25], Salkowski [23] claimed that for $\delta = \frac{m}{n} \neq \frac{1}{2}$ the only convex floating domains are the discs, but the proof is deficient.⁶ According to Auerbach [2], the ‘‘last theorem’’ in [23] states that the only convex floating domains with $\delta = \frac{1}{n} \neq \frac{1}{2}$ are the discs, but that he (Auerbach) ‘‘unfortunately could not understand the proof’’. Furthermore, Auerbach mentions that Salkowski separately proves the special cases $\delta = 1/3, 1/4$ of the ‘‘last theorem’’, and makes no comment about the reliability of those proofs. On the other hand, Theorems 5 and 6 in [25] concern the cases $\delta = 1/3$ and $1/4$, respectively, of Salkowski’s

⁴ See, e.g., Equation (1.1) in [16].

⁵ The above construction of floating domains from polygons is also contained in [25], where it is ascribed to N. Petrunin.

⁶ The work [23] is not available to the author.

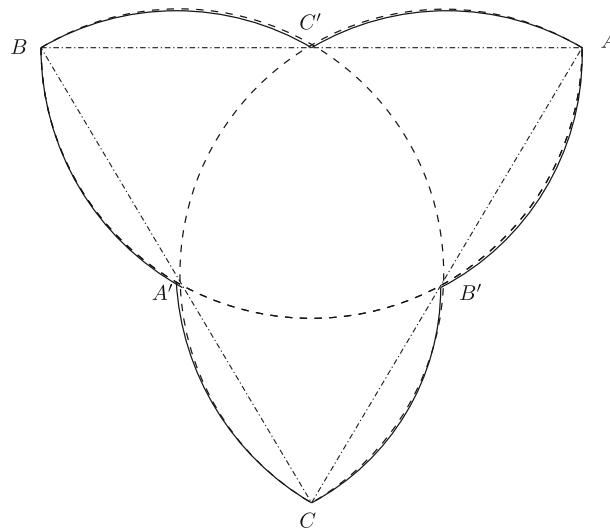


FIG. 3. The non-convex floating domain corresponding to the equilateral triangle

“last theorem”. The work [21] also contains a proof of the claim that ‘any convex floating domain is a disc’ in the case $\delta = 1/4$. However, since [21] proves this claim immediately after “proving” the same for $\delta = 1/2$, the extremely sketchy proof is likely to be deficient. There is also a considerable literature on non-convex floating domains with $\delta \neq 1/2$ [5, 26, 27], but we will not review it here. The interested reader may also consult the web site <http://www.tphys.uni-heidelberg.de/wegner/FI2mvs/Movies.html> maintained by Wegner.

Concluding remarks. It seems appropriate to add a few remarks of personal and social-historical character. I acknowledge the anonymous referee of [14] who attracted my attention to the literature on archimedean floating and mentioned that Eq. (1) comes up in the archimedean floating as well. See equations (14, 18) in [25]. This suggests the possibility of a nontrivial relationship between the Finn-Young floating and the archimedean floating. While visiting the Weizmann Institute of Science in December 2010, I took part in the miniconference dedicated to the 90th birthday of professor Victor Abramovich Zalgaller. This eventually led me to the Russian originals of [21] and [22]. The French translations [21, 22] omit most of the equations, as well as the figures. The transliteration *Zalgaller* is more faithful to the original than “Salgaller”, used in [22]. In the Russian alphabet the letter corresponding to “Z” precedes “K”, thus the coauthors in [22] are listed in the alphabetical order.

From Zalgaller’s email communications I learned that the archimedean floating problem was popular among older mathematics students of Leningrad University while he and his friend Piotr Kostelianets were sophomores. The results of the two sophomore friends, as well as those of A.N. Ruban⁷ were simultaneously submitted for publication in the Proceedings (Doklady) of the Soviet Academy of Sciences by A.A. Markov, the head of the Geometry Chair at the University.⁸

Shortly after, in June 1941, the war with the Nazi Germany began. All three authors of [21, 22] went to the front.⁹ Kostelianets did not come back from the war. Although Ruban survived, he became an invalid, no longer able to do mathematics. Zalgaller was the luckiest of the three. Although severely wounded, he was able to recover; he continued to fight in the front lines until the Victory Day. Zalgaller

⁷ He was a fifth year student.

⁸ Andrey Andreyevich Markov Jr. (1903–1979), the son of A. A. Markov of the “Markov chains” fame, and an outstanding mathematician as well.

⁹ Zalgaller and Kostelianets volunteered in July 1941.

resumed his mathematical career in Leningrad, and became a distinguished geometer. He co/authored many significant publications, including several well known books. Zalgaller emigrated to Israel in 1999.¹⁰

Herman Auerbach (1901–1942) was a Polish geometer who lived in Lwów, which was a distinguished mathematical center during the 20 some years between the two World Wars. He was an active participant in the mathematical activities at the Scottish Café [19]. It is plausible that Auerbach heard Question 2 personally from Ulam. In the years 1980–1984 I was friends with Marc Kac who studied in Lwów in the nineteen thirties. I much regret that I have never talked with Marc about the ‘golden years of Scottish Café’. Alas, now it is one of famous “missed opportunities” [7]. In September 1939 the Second World War broke out; Poland got invaded by the Nazi Germany from the West and by the Soviet Union from the East. Auerbach remained in Lwów which became a part of the Soviet Ukraine. He even published in the leading polish mathematical journal, where the papers were now required to have ukrainian summaries [3]. In 1941 German troops captured Lwów. Auerbach perished in the hands of Gestapo in 1942. In 1992 the Polish Mathematical Society published a tribute to Herman Auerbach and his work [4, 6, 24]. Auerbach wrote the manuscript of [4] in captivity, shortly before execution.

Acknowledgments. I thank professor Zalgaller of Rehovot, Israel, who sent me a scan of his personal copy of the original of [22] and patiently answered my questions. The historical and personal information pertaining to [22] and [21] came from these communications, as well as from [1] and [29]. I thank professor Alexey Borisov of Izhevsk, Russia, who provided me with the originals of [21, 22]. I thank professor Robert Finn of Palo Alto, California, who pointed out to me a connection between the results of [12] and the capillary floating and encouraged me to publish these results. The work was partially supported by MNiSzW Grant N N201 384834 and by NCN Grant DEC-2011/03/B/ST1/00407. A preliminary version of the present work is [15].

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References

- [1] Aleksandrov, A.D. et al.: Victor Abramovich Zalgaller (on his seventieth birthday). *Russ. Math. Surv.* **46**, 257–259 (1991)
- [2] Auerbach, H.: Sur un problème de M. Ulam concernant l’équilibre des corps flottants. *Studia Mathematica* **7**, 121–142 (1938)
- [3] Auerbach, H.: Sur une propriété caractéristique de l’ellipsoïde (Ukrainian summary). *Studia Mathematica* **9**, 17–22 (1940)
- [4] Auerbach, H.: Geometry of the triangle (Polish). *Wiadom. Mat.* **29**, 234–251 (1992)
- [5] Bracho, J., Montejano, L., Oliveros, D.: Carousels, Zindler curves and the floating body problem. *Period. Math. Hungar* **49**, 9–23 (2004)
- [6] Derkowska, A., Mikosz, M., Neugebauer, A.: Herman Auerbach (1901–1942) (Polish). *Wiadom. Mat.* **29**, 225–231 (1992)
- [7] Dyson, F.J.: Missed opportunities. *Bull. Am. Math. Soc.* **78**, 635–652 (1972)
- [8] Finn, R.: Floating bodies subject to capillary attractions. *J. Math. Fluid Mech.* **11**, 443–458 (2009)
- [9] Finn, R.: Remarks on “Floating bodies in neutral equilibrium”. *J. Math. Fluid Mech.* **11**, 466–467 (2009)
- [10] Finn, R.: Criteria for floating I. *J. Math. Fluid Mech.* **13**, 103–115 (2011)
- [11] Gericke, H.: Einige kennzeichnende Eigenschaften des Kreises. *Math. Z.* **40**, 417–420 (1936)
- [12] Gutkin, E.: Billiard tables of constant width and dynamical characterizations of the circle. Workshop on dynamics and related questions, pp. 1–4, PennState University (1993)
- [13] Gutkin, E.: Billiard dynamics: a survey with the emphasis on open problems. *Reg. Chaot. Dyn.* **8**, 1–13 (2003)
- [14] Gutkin, E.: Capillary floating and the billiard ball problem. *J. Math. Fluid Mech.* **14**, 363–382 (2012)
- [15] Gutkin, E.: Addendum to: Capillary floating and the billiard ball problem. arxiv:1206.2272 (2012)
- [16] Gutkin, E., Katok, A.: Caustics for inner and outer billiards. *Commun. Math. Phys.* **173**, 101–133 (1995)
- [17] Kovalev, M.D.: A characteristic property of a disc. *Trudy Mat. Inst. Steklov* **152**, 124–137 (1980)
- [18] de Laplace, P.S.: *Traité de Mécanique Céleste* Vol 4, Suppléments au Livre X. Gauthier-Villars, Paris (1806)
- [19] Mauldin, R. (ed.): *The Scottish Book. Mathematics from the Scottish Café*. Birkhäuser, Boston (1981)
- [20] Raphaël, E., di Meglio, J.-M., Berger, M., Calabi, E.: Convex particles at interfaces. *J. Phys. I France* **2**, 571–579 (1992)
- [21] Ruban, A.N.: Sur le problème du cylindre flottant. *C. R. (Doklady) Acad. Sci. URSS* **25**, 350–352 (1939)

¹⁰ See <http://en.wikipedia.org/wiki/VictorZalgaller> and the book [29] for more information about Zalgaller.

- [22] Salgaller, V., Kostelianetz, P.: Sur le problème du cylindre flottant. C. R. (Doklady) Acad. Sci. URSS **25**, 353–355 (1939)
- [23] Salkowski, E.: Eine kennzeichnende Eigenschaft des Kreises. Sitz. Heidelb. Akad. Wissensch., Math.-Nat. Klasse, pp. 57–62 (1934)
- [24] Szurek, M.: An introduction to Herman Auerbach’s paper “Geometry of the triangle” (Polish). *Wiadom. Mat.* **29**, 232–234 (1992)
- [25] Tabachnikov, S.: Tire track geometry: variations on a theme. *Isr. J. Math.* **151**, 1–28 (2006)
- [26] Varkonyi, P.L.: Floating body problems in two dimensions. *Stud. Appl. Math.* **122**, 195–218 (2009)
- [27] Wegner, F.: Floating bodies of equilibrium. *Stud. Appl. Math.* **111**, 167–183 (2003)
- [28] Young, T.: An essay on the cohesion of fluids. *Philos. Trans. R. Soc. Lond.* **95**, 65–87 (1805)
- [29] Zalgaller, V.A.: The daily grind of war. manuscript in Russian (available on the internet)
- [30] Zindler, K.: Über konvexe Gebilde. *Monatsh. Math. Physik* **30**, 87–102 (1920)

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