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Weighted Inequalities for the Dyadic Square Function

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Abstract. We study Fefferman–Stein inequalities for the dyadic square function associated with an integrable, Hilbert-space-valued function on the interval $[0, 1)$. The proof rests on a Bellman function method: the estimates are deduced from the existence of certain special functions enjoying appropriate majorization and concavity.

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1. Introduction

Let w be a weight (i.e., a nonnegative, locally integrable function) on $\mathbb{R}^d$ and let M be the Hardy–Littlewood maximal operator. In 1971, Fefferman and Stein [9] proved the existence of a finite constant c depending only on the dimension such that

$$w\left(\left\{x \in \mathbb{R}^d : Mf(x) \geq 1\right\}\right) \leq c\|f\|_{L^1(Mw)}$$

(throughout the paper, we use the standard notation $w(E) = \int_E w(x)\mathrm{d}x$ and $\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^d} |f(x)|^p w(x)\mathrm{d}x\right)^{1/p}$, $1 \leq p < \infty$). This gave rise to the following natural question, formulated by Muckenhoupt and Wheeden in the seventies. Suppose that T is a Calderón–Zygmund singular integral operator. Is there a constant c, depending only on T and d, such that

$$w\left(\left\{x \in \mathbb{R}^d : |Tf(x)| \geq 1\right\}\right) \leq c\|f\|_{L^1(Mw)}? \tag{1.1}$$

This problem, called the Muckenhoupt–Wheeden conjecture, remained open for a long time and in 2010 it was proved to be false: see the counterexamples for the Hilbert transform provided by Reguera, Thiele, Nazarov, Reznikov, Vasyunin and Volberg in [14,18,19].

In the mean-time, many partial or related results in this direction were obtained. In particular, Buckley [2] showed that the conjecture is true for the weights $w_\delta(x) = |x|^{-d(1-\delta)}$, $0 < \delta < 1$. See also Pérez' paper [17] as well as a series of works [10–12] by Lerner, Ombrosi and Pérez devoted to a little weaker statements related to (1.1). Chang et al. [5], and Chanillo and Wheeden [6] studied the above problem in the context of square functions. For a given $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \varphi = 0$, put $\varphi_t(x) = t^{-d}\varphi(x/t)$, $t > 0$, and define the associated area function by the formula

$$S_\varphi(f)(x) = \left(\int_{|x-y|<t} |f \star \varphi_t(y)|^2 \frac{\mathrm{d}t\mathrm{d}y}{t^{d+1}}\right)^{1/2}.$$

Then, as proved in [5], there is a constant $C(d, \varphi)$ depending only on the parameters indicated, such that

$$\|S_\varphi(f)\|_{L^2(w)} \leq C(d,\varphi)\|f\|_{L^2(Mw)}. \tag{1.2}$$

In [6], Chanillo and Wheeden generalized this result in several directions. First, they showed the corresponding weak-type $(1, 1)$ estimate: there is a finite constant $c(d, \varphi)$ such that

$$w(\{x \in \mathbb{R}^d : S_\varphi(f)(x) \geq 1\}) \leq c(d,\varphi)\|f\|_{L^1(Mw)}. \tag{1.3}$$

Furthermore, the inequality (1.2) extends naturally to L^p, $1 < p < 2$: we have

$$\|S_\varphi(f)\|_{L^p(w)} \leq C(p,d,\varphi)\|f\|_{L^p(Mw)} \tag{1.4}$$

for some $C(p, d, \varphi)$ independent of f and w. A very interesting fact is that (1.4) *does not* hold for $p > 2$. Chanillo and Wheeden offered the following substitution:

$$\|S_\varphi(f)\|_{L^p(w)} \leq C(p,d,\varphi)\|f\|_{L^p((Mw)^{p/2}w^{1-p/2})}, \quad 2 < p < \infty, \tag{1.5}$$

for some $C(p, d, \varphi)$ depending only on the parameters in the brackets.

Our contribution is to study related two-weight inequalities for the dyadic square function associated with an integrable, Hilbert-space-valued function on $[0, 1)$. Let us introduce the necessary background and notation. In what follows, $\mathcal{H}$ stands for the separable Hilbert space, with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$ (with no loss of generality, we may and do assume that $\mathcal{H} = \ell_2$), and $\mathcal{D}$ is the collection of all dyadic subintervals of $[0, 1)$. Let $(h_n)_{n\geq 0}$ be the standard Haar system, given by

$$\begin{aligned} &h_0 = \chi_{[0,1)}, \quad h_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}, \\ &h_2 = \chi_{[0,1/4)} - \chi_{[1/4,1/2)}, \quad h_3 = \chi_{[1/2,3/4)} - \chi_{[3/4,1)}, \\ &h_4 = \chi_{[0,1/8)} - \chi_{[1/8,1/4)}, \quad h_5 = \chi_{[1/4,3/8)} - \chi_{[3/8,1/2)}, \end{aligned}$$

and so on. For any $I \in \mathcal{D}$ and an integrable function $f : [0, 1) \to \mathcal{H}$, we will write $\langle f\rangle_I$ for the average of f over I: that is, $\langle f\rangle_I = \frac{1}{|I|}\int_I f$ (throughout, unless stated otherwise, the integration is with respect to the Lebesgue's measure). Furthermore, for any such f and any nonnegative integer n, we use the notation

$$f_n = \sum_{k=0}^n \frac{1}{|I_k|}\int_{[0,1)} f(s)h_k(s)\mathrm{d}s\, h_k$$

for the projection of f on the subspace generated by the first $n+1$ Haar functions (I_k is the support of h_k). Then the dyadic square function of f is given by

$$S(f)(x) = \left(\sum \left| \frac{1}{|I_n|} \int_{[0,1)} f(s) h_n(s) \mathrm{d}s \right|^2 \right)^{1/2},$$

where the summation runs over all nonnegative integers n such that $x \in I_n$. Furthermore, the dyadic maximal operator M_d acts on f by the formula

$$M_d f(x) = \sup \left\{ \frac{1}{|I|} \int_I |f(s)| \mathrm{d}s \,:\, x \in I \in \mathcal{D} \right\} = \sup_{n \geq 0} |f|_n(x).$$

We will also need the truncated versions of the above operators; for any nonnegative integer m, set

$$S_m(f)(x) = \left(\sum \left| \frac{1}{|I_n|} \int_{[0,1)} f(s) h_n(s) \mathrm{d}s \right|^2 \right)^{1/2},$$

where the summation runs over all $n \leq m$ such that $x \in I_n$, and

$$M_{d,m} f(x) = \sup_{0 \leq n \leq m} |f|_n(x).$$

In all the considerations below, the symbol w will denote a weight on $[0,1)$, i.e., an integrable function $w : [0,1) \to [0,\infty)$.

We are ready to formulate our main results.

Theorem 1.1. *Let w be a weight on $[0,1)$. Then for any integrable function $f : [0,1) \to \mathcal{H}$ we have*

$$w(\{x \in [0,1) : S(f)(x) \geq 1\}) \leq 2\|f\|_{L^1(M_d w)}, \tag{1.6}$$

$$\|S(f)\|_{L^p(w)} \leq \frac{6p}{p-1} \|f\|_{L^p(M_d w)}, \quad 1 < p < 2, \tag{1.7}$$

and

$$\|S(f)\|_{L^p(w)} \leq \sqrt{\frac{p}{2}} \|f\|_{L^p((M_d w)^{p/2} w^{1-p/2})}, \quad p \geq 2. \tag{1.8}$$

As in the context of area functions, we will show that the inequality of the form (1.7) does not hold with any finite constant when $2 < p < \infty$. We will also establish the following "mixed-weight" version of (1.8) in the case $1 < p < 2$.

Theorem 1.2. *Let w be a weight on $[0,1)$. Then for any $1 < p < 2$ and any integrable function f on $[0,1)$ we have*

$$\|S(f)\|_{L^p(w)} \leq (p-1)^{-1} \|f\|_{L^p((M_d w)^p w^{1-p})}. \tag{1.9}$$

Clearly, this result is qualitatively weaker than (1.7), however, we have decided to include it here since its proof exploits a novel interpolation-type argument which is of independent interest.

We would like to point out that even in the unweighted setting (i.e., for $w \equiv 1$), the above estimates are quite tight. The best constant in the unweighted version of (1.6) is equal to $C = 1.4623\ldots$, as shown by Bollobás [1] and the author [16]. The constant in (1.7) is of order $(p-1)^{-1}$: see Burkholder [3] and Davis [7]. Finally, as $p \to \infty$, the order $\sqrt{p}$ in (1.8) is optimal: see e.g. Davis [7] for the description of the best constants.

Typically, proofs of weighted inequalities in the analytic context depend heavily on extrapolation and interpolation arguments. Our approach will rest entirely on the so-called Bellman function method. More precisely, we will deduce the validity of the above L^p estimates from the existence of certain special functions, enjoying appropriate majorizations and concavity. The technique is described in detail in the next section. Section 3 is devoted to the weak-type inequality (1.6), while Sects. 4 and 5 contain the proof of the L^p inequalities in the case $1 < p < 2$ and $p \geq 2$. The final part of the paper discusses the probabilistic versions of the above results.

2. On the Method of Proof

Let us describe the technique which will be used to obtain the results announced in the preceding section. Throughout the paper, we will use the notation

$$\mathfrak{D} = \{(x, y, u, v) \in \mathcal{H} \times [0, \infty) \times [0, \infty) \times (0, \infty) : u \leq v\}.$$

Suppose that $V : \mathfrak{D} \to \mathbb{R}$ is a given function and assume that we want to establish the inequality

$$\int_0^1 V(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s \leq 0, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

for any integrable function $f : [0, 1) \to \mathcal{H}$ and any nonzero weight w (i.e., satisfying $\langle w\rangle_{[0,1)} > 0$). For instance, the choice $V(x, y, u, v) = u\chi_{\{y\geq 1\}} - 2|x|v$ leads to the weak type inequality (1.6), after a simple limiting argument (see Sect. 3 below); similarly, the function $V(x, y, u, v) = y^p u - C_p^p |x|^p v$ corresponds to the strong-type estimates. A key idea in the study of (2.1) is to consider a function $U : \mathfrak{D} \to (-\infty, \infty]$, which satisfies the following properties:

1° For any $x \in \mathbb{R}$ and any $u > 0$ we have

$$U(x, |x|, u, u) \leq 0. \tag{2.2}$$

2° For any $(x, y, u, v) \in \mathfrak{D}$,

$$U(x, y, u, v) \geq V(x, y, u, v). \tag{2.3}$$

3° For any $(x, y, u, v) \in \mathfrak{D}$ and any $d \in \mathcal{H}$, $e \in [-u, u]$ we have

$$\begin{aligned} U(x, y, u, v) \geq \frac{1}{2}\Big[&U\big(x - d, \sqrt{y^2 + |d|^2}, u - e, v \vee (u - e)\big) \\ &+ U\big(x + d, \sqrt{y^2 + |d|^2}, u + e, v \vee (u + e)\big)\Big]. \end{aligned} \tag{2.4}$$

(Here and below, $a \vee b$ stands for the maximum of the numbers a and b.)

The conditions 1° and 2° can be regarded as certain majorizations for U; the function can be neither too small nor too big. The most mysterious condition is the third one, and it can be understood as a concavity-type property.

The interplay between the existence of such a function and the validity of (2.1) is described in the two statements below.

Theorem 2.1. *If there exists U satisfying the properties* $1°$, $2°$ *and* $3°$, *then the inequality* (2.1) *holds true.*

Proof. Fix a nonzero weight w and an integrable function $f : [0,1) \to \mathcal{H}$. First we will show that the sequence $(\int_0^1 U(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s)_{n\geq 0}$ is nonincreasing (there is no problem with the existence of the integral for each n, since f_n, $S_n(f)$, w_n and $M_{d,n}w$ take only a finite number of values). To achieve this, fix an integer n and let I_{n+1} be the support of h_{n+1}. Then the functions $U(f_n, S_n(f), w_n, M_{d,n}w)$ and $U(f_{n+1}, S_{n+1}(f), w_{n+1}, M_{d,n+1}w)$ coincide outside I_{n+1} (since so do the pairs f_n, f_{n+1}; $S_n(f)$, $S_{n+1}(f)$; w_n, w_{n+1}; and $M_{d,n}w$, $M_{d,n+1}w$), and hence

$$\int_{[0,1)\setminus I_{n+1}} U(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s \\ = \int_{[0,1)\setminus I_{n+1}} U(f_{n+1}, S_{n+1}(f), w_{n+1}, M_{d,n+1}w)\mathrm{d}s.$$

Thus, we only need to show an appropriate bound for the integrals over I_{n+1}. To do this, note that f_n, $S_n(f)$, w_n and $M_{d,n}w$ are constant on I_{n+1}. Denote the corresponding values by x, y, u and v; then, clearly, we have $(x, y, u, v) \in \mathfrak{D}$ (the fact that $v > 0$ follows directly from the assumption $\langle w\rangle_{[0,1)} > 0$). Next, let I_-, I_+ be the left and the right half of I_{n+1}; then f_{n+1} and w_{n+1} are constant on $I_\pm$ and the corresponding values can be denoted by $x \pm d$ and $u \pm e$, for some $d \in \mathcal{H}$ and $e \in [-u, u]$. Furthermore, directly from the definition of the truncated square and maximal functions, we see that $S_{n+1}(f) = \sqrt{y^2 + |d|^2}$ and $M_{d,n+1}w = v \vee (u \pm e)$ on $I_\pm$. Consequently, the inequality

$$\int_{I_{n+1}} U(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s \\ \geq \int_{I_{n+1}} U(f_{n+1}, S_{n+1}(f), w_{n+1}, M_{d,n+1}w)\mathrm{d}s$$

is equivalent to (2.4), and therefore the desired monotonicity of the sequence $(\int_0^1 U(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s)_{n\geq 0}$ is established. Combining this property with (2.3), we get

$$\int_0^1 V(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s \leq \int_0^1 U(f_n, S_n(f), w_n, M_{d,n}w)\mathrm{d}s \\ \leq \int_0^1 U(f_0, S_0(f), w_0, M_{d,0}w)\mathrm{d}s \leq 0. \quad (2.5)$$

To see why the latter bound holds, note that $|f_0| = S_0(f)$ and $w_0 = M_{d,0}w$, so in fact we even have the pointwise estimate $U(f_0, S_0(f), w_0, M_{d,0}w) \leq 0$, due to (2.2). This proves the claim. □

A beautiful fact is that the implication of the above theorem can be reversed; though we will not exploit it, we believe it is worth to be stated and proved.

Theorem 2.2. *If the inequality* (2.1) *holds true, then there exists* U *satisfying the properties* 1°, 2° *and* 3°.

Proof. There is an abstract formula for the special function. Namely, for any $(x, y, u, v) \in \mathfrak{D}$, put

$$U(x,y,u,v) = \sup\left\{\int_0^1 V\big(f_n, \sqrt{y^2 - |x|^2 + S_n^2(f)}, w_n, (M_{d,n}w) \vee v\big)\mathrm{d}s\right\},$$

where the supremum is taken over all n, all integrable functions f satisfying $\langle f\rangle_{[0,1)} = x$ and all nonzero weights w with $\langle w\rangle_{[0,1)} = u$. Let us verify that this object has all the required properties. The first condition is easy to check: we have $\sqrt{|x|^2 - |x|^2 + S_n^2(f)} = S_n(f)$ and $(M_{d,n}w) \vee u = M_{d,n}w$ provided $\langle w\rangle_{[0,1)} = u$. Thus, $U(x, |x|, u, u) \le 0$, since in the light of (2.1), any integral appearing under the supremum defining $U(x, |x|, u, u)$ is nonpositive. The majorization 2° is also straightforward: considering constant functions $f \equiv x$ and $w \equiv u$, we see that for all n,

$$\int_0^1 V\big(f_n, \sqrt{y^2 - |x|^2 + S_n^2(f)}, w_n, (M_{d,n}w) \vee v\big)\mathrm{d}s = V(x, y, u, v)$$

and hence, by the very definition, $U(x, y, u, v) \ge V(x, y, u, v)$. To check 3°, fix x, y, u, v, d and e as in its formulation. For a fixed $\varepsilon > 0$, there are a function f^+, a nonzero weight w^+ and a nonnegative integer n such that $\langle f^+\rangle_{[0,1)} = x + d$, $\langle w^+\rangle_{[0,1)} = u + e$ and

$$\begin{aligned} &U(x+d, \sqrt{y^2+|d|^2}, u+e, v \vee (u+e)) - \varepsilon \\ &\quad \le \int_0^1 V\big(f_n^+, \sqrt{y^2+|d|^2-|x+d|^2+S_n^2(f^+)}, w_n^+, (M_{d,n}w^+) \vee v \vee (u+e)\big)\mathrm{d}s \end{aligned} \tag{2.6}$$

(i.e., f^+, w^+, n are parameters for which the supremum in the definition of $U\big(x+d, \sqrt{y^2+|d|^2}, u+e, v \vee (u+e)\big)$ is almost attained, up to ε). Similarly, pick f^-, w^-, m satisfying $\langle f^-\rangle_{[0,1)} = x - d$, $\langle w^-\rangle_{[0,1)} = u - e$ and

$$\begin{aligned} &U(x-d, \sqrt{y^2+|d|^2}, u-e, v \vee (u-e)) - \varepsilon \\ &\quad \le \int_0^1 V\big(f_m^-, \sqrt{y^2+|d|^2-|x-d|^2+S_m^2(f^-)}, w_m^-, (M_{d,m}w^-) \vee v \vee (u-e)\big)\mathrm{d}s. \end{aligned} \tag{2.7}$$

Replacing $f^\pm$ by f_n^+ and f_m^- if necessary, we may assume that $f_n^+ = f_{n+1}^+ = f_{n+2}^+ = \ldots$ and $f_m^- = f_{m+1}^- = f_{m+2}^- = \cdots$. Using the same cutting-off procedure, we may also assume that $w_n^+ = w_{n+1}^+ = w_{n+2}^+ = \cdots$ and $w_m^- = w_{m+1}^- = w_{m+2}^- = \cdots$. Now, let us splice the functions $f^\pm$ into one function f and the weights $w^\pm$ into one weight w, with the use of the formula

$$(f(s), w(s)) = \begin{cases} (f^+(2s), w^+(2s)) & \text{if } s \in [0, 1/2), \\ (f^-(2s-1), w^-(2s-1)) & \text{if } s \in [1/2, 1). \end{cases}$$

That is, we "squeeze" the domain of f^+, w^+ into the interval $[0, 1/2)$, we "squeeze" the domain of f^- and w^- into the interval $[1/2, 1)$, and then we "glue" $f^\pm$ into one function, and $w^\pm$ into one weight on $[0, 1)$. Clearly, we have

$$\langle f\rangle_{[0,1)} = \frac{1}{2}\langle f^+\rangle_{[0,1)} + \frac{1}{2}\langle f^-\rangle_{[0,1)} = x, \ \ \langle w\rangle_{[0,1)} = \frac{1}{2}\langle w^+\rangle_{[0,1)} + \frac{1}{2}\langle w^-\rangle_{[0,1)} = u$$

and hence, for any k,

$$\begin{aligned} U(x, y, u, v) &\geq \int_0^1 V\left(f_k, \sqrt{y^2 - |x|^2 + S_k^2(f)}, w_k, (M_{d,k}w) \vee v\right) \mathrm{d}s \\ &= \int_0^{1/2} V\left(f_k, \sqrt{y^2 - |x|^2 + S_k^2(f)}, w_k, (M_{d,k}w) \vee v\right) \mathrm{d}s \\ &\quad + \int_{1/2}^1 V\left(f_k, \sqrt{y^2 - |x|^2 + S_k^2(f)}, w_k, (M_{d,k}w) \vee v\right) \mathrm{d}s. \end{aligned}$$

Using the structural properties of the Haar system, it is not difficult to check that if k is sufficiently large, then the last two integrals are equal to right-hand sides or (2.6) and (2.7). Since ε was arbitrary, we obtain the desired condition $3°$. $\square$

3. A Weak-Type Inequality

We turn our attention to the weak-type estimate (1.6) which, as we will see, corresponds to the choice $V_1(x, y, u, v) = u\chi_{\{y \geq 1\}} - 2|x|v$. To define the associated special function, consider the splitting of the domain $\mathfrak{D}$ into the sets

$$\begin{aligned} D_1 &= \left\{(x, y, u, v) \in \mathfrak{D} : y < 1 \text{ and } |x| + \sqrt{1 + \frac{u}{v}(y^2 - 1)} \leq 1\right\}, \\ D_2 &= \mathfrak{D} \backslash D_1. \end{aligned}$$

Let $U_1 : \mathfrak{D} \to \mathbb{R}$ be given by

$$U_1(x, y, u, v) = \begin{cases} y^2 u - |x|^2 v & \text{if } (x, y, u, v) \in D_1, \\ u - 2|x|v & \text{if } (x, y, u, v) \in D_2. \end{cases}$$

We start the analysis with the following majorizations.

Lemma 3.1. (i) *The functions U_1 and V_1 satisfy $1°$ and $2°$.*

(ii) *For any $(x, y, u, v) \in \mathfrak{D}$ we have*

$$U_1(x, y, u, v) \leq u - 2|x|v. \tag{3.1}$$

(iii) *If $y < 1$ and $|x| \leq 1 + \sqrt{1 + \frac{u}{v}(y^2 - 1)}$, then*

$$U_1(x, y, u, v) \leq y^2 u - |x|^2 v. \tag{3.2}$$

Proof. (i) The condition 1° is easy. Indeed, if $(x, |x|, u, u) \in D_1$, then we have $U_1(x, |x|, u, u) = 0$; on the other hand, if $(x, |x|, u, u) \in D_2$, then $|x| > 1/2$ and hence $U_1(x, |x|, u, u) \leq 0$. The majorization 2° is also simple. Clearly, it is enough to handle the case when $(x, y, u, v) \in D_1$; but then $|x| \leq 1$ and hence

$$y^2 u - |x|^2 v \geq -|x|^2 v \geq -2|x|v,$$

as desired.

(ii) We may assume that $(x, y, u, v) \in D_1$, since otherwise both sides are equal. Then the inequality is equivalent to $y^2 u - u \leq |x|^2 v - 2|x|v$, which can be further transformed into

$$1 + \frac{u}{v}(y^2 - 1) \leq (1 - |x|)^2.$$

This bound follows directly from the definition of D_1.

(iii) If $|x| \leq 1 - \sqrt{1 + \frac{u}{v}(y^2 - 1)}$, then both sides are equal. On the other hand, if $|x| > 1 - \sqrt{1 + \frac{u}{v}(y^2 - 1)}$, then the inequality becomes

$$(1 - |x|)^2 \leq 1 + \frac{u}{v}(y^2 - 1).$$

This is equivalent to saying that

$$1 - \sqrt{1 + \frac{u}{v}(y^2 - 1)} \leq |x| \leq 1 + \sqrt{1 + \frac{u}{v}(y^2 - 1)},$$

which holds true due to the above assumptions on x, y, u and v. □

Let us check the concavity-type condition.

Lemma 3.2. *The function U_1 satisfies the property* 3°.

Proof. Let us first consider the case $(x, y, u, v) \in D_2$, which is much easier. Using (3.1), we write

$$\begin{aligned} &U_1\big(x - d, \sqrt{y^2 + |d|^2}, u - e, v \vee (u - e)\big) \\ &\quad + U_1\big(x + d, \sqrt{y^2 + |d|^2}, u + e, v \vee (u + e)\big) \\ &\leq (u - e) - 2|x - d|(v \vee (u - e)) + (u + e) - 2|x + d|(v \vee (u + e)) \\ &\leq 2u - 2|x - d|v - 2|x + d|v \\ &\leq 2u - 4|x|v \\ &= 2U_1(x, y, u, v). \end{aligned}$$

Now we turn to the analysis of the more difficult case $(x, y, u, v) \in D_1$. Suppose first that $y^2 + |d|^2 \geq 1$. Then (2.4) is equivalent to

$$(y^2 - 1)u \geq |x|^2 v - |x - d|(v \vee (u - e)) - |x + d|(v \vee (u + e)). \tag{3.3}$$

We will show a slightly stronger estimate

$$y^2 - 1 \geq |x|^2 - |x - d| - |x + d|. \tag{3.4}$$

To see that (3.3) can be deduced from it, note that $y < 1$ and hence

$$\begin{aligned} (y^2 - 1)u \geq (y^2 - 1)v &\geq |x|^2 v - |x - d|v - |x + d|v \\ &\geq |x|^2 v - |x - d|(v \vee (u - e)) - |x + d|(v \vee (u + e)), \end{aligned}$$

as desired. To show (3.4), we note that $|x| \leq 1-y$ in D_1. Using this inequality, our assumption $y^2+|d|^2 \geq 1$ and the triangle inequality $|x+d|+|x-d| \geq 2|d|$, we get

$$|x|^2 - y^2 + 1 \leq 2(1-y) \leq 2\sqrt{1-y^2} \leq 2|d| \leq |x+d| + |x-d|.$$

It remains to consider the case $(x, y, u, v) \in D_1$ and $y^2 + |d|^2 < 1$. First we will prove that

$$\begin{aligned} &U_1(x+d, \sqrt{y^2+|d|^2}, u+e, v \vee (u+e)) \\ &\quad \leq (y^2+|d|^2)(u+e) - |x+d|^2(v \vee (u+e)), \\ &U_1(x-d, \sqrt{y^2+|d|^2}, u-e, v \vee (u-e)) \\ &\quad \leq (y^2+|d|^2)(u-e) - |x-d|^2(v \vee (u-e)). \end{aligned} \tag{3.5}$$

It is enough to show the first estimate, the second one follows from the change of signs of d and e. In the light of (3.2), this bound will hold true if we show that

$$|x+d| \leq 1 + \sqrt{1 + \frac{u+e}{v \vee (u+e)}(y^2+|d|^2-1)}.$$

But this is simple: $(x, y, u, v) \in D_1$ implies $|x| \leq 1$, so

$$\begin{aligned} |x+d| \leq |x| + |d| \leq 1 + \sqrt{y^2+|d|^2} \\ \leq 1 + \sqrt{1 + \frac{u+e}{v \vee (u+e)}(y^2+|d|^2-1)}. \end{aligned}$$

Now add the two inequalities in (3.5). We get that the right hand side of (2.4) does not exceed

$$\begin{aligned} &(y^2+|d|^2)u - \frac{1}{2}\left[|x+d|^2(v \vee (u+e)) + |x-d|^2(v \vee (u-e))\right] \\ &\leq (y^2+|d|^2)u - \frac{1}{2}\left[|x+d|^2 v + |x-d|^2 v\right] \\ &= y^2 u - |x|^2 v + |d|^2(u-v) \\ &\leq U_1(x, y, u, v). \end{aligned}$$

This completes the proof. □

Proof of (1.6). Applying Theorem 2.1 to the functions U_1 and V_1 defined above, we obtain

$$\int_0^1 w_n \chi_{\{S_n(f)\geq 1\}} - 2|f_n| M_{d,n} w \mathrm{d}s = \int_0^1 V_1(f_n, S_n(f), w_n, M_{d,n}w) \mathrm{d}s \leq 0 \tag{3.6}$$

for all n. However, w_n is the projection of w onto the space generated by h_0, $h_1, \ldots, h_n$, and similarly, f_n is the projection of f. This implies

$$\int_0^1 w_n \chi_{\{S_n(f)\geq 1\}} \mathrm{d}s = \int_0^1 w \chi_{\{S_n(f)\geq 1\}} \mathrm{d}s$$

and

$$\int_0^1 |f_n| M_{d,n} w \mathrm{d}s \leq \int_0^1 |f| M_{d,n} w \mathrm{d}s.$$

Consequently, (3.6) implies

$$\int_0^1 w\chi_{\{S_n(f)\geq 1\}} \mathrm{d}s \leq 2\int_0^1 |f| M_{d,n} w \mathrm{d}s \leq 2\int_0^1 |f| M_d w \mathrm{d}s.$$

Let n go to infinity and apply Lebesgue's monotone convergence theorem to obtain

$$\int_0^1 w\chi_{\{S(f)>1\}} \mathrm{d}s \leq 2\int_0^1 |f| M_d w \mathrm{d}s.$$

To get the non-strict inequality under the indicator function, fix $\eta \in (0,1)$ and apply the above bound to f/η; then

$$\int_0^1 w\chi_{\{S(f)\geq 1\}} \mathrm{d}s \leq \int_0^1 w\chi_{\{S(f)>\eta\}} \mathrm{d}s \leq 2\eta^{-1}\int_0^1 |f| M_d w \mathrm{d}s.$$

Letting $\eta \to 1$ completes the proof. □

4. Moment Inequality, $1 < p < 2$

4.1. On the Estimate (1.7)

Let us start with some technical statements.

Lemma 4.1. *Let $p \in (1,2)$ and x, $d \in \mathcal{H}$.*

(i) *If $|d| < |x|$, then*

$$|x+d|^p + |x-d|^p - 2|x|^p \geq p(p-1)|x|^{p-2}|d|^2.$$

(ii) *If $|d| \geq |x|$, then*

$$|x+d|^p + |x-d|^p - 2|x|^p \geq (2^p-2)|d|^p.$$

Proof. (i) We have

$$|x+d|^p + |x-d|^p = \big(|x|^2+|d|^2-2\langle x,d\rangle\big)^{p/2} + \big(|x|^2+|d|^2+2\langle x,d\rangle\big)^{p/2}$$

and the function $t \mapsto t^{p/2}$ is concave on $[0,\infty)$. This implies that if we keep $|x|$ and $|d|$ fixed, the left-hand side of the desired inequality is the least when $|\langle x,d\rangle|$ is the largest. In other words, it suffices to show the estimate for $\mathcal{H} = \mathbb{R}$, and we may clearly assume that both x and d are nonnegative. Having fixed such an x, consider the function $F : [0,x] \to \mathbb{R}$ given by

$$F(t) = (x+t)^p + (x-t)^p - 2x^p - p(p-1)x^{p-2}t^2.$$

We have $F(0) = F'(0+) = 0$ and, for $t \in (0,x)$,

$$F''(t) = p(p-1)\big[(x+t)^{p-2} + (x-t)^{p-2} - 2x^{p-2}\big] \geq 0,$$

since the function $t \mapsto t^{p-2}$ is convex on $(0,\infty)$. This implies that F is nonnegative, which is exactly what we need.

(ii) Arguing as above, we may assume that $\mathcal{H} = \mathbb{R}$ and x, $d \geq 0$. Fix x and consider $G : [x, \infty) \to \mathbb{R}$ defined by the formula

$$G(t) = (x+t)^p + (t-x)^p - 2x^p - (2^p - 2)t^p.$$

Clearly, we have $G(0) = 0$ and

$$G'(t) = p\big[(x+t)^{p-1} + (t-x)^{p-1} - (2^p-2)t^{p-1}\big].$$

Now, keep t fixed and denote the expression in the square brackets by $H(x)$. Then for any $s \in (0, t)$ we have

$$H'(s) = (p-1)[(s+t)^{p-2} - (t-s)^{p-2}] < 0.$$

This yields

$$G'(t) \geq p\big[2^{p-1}t^{p-1} - (2^p-2)t^{p-1}\big] \geq 0,$$

where the latter bound, equivalent to $2^{p-1} + 2 \geq 2^p$, follows from

$$2^{p-2} + 1 \geq 2 \cdot 2^{(p-2)/2} = 2^{p/2} \geq 2^{p-1}.$$

This implies that G is nondecreasing on $[x, \infty)$, and hence it is nonnegative. $\square$

As we will see, the inequality (1.7) follows from (2.1) with $V_p(x, y, u, v) = y^p u - \left(\frac{6p}{p-1}\right)^p |x|^p v$. Introduce the function $U_p : \mathfrak{D} \to \mathbb{R}$ by

$$U_p(x,y,u,v) = \begin{cases} y^p u - C_p^p |x|^p v & \text{if } y \geq \beta_p |x|, \\ y^p u - (C_p^p |x|^p + \beta_p |x| y^{p-1} - y^p)v & \text{if } y < \beta_p |x|, \end{cases}$$

where $C_p^p = \frac{2}{2^{p-1}-1}$ and $\beta_p = \frac{2\sqrt{2}p}{p-1}$. The reason why we choose these particular special constants, will be clarified below, in Remark 4.4. Of course, we have

$$U_p(x,y,u,v) = y^p u - \max\big\{C_p^p|x|^p, C_p^p|x|^p + \beta_p|x|y^{p-1} - y^p\big\}v. \tag{4.1}$$

Let us now verify that U_p, V_p enjoy all the required properties.

Lemma 4.2. *The functions U_p, V_p satisfy $1°$ and $2°$.*

Proof. The first inequality is equivalent to $u - (C_p^p + \beta_p - 1)u \leq 0$, which holds trivially. To show the second bound, observe first that

$$\left(\frac{6p}{p-1}\right)^p = 2^p \left(\frac{3p}{p-1}\right)^p \geq C_p^p + \beta_p^p,$$

which holds due to

$$C_p^p = \frac{2}{2^{p-1}-1} \leq \frac{2}{(p-1)\ln 2} \leq \frac{3p}{p-1} \leq \left(\frac{3p}{p-1}\right)^p \quad \text{and} \quad \beta_p \leq \frac{3p}{p-1}.$$

So, the majorization is evident for $y \geq \beta_p|x|$; for remaining (x, y), the estimate is equivalent to

$$\left(\frac{6p}{p-1}\right)^p |x|^p \geq C_p^p|x|^p + \beta_p|x|y^{p-1} - y^p.$$

But $y < \beta_p|x|$, so

$$C_p^p|x|^p + \beta_p|x|y^{p-1} - y^p \le C_p^p|x|^p + \beta_p|x|y^{p-1} \le C_p^p|x|^p + \beta_p^p|x|^p \le \left(\frac{6p}{p-1}\right)^p |x|^p.$$

The proof is complete. □

We turn our attention to the main, concavity-type property.

Lemma 4.3. *The function U_p satisfies* 3°.

Proof. Assume first that $y \ge \beta_p|x|$. Directly from (4.1), we see that the right-hand side of (2.4) does not exceed

$$(y^2 + |d|^2)^{p/2}u - C_p^p\left[\frac{|x+d|^p + |x-d|^p}{2}\right]v.$$

Hence it is enough to prove that

$$(y^2 + |d|^2)^{p/2} - y^p \le C_p^p\left[\frac{|x+d|^p + |x-d|^p - 2|x|^p}{2}\right].$$

We consider two cases: if $|d| < |x|$, then we apply the mean-value property to obtain

$$(y^2 + |d|^2)^{p/2} - y^p \le \frac{p}{2}y^{p-2}|d|^2 \le \frac{p}{2}\beta_p^{p-2}|x|^{p-2}|d|^2 \le \frac{C_p^p p(p-1)}{2}|x|^{p-2}|d|^2 \le C_p^p\left[\frac{|x+d|^p + |x-d|^p - 2|x|^p}{2}\right],$$

since $C_p^p(p-1) \ge \beta_p^{p-2}$, as one easily checks. In the second case $|d| \ge |x|$, we write

$$(y^2 + |d|^2)^{p/2} - y^p \le |d|^p \le C_p^p(2^{p-1} - 1)|d|^p \le C_p^p\left[\frac{|x+d|^p + |x-d|^p - 2|x|^p}{2}\right],$$

since $C_p^p(2^{p-1} - 1) \ge 1$.

It remains to establish (2.4) for $y < \beta_p|x|$. Observe that the right-hand side of the estimate does not exceed

$$\frac{1}{2}\Big[U_p\big(x-d, \sqrt{y^2+|d|^2}, u-e, v\big) + U_p\big(x+d, \sqrt{y^2+|d|^2}, u+e, v\big)\Big],$$

since the increase in the variable v makes the function U_p smaller. In the light of (4.1), this is not larger than

$$(y^2 + |d|^2)^{p/2}u - C_p^p\left[\frac{|x+d|^p + |x-d|^p}{2}\right]v$$
$$-\beta_p|x|(y^2 + |d|^2)^{(p-1)/2}v + (y^2 + |d|^2)^{p/2}v$$

(we have used here the triangle inequality $2|x| \le |x+d| + |x-d|$). Hence, we will be done if we prove that

$$\begin{aligned}&((y^2+|d|^2)^{p/2}-y^p)u\\ &\le \Bigg\{ C_p^p\left[\frac{|x+d|^p+|x-d|^p-2|x|^p}{2}\right]\\ &\quad +\beta_p|x|((y^2+|d|^2)^{(p-1)/2}-y^{p-1})-((y^2+|d|^2)^{p/2}-y^p)\Bigg\}v.\end{aligned}$$

Actually, it is enough to show this bound for $u=v$. Indeed, having this done, we will get that the expression in the parentheses on the right is nonnegative; thus, taking v larger than u will make the right-hand side even larger than the left-hand side. So, from now on, we focus on the estimate

$$\begin{aligned}2((y^2+|d|^2)^{p/2}-y^p) \le C_p^p&\left[\frac{|x+d|^p+|x-d|^p-2|x|^p}{2}\right]\\ &+\beta_p|x|((y^2+|d|^2)^{(p-1)/2}-y^{p-1}).\end{aligned}$$

If $\beta_p|x| \ge \frac{2p}{p-1}(y^2+|d|^2)^{1/2}$, the above bound holds true, since, by the mean-value property for the convex function $t \mapsto t^{p/(p-1)}$,

$$2((y^2+|d|^2)^{p/2}-y^p) \le \frac{2p}{p-1}(y^2+|d|^2)^{1/2}((y^2+|d|^2)^{(p-1)/2}-y^{p-1}).$$

So, assume that $\beta_p|x| < \frac{2p}{p-1}(y^2+|d|^2)^{1/2}$. If $|d| \ge |x|$, then

$$\begin{aligned}2((y^2+|d|^2)^{p/2}-y^p) \le 2|d|^p &= C_p^p(2^{p-1}-1)|d|^p\\ &\le C_p^p\left[\frac{|x+d|^p+|x-d|^p-2|x|^p}{2}\right].\end{aligned}$$

On the other hand, if $|d| < |x|$, then $y \ge |x|$ (otherwise, the estimate $\beta_p|x| < \frac{2p}{p-1}(y^2+|d|^2)^{1/2}$ would not hold) and, by the mean-value property,

$$\begin{aligned}2((y^2+|d|^2)^{p/2}-y^p) &\le 2((|x|^2+|d|^2)^{p/2}-|x|^p)\\ &\le p|x|^{p-2}|d|^2 \le C_p^p\left(\frac{|x+d|^p+|x-d|^p-2|x|^p}{2}\right),\end{aligned}$$

since $C_p^p\frac{p(p-1)}{2} \ge p$. This completes the proof of the lemma. □

Proof of (1.7). By Theorem 2.1, we get that for each n,

$$\int_0^1 S_n(f)^p w_n \mathrm{d}s \le \left(\frac{6p}{p-1}\right)^p \int_0^1 |f_n|^p M_{d,n} w \mathrm{d}s.$$

Since w_n is the projection of w and f_n is the projection of f, the above estimate implies, in the light of Jensen's inequality,

$$\begin{aligned}\int_0^1 S_n(f)^p w \mathrm{d}s &\le \left(\frac{6p}{p-1}\right)^p \int_0^1 |f|^p M_{d,n} w \mathrm{d}s\\ &\le \left(\frac{6p}{p-1}\right)^p \int_0^1 |f|^p M_d w \mathrm{d}s.\end{aligned}$$

It remains to let n go to infinity and apply Lebesgue's monotone convergence theorem to get the assertion. $\square$

Remark 4.4. Let us now comment why we have chosen the above values for the constants C_p and β_p. To this end, let us gather all the properties of C_p and β_p which were needed in the above considerations: first, in the proof of the majorization 2° we have needed the estimate

$$C_p^p + \beta_p^p \leq \left(\frac{6p}{p-1}\right)^p. \tag{4.2}$$

Furthermore, in the later calculations we have required the following five inequalities: $C_p^p(p-1) \geq \beta_p^{p-2}$, $C_p^p(2^{p-1}-1) \geq 1$, $C_p^p(2^{p-1}-1) \geq 2$, $\beta_p \geq \frac{2\sqrt{2}p}{p-1}$ and $C_p^p \frac{p(p-1)}{2} \geq p$. Clearly, in the light of (4.2), we should take the smallest β_p allowed: $\beta_p = \frac{2\sqrt{2}p}{p-1}$. From the remaining four lower bounds for C_p, the condition $C_p^p(2^{p-1}-1) \geq 2$ is the strongest; this leads to the choice $C_p^p = 2/(2^{p-1}-1)$ used above.

4.2. A Related Result

This subsection is devoted to the weighted L^p-estimate (1.9), which, in comparison to (1.7), has a different weight standing on the right. The corresponding special function is obtained as a combination of U_1 (the function leading to the weak-type estimate) and a certain interpolation-type argument. Let us start with a technical observation.

Lemma 4.5. *Let $(x, y, u, v) \in \mathfrak{D}$ with $u > 0$. The condition*

$$y \leq 1 \quad \text{and} \quad |x| + \sqrt{1 + \frac{u}{v}(y^2-1)} \leq 1, \tag{4.3}$$

appearing in the definition of D_1, is equivalent to saying that

$$\frac{v}{u}|x| + \sqrt{|x|^2\left(\frac{v^2}{u^2} - \frac{v}{u}\right) + y^2} \leq 1. \tag{4.4}$$

Proof. Suppose that (4.3) holds. Then $1 \geq |x| + \sqrt{1 + \frac{u}{v}(y^2-1)} \geq |x| + \sqrt{1 - \frac{u}{v}}$, and hence

$$|x| \leq 1 - \sqrt{1 - \frac{u}{v}} \leq \frac{u}{v}. \tag{4.5}$$

Furthermore, the second inequality in (4.3) implies $1 + \frac{u}{v}(y^2-1) \leq (1-|x|)^2$, which is equivalent to $|x|^2(\frac{v^2}{u^2} - \frac{v}{u}) + y^2 \leq (1 - \frac{v}{u}|x|)^2$. By (4.5), this yields the validity of (4.4). The reasoning in the reverse direction is similar and left to the reader. $\square$

Thus, if we substitute $X = \frac{v}{u}x$, $Y = \sqrt{|x|^2(\frac{v^2}{u^2} - \frac{v}{u}) + y^2}$, then the function of Sect. 2 is given by

$$U_1(x, y, u, v) = u \begin{cases} Y^2 - |X|^2 & \text{if } |X| + Y \leq 1, \\ 1 - 2|X| & \text{if } |X| + Y > 1. \end{cases}$$

The expression on the right is the famous special function invented by Burkholder (see page 21 in [4] for a slightly transformed formula: one can actually find the function $L(x, y) = 1 - U(x, y)$ there), who used it in the proof of the weak-type inequality for martingale transforms. Now, for any $t > 0$, we have

$$U_1(x/t, y/t, u, v) = u \begin{cases} t^{-2}(Y^2 - |X|^2) & \text{if } |X| + Y \le 1, \\ 1 - 2|X|/t & \text{if } |X| + Y > 1. \end{cases}$$

If we define

$$U_p(x, y, u, v) = \frac{p(p-1)(2-p)}{2} \int_0^\infty t^{p-1} U_1(x/t, y/t, u, v) \mathrm{d}t,$$

then a little calculation (cf. [15]) reveals that

$$\begin{aligned} & U_p(x, y, u, v) \\ & = \frac{p(p-1)(2-p)}{2} u \left[\int_0^{|X|+Y} t^{p-1}(1-2X/t)\mathrm{d}t + \int_{|X|+Y}^\infty t^{p-3}(Y^2 - X^2)\mathrm{d}t \right] \\ & = u(Y - (p-1)^{-1}|X|)(|X| + Y)^{p-1}. \end{aligned}$$

We will require the following majorization.

Lemma 4.6. *For any $(x, y, u, v) \in \mathfrak{D}$ with $u > 0$ we have*

$$U_p(x, y, u, v) \ge p^{p-2}(y^p u - (p-1)^{-p}|x|^p v^p u^{1-p}). \tag{4.6}$$

Proof. Divide both sides by u. Since $y \le Y$, it is enough to show the inequality

$$(Y - (p-1)^{-1}|X|)(|X| + Y)^{p-1} \ge p^{p-2} \left(Y^p - (p-1)^{-p}|X|^p \right).$$

This estimate can be found on p. 17 of Burkholder's survey [4]. □

Proof of (1.9). By a straightforward approximation, we may assume that the weight w is positive almost everywhere. Fix a nonnegative integer n and a positive number t. We know that

$$\int_0^1 U_1(f_n/t, S_n(f)/t, w_n, M_{d,n}w)\mathrm{d}s \le 0,$$

by (2.5) applied to the function f/t. Therefore, by Fubini's theorem and the definition of U_p, we get

$$\int_0^1 U_p(f_n/t, S_n(f)/t, w_n, M_{d,n}w)\mathrm{d}s \le 0$$

(the use of Fubini's theorem is permitted since the functions f_n, $S_n(f)$, w_n and $M_{d,n}w$ take values in a finite set). Combining this with (4.6), we obtain

$$\int_0^1 S_n(f)^p w_n \mathrm{d}s \le (p-1)^{-p} \int_0^1 |f_n|^p (M_{d,n}w)^p w_n^{1-p} \mathrm{d}s.$$

Now, the function $(r, s) \mapsto r^p s^{1-p}$ is convex on $[0, \infty) \times (0, \infty)$: indeed, its Hessian matrix

$$\begin{bmatrix} p(p-1)r^{p-2}s^{1-p} & -p(p-1)r^{p-1}s^{-p} \\ -p(p-1)r^{p-1}s^{-p} & p(p-1)r^p s^{-1-p} \end{bmatrix}$$

is nonnegative definite. Therefore, since (f_n, w_n) is the projection of (f, w), Jensen's inequality implies

$$\int_0^1 |f_n|^p (M_{d,n}w)^p w_n^{1-p} \mathrm{d}s \le \int_0^1 |f|^p (M_{d,n}w)^p w^{1-p} \mathrm{d}s$$
$$\le \int_0^1 |f|^p (M_d w)^p w^{1-p} \mathrm{d}s.$$

It remains to note that

$$\int_0^1 S_n(f)^p w_n \mathrm{d}s = \int_0^1 S_n(f)^p w \mathrm{d}s$$

and let $n \to \infty$ to obtain the assertion. □

5. Moment Inequality, $p \ge 2$

5.1. On the Estimate (1.8)

This time, the function $V_p : \mathfrak{D} \to \mathbb{R}$ is given by

$$V_p(x, y, u, v) = \frac{2}{p}\left(y^p u - \left(\frac{p}{2}\right)^{p/2} |x|^p v^{p/2} u^{1-p/2}\right),$$

while the special function $U_p : \mathfrak{D} \to \mathbb{R}$ is

$$U_p(x, y) = y^p u - \frac{p}{2} y^{p-2} |x|^2 v.$$

Let us verify that the above functions have all the necessary properties.

Lemma 5.1. *The functions U_p and V_p enjoy* $1°$ *and* $2°$.

Proof. The first inequality is obvious: $U_p(x, |x|, u, u) = \left(1 - \frac{p}{2}\right)|x|^p u \le 0$. The second estimate follows at once from the mean-value property applied to the function $t \mapsto t^{p/2}$, since

$$U_p(x, y, u, v) = (yu^{1/p})^p - \frac{p}{2}(yu^{1/p})^{p-2}|x|^2 v u^{-1+2/p}.$$ □

As previously, the main difficulty lies in proving the concavity-type condition. However, this time the calculations are relatively short and simple.

Lemma 5.2. *The function U_p satisfies the property* $3°$.

Proof. The right-hand side of (2.4) is equal to

$$\begin{aligned}
&(y^2 + |d|^2)^{p/2} u \\
&\quad - \frac{p}{4}(y^2 + |d|^2)^{p/2-1}(|x + d|^2(v \vee (u + e)) + |x - d|^2(v \vee (u - e))) \\
&\le (y^2 + |d|^2)^{p/2} u - \frac{p}{4}(y^2 + |d|^2)^{p/2-1}(|x + d|^2 v + |x - d|^2 v) \\
&= (y^2 + |d|^2)^{p/2} u - \frac{p}{2}(y^2 + |d|^2)^{p/2-1}(|x|^2 + |d|^2) v \\
&= U_p(x, y, u, v) \\
&\quad + [(y^2 + |d|^2)^{p/2} - y^p]u - \frac{p}{2}[(y^2 + |d|^2)^{p/2-1}(|x|^2 + |d|^2) - y^{p-2}|x|^2]v.
\end{aligned}$$

Since $(y^2+|d|^2)^{p/2}-y^p\geq 0$ and $u\leq v$, we will be done if we show that

$$(y^2+|d|^2)^{p/2}-y^p\leq \frac{p}{2}[(y^2+|d|^2)^{p/2-1}(|x|^2+|d|^2)-y^{p-2}|x|^2],$$

or

$$(y^2+|d|^2)^{p/2-1}\left[y^2+|d|^2-\frac{p}{2}(|x|^2+|d|^2)\right]\leq y^{p-2}\left[y^2-\frac{p}{2}|x|^2\right]. \quad (5.1)$$

To prove this, it is enough to show that the function

$$F(s)=(y^2+s)^{p/2-1}\left[y^2+s-\frac{p}{2}(|x|^2+s)\right]$$

is nonincreasing on $[0,\infty)$. But this is straightforward: a direct differentiation shows that

$$F'(s)=-\frac{p}{2}\left(\frac{p}{2}-1\right)(y^2+s)^{p/2-2}(|x|^2+s)\leq 0. \quad \square$$

Proof of (1.8). The argument goes along the same lines as that in the proof of (1.9). We leave the straightforward modifications to the reader. □

Let us make a comment analogous to Remark 4.4 above.

Remark 5.3. There is a natural question whether the parameter $p/2$ appearing in the definition of U_p can be decreased: this would lead to the improvement of the L^p-constant in our main estimate. It turns out that if one replaces this parameter with some $\alpha<p/2$, then the above proof does not work. Indeed, the resulting function

$$F(s)=(y^2+s)^{p/2-1}\left[y^2+s-\alpha(|x|^2+s)\right]$$

satisfies

$$F'(0+)=\left(\frac{p}{2}-\alpha\right)y^{p-2}-\alpha\left(\frac{p}{2}-1\right)y^{p-4}x^2$$

which is positive for some x, y; therefore, (5.1) is not satisfied for some x, y and d.

5.2. Lack of Fefferman–Stein Inequalities for $p>2$

Now we will present a counterexample showing that for any $p>2$ there is no finite C_p such that

$$\|S(f)\|_{L^p(w)}\leq C_p\|f\|_{L^p(M_dw)}$$

for all f and w, even if $\mathcal{H}=\mathbb{R}$. Actually, there is an example which works for all p simultaneously. Fix a (large) integer N and set

$$w=2^N\chi_{[0,2^{-N})}=h_0+h_1+2h_2+4h_4+\cdots+2^{N-1}h_{2^{N-1}}.$$

We have $w_0=\chi_{[0,1)}$ and $w_{2^k}=2^{k+1}\chi_{[0,2^{-k-1})}$ for any $0\leq k\leq N-1$. Furthermore, for any $0\leq k\leq N-2$ and $\ell\in\{2^k,2^k+1,\ldots,2^{k+1}-1\}$ we have $w_\ell=w_{2^k}$; and $w_\ell=w_{2^{N-1}}$ for $\ell\geq 2^{N-1}$. Consequently, we see that

$$M_dw=2^N\chi_{[0,2^{-N})}+\sum_{k=0}^{N-1}2^k\chi_{[2^{-k-1},2^{-k})}$$

and hence $\int_0^1 M_dw=1+N/2$. Next, set $f=h_0-h_1+h_2-h_4+h_8-\cdots+(-1)^Nh_{2^{N-1}}$. On $[2^{-k-1},2^{-k})$, we have $h_0=h_1=\cdots=h_{2^{k-1}}=1$ and

$h_{2^k} = -1$, so $|f| \leq 2$ there; similarly, on $[0, 2^{-N})$ we have $h_0 = h_1 = h_2 = h_4 = \cdots = h_{2^{N-1}} = 1$, so $|f| \leq 2$ there. Thus,

$$\int_0^1 |f|^p M_d w \leq 2^p(1 + N/2). \tag{5.2}$$

On the other hand, on $[0, 2^{-N})$ we have $S(f) = \sqrt{N+1}$ and hence

$$\int_0^1 S(f)^p w = 2^{-N} \cdot (N+1)^{p/2} \cdot 2^N = (N+1)^{p/2}.$$

Comparing this to (5.2), we conclude that there are no Fefferman–Stein inequalities for the square function in the case $p > 2$.

6. Martingale Inequalities

All the results established above have their counterparts in the martingale theory. Let us start with introducing the necessary background and notation. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with the continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$. We assume that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$ be an adapted continuous-path and uniformly integrable martingale taking values in $\mathcal{H}$ (recall that we have assumed that $\mathcal{H} = \ell_2$). Denote by $[X] = ([X]_t)_{t\geq 0}$ the square bracket (quadratic variation) associated to X; see Dellacherie and Meyer [8] for the definition in the case when X takes values in $\mathbb{R}$, and extend it to the vector setting by the formula $[X]_t = \sum_{j=1}^\infty [X^j]_t$, where X^j stands for the j-th coordinate of X. Next, let $W = (W_t)_{t\geq 0}$ be a weight, i.e., a nonnegative, continuous-path and uniformly integrable martingale, and let $W^* = \sup_{s\geq 0} W_s$, $W^*_t = \sup_{0\leq s\leq t} W_s$ be the maximal and truncated maximal functions of W.

We are ready to formulate the main result of this section. In analogy to the notation used in the previous sections, we set $\|X_\infty\|_{L^p(W_\infty)} = (\mathbb{E}|X_\infty|^p W_\infty)^{1/p}$ for all $1 \leq p < \infty$.

Theorem 6.1. *For any X, W as above, we have the following*:

$$\mathbb{E}1_{\{[X]_\infty \geq 1\}} W_\infty \leq 2\|X_\infty\|_{L^1(W^*)}, \tag{6.1}$$

$$\|[X]_\infty^{1/2}\|_{L^p(W_\infty)} \leq \frac{6p}{p-1}\|X_\infty\|_{L^p(W^*)}, \quad 1 < p < 2, \tag{6.2}$$

$$\|[X]_\infty^{1/2}\|_{L^p(W_\infty)} \leq (p-1)^{-1}\|X_\infty\|_{L^p((W^*)^p W_\infty^{1-p})}. \tag{6.3}$$

and

$$\|[X]_\infty^{1/2}\|_{L^p(W_\infty)} \leq \sqrt{\frac{p}{2}}\|X_\infty\|_{L^p((W^*)^{p/2} W_\infty^{1-p/2})}, \quad p \geq 2. \tag{6.4}$$

Proof. We will focus on proving (6.1); the reasoning leading to the remaining estimates is similar and left to the reader. It is convenient to split the proof into two parts.

Step 1. Reductions. With no loss of generality, we may assume that X, $[X]$ and W are bounded. Indeed, fix a large positive number M, consider the stopping time $\tau_M = \inf\{t : |X_t| + [X]_t + W_t \geq M\}$ and the bounded processes

$X_t^M = X_t 1_{\{\tau_M>0\}}$, $W_t^M = W_t 1_{\{\tau_M>0\}}$, $t \geq 0$. Having established (6.1) for X^M and W^M, we let $M \to \infty$ and obtain the weak-type estimate in full generality. The proof of (6.1) rests on Itô's formula applied to the function U_1 and the process $(X,[X],W,W^*)$. However, there are two problems which need to be overcome. First, there is no Itô's formula for processes taking values in infinite-dimensional Hilbert spaces, see [13]; furthermore, the function U_1 does not have the necessary regularity (it is not of class C^2). The first obstacle is removed very easily: clearly, it is enough to show the weak-type bound for $\mathbb{R}^n$-valued martingales, and then let $n \to \infty$. To handle the second issue, we will use an additional mollification argument. Let $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be a C^∞ function, supported on the unit ball of $\mathbb{R}^{n+3}$ and satisfying $\int_{\mathbb{R}^{n+3}} g = 1$. Given $\delta > 0$, define a function $U_1^\delta : \{(x,y,u,v) \in \mathbb{R}^n \times [0,\infty) \times [0,\infty) \times [0,\infty) : u \leq v\} \to \mathbb{R}$ by the convolution

$$\begin{aligned} &U_1^\delta(x,y,u,v)\\ &\quad= \int_{\mathbb{R}^{n+3}} U_1(x+\delta s_1, y+\delta+\delta s_2, u+\delta+\delta s_3, v+5\delta+\delta s_4) g(s_1,s_2,s_3,s_4)\mathrm{d}s. \end{aligned}$$

This function is of class C^∞. Furthermore, we see that

$$U_{1v}^\delta(x,y,u,v) \leq 0, \tag{6.5}$$

since U_1 is nonincreasing as a function of its fourth variable. Finally, as we will show now, the property (2.4) (which is satisfied by U_1) implies a certain differential inequality for U_1^δ. Namely, take $(x,y,u,v) \in \mathbb{R}^n \times [0,\infty) \times (0,\infty)^2$ satisfying $u \leq v$, and fix $d \in \mathbb{R}^n$, $e \in \mathbb{R}$ and $t > 0$. If t is sufficiently small (so that $u \pm te \geq 0$), then (2.4) for U_1 guarantees

$$\begin{aligned} &U_1^\delta(x,y,u,v)\\ &\quad\geq \frac{1}{2}\Big[U_1^\delta(x-td, \sqrt{y^2+t^2|d|^2}, u-te, v) + U_1^\delta(x+td, \sqrt{y^2+t^2|d|^2}, u+te, v)\Big]. \end{aligned}$$

Now put all the terms on the right-hand side, divide throughout by t^2 and let t go to zero. As the result, we get that

$$\begin{aligned} &\langle U_{1xx}^\delta(x,y,u,v)d,d\rangle + U_{1y}^\delta(x,y,u,v)|d|^2 + U_{1uu}^\delta(x,y,u,v)e^2\\ &\quad+ 2\langle U_{1xu}^\delta(x,y,u,v)e,d\rangle \leq 0. \end{aligned} \tag{6.6}$$

Step 2. Application of Itô's formula. Introduce the auxiliary process $Z_t = (X_t, [X]_t^{1/2}, W_t, W_t^*)$, $t \geq 0$. By the above assumptions, we see that Z is bounded and has continuous paths. If we fix $t \geq 0$ and apply Itô's formula, we get

$$U^\delta(Z_t) = U^\delta(Z_0) + I_1 + I_2 + I_3/2, \tag{6.7}$$

where

$$I_1 = \int_0^t U^\delta_{1x}(Z_s)\mathrm{d}X_s + \int_0^t U^\delta_{1u}(Z_s)\mathrm{d}W_s,$$
$$I_2 = \int_0^t U^\delta_{1v}(Z_s)\mathrm{d}W^*_s,$$
$$I_3 = \int_0^t U^\delta_{1y}(Z_s)\mathrm{d}[X]_s + \frac{1}{2}\int_0^t D^2_{xu}U^\delta_1(Z_s)\mathrm{d}[X,W]_s.$$

Here the second integral in I_3 is the shortened notation for the sum of all the relevant second-order terms, i.e.,

$$\begin{aligned}\int_0^t D^2_{xu}U^\delta_1(Z_s)\mathrm{d}[X,W]_s = &\sum_{j,k=1}^n \int_0^t U^\delta_{1x_jx_k}(Z_s)\mathrm{d}[X^j,X^k]_s\\ &+2\sum_{j=1}^n \int_0^t U^\delta_{1x_ju}(Z_s)\mathrm{d}[X^j,W]_s\\ &+\int_0^t U^\delta_{1uu}(Z_s)\mathrm{d}[W]_s.\end{aligned}$$

Let us look at the terms I_1, I_2 and I_3 separately. By properties of stochastic integrals, the first term is a martingale (since X and W are bounded). The second term is nonpositive, by (6.5) and the fact that the process W^* is nondecreasing. Finally, I_3 is also nonpositive, which follows directly from (6.6). Indeed, pick the sequence $s_k = jt/N$, $j = 0, 1, 2, \ldots, N$, and, for any j, apply (6.6) to $x = X_{s_{j-1}}$, $y = [X]^{1/2}_{s_{j-1}}$, $u = W_{s_{j-1}}$, $v = W^*_{s_{j-1}}$, $d = X_{s_j} - X_{s_{j-1}}$ and $e = W_{s_j} - W_{s_{j-1}}$. Summing the obtained inequalities over j and letting $N \to \infty$ we get $I_3 \leq 0$.

Therefore, coming back to (6.7), we see that we have proved the estimate

$$\mathbb{E}U^\delta_1(Z_t) \leq \mathbb{E}U^\delta_1(Z_0).$$

But U_1 is a continuous function, so $U^\delta_1 \to U_1$ pointwise as $\delta \to 0$. Therefore, if we combine this with the boundedness of the process Z, we get

$$\mathbb{E}U_1(X_t, [X]^{1/2}_t, W_t, W^*_t) \leq \mathbb{E}U_1(X_0, [X]^{1/2}_0, W_0, W^*_0).$$

However, we have $[X_0]^{1/2} = |X_0|$ and $W^*_0 = W_0$; thus, by (2.2), the right-hand side above is nonpositive. Plugging the majorization (2.3) yields

$$\mathbb{E}W_t 1_{\{[X]_t \geq 1\}} \leq 2\mathbb{E}|X_t|W^*_t.$$

It remains to carry out the appropriate limiting procedure, which goes along the same lines as in the analytic setting. Since $W_t = \mathbb{E}(W_\infty|\mathcal{F}_t)$ and $X_t = \mathbb{E}(X_\infty|\mathcal{F}_t)$, the above estimate implies that $\mathbb{E}W_\infty 1_{\{[X]_t\geq 1\}} \leq 2\mathbb{E}|X_\infty|W^*_t \leq 2\mathbb{E}|X_\infty|W^*$, by conditional Jensen's inequality. Letting $t \to \infty$ gives

$$\mathbb{E}W_\infty 1_{\{[X]_\infty > 1\}} \leq 2\mathbb{E}|X_\infty|W^*_t \leq 2\mathbb{E}|X_\infty|W^*,$$

and to get the non-strict estimate under the indicator, one applies the latter bound to X/η, $\eta \in (0, 1)$, and lets $\eta \to 1$ (see the proof of (1.6) above). $\square$

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