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Integral Equations and Operator Theory

Erratum

Erratum to: Closed-Range Composition Operators on $\mathbb{A}^2$ and the Bloch Space

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1. Introduction

We thank Nina Zorboska for pointing out an error in the proof of [2, Proposition 3.1]; a faux pas in our application of Julia's Theorem. Indeed, we find that Proposition 3.1 does not hold in general (see Example 2.4 in this erratum). This impacts two operator-theoretic results in Section 3 of [2]; namely, Theorems 3.5 and 3.7. Theorem 3.5 (in [2]) does not hold in general. While statements (i) and (ii) of Theorem 3.7 are equivalent (cf. [1, Theorem 2.4]), statement (iii) implies, but is not a consequence of (i) and (ii) (in general). Both of these theorems hold under an additional hypothesis. We make all of this clear in our work here. And, although [2, Corollary 3.6] was established using Theorem 3.5, we find that this corollary holds in general; see the proof of Corollary 1.6 in this erratum. In order that our presentation be somewhat self-contained, we begin by reviewing the terminology of [2]. Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and let $\mathbb{T}$ denote its boundary $\{z : |z| = 1\}$. Let A denote normalized two-dimensional Lebesgue measure on $\mathbb{D}$ and let m denote normalized Lebesgue measure on $\mathbb{T}$; normalized so that these are probability measures. The Bergman space $\mathbb{A}^2$ is the Hilbert space of functions f that are analytic in $\mathbb{D}$ such that

$$||f||^2_{\mathbb{A}^2} := \int_{\mathbb{D}} |f|^2 \, dA < \infty.$$

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And the Bloch space $\mathcal{B}$ is the Banach space of functions f that are analytic in $\mathbb{D}$ such that

$$|||f||_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

A function φ that is analytic in $\mathbb{D}$ and that satisfies $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ is called an *analytic self-map* of $\mathbb{D}$. Any such function gives rise to a bounded composition operator $C_\varphi(f) := f \circ \varphi$ on both $\mathbb{A}^2$ and $\mathcal{B}$. With φ as above, and for $\varepsilon > 0$, let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$, let $\Lambda_\varepsilon = \{z \in \mathbb{D} : \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} > \varepsilon\}$, let $G_\varepsilon = \varphi(\Omega_\varepsilon)$, let $F_\varepsilon = \varphi(\Lambda_\varepsilon)$ and let $K = \mathbb{T} \cap \overline{\Omega}_\varepsilon$. By Julia's Theorem (cf. [9, page 63]), φ has a nontangential limit $\varphi^*(\zeta)$ at each point ζ in K and by the Julia-Carathéodory Theorem (cf. [9, page 57]), φ has an angular derivative $\varphi'(\zeta)$ at every such point; bounded above, in modulus, by $1/\varepsilon$. For any point ζ in $\mathbb{T}$ and any $\theta, 0 < \theta < \pi$, let $S(\zeta, \theta)$ denote the interior of closed convex hull of $\{\zeta\} \cup \{z : |z| \le \sin(\frac{\theta}{2})\}$; the so-called *Stolz region* based at ζ with vertex angle θ. For such θ, define $W_\varepsilon(\theta)$ by:

$$W_\varepsilon(\theta) = \bigcup_{\zeta\in K} S(\zeta, \theta);$$

and let W_ε denote $W_\varepsilon(\frac{\pi}{2})$. For any point z in $\mathbb{D}$ and any $s, 0 < s < 1$, let $D(z, s) = \{w \in \mathbb{D} : \rho(z, w) < s\}$, where $\rho(z, w) := |\frac{z-w}{1-\overline{w}z}|$ is the *pseudohyperbolic* distance between z and w. One may consult [3], [4] or [7] as good general references for material related to our work here.

As we mentioned earlier, the proof of [2, Proposition 3.1] has an error in it relating to our application of Julia's Theorem. The statement of this proposition is as follows.

Proposition 1.1. *Given the terminology of the above discussion, φ is continuous on $\overline{\Omega}_\varepsilon$ and φ' is continuous on K.*

The continuity of φ on $\overline{\Omega}_\varepsilon$ is not in question here, as it was established in [1]; see Remark 2.6 in this reference. If $|\varphi'|$ were continuous on K, then, by the continuity of φ on $\overline{\Omega}_\varepsilon$, we would find that φ' itself is continuous on K. By the Julia-Carathéodory Theorem, $|\varphi'|$ is lower semi-continuous on K; but no more can be said than this, in general (see Example 2.4 in this erratum). However, if K is the union of finitely many closed subarcs of $\mathbb{T}$, then φ' is continuous on K and [2, Proposition 3.1] holds, as does [2, Theorem 3.4], whose statement is given below; see the proof of Theorem 2.1 in this erratum.

Theorem 1.2. *Assuming the terminology of Discussion* 3.2, *φ' is continuous on $\overline{W}_\varepsilon$.*

Two operator-theoretic results (in Section 3 of [2]) are effected by our revision of [2, Proposition 3.1]; namely, Theorems 3.5 and 3.7. Their statements are as follows.

Theorem 1.3. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed-range on $\mathbb{A}^2$, then there exist ε and $s, 0 < \varepsilon, s < 1$, such that $\{z : s \le |z| < 1\} \subseteq F_\varepsilon$.*

Theorem 1.4. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

(i) *C_φ is closed-range on $\mathbb{A}^2$.*
(ii) *There exist ε, s and $c, 0 < \varepsilon, s, c < 1$, such that*

$$A(G_\varepsilon \cap D(z,s)) \geq cA(D(z,s)),$$

for all z in $\mathbb{D}$.
(iii) *There exist ε and $s, 0 < \varepsilon, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq G_\varepsilon$.*

The first of these does not hold in general; see Example 2.4 in this erratum. The equivalence of statements (i) and (ii) in the second holds in general; this is the content of [1, Theorem 2.4]. And, in general, statement (iii) implies statements (i) and (ii). But the converse of this does not hold in general; once again, see Example 2.4 in this erratum. Under an additional and quite natural hypothesis (see "Additional Hypothesis" and Remark 1.5 below), these results (i.e., Theorems 3.5 and 3.7 of [2]) hold. We proceed to describe this additional hypothesis, beginning with the assumption that C_φ is closed-range on $\mathbb{A}^2$, which we know is equivalent to: there exist ε, s and $c, 0 < \varepsilon, s, c < 1$, such that

$$A(G_\varepsilon \cap D(z,s)) \geq cA(D(z,s)),$$

for all z in $\mathbb{D}$. From this we find that φ has an angular derivative at every point of $K := \mathbb{T} \cap \overline{\Omega}_\varepsilon$ that is bounded above, in modulus, by $1/\varepsilon$ and $\varphi^*(K) = \mathbb{T}$; for this, see in [1] both the proof of Theorem 2.5 and Remark 2.6.

Additional Hypothesis. If one can choose $\varepsilon > 0$ sufficiently small so that K contains finitely many closed subarcs $\{J_\nu\}_{\nu=1}^N$ of $\mathbb{T}$ such that $\varphi^*(\cup_{\nu=1}^N J_\nu) = \mathbb{T}$, then the conclusion of [2, Theorem 3.5] holds as does statement (iii) of [2, Theorem 3.7]; see Theorem 2.1 in this erratum and the proof of [2, Theorem 3.5].

Remark 1.5. Let B be a Blaschke product and let σ_B be the compact subset of $\mathbb{T}$ consisting of the set of accumulation points of the zeros of B. If I is a component of $\mathbb{T} \setminus \sigma_B$, then I is an open subarc of $\mathbb{T}$ and so also is $B^*(I)$; see the proof of [1, Lemma 3.1]. Therefore, if $B^*(\mathbb{T} \setminus \sigma_B) = \mathbb{T}$, then, by a compactness argument, there are finitely many components $\{I_\nu\}_{\nu=1}^N$ of $\mathbb{T} \setminus \sigma_B$ such that $B^*(\cup_{\nu=1}^N I_\nu) = \mathbb{T}$. Now any closed subarc J_ν of I_ν $(1 \leq \nu \leq N)$ is contained in $K := \mathbb{T} \cap \overline{\Omega}_\varepsilon$, provided $\varepsilon > 0$ is sufficiently small. Therefore, if $B^*(\mathbb{T} \setminus \sigma_B) = \mathbb{T}$, then one can choose $\varepsilon > 0$ sufficiently small so that K contains finitely many closed subarcs $\{J_\nu\}_{\nu=1}^N$ of $\mathbb{T}$ such that $B^*(\cup_{\nu=1}^N J_\nu) = \mathbb{T}$. Thus, the conclusion of [2, Theorem 3.5] holds as does statement (iii) of [2, Theorem 3.7], if φ is a Blaschke product B satisfying: $B^*(\mathbb{T} \setminus \sigma_B) = \mathbb{T}$.

In [2], the authors made use of Theorem 3.5 (i.e., Theorem 1.3 above) to prove Corollary 3.6; whose statement appears below.

Corollary 1.6. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed-range on $\mathbb{A}^2$, then it is also closed-range on $\mathcal{B}$.*

This corollary holds in general; without any modifications. We now give a proof of this result that is independent of the work in [2, Section 3].

Proof. Since C_φ is closed-range on $\mathcal{B}$ if and only if $C_{\varphi_\alpha \circ \varphi}$ is, where $\varphi_\alpha(z) := \frac{\alpha - z}{1-\bar{\alpha}z}$ for some point α in $\mathbb{D}$, we may assume that $\varphi(0) = 0$. Now, suppose that C_φ is closed-range on $\mathbb{A}^2$. Then, by [1, Theorem 2.4], there exist ε, s and $c, 0 < \varepsilon, s, c < 1$, such that

$$A(G_\varepsilon \cap D(a,s)) \geq cA(D(a,s)), \tag{1.6.1}$$

for any point a in $\mathbb{D}$. By [2, Corollary 2.3], our goal is reached if we show that there is a positive lower bound for

$$||\varphi_\alpha \circ \varphi||_{\mathcal{B}/C} := \sup_{z\in\mathbb{D}} |(\varphi_\alpha \circ \varphi)'(z)|(1-|z|^2),$$

independent of α in $\mathbb{D}$. To this end, choose α in $\mathbb{D}$ and let $\Delta_\alpha = \{z \in \Omega_\varepsilon : \varphi(z) \in D(\alpha, s)\}$. Now,

$$\begin{aligned} ||\varphi_\alpha \circ \varphi||_{\mathcal{B}/C} &= \sup_{z\in\mathbb{D}} |\varphi_\alpha'(\varphi(z))||\varphi'(z)||(1-|z|^2) \\ &\geq \sup_{z\in\Delta_\alpha} |\varphi_\alpha'(\varphi(z))||\varphi'(z)||(1-|z|^2) \\ &\geq \sup_{z\in\Delta_\alpha} \varepsilon|\varphi_\alpha'(\varphi(z))||\varphi'(z)||(1-|\varphi(z)|^2) \\ &\geq \beta \cdot \sup_{z\in\Delta_\alpha} |\varphi'(z)|, \end{aligned}$$

where β is a positive real number independent of α. We proceed to show that there is a positive lower bound for $\sup_{z\in\Delta_\alpha} |\varphi'(z)|$, independent of α in $\mathbb{D}$. Let $c_\alpha = \sup_{z\in\Delta_\alpha} |\varphi'(z)|^2$. Now, by (1.6.1),

$$cA(D(\alpha,s)) \leq A(\varphi(\Delta_\alpha)) \leq \int_{\Delta_\alpha} |\varphi'(z)|^2 dA(z) \leq c_\alpha A(\Delta_\alpha). \tag{1.6.2}$$

Claim. There is a positive constant M, independent of α, such that

$$A(\Delta_\alpha) \leq M \cdot A(D(\alpha,s)).$$

To establish this claim we turn to the discussion on page 313 of [10] to find that $C_\varphi^* C_\varphi$ is the Toeplitz operator T_μ on $\mathbb{A}^2$ whose symbol μ is the Borel measure on $\mathbb{D}$ given by: $\mu(E) := A(\varphi^{-1}(E))$. Clearly T_μ is bounded on $\mathbb{A}^2$, since it is the composition of two bounded operators. Thus, by [10, Theorem 7.5], μ is a Carleson measure for $\mathbb{A}^2$. Hence, by [10, Theorem 7.4], and since $\Delta_\alpha \subseteq \varphi^{-1}(D(\alpha,s))$, there is a positive constant M, independent of α, such that

$$A(\Delta_\alpha) \leq A(\varphi^{-1}(D(\alpha,s))) = \mu(D(\alpha,s)) \leq M \cdot A(D(\alpha,s)).$$

So, our claim holds and hence, by (1.6.2), there is a positive lower bound for c_α, independent of α; which completes our proof. □

The rest of [2, Section 3], which includes the examples given there, still stands and needs no revision. The same applies to the results in Sections 2 and 4 of [2], all of which were established independent of the aforementioned results of Section 3.

2. A Resolution of Things Pertaining to Ω_ε

In this, the last section of the erratum, we establish results that support the revisions we outlined in Section 1; keeping the same notation.

Theorem 2.1. *Let φ be an analytic self-map of $\mathbb{D}$ and let F be a compact subset of $\mathbb{T}$ where φ has an angular derivative $\varphi'(\zeta)$ at each point ζ in F, such that $|\varphi'|$ is bounded on F. If F can be expressed as a finite union of closed subarcs of $\mathbb{T}$ and $\varphi^*(F) = \mathbb{T}$, then there exist ε and $s, 0 < \varepsilon, s < 1$, such that*

$$\{w : s < |w| < 1\} \subseteq G_\varepsilon.$$

Proof. We use the assumption that F can be expressed as a finite union of closed subarcs of $\mathbb{T}$ to show that φ' is continuous (and more) on F. Paradoxically, Julia's Theorem plays a central role in this. For $0 < \theta < \pi$, let

$$W_F(\theta) = \bigcup_{\zeta \in F} S(\zeta, \theta).$$

Claim. For any $\theta, 0 < \theta < \pi$, there exists $\varepsilon > 0$ such that $W_F(\theta) \subseteq \Omega_\varepsilon$.

By our hypothesis, there is a constant $M > 1$ such that $|\varphi'(\zeta)| \leq M$, for all ζ in F. For any particular point ζ in F, the nontangential limit of φ exists at ζ and its value $\xi := \varphi^*(\zeta) \in \mathbb{T}$. By the Julia-Carathéodory Theorem (cf. [9, page 57]) and Julia's Theorem (cf. [9, page 63]), $\varphi(H(\zeta, \lambda)) \subseteq H(\xi, M\lambda)$ for any $\lambda > 0$; where $H(\zeta, \lambda) := \{z : |1 - z\bar{\zeta}|^2 < \lambda(1 - |z|^2)\}$ is a *horodisk* based at ζ. Notice that for $0 < \lambda \leq 1$, there is a single point in $\partial H(\zeta, \lambda)$ of smallest modulus and its distance from $\mathbb{T}$ is $\frac{2\lambda}{1+\lambda}$. For $0 < \lambda \leq \frac{1}{M}$, let $\Gamma_\lambda(\zeta, \theta) = S(\zeta, \theta) \cap \partial H(\zeta, \lambda)$. By basic geometric considerations, there is a positive constant c, depending only on θ, such that $1 - |z| \geq 2c\lambda/(1 + \lambda)$ for all z in $\Gamma_\lambda(\zeta, \theta)$. Since $\varphi(H(\zeta, \lambda)) \subseteq H(\xi, M\lambda)$ and φ is continuous on $\mathbb{D}$, we find that $\varphi(\Gamma_\lambda(\zeta, \theta))$ is contained in the closure of $H(\xi, M\lambda)$ and hence:

$$\frac{1 - |z|}{1 - |\varphi(z)|} \geq \frac{c}{M},$$

whenever $z \in \Gamma_\lambda(\zeta, \theta)$ and $0 < \lambda \leq \frac{1}{M}$. From this it follows that there exists $a, 0 < a < 1$, such that $\{z \in W_F(\theta) : a < |z| < 1\} \subseteq \Omega_{\frac{c}{2M}}$. Thus, if $\varepsilon > 0$ is sufficiently small, then $W_F(\theta) \subseteq \Omega_\varepsilon$; and our claim holds. Notice that this claim holds whether or not F is a finite union of closed subarcs of $\mathbb{T}$. Define $\tilde{\varphi}$ on $\overline{\Omega}_\varepsilon$ by: $\tilde{\varphi}(z) = \varphi(z)$, if $z \in \mathbb{D}$ and $\tilde{\varphi}(\zeta) = \varphi^*(\zeta)$ if $\zeta \in \mathbb{T}$. Now, since $F \subseteq \overline{W}_F(\theta) \subseteq \overline{\Omega}_\varepsilon$ and $\tilde{\varphi}$ is continuous on $\overline{\Omega}_\varepsilon$ (see [1, Remark 2.6]), φ^* is continuous on F. Therefore, since $\varphi^*(F) = \mathbb{T}$, we may assume that each of the finitely many closed subarcs of $\mathbb{T}$ whose union is F is nondegenerate; that is, of the form: $J := \{e^{it} : t_1 \leq t \leq t_2\}$, where $0 < t_2 - t_1 \leq 2\pi$. Since φ is an analytic self-map of $\mathbb{D}$, it can be expressed as a product $fS_\mu B$, where f is an outer function whose modulus on $\mathbb{D}$ is bounded by $1, S_\mu$ is a singular inner function and B is a Blaschke product. And since φ has an angular derivative at each point in $J, \mu(J) = 0$ and $|f^*(\zeta)| = 1$ for all ζ in J. Moreover, by our claim above, $W_F(\theta)$ contains at most finitely many zeros of B. Thus, φ has an analytic continuation to $\mathcal{O}_d := \{re^{it} : \frac{1}{d} < r < d$ and

$t_1 < t < t_2\}$, for some constant $d > 1$. Let ζ_0 be one of the endpoints of J, either e^{it_1} or e^{it_2}. By the Julia-Carathéodory Theorem (cf. [9, page 57]), φ as an analytic self-map of $\mathbb{D}$ is conformal at ζ_0. But also, by [8, Proposition 4.9], φ as a function on $\mathcal{O}_d$ is conformal at ζ_0. This tells us that $\varphi(W_F(\theta))$ contains a set of the form $\{re^{it} : b < r < 1 \text{ and } e^{it} \in \varphi^*(J)\}$, for some constant $b, 0 < b < 1$. Piecing these things together, there exists $s, 0 < s < 1$, such that $\{w : s < |w| < 1\} \subseteq \varphi(W_F(\theta)) \subseteq G_\varepsilon$; and hence our proof is complete. □

Corollary 2.2. *Let φ be an analytic self-map of $\mathbb{D}$. If there exists $\varepsilon > 0$ such that $\mathbb{T} \cap \overline{\Omega}_\varepsilon$ contains a compact set F that is the union of finitely many closed subarcs of $\mathbb{T}$, such that $\varphi^*(F) = \mathbb{T}$, then there exist $\varepsilon^* > 0$ and $s, 0 < s < 1$, such that $\{w : s < |w| < 1\} \subseteq G_{\varepsilon^*}$.*

Proof. Now, by the Julia-Carathéodory Theorem (cf. [9, page 57]), φ has an angular derivative $\varphi'(\zeta)$ at each point ζ in F, and $|\varphi'|$ is bounded on F by $1/\varepsilon$. We can now apply Theorem 2.1 above to find $\varepsilon^* > 0$ and $s, 0 < s < 1$, such that $\{w : s < |w| < 1\} \subseteq G_{\varepsilon^*}$. □

Remark 2.3. Under the hypothesis of Theorem 2.1 above, we find that φ' is continuous on $\overline{W}_F(\theta)$; review the proof of this result. Therefore, Corollary 2.2 not only gives us conditions under which [2, Theorem 3.7] holds, but also conditions under which [2, Theorem 3.5] holds.

We end our analysis of Ω_ε by showing that the "finitely many" assumption in the results above is sharp. In fact, the example we give shows that this and another assumption in a quite separate result are both sharp. Recall that if φ is a univalent analytic self-map of the unit disk, then C_φ is closed-range on $\mathbb{A}^2$ if and only if φ has the form: $\varphi(z) = e^{i\theta}\frac{\alpha - z}{1 - \bar{\alpha}z}$, where α is some point in $\mathbb{D}$; cf. [1, Theorem 2.5]. Therefore, in the case that φ is univalent, [2, Theorems 3.5 and 3.7] hold with a vengeance. Our next example shows that there is a two-valent analytic self-map φ of the unit disk for which [2, Theorem 3.5] does not hold and for which statement (iii) of Theorem 3.7 (in [2]) is not equivalent to statements (i) and (ii); and by two-valent we mean that $\varphi^{-1}(\{w\})$ contains at most two points, for every point w in $\mathbb{D}$. We add in passing that, for any analytic self-map φ of the unit disk and for any $\varepsilon > 0, \varphi$ is boundedly-valent on Ω_ε; that is, there is a positive integer N such that for any point w in $\mathbb{D}, \Omega_\varepsilon \cap \varphi^{-1}(\{w\})$ contains at most N points. This follows quite easily from the definition of Ω_ε and the fact that $N_\varphi(w) = O(1 - |w|)$, where N_φ is the Nevanlinna counting function of φ.

Example 2.4. We now construct an example of a two-valent analytic self-map φ of $\mathbb{D}$, such that C_φ is closed-range on $\mathbb{A}^2$ and yet $\{w : s < |w| < 1\} \not\subseteq \varphi(\mathbb{D})$—whence, $\{w : s < |w| < 1\} \not\subseteq F_\varepsilon \cup G_\varepsilon$—for $0 < s < 1$ (and $\varepsilon > 0$). This function φ is the square of a univalent analytic self-map ϕ of $\mathbb{D}$. Before we describe the image of $\mathbb{D}$ under ϕ, we need to establish some preliminaries. In what follows, let $\mathbb{H}$ denote the upper half-plane $\{w : \text{Im}(w) > 0\}$ and let E be the closed subset of $\mathbb{C}$ given by:

$$E = \{w : |w - i/2| \leq 1/4\} \cup \{w : 0 \leq \text{Im}(w) \leq 1/2 \text{ and } -\delta \leq \text{Re}(w) \leq \delta\},$$

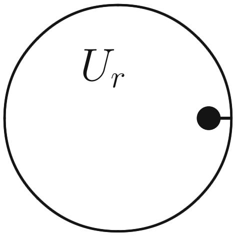

Figure 1.

where δ is some very small positive constant (e.g., 10^{-3}). Notice that $\partial(\mathbb{H}\setminus E)$ is piecewise smooth and has four "corners", that are located at junction points of $\{w : 0 \leq \mathrm{Im}(w) \leq 1/2$ and $-\delta \leq \mathrm{Re}(w) \leq \delta\}$ with $\{w : |w - i/2| \leq 1/4\}$ and with the real line $\mathbb{R}$. For any complex number a and any $r > 0$, let $\Delta(a, r) = \{w \in \mathbb{C} : |w - a| < r\}$. For the same very small positive constant δ, let $\mathcal{E} = \{w \in \mathbb{H}\setminus E : \Delta(w, \delta) \subseteq \mathbb{H}\setminus E\}$ and let $V_1 = \cup_{w\in\mathcal{E}}\Delta(w, \delta)$. Notice that V_1 is the same as $\mathbb{H}\setminus E$, but with the four aforementioned corners smoothed. For $0 < r \leq 1$, let $V_r = rV_1 := \{rw : w \in V_1\}$ and let $U_r = S(V_r)$, where $S(w) := \frac{i-w}{i+w}$. Thus, U_r is a smoothly bounded Jordan subregion of $\mathbb{D}$ that is symmetric with respect to $\mathbb{R}$ and that looks like $\mathbb{D}$ with a "lollipop" L_r (of diameter $6r/(4 + 3r)$) deleted at 1 (see Fig. 1). Since U_r is symmetric with respect to $\mathbb{R}$ and contains 0, there is a conformal mapping σ_r from U_r onto $\mathbb{D}$ such that $\sigma_r(\bar{z}) = \overline{\sigma_r(z)}$ on U_r, σ_r fixes -1 and 0, and $\sigma_r'(0) > 0$. Now, by *smoothly bounded* we mean that the Jordan curve under consideration is $C^{1+\alpha}$, for some $\alpha > 0$; see the definition of this on page 62 of [6]. Since ∂U_r consists of finitely many arcs of circles joined at points of tangency, ∂U_r is in fact C^{1+1} and thus [6, Theorem 4.3, page 62] tells us that σ_r' extends continuously to $\overline{U}_r$ and is nonzero there.

Lemma 2.5. *Using the notation above, there is a constant $C > 1$, independent of $r > 0$, such that*

$$1/C \leq |\sigma_r'(z)| \leq C,$$

for all z in $\overline{U}_r$.

Proof. Since σ_r' extends continuously to $\overline{U}_r$ (and is nonzero there), we need only establish this inequality for all z in U_r. By [6, Theorem 4.3, page 62], there is a constant $M > 1$ such that

$$1/M \leq |\sigma_1'(z)| \leq M, \tag{2.5.1}$$

for all z in U_1. Notice that $\sigma_r = S_r \circ \sigma_1 \circ T_r$, where $T_r(z) := \frac{z-a_r}{1-a_r z}$ with $a_r = \frac{1-r}{1+r}$, and $S_r(z) := \frac{z-\sigma_1(-a_r)}{1-\sigma_1(-a_r)z}$. Therefore, by (2.5.1),

$$|\sigma_r'(z)| = |S_r' \circ \sigma_1 \circ T_r(z)||\sigma_1' \circ T_r(z)||T_r'(z)| \sim |S_r' \circ \sigma_1 \circ T_r(z)||T_r'(z)|, \tag{2.5.2}$$

in U_r; where we use the symbol "$\sim$" to indicate bounded equivalence, independent of r. We now look for a good approximation to $\sigma_1(-a_r)$. Notice that

σ_1^{-1} is an analytic self-map of $\mathbb{D}$ that fixes 0 (and -1), but is not equal to z. Therefore, by Schwarz's Lemma, $|\sigma_1^{-1}(z)| < |z|$ in $\mathbb{D}$. Moreover, by the geometry of U_1, we can apply the Schwarz Reflection Principle to find that σ_1 has an analytic continuation across the relative interior of $\mathbb{T}\cap\partial U_1$, which contains -1. It now follows that $\beta := \sigma_1'(-1)$ exists, and $0 < \beta \leq 1$. Thus, $\sigma_1(z) \approx \beta(z+1)-1$, for z in $\mathbb{D}$ near -1. Consequently, $\sigma_1(-a_r) \approx \beta(-a_r+1)-1$. Thus, there exists t (independent of r), $0 < t < 1$, such that $\rho(-a_r, \sigma_1(-a_r)) \leq t$. So, if we let $\tau_r = T_r^{-1}$, then we find that $|S_r'(z)| \sim |\tau_r'(z)|$ in $\mathbb{D}$ and hence (by (2.5.2)):

$$|\sigma_r'(z)| \sim |\tau_r' \circ \sigma_1 \circ T_r(z)||T_r'(z)| = |\tau_r' \circ \sigma_1 \circ T_r(z)|/|\tau_r' \circ T_r(z)|,$$

in U_r. What remains to be shown is that $|\tau_r' \circ \sigma_1(w)| \sim |\tau_r'(w)|$, in U_1. Now $\tau_r(w) = \frac{w+a_r}{1+a_r w}$ and so we need only show that there is a constant $N > 1$ (independent of $r, 0 < r < 1$) such that

$$|1/a_r + w|/N \leq |1/a_r + \sigma_1(w)| \leq N|1/a_r + w|,$$

for all w in U_1. And this only needs to be shown for small values of r and for w in U_1 near -1. For such w, $\sigma_1(w) \approx \beta(w+1) - 1$, where $0 < \beta \leq 1$. Thus, for such w,

$$\begin{aligned} 4|1/a_r + w| \geq 2|1/a_r + \sigma_1(w)| &= 2|\sigma_1(w) - (-1/a_r)| \\ &\geq 2|\sigma_1(w) - [\beta(-1/a_r + 1) - 1]| \\ &\geq \beta|1/a_r + w|; \end{aligned}$$

and so our goal is reached. □

Remark 2.6. (*Concerning harmonic measure*) Let G be a bounded region in $\mathbb{C}$ for which the Dirichlet problem is solvable, and suppose $z_0 \in G$. Define Υ_{z_0} on $C_{\mathbb{R}}(\partial G)$ by: $\Upsilon_{z_0}(u) = \hat{u}(z_0)$, where $\hat{u}$ is the continuous function on $\overline{G}$ that is harmonic in G and has boundary values u. By the Maximum Principle, Υ_{z_0} defines a bounded (positive) linear functional (of norm 1) on $C_{\mathbb{R}}(\partial G)$. Thus, by the Riesz Representation Theorem, there is a unique positive Borel (probability) measure $\omega(\cdot, G, z_0)$ supported in ∂G such that

$$\hat{u}(z_0) = \int_{\partial G} u(\zeta) d\omega(\zeta, G, z_0)$$

for all u in $C_{\mathbb{R}}(\partial G)$. This measure is called *harmonic measure* on ∂G for evaluation at z_0. If w_0 is any other point in G, then, by Harnack's Inequality, $\omega(\cdot, G, w_0)$ is boundedly equivalent to $\omega(\cdot, G, z_0)$ on ∂G. Let H be another bounded region for which the Dirichlet problem is solvable such that $G \subseteq H$, and let B be a Borel subset of $(\partial G)\cap(\partial H)$. Then, by the Maximum Principle, $\omega(B, G, z_0) \leq \omega(B, H, z_0)$. We now suppose that G is a smoothly bounded Jordan region. In this case, $d\omega(\zeta, G, z_0) = |f'(\zeta)||d\zeta|/2\pi$, where f is a conformal mapping from G onto $\mathbb{D}$ that sends z_0 to 0. Let H be another smoothly bounded Jordan region such that $z_0 \in G \subseteq H$. If f and g are conformal mappings from G and H (respectively) onto $\mathbb{D}$ such that $f(z_0) = g(z_0) = 0$, then, by our discussion above, $|f'| \leq |g'|$ on $(\partial G) \cap (\partial H)$. Lastly, we recall a well-known probabilistic description of the distribution of harmonic measure. If G is any bounded region for which the Dirichlet problem is solvable, $z_0 \in G$ and B is a Borel subset of ∂G, then $\omega(B, G, z_0)$ is the probability that a Brownian motion path starting at z_0 will first exit G through a point in B. Interpreting Lemma 2.5 in these terms gives us: for any Borel subset B of ∂U_r, the probability that a Brownian motion path starting at 0 will first exit U_r through a point in B is boundedly equivalent to the Hausdorff-one measure of B; where the bound is independent of r.

We need one more preliminary result to help us with the construction, whose proof makes use of Lemma 2.5 and another basic inequality. Suppose that $0 < \theta_2 - \theta_1 \leq \pi/2$, and let I be the closed subarc of $\mathbb{T}$ given by $I := \{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$. For each $\theta, \theta_1 \leq \theta \leq \theta_2$, the *horodisk* $H(e^{i\theta}, 1/2) := \{z : 2|1 - ze^{-i\theta}|^2 < 1 - |z|^2\}$ is just the open disk of radius 1/3 whose boundary is internally tangent to $\mathbb{T}$ at $e^{i\theta}$; namely, $\Delta(2e^{i\theta}/3, 1/3)$. Let P_I be the smoothly bounded Jordan region:

$$P_I := \bigcup_{\theta \in [\theta_1, \theta_2]} H(e^{i\theta}, 1/2).$$

Since ∂P_I consists of I (an arc of $\mathbb{T}$) along with three arcs of circles of radius 1/3, joined at points of tangency, there is a constant $R > 1$, independent of I (as above) and independent of $\theta, \theta_1 \leq \theta \leq \theta_2$, such that

$$1/R \leq d\omega(\zeta, P_I, 2e^{i\theta}/3)/|d\zeta| \leq R, \qquad (*)$$

on ∂P_I; cf. [6, Corollary 4.7, page 65]. That R can be chosen independent of $\theta, \theta_1 \leq \theta \leq \theta_2$, is by Harnack's Inequality and that it can be chosen independent of I follows from the fact that ∂P_I is essentially the same, independent of I. Indeed, one can obtain (*) in a more elementary way via Harnack's Inequality, the conformal invariance of harmonic measure and Remark 2.6.

Lemma 2.7. *There is an absolute constant $Q > 1$ such that the following holds. Let $\mathcal{J}$ be a smoothly bounded Jordan subregion of $\mathbb{D}$ that contains $\Delta(0, 2/3)$ and let I be a closed subarc of $\mathbb{T}$ that traces out at most $\pi/2$ radians and that contains a point η in its relative interior. Further, suppose that $P_I \subseteq \mathcal{J}$ and that $(\partial\mathcal{J}) \cap (\mathbb{D} \setminus P_I)$ is a positive distance from P_I. Then, for sufficiently small $r > 0$, any conformal mapping f from the (Jordan) region $\mathcal{J}^{\#} := \mathcal{J} \cap \eta U_r$ onto $\mathbb{D}$ that fixes zero, satisfies: for z in $\overline{P}_I \cap \partial\mathcal{J}^{\#}$ $(= \overline{P}_I \cap \partial(\eta U_r))$,*

$$1/Q \leq |f'(z)| \leq Q.$$

Proof. Since η is in the relative interior of I and $P_I \subseteq \mathcal{J}$, we find that, for sufficiently small $r > 0, \mathcal{J}^{\#}$ is itself a smoothly bounded Jordan subregion of $\mathbb{D}$. One can establish this lemma via a Brownian motion argument that incorporates Lemma 2.5 and the discussion above. We outline a more elementary approach here. Recall that σ_r maps U_r conformally onto $\mathbb{D}$, fixes zero and satisfies: $\sigma_r'(0) > 0$. By the Schwarz Reflection Principle, a Riemann sphere version of Theorem 1 on page 55 of [5] and the special geometry of $U_r, \sigma_r(z)$ converges uniformly to z on $\overline{U}_r$, as $r \to 0$. Hence, $\sigma_{r,\eta}(z) := \eta\sigma_r(\overline{\eta}z)$ converges uniformly to z on the closure of ηU_r, as $r \to 0$. Among other things, this tells us that $I_r := \sigma_{r,\eta}(\overline{P}_I \cap \partial(\eta U_r))$ converges to I, as $r \to 0$. Therefore, since $(\partial\mathcal{J}) \cap (\mathbb{D} \setminus P_I)$ is a positive distance from P_I, we find that, for sufficiently small $r, P_{I_r} \subseteq \sigma_{r,\eta}(\mathcal{J}^{\#})$; and also that $\Delta(0, 1/2) \subseteq \sigma_{r,\eta}(\mathcal{J}^{\#})$. Therefore, by (*), Remark 2.6 and Harnack's Inequality, there is a constant $M > 1$ (dependent only on $r > 0$ being sufficiently small), such that if g is a conformal mapping from $\sigma_{r,\eta}(\mathcal{J}^{\#})$ onto $\mathbb{D}$ that fixes zero, then

$$1/M \leq |g'(z)| \leq M,$$

for all z in I_r. Since $g \circ \sigma_{r,\eta}$ maps $\mathcal{J}^{\#}$ onto $\mathbb{D}$ and fixes zero, we can now apply Lemma 2.5 and find that $Q := CM$ satisfies the conclusion of this lemma. □

We now have the tools we need to construct our univalent, analytic self-map ϕ of $\mathbb{D}$ such that $\varphi := \phi^2$ satisfies: C_φ is closed-range on $\mathbb{A}^2$ and yet, for $0 < s < 1, \{z : s < |z| < 1\} \not\subseteq \varphi(\mathbb{D})$. We proceed to describe the image of $\mathbb{D}$ under ϕ. Recall that U_r is the unit disk with a "lollipop" L_r deleted. The radial projection of L_r on $\mathbb{T}$ has the form: $\{e^{i\theta} : -\theta_r \leq \theta \leq \theta_r\}$, for some (small, positive) value θ_r. Let U_r^* be the rotation of U_r given by: $U_r^* = e^{i(\pi+\theta_r)}U_r$. Then, $U_r \cap U_r^*$ is the unit disk $\mathbb{D}$ with two equally sized lollipops deleted; one based at 1 and the other "nearly" based at -1. For positive integers n, let $\theta_n = \pi/2 - \pi/2^n$, and let $\{r_n\}_{n=1}^{\infty}$ be a decreasing sequence in the interval $(0, 1]$, that converges quickly to zero; where the rate of convergence shall be specified later. Let U be the (Jordan) subregion of $\mathbb{D}$ given by:

$$U := \bigcap_{n=1}^{\infty} e^{i\theta_n}[U_{r_n} \cap U_{r_n}^*];$$

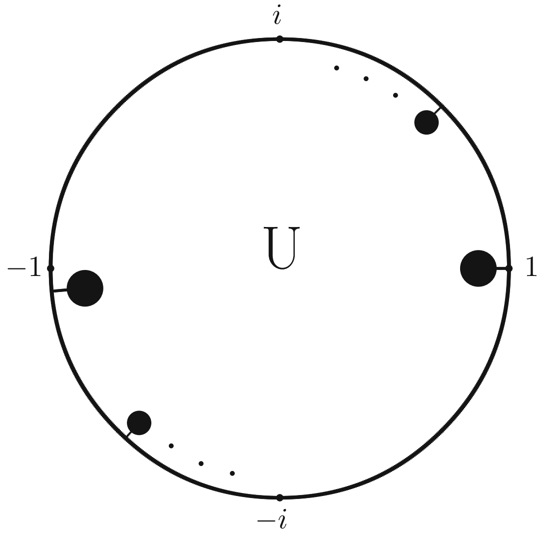

FIGURE 2.

see Fig. 2. And, for any positive integer N, let U^N be the smoothly bounded (Jordan) subregion of $\mathbb{D}$:

$$U^N := \bigcap_{n=1}^{N} e^{i\theta_n}[U_{r_n} \cap U^*_{r_n}].$$

If ϕ is a conformal mapping from $\mathbb{D}$ onto U (as described above), then it is straightforward that $\phi^2(\mathbb{D})$ looks like the unit disk $\mathbb{D}$ with a sequence of holes, starting near 1 and converging to -1. Thus, for $0 < s < 1, \{z : s < |z| < 1\} \nsubseteq \phi^2(\mathbb{D})$. What remains to be shown is that if $\{r_n\}_{n=1}^{\infty}$ converges to zero sufficiently fast, then C_{ϕ^2} is closed-range on $\mathbb{A}^2$. For positive integers n, let $\lambda_n = \pi(2^n - 3)/2^{n+1}$, let $I_n = \{e^{i\theta} : \lambda_n \leq \theta \leq \lambda_{n+1}\}$ and let $J_n = e^{i\pi} I_n$. Let $I_0 = \{e^{i\theta} : -\pi/2 \leq \theta \leq -\pi/4\}$ and let $J_0 = e^{i\pi} I_0$. Notice that $e^{i\theta_n}$ is well within the relative interior of I_n, for all positive integers n. And if $\{r_n\}_{n=1}^{\infty}$ converges to zero at a sufficiently fast rate, then, likewise, $e^{i(\pi+\theta_n+\theta_{r_n})}$ is well within the relative interior of J_n, for all positive integers n. We choose $\{r_n\}_{n=1}^{\infty}$ convergent to zero at a rate sufficiently fast so that, in addition, we have the following.

(i) $P_{I_0} \subseteq U$ and $P_{J_0} \subseteq U$.
(ii) For any positive integer n, P_{I_n} (respectively, P_{J_n}) "contains" just one lollipop of U; and the other lollipops have no intersection with $\overline{P}_{I_n}$ (respectively, $\overline{P}_{J_n}$).
(iii) There is a constant $\lambda > 0$ such that, for any positive integer N,

$$|\psi'_N| \geq \lambda \quad \text{on } \partial U^N,$$

where ψ_N is the conformal mapping from U^N onto $\mathbb{D}$ that fixes zero and satisfies: $\psi'_N(0) > 0$.

Conditions (i) and (ii) are just geometric and they alone force a certain rate of convergence of $\{r_n\}_{n=1}^{\infty}$ to zero. Condition (iii) is achievable via (*) above and an inductive construction using Lemma 2.7. Since U^N is smoothly bounded, ψ'_N extends continuously from U^N to its closure and hence, by the Minimum Modulus Principle, $|\psi'_N| \geq \lambda$ on U^N. Let ψ be the conformal mapping from U onto $\mathbb{D}$ that fixes zero and satisfies: $\psi'(0) > 0$. Then ψ_N converges to ψ uniformly on compact subsets of U (cf. [5, Theorem 1, page 55]; and thus the same can be said for ψ'_N and ψ'. Therefore, we find that $|\psi'| \geq \lambda$ on U. For z in U, let $d(z) = \mathrm{dist}(z, \partial U)$. For the same very small positive constant δ mentioned at the beginning of this example, let $U^{\#}$ be the subregion of U given by:

$$U^{\#} := \{z \in U : d(z)/(1 - |z|) > \delta\}.$$

Notice that $U^{\#}$ is all of U except for a narrow sheath of points around each lollipop of U; where the "thickness" of the sheath around a lollipop $e^{i\theta_n} L_{r_n}$ (or $e^{i(\pi+\theta_n+\theta_{r_n})} L_{r_n}$) is no greater than $6\delta r_n/(4 + 3r_n)$ ($< 2\delta r_n$). Let ϕ denote the inverse mapping of ψ from $\mathbb{D}$ onto U and let $\varphi = \phi^2$. Therefore, if $w \in \psi(U^{\#})$, then $d(\phi(w))/(1 - |\phi(w)|) > \delta$. So, by the Koebe Distortion Theorem (cf. [8, Corollary 1.4], or [6, Theorem 4.3, page 19]),

$$(1 - |w|^2)|\phi'(w)|/(1 - |\phi(w)|^2) > \delta/2,$$

for all w in $\psi(U^{\#})$. Since $|\phi'|$ is bounded above by $1/\lambda$ on $\mathbb{D}$, we find that

$$(1-|w|^2)/(1-|\phi(w)|^2) > \delta\lambda/2,$$

for all such w. We then have:

$$(1-|w|^2)/(1-|\varphi(w)|^2) > \delta\lambda/4,$$

for all w in $\psi(U^{\#})$. Thus, $\psi(U^{\#}) \subseteq \Omega_{\delta\lambda/4} := \{w \in \mathbb{D} : \frac{1-|w|^2}{1-|\varphi(w)|^2} > \delta\lambda/4\}$. Whence, $G_{\delta\lambda/4} := \varphi(\Omega_{\delta\lambda/4}) \supseteq \varphi(\psi(U^{\#})) = \{z^2 : z \in U^{\#}\}$, which is the unit disk with a sequence of holes that tend to -1, tangentially. And these holes are essentially the size of the images of the disk parts of the lollipops under the mapping $z \mapsto z^2$. Thus, there is a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ that converges to -1 and there is a constant $t, 0 < t < 1$, such that each hole is contained in $D(z_n, t)$, for some n, and $\inf\{\rho(z_k, z_n) : k \neq n\} \longrightarrow 1$, as $n \to \infty$. We conclude that $G_{\delta\lambda/4}$ satisfies the reverse Carleson condition, which implies that C_φ is closed-range on $\mathbb{A}^2$; cf. [1, Theorem 2.4]. We observe that the image, under ψ, of the closure of $U^{\#}$ in $\mathbb{T}$—which is essentially the set K (corresponding to φ) mentioned in the first section of this erratum—is the union of a countable collection of pairwise disjoint closed arcs of $\mathbb{T}$. Thus, the "finitely many" assumption of Section 1 is sharp. What is a little less obvious is that φ' fails to be continuous on this image; but that indeed is the case. Summarizing the main properties here: There is a two-valent analytic self-map φ of $\mathbb{D}$ such that C_φ is closed-range on $\mathbb{A}^2$, yet, for $0 < s < 1, \{z : s < |z| < 1\} \nsubseteq \varphi(\mathbb{D})$.

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References

[1] Akeroyd, J.R., Ghatage, P.G.: Closed-range composition operators on $\mathbb{A}^2$. Ill. J. Math. **52**, 533–549 (2008)

[2] Akeroyd, J.R., Ghatage, P.G., Tjani, M.: Closed-range composition operators on $\mathbb{A}^2$ and the Bloch space. Integr. Equ. Oper. Theory **68**(4), 503–517 (2010)

[3] Cowen, C.C., MacCluer, B.D.: Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton (1995)

[4] Garnett, J.B.: Bounded Analytic Functions. Academic Press, New York (1982)

[5] Goluzin, G.M.: Geometric Theory of Functions of a Complex Variable. Translations of Mathematical Monographs, vol. 26. American Mathematical Society, Providence (1969)

[6] Garnett, J.B., Marshall, D.E.: Harmonic Measure. Cambridge University Press, New York (2005)

[7] Luecking, D.H.: Inequalities on Bergman spaces. Ill. J. Math. **25**, 1–11 (1981)

[8] Pommerenke, Ch.: Boundary Behaviour of Conformal Maps. Springer, Berlin (1992)

[9] Shapiro, J.H.: Composition Operators and Classical Function Theory. Universitext: Tracts in Mathematics. Springer, New York (1993)

[10] Zhu, K.: Operator Theory in Function Spaces. Mathematical Surveys and Monographs, vol. 138, 2nd edn. American Mathematical Society, Providence (2007)

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