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Metric ultraproducts of classical groups

John S. Wilson

Abstract. Simple non-discrete metric ultraproducts of classical groups are geodesic spaces with respect to a natural metric.

1. Introduction. Metric ultraproducts of groups equipped with bi-invariant metrics have been studied in a number of recent papers; see [2,4–7]. We recall that a metric or pseudometric d on a group G is called bi-invariant if $d(gx, gy) = d(xg, yg) = d(x, y)$ for all $x, y, g \in G$. We fix an infinite index set I and an ultrafilter $\mathcal{U}$ on I. Let $(G_i, d_i)_{i \in I}$ be a family of groups with bi-invariant pseudometrics and with diameters $\sup\{d_i(g_i, 1) \mid g_i \in G\}$ bounded independently of i. The Cartesian product $\prod_i G_i$ of this family has a pseudometric defined by

$$d((g_i), (h_i)) := \lim_{i \to \mathcal{U}} d_i(g_i, h_i),$$

and the metric ultraproduct $(G, d) := \prod_{i \to \mathcal{U}}(G_i, d_i)$ is defined to be the quotient group of $\prod_i G_i$ modulo the normal subgroup $\{(g_i) \mid \lim_{i \to \mathcal{U}} d_i(g_i, 1) = 0\}$; it is a metric group with the metric d induced by the pseudometric on the Cartesian product.

The paper [7] was concerned primarily with the case when each group G_i is a non-abelian finite simple group and G is a simple group for which the metric is not discrete. Arguments in that paper show that the group G is then (isomorphic to) a metric ultraproduct of alternating groups with their Hamming metrics, or an ultraproduct of simple classical groups of unbounded rank with natural metrics. It was also shown that if the index set I is countable, then G is path-connected, and, in some cases, is a geodesic space.

For the simple classical groups, there are several natural choices of metric. We shall work with a metric d_{cr}, the *classical rank metric*, defined below. We

shall see (in Lemma 3.1) that d_{cr} is asymptotically equivalent on the family of simple classical groups to the metric d_{pr} used in [7] and some other earlier papers, and so gives rise to the same metric ultraproduct with an equivalent metric.

We recall that a *geodesic* from one point x of a metric space (X, d) to another point y is an isometric embedding $p \colon [0, d(x, y)] \to X$ such that $p(0) = x$ and $p(d(x, y)) = y$, and that X is called a *geodesic space* if all pairs of points are connected by geodesics. Thus geodesic spaces are in particular path-connected. The path-connectedness of metric ultraproducts of classical groups of unbounded ranks was established in [7]; however the authors were unable to prove in general that they are geodesic spaces with respect to a natural metric. Here we prove the following result.

Theorem 1.1. *With respect to the metric induced by d_{cr}, the non-discrete metric ultraproducts of families of classical groups are complete geodesic metric spaces.*

2. Word metrics.   We shall use the following result.

Lemma 2.1. *Let (G_i) be a family of groups. For each i let X_i be a union of conjugacy classes in G_i with $\langle X_i \rangle = G_i$, and let w_i be the word metric on G_i with respect to X_i, defined by setting*

$$w_i(g, h) = \min\{r \mid gh^{-1} \text{ is a product of at most } r \text{ elements of } X_i \cup X_i^{-1}\}.$$

Thus each w_i is bi-invariant.

Suppose that $m_i = \sup\{w_i(g, 1) \mid g \in G_i\}$ is finite for each i and that $m_i \to_{\mathcal{U}} \infty$. Choose positive integers n_i such that the rationals m_i/n_i are bounded above and let $d_i = n_i^{-1} w_i$ for each i. Then the metric ultraproduct $(G, d) = \prod_{i \to \mathcal{U}} (G_i, d_i)$ is a geodesic space.

Proof. For the reader's convenience we include a proof of this particular case of a result of Wantiez [8, p. 147]. It suffices to prove that if $g \in G \setminus \{1\}$, then there is a geodesic from 1 to g. Let $\lambda = d(1, g)$, and let $(g_i) \in \prod G_i$ be a family that represents g. Thus $n_i^{-1} w_i(g_i) \to_{\mathcal{U}} \lambda$. We write each g_i as a product of $w_i(g_i)$ elements of $X_i \cup X_i^{-1}$. For each $r \in [0, \lambda]$ let $p_i(r)$ be the product of the first $\lfloor \lambda^{-1} r w_i(g_i) \rfloor$ terms in the product for g_i, and let $p(r)$ be the image of $(p_i(r))$ in G. Then p is a geodesic from 1 to g. $\qquad \square$

We pause briefly to discuss the ultraproducts of alternating groups A_n with $n \geqslant 5$. The alternating groups carry the normalized Hamming metric d_{H} defined by

$$d_{\mathrm{H}}(g, h) = \frac{|\{m \in \{1, \ldots, n\} \mid mg \neq mh\}|}{n} \quad \text{for all } g, h \in A_n.$$

Thus $n d_{\mathrm{H}}(g, h) - 1$ is equal to the word length of $g^{-1} h$ corresponding to the generating set of S_n consisting of the transpositions. Consider also the word metric d_w with respect to the set X of 3-cycles as generating set for A_n. Each even permutation that moves precisely r elements can be written as a product of at most $r/2$ elements of X, but not as a product of fewer than $r/3$ elements of X. Thus $\frac{1}{3} d_{\mathrm{H}}(g, h) \leqslant n^{-1} d_w(g, h) \leqslant \frac{1}{2} d_{\mathrm{H}}(g, h)$ for all $g, h \in A_n$. It follows

that on each family (A_{n_i}) of alternating groups with (n_i) unbounded the metrics $n_i^{-1}d_w$ and d_{H} are asymptotically equivalent, and so the corresponding ultraproducts coincide. The hypotheses of Lemma 2.1 evidently hold, and so we have the following assertion.

Proposition 2.2. *Let G be an ultraproduct of alternating groups of unbounded ranks with respect to their Hamming metrics. Then G is path-connected.*

In [7] it was proved that the above metric ultraproduct is in fact a geodesic space with respect to the metric induced by the Hamming metrics.

3. Metrics on simple classical groups. Throughout this section, V is an n-dimensional vector space over a field F, where either $n \geqslant 3$ or $n = 2$ and $|F| \geqslant 4$, and G is the derived group of $\widehat{G}$, where $\widehat{G}$ is the subgroup of $\mathrm{GL}(V)$ preserving one of the following: (1) the zero form on V, (2) a non-singular symplectic form, (3) a non-singular hermitian form with respect to a non-trivial involution on F, or (4) for char F odd, a non-singular symmetric form, and for char F even, a non-singular quadratic form. The centre Z of $\mathrm{GL}(V)$ consists of the scalar multiplications $\bar{\lambda}$ by elements λ of $F^\times = F \setminus \{0\}$. Each finite simple classical group arises as the group $G/(Z \cap G)$ for one of the above cases. The properties of classical groups that we use can be found in [3, Chapter 2].

For $g, h \in \widehat{G}$ we define

$$d_{\mathrm{pr}}(g, h) = \min_{\bar{\lambda} \in Z} \frac{\mathrm{rk}(g - \lambda h)}{n}, \qquad d_{\mathrm{cr}}(g, h) = \min_{\bar{\lambda} \in Z \cap \widehat{G}} \frac{\mathrm{rk}(g - \lambda h)}{n}.$$

Lemma 3.1. (a) *d_{pr} and d_{cr} are pseudometrics and they induce metrics in $\widehat{G}/(Z \cap \widehat{G})$ and $G/(Z \cap G)$.*

(b) *For $g, h \in \widehat{G}$ we have $d_{\mathrm{pr}}(g, h) \leqslant d_{\mathrm{cr}}(g, h)$, with equality if $d_{\mathrm{pr}}(g, h) < \frac{1}{2} \dim V$.*

Proof. (a) This is clear; for example the triangle inequality holds since (with obvious notation)

$$\mathrm{rk}(g_3 - \lambda_1\lambda_2 g_1) = \mathrm{rk}((g_3 - \lambda_2 g_2) + \lambda_2(g_2 - \lambda_1 g_1))$$
$$\leqslant \mathrm{rk}(g_3 - \lambda_2 g_2) + \mathrm{rk}(g_2 - \lambda_1 g_1).$$

(b) The first assertion is clear and so is the second in Case (1). In the remaining cases, let $(\ ,\ )$ be the non-singular sesquilinear form on V preserved by $\widehat{G}$ and $*$ the automorphism of F of order 1 or 2 with $(\lambda v, w) = (v, \lambda^* w)$ for all λ, v, w. Let $\lambda \in F^\times$ and $k \in \widehat{G}$. An easy calculation shows that $\mathrm{im}(k - \bar{\lambda}) \leqslant (\ker(k - \overline{\lambda^{*-1}}))^\perp$. Therefore

$$\mathrm{rk}(k - \bar{\lambda}) \leqslant \dim V - \dim \ker(k - \overline{\lambda^{*-1}}) = \mathrm{rk}(k - \overline{\lambda^{*-1}}),$$

and symmetry gives equality. It follows that if $\lambda \in F^\times$ and $\mathrm{rk}(g - \lambda h) < \frac{1}{2} \dim V$, then both $\ker(gh^{-1} - \bar{\lambda})$ and $\ker(gh^{-1} - \overline{\lambda^{*-1}})$ have dimension greater than $\frac{1}{2} \dim V$ and hence that $\lambda\lambda^* = 1$. But then for $v, w \in V$ we have $(\lambda v, \lambda w) = (v, \lambda\lambda^* w) = (v, w)$, and hence $\bar{\lambda} \in \widehat{G}$. The result follows. $\qquad\square$

Lemma 3.2. *Suppose that $n \geqslant 5$. Write $Y = \{g \in \widehat{G} \mid \mathrm{rk}(g-1) = 1\}$ and $X = (Z \cap \widehat{G})Y$, and let d_w be the pseudometric on $\langle X \rangle$ with respect to the generating set X. Let $g \in \widehat{G}$.*

(a) *If g is a product of r elements of Y, then $\ker(g-1)$ has codimension at most r and hence $\mathrm{rk}(g-1) \leqslant r$.*
(b) *If $g \in \widehat{G}$ and $\mathrm{rk}(g-1) \leqslant r$, then g can be written as a product of*
  (i) *at most $r+1$ elements of Y and an element of Z in Case (1);*
  (ii) *at most $r+1$ elements of Y in Cases (2)–(4).*
  In particular, $\widehat{G} = \langle X \rangle$.
(c) *$nd_{\mathrm{cr}}(g,h) \leqslant d_w(g,h) \leqslant nd_{\mathrm{cr}}(g,h) + 1$ for all $g, h \in \widehat{G}$.*
(d) *$\widehat{G}$ has a subgroup K with $\widehat{G} = GK$ and such that K acts trivially on a subspace of V of codimension at most 2.*

Proof. Assertion (a) is clear. Assertion (b) is a consequence of results of Dieudonné [1]: Theorem 1 of [1] gives the result for Case (1) above, Theorems 2, 3 for Case (2), and Theorems 4, 5 for Case (3) and for Case (4) with char F odd. The remaining groups are covered by Theorems 6, 7, which also explain why the restriction $n \geqslant 5$ is needed.

(c) Let $g, h \in \widehat{G}$. Write $d_w(g,h) = r$; thus gh^{-1} is a product of r elements of $(Z \cap \widehat{G})Y$ and hence of form λg_1 with $\lambda \in Z \cap \widehat{G}$ and with g_1 a product of r elements of Y. Thus $\mathrm{rk}(g_1 - 1) \leqslant r$ from (a), and so $nd_{\mathrm{cr}}(g,h) \leqslant \mathrm{rk}(gh^{-1} - \bar{\lambda}) \leqslant r$. Suppose that $d_{\mathrm{cr}}(g,h) = r_1/n$, and find $\bar{\mu} \in Z \cap \widehat{G}$ with $r_1 = \mathrm{rk}(gh^{-1} - \bar{\mu})$. Thus $\mu^{-1}gh^{-1} \in \widehat{G}$ and $\mathrm{rk}(\mu^{-1}gh^{-1} - 1) = r_1$. Therefore by (b) we have $d_w(g,h) = d_w(gh^{-1}, 1) \leqslant r_1 + 1$.

(d) In Case (1) above, since G is the kernel of the determinant map, we may take K to be the pointwise stabilizer of a fixed subspace of codimension 1 in V; in Case (2) we take $K = 1$. In Cases (3) and (4) we choose a 2-dimensional orthogonal summand W of V with orthogonal complement C and define K to be the group fixing C pointwise and inducing automorphisms of W.   □

4. *Proof of Theorem 1.1.* Let G be a simple metric ultraproduct of a family of simple classical groups of unbounded rank. Each simple group is of form $G_i/(Z_i \cap G_i)$ where G_i is the derived group of the group $\widehat{G}_i$ preserving a form on a space V_i of dimension n_i and Z_i is the centre of $\widehat{G}_i$. Thus

$$G = \prod_{i \to \mathcal{U}} (G_i/(Z_i \cap G_i), d_{\mathrm{cr}}).$$

Since G is not discrete, $n_i \to_{\mathcal{U}} \infty$. For each i let $Y_i = \{g_i \in \widehat{G}_i \mid \mathrm{rk}(g_i - 1) \leqslant 1\}$ and $X_i = (Z_i \cap \widehat{G}_i)Y_i$, let w_i be the word metric on $\widehat{G}$ with respect to X_i and choose a subgroup K_i with the property expressed in Lemma 3.2 (d). Write

$$M = \prod_{i \to \mathcal{U}} (\widehat{G}_i/(Z_i \cap \widehat{G}_i), d_{\mathrm{cr}}).$$

By Lemma 2.1 [and Lemma 3.2 (a)], $\prod_{i\to\mathcal{U}}(\widehat{G}_i/(Z_i\cap\widehat{G}_i), n_i^{-1}d_{w_i})$ is a geodesic space. However

$$\lim_{i\to\mathcal{U}} n_i^{-1}d_w(g_i, h_i) = \lim_{i\to\mathcal{U}} d_{\mathrm{cr}}(g_i, h_i) \qquad \text{for all } (g_i), (h_i) \in \prod \widehat{G}_i$$

by Lemma 3.2 (c). Therefore $n_i^{-1}d_{w_i}(g_i, 1) \to_{\mathcal{U}} 0$ if and only if $d_{\mathrm{cr}}(g_i, 1) \to_{\mathcal{U}} 0$, and hence $\prod_{i\to\mathcal{U}}(\widehat{G}_i/(Z_i\cap\widehat{G}_i), n_i^{-1}d_{w_i}) = M$; moreover the metrics $n_i^{-1}w_i$ and the classical rank metrics induce the same metric in M.

Since $\widehat{G}_i = G_iK_i$ for each i and since clearly $\prod K_i \leqslant \ker(\prod \widehat{G}_i \to M)$, the induced map $\theta \colon \prod G_i \to M$ is surjective. An element $(g_i) \in \prod G_i$ lies in $\ker\theta$ if and only if $d_{\mathrm{cr}}(g_i, 1) \to_{\mathcal{U}} 0$, and this is the case if and only if $(g_i) \in \ker(\prod G_i \to G)$. Since the induced map from G to M is a group isomorphism and evidently an isometry the theorem follows.

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John S. Wilson
Christ's College
Cambridge
United Kingdom
e-mail: `jsw13@cam.ac.uk`