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Positivstellensatz for homogeneous semialgebraic sets

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Abstract. We call a closed basic semialgebraic set $X \subset \mathbb{R}^n$ homogeneous if it is defined by a finite system of inequalities of the form $g(x) \geq 0$, where g is a homogeneous polynomial. We prove an effective version of the Putinar and Vasilescu Positivstellensatz for positive homogeneous polynomials on homogeneous semialgebraic sets.

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1. Introduction. We denote by $\mathbb{R}[x]$ the ring of real polynomials in $x = (x_1, \ldots, x_n)$ and by $\sum \mathbb{R}[x]^2$ the set of sums of squares of polynomials from $\mathbb{R}[x]$. A homogeneous polynomial f is called *positive definite* if $f(x) > 0$ for $x \in \mathbb{R}^n \setminus \{0\}$.

An important result concerning nonnegative polynomials is the solution of Hilbert's 17th problem by Artin [1], which states that if a polynomial f is nonnegative on $\mathbb{R}^n$, then f is a sum of squares of rational functions.

For positive definite homogeneous polynomials Reznick proved the following theorem (see [10, Theorem 3.12]). Let us start with some notation. We denote by $Q_{n,k}$ the set of finite sums of kth powers of linear functions. Let $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. Set

$$G_n^r = |x|^{2r}, \quad r \in \mathbb{N}.$$

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Let $p \in \mathbb{R}[x]$ be a positive definite homogeneous polynomial. Set

$$\epsilon(p) = \frac{\inf\{p(u) : u \in S\}}{\sup\{p(u) : u \in S\}},$$

where $S = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere.

Theorem 1 (Reznick). *Let $p \in \mathbb{R}[x]$ be a positive definite homogeneous polynomial of degree d. Then for any $r \in \mathbb{Z}$ such that*

$$r \geq \frac{nd(d-1)}{(4\log 2)\epsilon(p)} - \frac{n+d}{2},$$

we have $pG_n^r \in Q_{n,d+2r}$.

Scheiderer [11, Remark 4.6] gave a generalization of Reznick's theorem by showing that $|x|^2$ in the definition of G_n^r can be replaced by any positive definite form. Interesting contributions in this context are also [13, Theorem 5.1] and [15].

In real algebraic geometry, important problems concern nonnegative polynomials on closed semialgebraic sets. The deepest result in this topic is the Krivine Positivstellensatz [4] rediscovered by Stengle [16]. It states that if

$$X = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\},$$

where $g_1, \ldots, g_r \in \mathbb{R}[x]$, then every polynomial f that is strictly positive on X is of the form $h_1 f = 1 + h_2$ for some polynomials h_1, h_2 from the preordering

$$T(g_1, \ldots, g_r) := \left\{ \sum_{e=(e_1,\ldots,e_r)\in\{0,1\}^r} \sigma_e g_1^{e_1} \cdots g_r^{e_r} : \sigma_e \in \sum \mathbb{R}[x]^2 \text{ for } e \in \{0,1\}^r \right\}.$$

In the case of compact basic semialgebraic sets, an important result was obtained by Schmüdgen (see [12]): if the set X is compact, then every polynomial f that is strictly positive on X belongs to the preordering $T(g_1, \ldots, g_r)$. Moreover, as proved by Putinar [7], under some additional assumption, f belongs to the *quadratic module*

$$P(g_1, \ldots, g_r) := \left\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_r g_r : \sigma_i \in \sum \mathbb{R}[x]^2,\ i = 0, \ldots, r \right\}.$$

Jacobi found an algebraic proof of this fact [2, Theorem 7] (see also [3]). A version of the above results was obtained in [5]: if X is compact, f is strictly positive on X, and $g(x) := R^2 - |x|^2 \geq 0$ on X for some $R > 0$, then f belongs to

$$\mathcal{K}(g, g_1, \ldots, g_r) := T(g) + \left\{ \varphi(g_1)g_1 + \cdots + \varphi(g_r)g_r : \varphi \in \sum \mathbb{R}[t]^2 \right\}.$$

In this context for every closed basic semialgebraic set Putinar and Vasilescu proved the following Positivstellensatz (see [9, Theorem 4.2], [8]).

Theorem 2 (Putinar, Vasilescu). *Let $(p_1,\dots,p_m)$ be an m-tuple of real polynomials in $t\in\mathbb{R}^n$, and let*

$$\phi(t)=(1+|t|^2)^{-1},\quad t\in\mathbb{R}^n.$$

Let p be a real polynomial on $\mathbb{R}^n$. Suppose that the degrees of p_j's and p are all even.

Let $P_1,\dots,P_m,P$ be the homogenizations of the polynomials $p_1,\dots,p_m,p$ respectively and assume that $P(x)>0$ whenever $x\in\{x\in\mathbb{R}^{n+1}: P_i(x)\geq 0,\ i=1,\dots,m\}$, $x\neq 0$.

Then there exist an integer $b\geq 0$ and a finite collection of real polynomials $q_l, q_{kl}, l\in L, k=1,\dots,m$, such that

$$p(t)=\phi(t)^{2b}\left(\sum_{l\in L}q_l(t)^2+\sum_{k=1}^{m}\sum_{l\in L}p_k(t)q_{kl}(t)^2\right),\quad t\in\mathbb{R}^n. \tag{1}$$

The aim of this article is to simplify the representation (1) for any homogeneous polynomial p. We will show

Theorem 3. *Let $f\in\mathbb{R}[x]$ be a homogeneous polynomial of positive even degree d, and let $g_1,\dots,g_r\in\mathbb{R}[x]$ be homogeneous polynomials of even degrees. Set*

$$X=\{x\in\mathbb{R}^n: g_1(x)\geq 0,\dots,g_r(x)\geq 0\}. \tag{2}$$

If $f(x)>0$ for $x\in X\backslash\{0\}$, then there exist positive even integers D,N, a polynomial $q\in Q_{n,D}$, and $a,b\in\mathbb{R}$ such that

$$f(x)=|x|^{-D+d}\left(q+\sum_{i=1}^{r}|x|^{\alpha_i}(ag_i(x)+b|x|^{\deg g_i})^N g_i(x)\right), \tag{3}$$

where $\alpha_i=D-(N+1)\deg g_i$ for $i=1,\dots,r$ are nonnegative even numbers.

In the proof of Theorem 3, we will use the method from [5] and apply the Reznick theorem.

From Theorem 3 we immediately obtain a version of the Putinar–Vasilescu theorem.

Corollary 1. *Under the assumptions and notation of Theorem 2, there are even integers $b,D,N\geq 0$ such that $D-(N+1)\deg p_k\geq 0$ for $k=1,\dots,r$, and a finite collection of real polynomials q_l, $l\in L$, with $\deg q_l\leq 1$ and polynomials $q_{k,1}$, $k=1,\dots,r$, of the form*

$$q_{k,1}(t)=(1+|t|^2)^{\alpha_k}\left(\xi p_k(t)+\eta(1+|t|^2)^{\frac{\deg p_k}{2}}\right)^N$$

for some $\xi,\eta\in\mathbb{R}$, where $\alpha_k=\frac{D-(N+1)\deg p_k}{2}$ for $k=1,\dots,r$, such that

$$p(t)=\phi(t)^b\left(\sum_{l\in L}q_l^D(t)+\sum_{k=1}^{m}p_k(t)q_{k,1}\right),\quad t\in\mathbb{R}^n.$$

Theorem 3 and Corollary 1 provide an additional information about how the polynomials defining the basic closed semialgebraic set X are involved in the representation of f and p respectively (comparable to Schweighifer result [14, Lemma 8] and [5]).

2. Proof of Theorem 3. Let $f, g_1, \dots, g_r \in \mathbb{R}[x]$ be homogeneous polynomials of even degrees, and let $X \subset \mathbb{R}^n$ be of the form (2).

Lemma 1. *There exists a polynomial $\varphi \in \sum \mathbb{R}[t]^2$ of the form $\varphi(t) = (at+b)^N$, where t is a single variable and N is an even nonnegative integer, such that*

$$f(x) - \sum_{i=1}^{r} g_i(x)\varphi(g_i(x)) > 0 \quad \textit{for } x \in S. \tag{4}$$

Proof. In the proof we will use the method of Kurdyka and Spodzieja from [5, Lemma 1].

Let $M > 1$ and $A \geq 1$ be constants such that

$$f(x) \geq -M \quad \text{for} \quad x \in S \tag{5}$$

and

$$|g_i(x)| \leq A \quad \text{for} \quad x \in S, \qquad i = 1, \dots, r. \tag{6}$$

Let

$$G_1 := \{x \in S : f(x) > 0\}.$$

Then there exists $\eta > 0$ such that

$$G_2 := \{x \in S : \operatorname{dist}(x, X) \leq \eta\} \subset G_1$$

and

$$m := \min\{f(x) : x \in G_2\} > 0. \tag{7}$$

Since $X \cap S = \{x \in S : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ and S is compact, there exists $0 < \delta \leq 1$ such that

$$G_3 := \{x \in S : g_i(x) \geq -\delta \quad \text{for} \quad i = 1, \dots, r\} \subset G_2. \tag{8}$$

Take

$$\varepsilon := \frac{m}{(r+1)A}, \qquad B := A\frac{M + r\varepsilon}{\delta}.$$

Then there exist $a, b \in \mathbb{R}$, $N \in \mathbb{N}$, and a polynomial $\varphi \in \mathbb{R}[t]$ of the form $\varphi(t) = (at+b)^N$, where N is an even nonnegative integer, such that

$$\varphi(t) > B \qquad \text{for} \quad t \in [-A, -\delta], \tag{9}$$

$$\varphi(t) < \varepsilon \qquad \text{for} \quad t \in [0, A]. \tag{10}$$

We prove (4). Let $x \in X \cap S$. Then $g_i(x) \geq 0$ for $i = 1, \dots, r$. Since $\varphi(t) < \varepsilon$ for $t \in [0, A]$, by (6) we have

$$g_i(x) \cdot \varphi(g_i(x)) \leq A\varepsilon < \frac{m}{r+1} \quad \text{for} \quad i = 1, \dots, r.$$

Hence, by (7),

$$f(x) - \sum_{i=1}^{r} g_i(x) \cdot \varphi(g_i(x)) > m - r\frac{m}{r+1} > 0,$$

and the assertion (4) holds for $x \in X \cap S$.

Suppose now that $x \in G_3 \backslash X$. We may assume that

$$g_1(x), \dots, g_k(x) \geq 0 \quad \text{and} \quad g_{k+1}(x), \dots, g_r(x) < 0$$

for some $0 \leq k < r$. Since $\varphi(t) < \varepsilon$ for $t \in [0, A]$, we have

$$g_i(x) \cdot \varphi(g_i(x)) \leq A\varepsilon < \frac{m}{r} \quad \text{for} \quad i = 1, \dots, k.$$

Moreover from $\varphi(t) \geq 0$ for $t \in \mathbb{R}$, we have

$$g_i(x) \cdot \varphi(g_i(x)) < 0 \quad \text{for} \ \ i = k+1, \dots, r.$$

Therefore, $f(x) - \sum_{i=1}^{r} g_i(x) \cdot \varphi(g_i(x)) > m - k\frac{m}{r} > 0$, and (4) holds for $x \in G_3 \backslash X$.

Let now $x \in S \backslash G_3$. Then we may assume that

$$g_1(x), \dots, g_k(x) \geq 0, \quad 0 > g_{k+1}(x), \dots, g_l(x) \geq -\delta, \quad g_{l+1}(x), \dots, g_r(x) < -\delta,$$

where $0 \leq k \leq l < r$. Then

$$g_i(x) \cdot \varphi(g_i(x)) < \frac{m}{r+1} \quad \text{for} \ \ i = 1, \dots, k,$$

and

$$g_i(x) \cdot \varphi(g_i(x)) < 0 \quad \text{for} \ \ i = k+1, \dots, l.$$

Since $\varphi(t) > B$ for $t \in [-A, -\delta]$, we see that

$$g_i(x) \cdot \varphi(g_i(x)) < -\delta B = A(-M - r\varepsilon) = -AM - \frac{rm}{r+1} \quad \text{for} \ \ i = l+1, \dots, r.$$

Hence,

$$g_i(x) \cdot \varphi(g_i(x)) < -M - \frac{rm}{r+1} \quad \text{for} \ \ i = l+1, \dots, r,$$

since $A \geq 1$.

Therefore,

$$f(x) - \sum_{i=1}^{r} g_i(x) \cdot \varphi(g_i(x)) > -M - k\frac{m}{r+1} + (r-l)\left(M + \frac{rm}{r+1}\right) > 0.$$

This ends the proof of Lemma 1. □

Let $a, b \in \mathbb{R}$ and $N \in \mathbb{N}$ be an even number such that for $\varphi(t) = (at+b)^N$ the inequality (4) holds. Let $\deg g_i = d_i$ for $i = 1, \dots, r$, and let

$$D_0 := \max\{d, (N+1)d_1, \dots, (N+1)d_r\}.$$

Recall that $d = \deg f$ is an even number. Therefore D_0 is an even number too. Set

$$\alpha_i = D_0 - (N+1)d_i \quad \text{for} \ \ i = 1, \dots, r.$$

Obviously the α_i are nonnegative even numbers.

Lemma 2. *The function $F : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$F(x) = |x|^{D_0 - d} f(x) - \sum_{i=1}^{r} |x|^{\alpha_i} g_i(x)(a g_i(x) + b|x|^{d_i})^N \tag{11}$$

is a homogeneous polynomial of degree D_0.

Proof. We show that $F(tx) = t^{D_0}F(x)$. Indeed,

$$F(tx) = |tx|^{D_0-d}f(tx) - \sum_{i=1}^{r} |tx|^{\alpha_i} g_i(tx)(ag_i(tx) + b|tx|^{d_i})^N. \tag{12}$$

Since the α_i are even numbers and the polynomials f, $g_1, \dots, g_r$ are homogeneous, the right hand side of (12) is equal to

$$t^{D_0}|x|^{D_0-d}f(x) - \sum_{i=1}^{r} t^{\alpha_i+d_i+Nd_i}|x|^{\alpha_i} g_i(x)(ag_i(x) + b|x|^{d_i})^{2N}.$$

Hence, by the definition of the α_i, we deduce the assertion. □

Let $F \in \mathbb{R}[x]$ be the polynomial defined by (11).

Lemma 3. *For any $x \in \mathbb{R}^n$, $x \neq 0$, we have*

$$F(x) > 0. \tag{13}$$

Proof. If $x \neq 0$, then $x = tx_0$ for some $x_0 \in S$ and $t > 0$. Hence, $F(x) = t^{D_0}F(x_0)$ and by Lemma 1 we have $f(x_0) - \sum_{i=1}^{r} g_i(x_0)\varphi(g_i(x_0)) > 0$. Thus $F(x) > 0$. □

To sum up, we have shown that the polynomial F is homogeneous and it is positive for $x \in \mathbb{R}^n\backslash\{0\}$, so we can use the Reznick theorem. Therefore we can represent $|x|^{2r}F$, for some $r \in \mathbb{N}$, as a sum of even powers of linear polynomials.

Now we prove Theorem 3.

Proof of Theorem 3. Take a polynomial of the form (11) such that the assertion of Lemma 2 holds. By Lemma 3,

$$F(x) > 0 \quad \text{for} \quad x \in \mathbb{R}^n\backslash\{0\}.$$

Take $\epsilon(F) = \frac{\inf\{F(u): u \in S\}}{\sup\{F(u): u \in S\}}$, and let $r \in \mathbb{N}$ be such that

$$r \geq \frac{nD_0(D_0-1)}{(4\log 2)\epsilon(F)} - \frac{n+D_0}{2}.$$

So, for $D = D_0+2r$, by Theorem 1 there exist linear functions $q_1, \dots, q_j \in \mathbb{R}[x]$ such that

$$|x|^{2r}F(x) = q_1^D + \cdots + q_j^D.$$

Hence

$$|x|^{D-d}f(x) - \sum_{i=1}^{r} |x|^{\alpha_i+2r} g_i(x)(ag_i(x) + b|x|^{d_i})^N = \sum_{i=1}^{j} q_i^D$$

and

$$f(x) = |x|^{-D+d}\left(\sum_{i=1}^{j} q_i^D + \sum_{i=1}^{r} |x|^{\alpha_i+2r} g_i(x)(ag_i(x) + b|x|^{d_i})^N\right),$$

which completes the proof of Theorem 3. □

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