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Fixed points of automorphisms preserving the length of words in free solvable groups

Witold Tomaszewski

Abstract. Let δ be an automorphism of prime order p of the free group F_n. Suppose δ has no fixed points and preserves the length of words. By $\sigma := \delta^{(m)}$ we denote the automorphism of the free solvable group $F_n/F_n^{(m)}$ induced by δ. We show that every fixed point of σ has the form $cc^{\sigma}\dots c^{\sigma^{p-1}}$, where $c \in F_n^{(m-1)}/F_n^{(m)}$. This is a generalization of some known results, including the Macedońska–Solitar Theorem [10].

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1. Introduction. Motivation and the main result. If G is an arbitrary group, then as usual $G^{(m)}$ is the m-th term of the derived series of G, that is $G^{(0)} = G$, $G^{(1)} = G' = [G, G]$, and for $m > 1$ we have $G^{(m+1)} = [G^{(m)}, G^{(m)}]$. If g, h are elements of a group, then $g^h = h^{-1}gh$ and $[g, h] = g^{-1}h^{-1}gh$. We denote the free group of finite or infinite rank by F and by F_n the free group of rank n, freely generated by $x_1, \dots, x_n$. If $w \in F_n$, then $|w|$ is the length of w in the variables $x_1, \dots, x_n$. The group $F_n/F_n^{(m)}$ is the free solvable group of rank n, freely generated by $g_i = x_iF_n^{(m)}$ for $i \in \{1, \dots, n\}$. Throughout this paper I and J are sets consisting of integers, and we assume that 1 belongs to both sets. For a family of groups $\{G_i\}_{i\in I}$, let $\prod_{i\in I} G_i$ and $\prod^*_{i\in I} G_i$ be respectively the direct and the free product of the groups of this family.

Let δ be an automorphism of F_n. We say that δ **preserves the length** of words in the variables $x_1, \dots, x_n$ if for every word $w \in F_n$ we have $|w^{\delta}| = |w|$.

An automorphism δ preserves the length of words in the variables $x_1, \dots, x_n$ if and only if there is a permutation $\sigma \in S_n$ and an n-tuple $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$ such that $x_i^{\delta} = x_{i^{\sigma}}^{\varepsilon_i}$. The set of all such automorphisms forms a subgroup H of $\mathrm{Aut}(F_n)$. The subgroup H has two natural subgroups. The first,

K, is isomorphic to Z_2^n, the elementary abelian group of order 2^n. We associate an n-tuple $(\varepsilon_1, \ldots, \varepsilon_n) \in Z_2^n$ with the automorphism $\xi \in K$ acting on free generators as follows $x_i^{\xi} = x_i^{(-1)^{\varepsilon_i}}$ for $i = 1, \ldots, n$. The second subgroup, L, is isomorphic to the symmetric group S_n. If $\sigma \in S_n$, then the corresponding automorphism $\bar{\sigma} \in L$ acts on generators as follows: $x_i^{\bar{\sigma}} = x_{i^\sigma}$ for $i = 1, \ldots, n$. We say that the automorphisms from L permute the generators. It is easy to see that K and L are normal subgroups in H and that, in fact, H is isomorphic to the direct product $K \times L$ and to the direct product $Z_2^n \times S_n$. Hence every automorphism δ preserving the length of words can be uniquely decomposed as a product $\delta = \xi\sigma = \sigma\xi$, where $\xi \in K$ and $\sigma \in L$.

Let δ be an automorphism of F_n without nontrivial fixed points and preserving the length of words. The automorphism δ induces an automorphism $\delta^{(m)}$ of the free solvable group $F_n/F_n^{(m)}$ by the action $(wF_n^{(m)})^{\delta^{(m)}} = w^{\delta}F_n^{(m)}$. The aim of this work is to describe the subgroup of fixed points of $\delta^{(m)}$, that is

$$S(\delta^{(m)}) = \{w \in F_n/F_n^{(m)} : \; w^{\delta^{(m)}} = w\}.$$

Let $S^{(m)}(\delta)$ be the preimage of $S(\delta^{(m)})$ in F_n, that is

$$S^{(m)}(\delta) = \{w \in F_n : w^{-1}w^{\delta} \in F_n^{(m)}\}.$$

This topic is connected with the notion of symmetric words in groups and the Marczewski–Płonka problem. We say that a word $w(x_1, \ldots, x_n)$ in F_n is symmetric in a group G if for any permutation $\alpha \in S_n$, $w(x_1, \ldots, x_n) = w(x_{1^\alpha}, \ldots, x_{n^\alpha})$ is the identity in G. If w is an n-symmetric word in G, then the function $f : G^n \to G$ given by $f(g_1, \ldots, g_n) = w(g_1, \ldots, g_n)$ for every n-tuple $(g_1, \ldots, g_n) \in G^n$ is called a symmetric operation in G. Let $\mathcal{K}$ be a class of all groups G such that the group operation xy is a composition of symmetric operations. It is clear that all abelian groups are in $\mathcal{K}$. In 1967 Marczewski asked whether $\mathcal{K}$ consists only of abelian groups (see [11]). In 1970 Płonka gave in [12] an example of a non-abelian group which belongs to $\mathcal{K}$. But it is still an open question which groups belong to $\mathcal{K}$ (see [13]). In [15] Płonka described symmetric words in nilpotent groups of class ≤ 3, and it follows from his description that non-abelian nilpotent groups do not belong to $\mathcal{K}$. In the series of papers [4–6] Hołubowski described symmetric words in free nilpotent groups of class 4 and 5, 2- and 3-symmetric words in free metabelian groups and in free metabelian nilpotent groups of any class. In [3] Gupta and Hołubowski found all 2-symmetric words in free nilpotent-by-abelian groups and free centre-by-metabelian groups. In [10] Macedońska and Solitar characterized 2-symmetric words in free metabelian and solvable groups of derived length 3. I presented in [16] a description of 2-symmetric words in free solvable groups of any derived length, and in cooperation with Bagiński, I described in [1] fixed points of the automorphism cyclically permuting generators in free metabelian groups. In papers the [7–9, 14, 15], Płonka, Krstić, and Macedońska showed that if G is a (free) nilpotent group, then the function $\delta_{n-1}^n(w(x_1, \ldots, x_n)) = w(x_1, \ldots, x_{n-1}, 1)$ is an isomorphism of the group of n-symmetric words onto the group of $(n-1)$-symmetric words.

Let $G = F_n/V$ be a relatively free group, freely generated by elements $g_1 = x_1V, \dots, g_n = x_nV$. Then $w(x_1,\dots,x_n) \in F_n$ is a symmetric word in G if its image $w(g_1,\dots,g_n)$ in G is a fixed point for all automorphisms permuting the generators $g_1,\dots,g_n$. Such automorphisms are induced by automorphisms permuting the generators $x_1,\dots,x_n$ in F_n. In fact, a word $w(x_1,\dots,x_n) \in F_n$ is the symmetric word in G if and only if its image $w(g_1,\dots,g_n)$ is a fixed point of two automorphisms. The one that interchanges g_1 and g_2 and acts identically on the rest of the generators and the other which cyclically permutes generators, i.e. acts on them as follows: $g_1 \to g_2 \to \cdots \to g_n \to g_1$.

The main result of this paper is the following:

Main Theorem. *Let δ be a length preserving automorphism of F_n of prime order p and without nontrivial fixed points. Let $\delta^{(m)}$ be an automorphism of the free solvable group $G = F_n/F_n^{(m)}$ induced by δ. Then every fixed point of $\sigma = \delta^{(m)}$ has the form $cc^\sigma \dots c^{\sigma^{p-1}}$, where $c \in G^{(m-1)} = F_n^{(m-1)}/F_n^{(m)}$.*

It is easy to see that every element of the form $cc^\sigma \cdots c^{\sigma^{p-1}}$, where $c \in G^{(m-1)} = F_n^{(m-1)}/F_n^{(m)}$, is a fixed point for $\sigma = \delta^{(m)}$, but it is not obvious that only such elements are the fixed points.

The Main Theorem is a generalization of a result of Macedońska and Solitar [10], who described the form of fixed points for automorphisms permuting generators in the 2-generator free metabelian group and free solvable group of derived length 3. This was later generalized by the author [16] to include free solvable groups of any derived length. The Main Theorem is also a generalization of a result of Bagiński and the author [1] which gives a description of the fixed points of the automorphism cyclically permuting generators in free metabelian groups.

The Main Theorem gives the full description of 2-symmetric words in free solvable groups, which we formulate as follows:

Corollary 1. *Let $w(x,y) \in F_2$ be a 2-symmetric word in any solvable group of derived length m. Then w has the form $w = c(x,y)c(y,x)\xi$, where $c(x,y) \in F_2^{(m-1)}$ and $\xi \in F_2^{(m)}$.*

Proof. If $w(x,y)$ is a 2-symmetric word in any solvable group of the derived length m, then it is a 2-symmetric word in $F_2/F_2^{(m)}$. So by the Main Theorem w has the required form. □

The next result follows directly from the Main Theorem

Corollary 2. *The free, solvable, non-abelian groups of finite rank do not belong to $\mathcal{K}$.*

Proof. Let $G = F_n/F_n^{(m)}$ be a free, non-abelian, solvable group, freely generated by $g_1,\dots,g_n$. Let α be the automorphism cyclically permuting the generators of the free group F_n that is acting on generators of the free group as follows: $x_1 \to x_2 \to \cdots \to x_n \to x_1$. There exists a positive integer k such that $\beta = \alpha^k$ has no nontrivial fixed points in F_n and has prime order p. If $w \in F_n$ is an n-symmetric word in G, then its image $\bar{w}$ in G has to be a fixed point of $\bar{\beta}$, so by the Main Theorem, $\bar{w}$ belongs to $G^{(m-1)}$, and since G is non-abelian, we have $m > 1$, so $\bar{w}$ belongs to G'. If xy were a composition of symmetric operations, it would belong to G'. However, this would be impossible. □

2. Proofs. Throughout this section we assume that δ is an automorphism of F_n preserving the length of words and that δ has no nontrivial fixed points in F_n. The proof of the Main Theorem is based on Dyer–Scott's Theorem on automorphisms of prime order in a free group [2].

Proposition 1. *Let δ be an automorphism of F_n of order k that has no nontrivial fixed points and preserves the length of words in the variables $x_1, \dots, x_n$. Let w and c be elements of F_n.*

(i) *The equation $cc^{\delta} \dots c^{\delta^{k-1}} = 1$ holds if and only if there exists $u \in F_n$ such that $c = u^{-1}x^{\varepsilon}u^{\delta}$, where $x \in \{x_1, \dots, x_n\}$ is such that $x^{\delta} = x^{-1}$ and $\varepsilon \in \{-1, 0, 1\}$. Moreover, if k is odd, then $\varepsilon = 0$ and $c = u^{-1}u^{\delta}$.*

(ii) *There are no $a \in F_n$ and $x \in \{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$ which satisfy $x^{\delta} = x^{-1}$ and $a^{\delta} = xax$.*

(iii) *If $w^{\delta} = c^{-1}wc$ and $cc^{\delta} \dots c^{\delta^{k-1}} = 1$, then $w = 1$.*

Proof. (i) Note that if k is odd, then there is no $x \in \{x_1, \dots, x_n\}$ such that $x^{\delta} = x^{-1}$, so in this case $\varepsilon = 0$. If $c = u^{-1}x^{\varepsilon}u^{\delta}$, then

$$\begin{aligned} cc^{\delta} \dots c^{\delta^{p-1}} &= u^{-1}x^{\varepsilon}u^{\delta}(u^{-1}x^{\varepsilon}u^{\delta})^{\delta}(u^{-1}x^{\varepsilon}u^{\delta})^{\delta^2} \dots (u^{-1}x^{\varepsilon}u^{\delta})^{\delta^{k-1}} \\ &= u^{-1}x^{\varepsilon}u^{\delta}u^{-\delta}x^{-\varepsilon}u^{\delta^2}u^{-\delta^2}x^{\varepsilon}u^{\delta^3} \dots u^{-\delta^{k-1}}x^{(-1)^k\varepsilon}u^{\delta^k} = u^{-1}u^{\delta^k} = u^{-1}u = 1. \end{aligned}$$

For the converse, we use induction on the length of a word c. If c has the length 1, then $c \in \{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$, and it satisfies $cc^{\delta} \dots c^{\delta^{k-1}} = 1$ only if $c^{\delta} = c^{-1}$. If $|c| = 2$, then $c = x_i^{\varepsilon_1}x_j^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 = \pm 1$. For the word $cc^{\delta} \dots c^{\delta^{k-1}}$ to cancel, we need $(x_i^{\varepsilon_1})^{\delta} = x_j^{-\varepsilon_2}$ and then $c = u^{-1}u^{\delta}$ where $u = x_i^{-\varepsilon_1}$. Let the statement be true for every word of length less than $|c| > 2$. Then $c = x_i^{\varepsilon_1}c_1x_j^{\varepsilon_2}$ and $cc^{\delta} \dots c^{\delta^{k-1}} = x_i^{\varepsilon_1}c_1x_j^{\varepsilon_2}(x_i^{\varepsilon_1}c_1x_j^{\varepsilon_2})^{\delta} \dots (x_i^{\varepsilon_1}c_1x_j^{\varepsilon_2})^{\delta^{k-1}}$.

For the word $cc^{\delta} \dots c^{\delta^{k-1}}$ to cancel, we need again $(x_i^{\varepsilon_1})^{\delta} = x_j^{-\varepsilon_2}$ and then $1 = cc^{\delta} \dots c^{\delta^{k-1}} = x_i^{\varepsilon_1}c_1c_1^{\delta} \dots c_1^{\delta^{k-1}}x_i^{-\varepsilon_1}$. So c_1 satisfies the equation $c_1c_1^{\delta} \dots c_1^{\delta^{k-1}} = 1$, and by the inductional assumption it has the form $c_1 = u_1^{-1}x^{\varepsilon}u_1^{\delta}$. Hence c has the form $c = u^{-1}x^{\varepsilon}u^{\delta}$, where $u = u_1x_i^{-\varepsilon_1}$.

(ii) Assume that $a^{\delta} = xax$, where x is a generator or its inverse, satisfying $x^{\delta} = x^{-1}$. We have $|a| = |a^{\delta}| = |xax|$, so we must have $a = x^{-1}b$ or $a = bx^{-1}$ for some b such that $|b| = |a| - 1$. Consider the first case, the second is analogous. We have $a^{\delta} = (x^{-1}b)^{\delta} = xax = bx$ and hence $xb^{\delta} = bx$. So, $b^{\delta} = x^{-1}bx$. Again $|b| = |x^{-1}bx|$, which means that $b = ux^{-1}$ or $b = xu$ for some u such that $|u| = |b| - 1 = |a| - 2$. The case $b = ux^{-1}$ is impossible because then $a^{\delta} = bx = u$ and $|a| = |a^{\delta}| = |u| = |a| - 2$. The second case is also impossible because $a^{\delta} = xax = bx = xux$. It follows from this that $a = u$, but a and u have different lengths.

(iii) We can assume that a word satisfying these assumptions is cyclically reduced. If not, then we can consider a cyclically reduced word w^{v}, for which

$$(w^v)^\delta = (w^\delta)^{v^\delta} = v^{-\delta}c^{-1}wcv^\delta = (v^{-\delta}c^{-1}v)w^v(v^{-1}cv^\delta),$$

and we have

$$\begin{aligned}&(v^{-1}cv^\delta)(v^{-1}cv^\delta)^\delta(v^{-1}cv^\delta)^{\delta^2}\dots(v^{-1}cv^\delta)^{\delta^{k-2}}(v^{-1}cv^\delta)^{\delta^{k-1}}\\&= v^{-1}cv^\delta v^{-\delta}c^\delta v^{\delta^2}v^{-\delta^2}c^{\delta^2}v^{\delta^3}\dots v^{-\delta^{k-2}}c^{\delta^{k-2}}v^{\delta^{k-1}}v^{-\delta^{k-1}}c^{\delta^{k-1}}v^{\delta^k}\\&= v^{-1}\underbrace{cc^\delta\dots c^{\delta^{k-2}}c^{\delta^{k-1}}}_{=1}v^{\delta^k} = v^{-1}v = 1.\end{aligned}$$

So w^v also satisfies the assumptions of the Proposition for $c_1 = v^{-\delta}c^{-1}v$. Moreover, since δ does not change the length of words, we have $|w| = |w^c|$. So there exists v such that $w = cv$ or $w = vc^{-1}$. It is enough to consider the first equality because the reasoning for the second is analogous. If $w = cv$, then $(cv)^\delta = (cv)^c = vc$. Hence, $vc = c^\delta v^\delta$ and $v^\delta = c^{-\delta}vc$. Using (i) we have that $c = u^{-1}x^\varepsilon u^\delta$, and so $v^\delta = (u^{-1}x^\varepsilon u^\delta)^{-\delta}vu^{-1}x^\varepsilon u^\delta = u^{-\delta^2}x^\varepsilon u^\delta vu^{-1}x^\varepsilon u^\delta$. So we get:

$$x^\varepsilon u^\delta vu^{-1}x^\varepsilon = u^{\delta^2}v^\delta u^{-\delta} = (u^\delta vu^{-1})^\delta.$$

If $\varepsilon \neq 0$, then for $a = u^\delta vu^{-1}$ we get $a^\delta = xax$, which by (ii) is impossible. If $\varepsilon = 0$, then $u^\delta vu^{-1}$ is a fixed point of δ. But δ has no nontrivial fixed points, so $u^\delta vu^{-1} = 1$. Hence $v = u^{-\delta}u = (u^{-1}u^\delta)^{-1} = c^{-1}$ and $w = cv = cc^{-1} = 1$, as required.

□

Lemma 1. *Let F_n be a free group, freely generated by $x_1, \dots, x_n$, and let δ be an automorphism of F_n of order p which has no nontrivial fixed points and which does not change the length of words. Then for every positive integer m, the subgroup $F_n^{(m)}$ is a free product $\prod^*_{i\in I}\langle a_{i1}, \dots, a_{ip}\rangle * \prod^*_{j\in J}\langle c_{j1}, \dots, c_{j(p-1)}\rangle$, where I, J are sets of positive integers, and for all $i \in I$ and $j \in J$: $a_{i1}^\delta = a_{i2}, a_{i2}^\delta = a_{i3}, \dots, a_{i(p-1)}^\delta = a_{ip}, a_{ip}^\delta = a_{i1}$, $c_{j1}^\delta = c_{j2}, c_{j2}^\delta = c_{j3}, \dots, c_{j(p-1)}^\delta = c_{j(p-1)}^{-1}c_{j(p-2)}^{-1}\dots c_{j1}^{-1}$.*

Proof. Let δ_m be a restriction of δ to $F_n^{(m)}$. Then δ_m is an automorphism of prime order p of $F_n^{(m)}$. Dyer and Scott proved [2, Theorem 3] that then $F_n^{(m)}$ is a free product $F^{\langle\delta_m\rangle} * \prod^*_{i\in I}F_i * \prod^*_{j\in J}F_j$, where $F^{\langle\delta_m\rangle}$ is a subgroup of fixed points of δ_m, $F_i = \langle a_{i1},\dots, a_{ip}\rangle$, $a_{i1}^\delta = a_{i2}, a_{i2}^\delta = a_{i3},\dots, a_{i(p-1)}^\delta = a_{ip}, a_{ip}^\delta = a_{i_1}$, $F_j = \langle c_{j1},\dots, c_{j(p-1)}, w_{k,j}, k\in J_j\rangle$, where $c_{j1}^\delta = c_{j2}, c_{j2}^\delta = c_{j3}, \dots, c_{j(p-1)}^\delta = c_{j(p-1)}^{-1}c_{j(p-2)}^{-1}\dots c_{j1}^{-1}$, and for every $k \in J_j$ we have $w_{k,j}^\delta = c_{j1}^{-1}w_{k,j}c_{j1}$. It is clear that $F^{\langle\delta_m\rangle}$ is trivial (δ has no nontrivial fixed points and neither does δ_m). The word c_{j1} satisfies the equation

$$c_{j1}c_{j1}^\delta\dots c_{j1}^{\delta^{p-1}} = c_{j1}c_{j2}\dots c_{j(p-1)}c_{j(p-1)}^{-1}c_{j(p-2)}^{-1}\dots c_{j1}^{-1} = 1,$$

so by Proposition 1 (iii) the words $w_{k,j}$ have to be trivial, so $F_j = \langle c_{j1},\dots c_{j(p-1)}\rangle$, and the statement of the Lemma follows. □

The similar basis as in Lemma 1 for automorphism interchanging generators in F_2 was constructed by the author in [16]. This construction uses the special basis in free groups of countable rank described in [16] or in [17]. But the author cannot use this technique for automorphisms permuting generators in free groups of rank greater than 2.

Lemma 2. *Let $A = \prod_{i\in I}\langle A_i\rangle \times \prod_{j\in J}\langle C_j\rangle$ be a free abelian group, freely generated by the union of sets $A_i = \{\alpha_{i1},\ldots,\alpha_{ip}\}$ and $C_j = \{\gamma_{j1},\ldots,\gamma_{j(p-1)}\}$ for $i \in I$, $j \in J$, and let φ be an automorphism of A acting on subgroups $\langle A_i\rangle$ and $\langle C_j\rangle$ as follows: $\alpha_{i1}^{\varphi} = \alpha_{i2}, \alpha_{i2}^{\varphi} = \alpha_{i3},\ldots,\alpha_{i(p-1)}^{\varphi} = \alpha_{ip}, \alpha_{ip}^{\varphi} = \alpha_{i1}$, $\gamma_{j1}^{\varphi} = \gamma_{j2}, \gamma_{j2}^{\varphi} = \gamma_{j3},\ldots,\gamma_{j(p-1)}^{\varphi} = \gamma_{j(p-1)}^{-1}\gamma_{j(p-2)}^{-1}\cdots\gamma_{i1}^{-1}$. Then every fixed point a of φ has the form $a = cc^{\varphi}\ldots c^{\varphi^{p-1}}$, where $c \in \prod_{i\in I} A_i$.*

Proof. It is easy to see that the statement is true if $a \in A_i$ for any $i \in I$ and that φ has no nontrivial fixed points in C_j for any $j \in J$. The Lemma follows since φ acts independently on direct summands of A. □

Remark 1. *Without loss of generality, we can assume that every fixed point of φ satisfying the assumptions of the previous lemma has the form $cc^{\varphi}\ldots c^{\varphi^{p-1}}$, where $c = \alpha_{11}^{d_1}\ldots\alpha_{k1}^{d_k}$ for some k, where $a_{i1} \in A_i$.*

Remark 2. *It follows from Lemma 1 that for every m, a group $F_n^{(m)}/F_n^{(m+1)}$ has a basis such that the automorphism $\varphi = \bar{\delta}_m$ satisfies the assumptions of Lemma 2, where $\bar{\delta}_m$ is a restriction of $\bar{\delta}$ onto $F_n^{(m)}/F_k^{(m+1)}$. We will use this remark both for $F_n^{(m)}/F_n^{(m+1)}$ and*

$$F_n^{(m-1)}/F_n^{(m)} \simeq \left(F_n^{(m-1)}/F_n^{(m+1)}\right)\Big/\left(F_n^{(m)}/F_n^{(m+1)}\right).$$

Lemma 3. *For every integer $m > 0$ we have:*

(i) $S^{(m+1)}(\delta) \subseteq S^{(m)}(\delta)$,
(ii) $F_n^{(m)} \subseteq S^{(m)}(\delta) \subseteq F_n^{(m-1)}$,
(iii) $S^{(m+1)}(\delta)$ *is a normal subgroup of* $S^{(m)}(\delta)$.

Proof. Let us be reminded that

$$S^{(m)}(\delta) = \{w \in F_n : w^{-1}w^{\delta} \in F_n^{(m)}\}.$$

(i) If w belongs to $S^{(m+1)}(\delta)$, then $w^{-1}w^{\delta} \in F_n^{(m+1)} \subseteq F_n^{(m)}$, so $w \in S^{(m)}(\delta)$.
(ii) The inclusion $F_n^{(m)} \subseteq S^{(m)}(\delta)$ is trivial, so we shall prove that $S^{(m)}(\delta) \subseteq F_n^{(m-1)}$. We use an induction on m. For $m = 1$ the situation is clear. Now suppose that the statement is true for some $m > 0$. By part (i) and the induction hypothesis $S^{(m+1)}(\delta) \subseteq S^{(m)}(\delta) \subseteq F_n^{(m-1)}$. Let $u \in S^{(m+1)}(\delta)$. We know that $u \in S^{(m)}(\delta) \cap F_n^{(m-1)}$, and we want to show that $u \in F_n^{(m)}$. By Lemmas 1 and 2 we have that $u = aa^{\delta}\ldots a^{\delta^{p-1}}z$ with

$$a = a_1^{d_1}\ldots a_k^{d_k},$$

where $a_1, a_1^{\delta}, \ldots, a_1^{\delta^{p-1}}, \ldots, a_k, a_k^{\delta}, \ldots, a_k^{\delta^{p-1}}$ are among the free generators for $F_n^{(m-1)}$ and $z \in F_n^{(m)}$. Let ϕ_i be an endomorphism of $F_n^{(m-1)}$ that fixes the generators $a_i, a_i^{\delta}, \ldots, a_i^{\delta^{p-1}}$ and maps all other generators to 1. Clearly, ϕ_i commutes with the action of δ on $F_n^{(m-1)}$ and thus

$$v_i = u^{\phi_i} = bb^{\delta} \ldots b^{\delta^{p-1}} z^{\phi_i}$$

is also in $S^{(m+1)}(\delta)$, where $b = a_i^{d_i}$. Then $\bar{v}_i$, the image of v_i in $F_n^{(m-1)}/F_n^{(m+1)}$, is a fixed point of an automorphism cyclically permuting the generators of the free metabelian group $\langle \bar{a}_i, \bar{a}_i^{\delta}, \ldots, \bar{a}_i^{\delta^{p-1}} \rangle$. It follows from [1, Theorem 3] that $\bar{v}_i$ belongs to $\langle \bar{a}_i, \bar{a}_i^{\delta}, \ldots, \bar{a}_i^{\delta^{p-1}} \rangle'$, which means that $d_i = 0$. This shows that $d_1 = d_2 = \cdots = d_k = 0$ and thus $u \in F_n^{(m)}$.

(iii) Let $w \in S^{(m+1)}(\delta)$ and $v \in S^{(m)}(\delta)$. By assumption $v^{-1}v^{\delta} \in F_n^{(m)}$ and $w^{-1}w^{\delta} \in F_n^{(m+1)}$. By (ii) we know that $w \in F_n^{(m)}$, and thus w commutes with $v^{-1}v^{\delta}$ modulo $F_n^{(m+1)}$. Thus, modulo $F_n^{(m+1)}$, we have

$$\begin{aligned}(w^{v^{-1}})^{-1}(w^{v^{-1}})^{\delta} &= vw^{-1}v^{-1}v^{\delta}w^{\delta}(v^{\delta})^{-1} = vv^{-1}v^{\delta}w^{-1}w^{\delta}(v^{\delta})^{-1} \\ &= v^{\delta}(v^{\delta})^{-1} = 1,\end{aligned}$$

so $w^{v^{-1}} = vwv^{-1} \in S^{(m+1)}(\delta)$.

□

Proof of the Main Theorem. If u is a fixed point of $\delta^{(m+1)}$ in $F_n/F_n^{(m)}$, then by Lemma 3 (ii) u belongs to $F_n^{(m-1)}/F_n^{(m)}$, which is a free abelian group. By Remark 2, if φ is a restriction of $\delta^{(m+1)}$ to $F_n^{(m-1)}/F_n^{(m)}$, then φ satisfies the assumptions of Lemma 2. So u has the required form. □

There are many questions connected with the topic of this paper to which the author so far has no answers.

For example, is it true that $S^{(m)}(\delta)/S^{(m+1)}(\delta)$ is a free metabelian group?

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Witold Tomaszewski
Institute of Mathematics,
Silesian University of Technology,
Kaszubska 23, 44-100 Gliwice, Poland
e-mail: Witold.Tomaszewski@polsl.pl

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