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Nuclear ranges in implicative semilattices

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Abstract. A nucleus on a meet-semilattice A is a closure operation that preserves binary meets. The nuclei form a semilattice $\mathrm{N}A$ that is isomorphic to the system $\mathcal{N}A$ of all nuclear ranges, ordered by dual inclusion. The nuclear ranges are those closure ranges which are total subalgebras (l-ideals). Nuclei have been studied intensively in the case of complete Heyting algebras. We extend, as far as possible, results on nuclei and their ranges to the non-complete setting of implicative semilattices (whose unary meet translations have adjoints). A central tool are so-called r-morphisms, that is, residuated semilattice homomorphisms, and their adjoints, the l-morphisms. Such morphisms transport nuclear ranges and preserve implicativity. Certain completeness properties are necessary and sufficient for the existence of a least nucleus above a prenucleus or of a greatest nucleus below a weak nucleus. As in pointfree topology, of great importance for structural investigations are three specific kinds of l-ideals, called basic open, boolean and basic closed.

Mathematics Subject Classification. 06A15, 06D20.

Keywords. Closure operation, Closure range, Heyting lattice, l-ideal, Implicative semilattice, Nucleus, Nuclear range.

1. Introduction

Closure is a vital theme in quite diverse mathematical disciplines (see [16] for a survey and historical background). A convenient framework for closure theory is that of ordered sets (posets): a closure or hull operation on a poset A is an isotone (order-preserving), inflationary (extensive) and idempotent self-map on A, or in succinct terms, a unary operation g on A with

$$x \leq gy \Leftrightarrow gx \leq gy. \tag{C}$$

Presented by W. Wm. McGovern.

Extensions to the even more comprehensive categorical theory of reflections, monads [1, Ch. V-20] and closure operators [10, 12] are possible but will not concern us here. Closure may be described by several equivalent structures, the most obvious ones being the so-called closure ranges [15, 16], elsewhere also termed partial ordinals [2], closure subsets [9], or closure systems [11]. While by a closure range of a poset A we mean a subset C such that above each element of A there is a least element in C, we reserve the name "closure system" for subsets of power sets $\mathcal{P}X$ that are closed under arbitrary intersections (with $\bigcap\emptyset = X$), hence closure ranges in $\mathcal{P}X$. Sending each closure operation to its range (the fixpoint set), one obtains an isomorphism between the pointwise ordered set $\mathrm{C}A$ of all closure operations and the set $\mathcal{C}A$ of all closure ranges, ordered by dual inclusion.

Taking into account any *binary* operation on A that is isotone in both arguments, one calls g multiplicative if g preserves that operation. An early investigation of this rather general situation with many examples is due to Varlet [44, 45]. Specifically, motivated by the familiar notion of Kuratowski closure in topology, one considers closure operations on boolean algebras that preserve finite joins. This approach, intertwining topology with algebraic logic and lattice theory, was pursued by McKinsey and Tarski in their groundbreaking papers *The algebra of topology* [34] and *On closed elements in closure algebras* [35], which had a great number of followers. In the same vein as the lattices of open resp. closed sets in topological spaces are the fixpoint sets of topological interior resp. closure operators on power sets, the range of any interior operation on a boolean algebra is a Heyting algebra, and all Heyting algebras arise this way; see Esakia [20] for a categorical equivalence.

Another obvious choice is the binary meet operation of $\wedge$-semilattices with top, in this paper merely referred to as semilattices. At first glance, this choice does not look very promising, because on a boolean algebra the only $\wedge$-preserving closure operations are the unary join-operations $\gamma_a : x \mapsto a \vee x$. But $\wedge$-preserving closure operations, nowadays often called nuclei, turned out to be of fundamental importance in pointfree topology, logic, topos theory [27, 28], and other branches of mathematics. Perhaps the first account of that theme is Bergmann's 1952 paper [5]. Restricting the isomorphism between $\mathrm{C}A$ and $\mathcal{C}A$ to the poset $\mathrm{N}A$ of all nuclei on a semilattice A leads to an isomorphism between $\mathrm{N}A$ and the subsemilattice $\mathcal{N}A$ of $\mathcal{C}A$ consisting of all nuclear ranges (also termed nuclear systems [11] or strong ideals [40]).

In the paper at hand and its successor [17] we investigate nuclei on implicative semilattices (cf. [6, 7, 36, 44, 45]), that is, semilattices with a binary operation $\rightarrow$, called residuation, formal implication or relative pseudocomplementation (according to the respective interpretation), such that

$$x \wedge y \leq z \Leftrightarrow y \leq x \rightarrow z.$$

Other common notations for $x \rightarrow z$ are $x * z$ [36, 30, 45] or $z : x$ [9], but often it is more convenient to use the symbols x^z for $x \rightarrow z$ and x^{yz} for $(x^y)^z$ in order to avoid parentheses in iterated applications [8, 45]. According to that convention, it is consistent to write $x^\perp$ for the pseudocomplement $x^* = x \rightarrow \perp$ (provided a bottom element $\bot$ exists). In fact, $^\bot$ is a kind of orthocomplementation, in view of De Morgan's law $(x \vee y)^\bot = x^\bot \wedge y^\bot$ in Heyting algebras. Notice also the "powerful" equation

$$x^{zz} \wedge x^z = z$$

which assures that implicative semilattices are distributive: $x \wedge y \leq z$ implies $z = x' \wedge y'$ for some $x' \geq x$ and $y' \geq y$ (take $x' = x^{zz}$, $y' = x^z$). The exchanged notation z^x instead of x^z would be more suggestive, in view of the rules

$$\begin{aligned} x \rightarrow (y \wedge z) &= (x \rightarrow y) \wedge (x \rightarrow z), \\ (x \vee y) \rightarrow z &= (x \rightarrow z) \wedge (y \rightarrow z), \\ (x \wedge y) \rightarrow z &= x \rightarrow (y \rightarrow z), \end{aligned}$$

which in the z^x notation, oppressing the symbol $\wedge$ and writing $+$ for $\vee$, would turn into the familiar exponential rules $(yz)^x = y^x z^x$, $z^{x+y} = z^x z^y$, $z^{xy} = (z^y)^x$. However, this exchanged notation seems not to have prevailed in the lattice-theoretical literature, while it is common in the wider context of cartesian closed categories (see, e.g., [1, Ch. VII]).

Implicative semilattices are also called relatively pseudocomplemented or Brouwerian (see, e.g., Köhler [30,31]); some authors use the latter term for the dual structures (McKinsey and Tarski [35]). Picado, Pultr and Tozzi [40] speak of Heyting semilattices, while we reserve that term for bounded implicative semilattices (possessing a least element $\bot$). A Heyting lattice is then both a Heyting semilattice and a lattice, and the associated algebra with the operations $\vee, \wedge, \bot, \top$ and $\rightarrow$ is a Heyting algebra; for the origins of this concept, see the pioneering work of Glivenko [24] and Heyting [25] related to Brouwer's intuitionistic logic. Immense research has been devoted to that theme and its role in logic, algebra and topology (see, e.g., Esakia [20]).

Some of the results we shall prove for nuclei on semilattices are known for the case of lattices or at least of complete lattices, but the lack of certain joins or meets often requires new methods. By general properties of adjoint maps (see Sect. 2), binary meets in implicative semilattices distribute over all existing joins, and the complete Heyting lattices are the frames or locales [26,27,38,39,43], satisfying for all subsets Y the distributive law

$$x \wedge \bigvee Y = \bigvee \{x \wedge y \,|\, y \in Y\}.$$

Further examples of Heyting lattices are all products of bounded chains. The chain ω of natural numbers is *not* an implicative semilattice, missing a top element. On the contrary, the dual chain ω^{op} is a $\bigvee$-complete implicative lattice (but not a Heyting lattice), and $\mathcal{C}\omega^{op} = \mathcal{N}\omega^{op}$ is the closure system of all subsets containing the top element. Thus, it is a boolean frame. The following six structures all describe the same objects if their carriers are finite:

nonempty (bounded) distributive lattices, frames, locales,

Heyting semilattices, Heyting lattices, Heyting algebras.

Nuclei on Heyting algebras are also referred to as modal operators (Beazer and Macnab [4], Macnab [32,33]). Our perspective is slightly different from the classical one: we consider an implicative semilattice A with top element $\top$ as a general algebra with the binary meet operation $\wedge$, the nullary operation $\top$, and the family $\alpha = (\alpha_a : a \in A)$ of unary operations

$$\alpha_a : A \longrightarrow A \ \text{ with } \ \alpha_a y = a \to y,$$

which are related to the unary meet operations

$$\lambda_a : A \longrightarrow A \ \text{ with } \ \lambda_a x = a \wedge x$$

via the adjoining equivalence

$$\lambda_a x \leq y \Leftrightarrow x \leq \alpha_a y.$$

The subalgebras of $(A, \wedge, \top, \alpha)$ are what Köhler calls total subalgebras [31]. In accordance with a general ideal concept in universal algebra, we call them left $\to$-ideals, or l-ideals for short (Picado, Pultr and Tozzi merely speak of ideals [40]). Those l-ideals which are closure ranges are said to be nuclear, because they are exactly the ranges of nuclei. Under the changed perspective the appropriate alternative to the usual homomorphisms are residuated (that is, coadjoint) and top-preserving $\wedge$-homomorphisms between semilattices. We call them r-morphisms and their adjoints l-morphisms. A basic observation will be that the image of an implicative semilattice under an r- or l-morphism is implicative, too. For implicative semilattices, the unary meet operations λ_a induce r-morphisms from A onto $\downarrow a = \{x \in A \,|\, x \leq a\}$, and their adjoints α_a are nuclei inducing l-morphisms from $\downarrow a$ into A.

In the complete case, the r-morphisms are nothing but the frame homomorphisms, whereas the l-morphisms are the locale morphisms or localic maps [38], the natural morphisms in pointfree topology, corresponding to continuous maps between topological spaces [26,27,38]. The tools of r- and l-morphisms provide extensions of results from the realm of frames/locales to arbitrary Heyting algebras or even to implicative semilattices. Sometimes the existence of certain joins or meets is indispensable. But one also finds substantial results on nuclei in the non-complete setting, for example in the work of Macnab [32,33] on Heyting algebras; see also Varlet [44,45].

Section 2 provides the necessary fundaments concerning adjunctions and closure operations. In Section 3 we introduce some useful weak variants of nuclei and see that nuclei and their ranges are transported forth and back by suitable adjoint maps. Section 4 contains the relevant connections between nuclei, prenuclei and weak nuclei on implicative semilattices and their ranges. Certain completeness properties turn out to be necessary and sufficient in order that for each prenucleus there is a least nucleus above it, and for each weak nucleus there is greatest nucleus below it. Section 5 is devoted to three important kinds of l-ideals in implicative semilattices: the basic open l-ideals $\mathfrak{a}a = \{a \to x \,|\, x \in A\}$, the boolean l-ideals $\mathfrak{b}a = \{x \to a \,|\, x \in A\}$, and the basic closed l-ideals $\mathfrak{c}a = \{x \in A \,|\, x \geq a\}$. The first two kinds are always nuclear, whereas the $\mathfrak{c}a$'s are closure ranges only in lattices. The nuclear l-ideals $\mathfrak{b}a$ form a meet-dense subset of $\mathcal{N}A$ and are exactly those l-ideals which are boolean lattices. The basic open and the basic closed l-ideals are complementary in the frame $\mathcal{T}A$ of all l-ideals, and together they generate $\mathcal{N}A$ via joins of finite meets if A is a Heyting algebra [33].

In the second part [17] we use the results derived in this first part for a thorough study of the algebraic structure of $\mathcal{T}A$ (cf. Köhler [31]) and of $\mathcal{N}A$ (cf. Picado, Pultr and Tozzi [40]). Other applications occur in [19].

2. Closure operations, closure ranges, and adjoint maps

The letter A denotes a (partially) ordered set (*poset*), $\leq$ its order relation, $\geq$ the dual order, and A^{op} the opposite or dually ordered set. A least element (*bottom*) is denoted by $\bot$ or $\bot A$, and a greatest element (*top*) by $\top$ or $\top A$. A poset possessing a top is said to be *topped*. For $Y \subseteq A$,

$$\uparrow Y = \{x \in A \mid \exists\, y \in Y\,(x \geq y)\} \text{ and } \downarrow Y = \{x \in A \mid \exists\, y \in Y\,(x \leq y)\}$$

are the *upset* and the *downset* generated by Y, respectively. $\uparrow y = \uparrow\{y\}$ resp. $\downarrow y = \downarrow\{y\}$ is the *principal upset* resp. *principal downset* generated by $y \in A$. An equation $x = \bigvee Y$ means that x is the least upper bound (*join, supremum*) of Y; dually, *meets* (*infima*) are defined as greatest lower bounds and denoted by $\bigwedge Y$. The poset A is *complete* if all subsets have joins (or equivalently, all subsets have meets), and $\bigvee$-*complete* if all nonempty subsets have joins (or equivalently, all lower bounded subsets have meets). In a $\vee$-*semilattice*, the binary joins $x \vee y = \bigvee\{x, y\}$ exist for all elements x, y; dually, in $\wedge$-*semilattices* all binary meets $x \wedge y$ exist.

If maps are applied to elements, we omit parentheses and write fa or f_a for the image of an element a under a map f, and also fX for the image of a subset X of the domain, while the preimage of a subset Y of the codomain is denoted by $f^{\leftarrow}Y$. We write A^A for the pointwise ordered set of all self-maps or *unary operations* f on A, and we put

$$A_f = \{a \in A \mid fa = a\},\ A_f x = A_f \cap \uparrow x \text{ for } x \in A.$$

The *fixpoint set* A_f coincides with the *range* fA whenever f is idempotent ($ff = f$). The map f is *isotone* if $x \leq y$ implies $fx \leq fy$, and *inflationary* or an *inflation* if $x \leq fx$. By a *preclosure operation* we mean an isotone inflation, and by a *weak closure operation* [41] an idempotent inflation. A *retraction* (in [23]: *projection*) is an isotone idempotent map, and a *closure* or *hull operation* is an idempotent preclosure operation. Dual closure operations are called *coclosure* or *kernel operations*. An easy verification shows:

Lemma 2.1. *For all preclosure operations j and all weak closure operations g, passing to the ranges inverts the order: $j \leq g$ is equivalent to $A_g \subseteq A_j$.*

We call a subset C of A a *closure range* if for each $a \in A$ there is a least $c \in C$ with $a \leq c$, but apply the terms *closure operator* and *closure system* only to the case of power set lattices $A = \mathcal{P}X$. A *coclosure range* or *kernel range* in A is a closure range in the dual poset A^{op}. The term "closure range" is justified by the fact that associating with each closure operation its range yields an isomorphism between $\mathrm{C}A$, the pointwise ordered set of closure operations on

A, and $\mathcal{C}A$, the set of all closure ranges, ordered by dual inclusion [2,16,44]. Closure ranges are subsets that are closed under all existing meets; the converse holds in complete lattices. A basic construction of closure operations leans on the existence of certain meets [16]:

Lemma 2.2. *Let S be a any subset of A. If for each $x \in A$ the set $S \cap {\uparrow} x$ has a meet mx then the so defined map m is a closure operation on A, and its range is the closure range generated by S.*

Closure operations are connected with adjoint maps. Given two posets A, B and two maps $h : A \longrightarrow B$ and $f : B \longrightarrow A$ related by the equivalence

$$hx \leq y \Leftrightarrow x \leq fy$$

one calls f the (*right* or *upper*) *adjoint* of h, and h the *coadjoint* (*left* or *lower adjoint*) of f. In that situation, the resulting equations $fhf = f$ and $hfh = h$ ensure that fh is a closure operation, and hf a kernel operation. In accordance with [38] we chose the letter h because often h will be a homomorphism between semilattices or lattices. Observe that in [9] and elsewhere f stands for the *left* adjoint, while right adjoints are often denoted by g.

Every map h from A to B factors through its surjective corestriction h_0 from A onto hA and the inclusion map h^0 from hA into B. By the characteristic equivalence (C), g is a closure operation iff g^0 is adjoint to g_0. A map is left adjoint iff it is *residuated*, that is, preimages of principal downsets are principal downsets, and (right) adjoint iff it is *residual*, that is, preimages of principal upsets are principal upsets. Residuated maps preserve all existing joins, and residual maps all existing meets; the converse holds for maps between complete lattices. For more results on adjoint maps and closure operations refer to [9,15,16,18,23]. Note the following straightforward equivalences:

Lemma 2.3. *A map $h : A \longrightarrow B$ with adjoint $f : B \longrightarrow A$ is surjective iff f is injective iff $hf = id_B$ iff $hx = \bigvee Y$ for all subsets Y of B with $x = \bigvee fY$. Thus, for a residuated surjection h, if A is ($\bigvee$-)complete then so is $B = hA$.*

Lemma 2.4. *A map $h : A \longrightarrow B$ with adjoint f is injective iff f is surjective, and then $C \subseteq A$ is a principal upset iff its preimage $f^{\leftarrow}C$ is a principal upset.*

Proof. Suppose $f^{\leftarrow}C = {\uparrow} y$. Then $y \in f^{\leftarrow}C$, $fhfy = fy \in C$, $hfy \in f^{\leftarrow}C = {\uparrow} y$, and so $y = hfy$ (as $hfy \leq y$). Now, injectivity of h gives $fhx = x$, and then

$$\begin{aligned} fy \ \leq \ x &\Rightarrow y \leq hx \Rightarrow hx \in f^{\leftarrow}C \\ &\Rightarrow x = fhx \in C \Rightarrow y \leq hx \Rightarrow fy \leq fhx = x \end{aligned}$$

confirms the equation $C = {\uparrow} fy$. □

Closure operations and their ranges are transported by adjoint maps:

Proposition 2.5. *Let $h : A \longrightarrow B$ be residuated and $f : B \longrightarrow A$ its adjoint.*

(1) *For a closure operation j on B with range C, fjh is a closure operation on A with range fC. Hence, adjoint maps send closure ranges to closure ranges.*

(2) *For a closure operation g on A, the restriction g' of hgf to hA is a closure operation on hA if one of the following conditions is fulfilled:*
(a) *fB is an upset in A,* (b) *fh commutes with g,* (c) *$fhgf = gf$.*
In these cases the range of g' is the preimage of A_g under $f{\upharpoonright}hA$. In particular, preimages of closure ranges under injective adjoint maps whose range is an upset are again closure ranges.

Proof. (1) Since f, j and h are isotone, so is fjh, and $id_B \leq j$ gives the inequality $id_A \leq fh \leq fjh$; and $hf \leq id_B$ yields $fjhfjh \leq fjjh = fjh$, so that the map fjh is idempotent. Further, $jhx = \bot(C\cap\uparrow hx)$ implies $fjhx = \bot(fC\cap\uparrow x)$, whence fC is the range of fjh.
(2) (a) implies (c): if fB is an upset then for $y \in B$ there is some $z \in B$ with $gfy = fz$, whence $fhgfy = fhfz = fz = gfy$. That (b) implies (c) is also clear by the equation $fhf = f$. And if (c) holds then the restriction g' of hgf to hA is a closure operation on hA by the following equivalences:

$$hx \leq hgfhy \Leftrightarrow fhx \leq fhgfhy = gfhy \Leftrightarrow gfhx \leq gfhy \Leftrightarrow hgfhx \leq hgfhy.$$

Further,

$$g'hA = \{y \in hA \,|\, hgfy \leq y\} = \{y \in hA \,|\, gfy \leq fy\} = hA \cap f^{\leftarrow}A_g. \qquad \square$$

Partial completeness properties ensure the existence of $\overline{g}$, the least closure operation above g, or of g°, the greatest closure operation below g:

Proposition 2.6. *Let g be a unary operation on A such that for each $x \in A$ the set A_gx has a meet mx.*
(1) *The so defined map $m \in A^A$ is a closure operation.*
(2) *If g is a preclosure operation then $m = \overline{g}$.*
(3) *If g is a weak closure operation then $m = g^{\circ}$.*

Proof. (1) is clear by Lemma 2.2.
(2) $g \leq m$ follows from $gx \leq a = ga$ for $x \leq a \in A_g$. And if some $j \in \mathrm{C}A$ fulfils $g \leq j$ then $mx \leq jx$, as $x \leq a = jx$ and $ga \leq ja = jjx = jx = a \in A_gx$.
(3) $m \leq g$ holds, as $a = gx$ entails $x \leq a = ga \in A_g$, hence $mx \leq a = gx$. And if some $j \in \mathrm{C}A$ fulfils $j \leq g$ then $jx \leq a$ for all $a \in A_gx$, since $x \leq a = ga$ entails $jx \leq jga \leq gga = ga = a$. Thus, $jx \leq mx$ for all $x \in A$. $\square$

We call a map $g \in A^A$ *lower complete*, briefly *l-complete*, if each of the sets A_gx has a meet. Under weak assumptions on A, l-completeness is not only sufficient but also necessary in order that $\overline{g}$ resp. g° exists.

Theorem 2.7. *A preclosure operation g on a topped poset A is l-complete iff there is a least closure operation $\overline{g}$ above g, and then $\overline{g}x = \bot A_gx$.*

Proof. Suppose g is a preclosure operation on A such that $\overline{g}$ exists. Then $a \in A_{\overline{g}}$ implies $a \leq ga \leq \overline{g}a = a$, that is, $a \in A_g$. Conversely, assume $a \in A_g$. The map j on A defined by $jx = a$ if $x \leq a$ and $jx = \top A$ if $x \not\leq a$ is easily seen to be a closure operation with $g \leq j$. Thus, $\overline{g} \leq j$; in particular, $\overline{g}a \leq ja = a \in A_{\overline{g}}$. This proves the equations $A_g = A_{\overline{g}}$ and $\overline{g}x = \bot A_gx$. $\square$

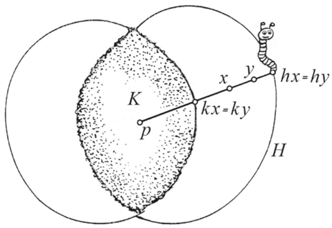
Similar constructions of closure operations $j \geq g$ with $ja = a$ show that 2.7 remains true for $\vee$- or $\wedge$-semilattices instead of topped posets.

Theorem 2.8. *A weak closure operation g on a $\vee$-semilattice is l-complete iff there is a greatest closure operation g° below g, and then $g^\circ x = \bigwedge A_g x$.*

Proof. Suppose g is a weak closure operation on a $\vee$-semilattice A such that g° exists. For any $x \in A$, $g^\circ x$ is a lower bound of $A_g x$, since $x \leq a = ga$ implies $g^\circ x \leq g^\circ a \leq ga = a$. Given any lower bound b of $A_g x$, put $jy = b \vee y$ if $x \leq y$ and $jy = y$ if $x \not\leq y$. Then the so defined map j on A is inflationary and satisfies $j \leq g$ (indeed, $jy = b \vee y \leq gy$ if $x \leq y$ and so $x \leq gy \in A_g x$, while $jy = y \leq gy$ if $x \not\leq y$). If $x \leq y \leq jz$ then $x \leq z$ (otherwise, $x \not\leq z = jz$, in contrast to $x \leq y \leq jz$) and so $jy = b \vee y \leq jz = b \vee z$, while $y \leq jz$ together with $x \not\leq y$ entails $jy = y \leq jz$. Hence, j is a closure operation on A with $j \leq g^\circ$, and therefore $b \leq b \vee x = jx \leq g^\circ x$. Thus, $g^\circ x = \bigwedge A_g x$. □

3. Nuclei and their generalizations

A *nucleus* on a $\wedge$-semilattice is a closure operation that preserves binary meets (cf. [6,7]). At first glance, the term "nucleus" looks a bit strange, because nucleus is the latin word for *kernel*; but it is justified by the one-to-one correspondence between nuclei and congruence kernels of residuated $\wedge$-homomorphisms (see the end of this section for more details). We reserve the term *interior operation* for kernel operations that preserve finite meets (having in mind the prototypes of topological interior operators).



A peach, its kernel, and its hull

Example 3.1. A peach P has a kernel K and a hull H, its skin.
Starting from a fixed inner point p of K, the peach P is partially ordered by $x \leq y$ if x lies closer to p than y on a radial ray, or $x = y$. In fact, P is a $\wedge$-semilattice with $x \wedge y = p$ if x and y are incomparable; but, clearly, P has no greatest element. Adding a universal upper bound $\top$ yields a complete lattice $P^\top = P \cup \{\top\}$, which however is not implicative. K deserves its name, being in fact a kernel range in P; its kernel operation k maps x to the nearest point of K on the ray from p to x. Whereas H is *not* a closure range, $H \cup \{p\}$ is a closure range indeed. The closure or hull operation h associated with $H \cup \{p\}$ maps each x distinct from p to the nearest point of H on the ray from p through x and leaves p fixed. Both k and h preserve binary meets, and so does the extension $h^\top$ of h to $P^\top$ with $h^\top \top = \top$. Thus, h and $h^\top$ are nuclei, and

their range is $H \cup \{p\}$ resp. $H^\top \cup \{p\}$, whereas the extension $K^\top = K \cup \{\top\}$ of the kernel K is the range of the interior operation $k^\top$.

The pointwise formed meet of two nuclei is again a nucleus, and the same holds for arbitrary meets, provided they exist (as in the complete case). Thus, the nuclei always form a $\wedge$-semilattice $\mathrm{N}A$ (with id_A as bottom, and the constant map $x \longmapsto \top A$ as top if $\top A$ exists). It is a challenging task to find out under what circumstances $\mathrm{N}A$ becomes an implicative (or Heyting) semilattice or lattice when A is one. A thorough analysis of this and related problems reveals that some major results in the theory of nuclei on semilattices depend on suitable completeness hypotheses (See [17]). It is our main purpose to detect where completeness assumptions are indispensable, and where they may be circumvented by alternate arguments.

We call a closure range C in a $\wedge$-semilattice A a *nuclear range* if for all $x, y \in A$ and $z \in C$ with $x \wedge y \leq z$ there exists a $c \in C$ with $x \leq c$ and $c \wedge y \leq z$. Recall that by a semilattice we always mean a $\wedge$-semilattice with top. A nonempty subset C of a semilattice is a nuclear range iff for all $x, y \in A$ and $z \in C$ with $x \wedge y \leq z$ there exists a *least* $c \in C$ with $x \leq c$ and $c \wedge y \leq z$. The following description of nuclear ranges, justifying our terminology, is due to Varlet [44], who speaks of *multiplicative closure*:

Proposition 3.2. *Associating with each nucleus on a $\wedge$-semilattice A its range, one obtains an isomorphism between $\mathrm{N}A$, the $\wedge$-semilattice of nuclei, and $\mathcal{N}A$, the $\wedge$-semilattice of nuclear ranges, ordered by dual inclusion.*

For a categorical treatment of nuclei on semilattices, a suitable morphism class is formed by so-called *r-morphisms*, that is, residuated semilattice homomorphisms preserving top elements. Their adjoints are called *l-morphisms*, having a left adjoint that preserves finite meets, and sometimes referred to as *localizations* (Bezhanishvili and Ghilardi [6]). In the category of locales, they are the *localic maps* (Johnstone [27], Picado and Pultr [38]).

An injective r- resp. l-morphism will be called an *r-* resp. *l-embedding*, and a surjective r- resp. l-morphism an *r-* resp. *l-surjection*. By an *r-* resp. *l-domain* of a semilattice we mean the domain of an inclusion map that is an r- resp. l-morphism. From basic connections between adjoint maps and closure operations (see [16, Ch. 3]) and the fact that composites of maps preserve finite meets if the factors do, one derives the following facts:

Proposition 3.3. *Let $h : A \longrightarrow B$ be an r-morphism having the range D, and $f : B \longrightarrow A$ its adjoint l-morphism with range C. Then $g = fh$ is a nucleus with range C, and $k = hf$ is an interior operation whose range D is isomorphic to C under the restriction $i = h_0 \restriction C$. Thereby, one obtains a factorization $h = k^0 i g_0$ into an r-embedding k^0, an isomorphism i and an r-surjection g_0. An analogous mono-iso-epi-factorization $f = g^0 i^{-1} k_0$ into l-morphisms holds in the opposite direction; see Figure 1.*

Furthermore, h is a nucleus iff f is an interior operation iff $h = fh$ iff $f = hf$.

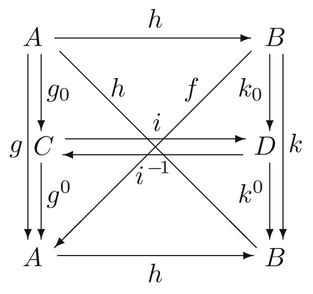


FIGURE 1. Factorization of r- and l-morphisms

Corollary 3.4. *The r-surjections are up to isomorphisms the surjective corestrictions of nuclei on semilattices, and the l-embeddings are up to isomorphisms the inclusion maps of nuclear ranges (l-domains) in semilattices.*

Corollary 3.5. *The poset of all r-morphisms between semilattices A and B is dual to the poset of all l-morphisms from B to A, but also isomorphic to the poset of all isomorphisms between l-domains of A and r-domains of B.*

The r-domains of frames are the *subframes*, while the l-domains of locales are the *sublocales* in the sense of [38]. In view of Corollary 3.4, afficionados of category theory alternately refer to the extremal r-epimorphisms, that is, r-surjections, as sublocales; for the sake of distinction, l-domains of locales are sometimes called *sublocale sets* [37].

It is useful to take into account several generalizations of nuclei, some of which also occur (under other names) in the ample work of Simmons on frames (see, e.g., [43]). Let A be a $\wedge$-semilattice. By a *subnucleus* on A we mean an inflation g on A satisfying

$$x \wedge gy \leq g(x \wedge y).$$

Following Banaschewski [3], we call an isotone subnucleus a *prenucleus* (some authors reserve that term for $\wedge$-preserving inflations; Simmons [43] calls prenuclei *stable inflators*). By a *weak nucleus* we mean an idempotent subnucleus. The nuclei are not only the idempotent prenuclei but also the isotone weak nuclei. Being $\wedge$-preserving and idempotent, any nucleus g satisfies

$$g(gx \wedge gy) = gx \wedge gy \leq g(x \wedge y).$$

We call an inflation fulfilling that condition a *pseudonucleus*. Pseudonuclei play a "central" role in the determination of the center (the boolean part) of the frame of all l-ideals, as demonstrated in the second part [17].

In Figure 2 we display the hierarchy among the operations introduced before. The bold framed classes are closed under composition and pointwise meets. The second property also holds for the class of nuclei, which however is not closed under composition, and $jg \in \mathrm{N}A$ is *not* equivalent to $gj \in \mathrm{N}A$.

Lemma 3.6. *For $g, j \in \mathrm{N}A$, the following equivalences and implications hold:*

$$jg \in \mathrm{N}A \Leftrightarrow jgj = jg \Leftrightarrow gjg = jg \Leftarrow jg = gj \Rightarrow gj = jgj \Leftrightarrow gj = gjg \Leftrightarrow gj \in \mathrm{N}A.$$

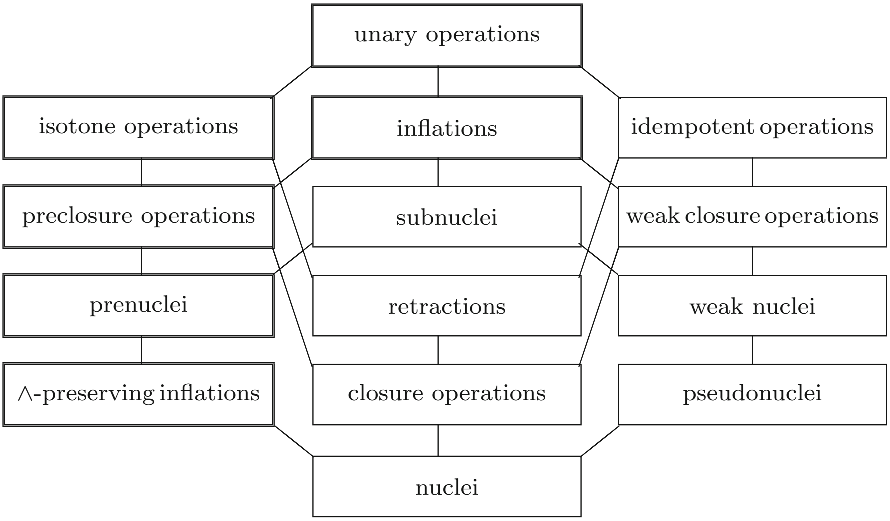


Figure 2. Generalizations of nuclei

On the other hand, if $g \in \mathrm{N}A$ *and* $j \in \mathrm{N}A_g$ *then* $g^0 j\, g_0 \in \mathrm{N}A$.

The last claim and the next proposition follow from Proposition 2.5.

Proposition 3.7. *Let* $h : A \longrightarrow B$ *be an r-morphism with adjoint* $f : B \longrightarrow A$.

(1) *For a nucleus* j *on* B *with range* C, fjh *is a nucleus on* A *with range* fC. *Thus, l-morphisms send nuclear ranges to nuclear ranges.*
(2) *If* fB *is an upset then for* $g \in \mathrm{N}A$ *the restriction* g' *of* hgf *to* hA *is a nucleus on* hA, *and the range of* g' *is the preimage of* A_g *under* $f \restriction hA$. *Thus, the preimage maps of l-embeddings whose range is an upset send nuclear ranges to nuclear ranges.*

Proposition 3.8. *Let* $h : A \longrightarrow B$ *be an r-morphism with adjoint* $f : B \longrightarrow A$.

(1) *If* A *is an implicative semilattice then so is* $hA \simeq fB$, *and* $f \restriction hA$ *is an l-embedding adjoint to* $h_0 : A \longrightarrow hA$ *and preserves the operation* $\to$.
(2) *If* B *is an implicative semilattice then so is* $fB \simeq hA$, *and* $h \restriction fB$ *is an r-embedding coadjoint to* $f_0 : B \longrightarrow fB$ *but need not preserve* $\to$.

Proof. Being surjective, h_0 has the injective adjoint $f \restriction hA$, and dually for f_0.
(1) $hx \wedge hy \le hz \Leftrightarrow h(x \wedge y) \le hz \Leftrightarrow x \wedge y \le fhz \Leftrightarrow x \le y \to fhz$
$\Rightarrow hx \le h(y \to fhz) \Rightarrow hx \wedge hy \le h((y \to fhz) \wedge y) \le hfhz = hz$.
Thus, $h(y \to fhz) = hy \to hz$ in hA. As fh is a nucleus, $f \restriction hA$ preserves $\to$:

$$x \le f(hy \to hz) \Leftrightarrow hx \le hy \to hz \Leftrightarrow x \le y \to fhz = fhy \to fhz.$$

(2) $fx \wedge fy \le fz \Leftrightarrow hfx \wedge hfy \le hfz \Leftrightarrow hfx \le hfy \to hfz$
$\Rightarrow fx \le f(hfy \to hfz) \Rightarrow fx \wedge fy \le f((hfy \to hfz) \wedge hfy) \le fhfz = fz$.
Thus, $f(hfy \to hfz)$ is $fy \to fz$, the relative pseudocomplement in fB.

An embedding of a three-element chain in a four-element boolean lattice is an r-embedding that does not preserve $\to$. □

Let us add a few words about congruences. An equivalence relation R on A is *isotone* if $x \leq y$ implies $R{\downarrow}x \subseteq {\downarrow}Ry$, a *weak closure equivalence* if each equivalence class has a top, and a *closure equivalence* if both conditions hold. An equivalence relation R on a $\wedge$-semilattice is a *weak congruence* if $x\,R\,y$ implies $x \wedge z \in {\downarrow}R(y \wedge z)$, and a *congruence* if $x\,R\,y$ implies $x \wedge z \in R(y \wedge z)$.

Proposition 3.9. *Sending each map g to the equivalence relation R defined by $x\,R\,y \Leftrightarrow gx = gy$, one obtains bijective correspondences between*

(1) *weak closure operations and weak closure equivalences,*
(2) *closure operations and closure equivalences,*
(3) *weak nuclei and weak congruences that are weak closure equivalences,*
(4) *nuclei and congruences that are closure equivalences,*
(5) *nuclei on frames and frame congruences.*

Proof. Except (3), these equivalences are known (see [9,22,38,41]), so we only prove (3). If g is a weak nucleus and $x\ R\ y$, then $t = g(y \wedge z) = \top R\,(y \wedge z)$ satisfies $x \wedge z \leq gx \wedge z = gy \wedge z \leq gt = t$ and $t\ R\ y \wedge z$, so R is a weak congruence, and a weak closure equivalence by (1).

Conversely, if R is assumed to be a weak congruence and a weak closure equivalence then for $y, z \in A$ and $x = gy = ggy$ we have $gx = gy$, that is, $x\,R\,y$, so there is a $t \in A$ satisfying $x \wedge z \leq t\,R\,y \wedge z$, that is, $gy \wedge z \leq t \leq gt = g(y \wedge z)$. Thus, g is a subnucleus, and a weak nucleus by (1). □

4. Nuclei, prenuclei and weak nuclei on implicative semilattices

From now on, we assume that

A is an implicative semilattice with top element $\top$ and residuation $\rightarrow$.

Nuclei on frames play an important role in pointfree topology. A comprehensive investigation of that concept, also in the non-complete case, is due to Macnab [4,32,33], who called nuclei on Heyting algebras *modal operators* and gave a nice description of them by a single equation, which extends to implicative semilattices: a map g on A is a nucleus iff

$$x \rightarrow gy = gx \rightarrow gy.$$

Any nucleus g on A fulfils the inequality (which may fail to be an equality)

$$g(x \rightarrow y) \leq gx \rightarrow gy,$$

A subset of A is said to be *left residually closed* or *l-closed* if it is invariant under all the unary residuation operations α_a with $\alpha_a x = a \rightarrow x$. If A is regarded as a general algebra $(A, \wedge, \top, \alpha)$ with $\alpha = (\alpha_a \,|\, a \in A)$ then the subalgebras are the l-closed subsemilattices. Köhler [31] calls them *total subalgebras*, while Picado, Pultr and Tozzi [40] call them merely *ideals*. In fact, in a general algebra with a distinguished binary operation ("a multiplication") m, a *left* (m-)*ideal* is a subalgebra containing $m(x, y)$ whenever it contains y. In order to avoid ambiguities, we call total subalgebras of implicative semilattices *left* $\rightarrow$-ideals or briefly *l-ideals*, referring to the binary residuation $\rightarrow$. The analogy to classical algebra is obvious; for example, the intuitionistic rule of importation and exportation, $(x \wedge y) \to z = x \to (y \to z)$, mimics the associative law $(xy) * z = x * (y * z)$; however, l-ideals are rarely two-sided. Order-theoretical filters (i.e. subsemilattices that are upsets) are l-ideals, but not conversely. Filters of implicative semilattices may be characterized as nonempty subsets F with $y \in F$ whenever $x \in F$ and $x \to y \in F$.

An l-ideal is said to be *nuclear* if it is a closure range. This terminology is justified by the next two propositions. In proofs we often write x^y for $x \to y$.

Proposition 4.1. *Let g be any unary operation on A.*

(1) *If g is a subnucleus then $g(x \to y) \le x \to gy$, and A_g is l-closed.*
(2) *g is a prenucleus iff g is a preclosure operation with $g(x \to y) \le x \to gy$.*
(3) *If g is a prenucleus or a pseudonucleus then A_g is an l-ideal.*
(4) *g is a nucleus iff A_g is a nuclear l-ideal and $gx = \bigwedge A_g x$.*

Proof. (1) $x \wedge g(x^y) \le x^{yy} \wedge g(x^y) \le g(x^{yy} \wedge x^y) = gy$ gives $g(x^y) \le x^{gy}$. And for $y \in A_g$, $g(x^y) \le x^{gy}$ entails $g(x^y) \le x^y \le g(x^y)$, hence $x^y \in A_g$.

(2) If g is a preclosure operation with $g(x^y) \le x^{gy}$ for all $x, y \in A$ then $y \le x^{x \wedge y}$ yields $gy \le g(x^{x \wedge y}) \le x^{g(x \wedge y)}$, hence $x \wedge gy \le g(x \wedge y)$.

(3) By (1), A_g is l-closed. For $y, z \in A_g$ we get $y \wedge z = y \wedge gz \le g(y \wedge z) \le gy \wedge gz = y \wedge z$ if g is a prenucleus, and $y \wedge z = gy \wedge gz = g(gy \wedge gz) = g(y \wedge z)$ if g is a pseudonucleus. In any case, $y \wedge z \in A_g$, whence A_g is an l-ideal.

(4) We use the bijection between closure operations and closure ranges. By (3), if $g \in \mathrm{N}A$ then A_g is a nuclear l-ideal with $gx = \bigwedge A_g x$. On the other hand, if that holds then g is a closure operation (Lemma 2.2). For $a = g(x \wedge y)$, $x \wedge y \le a \in A_g$ entails $x^a \in A_g$, $gy \le g(x^a) = x^a$ and so $x \wedge gy \le a$. Thus, g is a subnucleus and, being a closure operation, a nucleus. ☐

We supplement Lemma 2.2 by the following construction of nuclei:

Proposition 4.2. *For any l-closed subset C of A such that each of the sets $C \cap \uparrow x$ has a meet mx, the operation $n_C = m$ on A is a nucleus. Every nucleus g comes in that manner from a unique nuclear l-ideal, namely $C = A_g$.*

Proof. By Lemma 2.2, m is a closure operation. For $z \in C$, $x \wedge y \le z$ implies $y \le x^z \in C$, $my \le x^z$ and $x \wedge my \le z$.

Thus, $x \wedge my \le z$ for $z \in C \cap \uparrow(x \wedge y)$, and so $x \wedge my \le m(x \wedge y)$. The rest is clear by Proposition 4.1 (4). ☐

Let us denote by $\mathcal{S}lA$ the (inclusion-ordered) collection of all

l-domains = l-closed closure ranges = nuclear l-ideals = nuclear ranges.

Corollary 4.3. *$\mathcal{N}A$ is a semilattice dual to $\mathcal{S}lA$, and isomorphic to $\mathrm{N}A$ by virtue of the mutually inverse bijections $C \longmapsto n_C$ and $g \longmapsto A_g$.*

This correspondence was observed by several authors [6,32,33,40,45], at least in the setting of Heyting algebras or locales. Macnab calls nuclear ranges in Heyting algebras *modal subalgebras*, while Picado, Pultr and Tozzi [40] speak of *strong ideals*. Note that in implicative semilattices satisfying the descending chain condition all l-ideals are nuclear.

In a bounded chain A, which is a Heyting lattice anyway, every closure operation is a nucleus, every weak closure operation is a pseudonucleus, every subset containing $\top$ is an l-ideal, every closure range is nuclear, and the union of two nuclear ranges is their join in $\mathcal{S}lA$ (meet in $\mathcal{N}A$). But $\mathcal{S}lA$ resp. $\mathcal{N}A$ need not be a lattice and neither pseudocomplemented nor dually pseudocomplemented, nor distributive.

Example 4.4. Consider the chain $\mathbb{N} = \omega \setminus \{0\}$ of positive integers and the bounded rational chain

$$A = \{\pm\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \simeq \omega \oplus \omega^{op},$$

a simple example of a non-complete Heyting lattice. In the $\vee$-semilattice $\mathcal{S}lA$, which is dual to the semilattice $\mathcal{N}A$ of nuclear ranges, the set $\{B, C\}$ with

$$B = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{-\tfrac{1}{2n-1} \,|\, n \in \mathbb{N}\}$$
$$C = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{-\tfrac{1}{2n} \,|\, n \in \mathbb{N}\}$$

has no greatest lower bound. Indeed, each of the finite nuclear l-ideals

$$F_k = \{\tfrac{1}{n} \,|\, n \leq k\} \ (k \in \mathbb{N})$$

is a lower bound of $\{B, C\}$, and their union is the filter

$$F = \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\} = B \cap C,$$

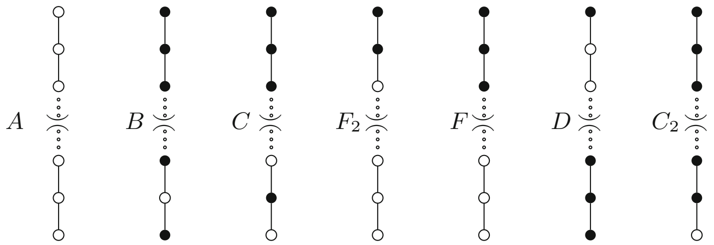
which fails to be a closure range. The complementary l-ideal

$$D = \{-\tfrac{1}{n} \,|\, n \in \mathbb{N}\} \cup \{1\}$$

is nuclear but has no pseudocomplement in $\mathcal{S}lA$, as F contains no greatest l-domain; and D has no pseudocomplement in $\mathcal{N}A$ either: the l-domains

$$C_k = F \cup \{-\tfrac{1}{n} \,|\, n \geq k\} \ (k \in \mathbb{N})$$

fulfil $C_k \cup D = A$, but there is no least l-domain E satisfying $E \cup D = A$. The semilattice $\mathcal{N}A = (\mathcal{S}lA)^{op}$ is not distributive: though B is contained in $A = C \cup D$, there are no l-domains $C' \subseteq C$ and $D' \subseteq D$ with $B = C' \cup D'$. This example also witnesses that pseudonuclei need not be nuclei, and that preimages of nuclear l-ideals under injective l-morphisms need not be nuclear. Indeed, the map g with $gx = 1$ for $x \in D$ and $gx = x$ otherwise is easily seen to be a pseudonucleus but not a nucleus; and the map f with $fx = x$ for $x > 0$ and $fx = x/2$ for $x < 0$ is an injective l-morphism. The range of g is the filter F, which is an l-ideal but not a closure range. The range of f is the l-domain C, and the preimage of the l-domain B under f is the filter F.



Now, we are in a position to determine for many unary operations the least nucleus above or the greatest nucleus below them.

Theorem 4.5. *For any $g \in A^A$ such that each of the sets $A_g x$ has a meet mx, the map $m \in A^A$ is a closure operation on A. Furthermore,*

(1) *if g is a subnucleus then m is a nucleus,*
(2) *if g is a prenucleus then $m = \overline{g}$ is the least nucleus above g,*
(3) *if g is a weak nucleus then $m = g^\circ$ is the greatest nucleus below g.*

Proof. The general closure claim holds by Proposition 2.6 (1).

(1) follows from Proposition 4.1 (1) and Proposition 4.2.

(2) By Proposition 2.6 (2), m is the least closure operation above g.

(3) By Proposition 2.6 (3), m is the greatest closure operation below g. □

Corollary 4.6. *For each prenucleus on a frame there is a least nucleus above it, and for each weak nucleus on a frame there is a greatest nucleus below it.*

The first of these two results was observed by Banaschewski [3], the second by Beazer and Macnab [4]. Combination of Theorem 4.5 with Theorems 2.7 and 2.8 leads to the following conclusions.

Theorem 4.7. (1) *A prenucleus g on an implicative semilattice A is l-complete iff $\overline{g}$ exists and is the least nucleus above g; in that case, $\overline{g}x = \bot A_g x$.*
(2) *A weak nucleus g on an implicative lattice A is l-complete iff g° exists and is the greatest nucleus below g; in that case, $g^\circ x = \bigwedge A_g x$.*

5. The ABC of nuclei and nuclear ranges

As before, A always denotes an implicative semilattice. For a better understanding of the relationships between nuclei and their ranges it is helpful to focus on three specific kinds of l-ideals, called *basic open*, *boolean* and *basic closed* (cf. [4,19,40], and for the complete case [26,27,32,38,43]).

For a topology $\mathcal{O}$, regarded as a frame (complete Heyting lattice), each open set $U \in \mathcal{O}$ gives rise to an induced topology $\mathcal{O}_U = \{U \cap V \mid V \in \mathcal{O}\}$, which is isomorphic to a sublocale (l-domain, nuclear l-ideal) of $\mathcal{O}$, viz.

$$\mathfrak{a}U = \{U \to V = (U^{\mathrm{c}} \cup V)^\circ \mid V \in \mathcal{O}\}$$

where ^{c} means set-theoretical complementation and $^\circ$ topological interior. Motivated by these prototypical examples, consider for any element a of A the unary residuation $\alpha_a\colon A \longrightarrow A$, which is defined by

$$\alpha_a x = a^x = a \to x,$$

adjoint to the unary meet operation $\lambda_a = a \wedge -$, and a nucleus by the rules $x \leq a^x = a^{a^x}$ and $a^{x\wedge y} = a^x \wedge a^y$. Consequently, its range

$$\mathfrak{a}a = \{a^x \mid x \in A\} = \{x \in A \mid a^x = x\} = \{x \in A \mid a^{xx} = \top\}$$

is a nuclear l-ideal. Thus, the surjective corestriction of α_a is not only *adjoint* to the restriction of λ_a to $\mathfrak{a}a$ but also *coadjoint* to the inclusion map of $\mathfrak{a}a$ in A. Moreover, α_a preserves $\to$. The following facts are found in [40].

Theorem 5.1. *The map* $\mathfrak{a}_A : A \longrightarrow \mathcal{S}lA$, $a \longmapsto \mathfrak{a}a$ *satisfies* $\mathfrak{a}(a \wedge b) = \mathfrak{a}a \cap \mathfrak{a}b$ *and is an embedding preserving all existing joins and finite meets* (*though* $\mathcal{S}lA$ *need not be a* $\wedge$*-semilattice*). *Hence,* $\alpha_A = \alpha$ *is a dual embedding of* A *in* $\mathrm{N}A$. *The sets* $\mathfrak{a}a$ *are exactly those nuclear ranges whose nuclei preserve the operation* $\to$ *and have residual surjective corestrictions.*

The second kind of nuclei to be considered are the maps $\beta_a \in A^A$ with

$$\beta_a x = x^{aa} = (x \to a) \to a$$

(denoted by w_a in Macnab's work [4,33,43]). For an early proof that each β_a is a nucleus see Varlet [45]. The range of β_a is

$$\mathfrak{b}a = \{x^a \mid x \in A\} = \{y \in A \mid y = y^{aa}\},$$

which is therefore a nuclear l-ideal. However, the map $\mathfrak{b}_A$ from A to $\mathcal{N}A$ with $\mathfrak{b}_A a = \mathfrak{b}a$ is not isotone unless A is a boolean lattice (see Theorem 5.9). Nevertheless, $\mathfrak{b}_A$ has a universal property for so-called *l-continuous maps*, defined by the requirement that preimages of principal upsets are l-ideals (see [14,16] for general background). If A has a bottom $\perp$ then $\mathfrak{b}\perp$ is the *booleanization* of A; that it is a boolean lattice is the content of the famous Glivenko-Frink Theorem [21,24]. Parts of the next more general theorem are known (cf. [9,44,45]), at least in the complete case [26,27,38].

Theorem 5.2. *For any* $a \in A$, *the range* $\mathfrak{b}a$ *of the nucleus* β_a *is the least l-ideal containing* a. *The sets* $\mathfrak{b}a$ *are exactly those* (*nuclear*) *l-ideals which are boolean lattices. Thus, all l-ideals are unions of boolean ones.*

Proof. $\top = a^a$ is the greatest and a the least element of $\mathfrak{b}a$, since $a = a^{aa} \in \mathfrak{b}a$ and $a \leq x^a$ for all $x \in A$. The join of x and y in $\mathfrak{b}a$ is $x \vee_a y = (x^a \wedge y^a)^a$; indeed, $x, y \leq (x^a \wedge y^a)^a$, and if $x, y \leq z \in \mathfrak{b}a$ then $z^a \leq x^a \wedge y^a$ and so $(x^a \wedge y^a)^a \leq z^{aa} = z$. The residuation in $\mathfrak{b}a$ is induced from that in A. And as $x \vee_a x^a = (x^a \wedge x^{aa})^a = a^a = \top$, x^a is the complement of x in $\mathfrak{b}a$.

Now let C be an arbitrary l-ideal of A. Then $a \in C$ implies $x^{aa} \in C$ for all $x \in A$, which shows that $\mathfrak{b}a$ is the least (nuclear) l-ideal containing a. And if C is a boolean lattice with $a = \perp C$ then, for each $y \in C$, the element

$y^a \in \mathfrak{b}a \subseteq C$ coincides with the complement z of y in C, because $y \wedge z = a$ implies $z \leq y^a$, and $y \vee z = \top$ (join in C) entails

$$y^a = y^a \wedge \top = (y^a \wedge y) \vee (y^a \wedge z) = a \vee (y^a \wedge z) \leq z,$$

hence $y^a = z$. As complements in C are unique, we get $y = y^{aa} \in \mathfrak{b}a$. Thus, $C = \mathfrak{b}a$. □

$\mathcal{T}A$, the closure system of all l-ideals, is always a frame [31,40]; hence, complements in $\mathcal{T}A$ are pseudocomplements. From Theorem 5.2, we deduce:

Corollary 5.3. *A subset $\mathcal{Y}$ of $\mathcal{S}lA$ has a meet in $\mathcal{S}lA$ iff $\bigcap \mathcal{Y} \in \mathcal{S}lA$, and then $\bigcap \mathcal{Y}$ is that meet. Thus, not only finite joins but also all existing meets and complements in $\mathcal{S}lA$ coincide with those in $\mathcal{T}A$.*

The third kind of nuclei to be considered exist only under the proviso that there are enough binary joins. For each $a \in A$, the principal upset

$$\mathfrak{c}a = \uparrow a$$

is an l-ideal but need not be nuclear, that is, a closure range. However, in $\vee$-semilattices, and only in these, each $\mathfrak{c}a$ is the range of the nucleus γ_a with $\gamma_a x = a \vee x$. Consider the topological case of an open set U, that is, a member of a topology $\mathcal{O}$, regarded s a frame. Here, the nuclear range $\mathfrak{a}U$ is isomorphic to the induced topology $\mathcal{O}_U$, while the nuclear range $\mathfrak{c}U$ is isomorphic to the induced topology $\mathcal{O}_C$ on a closed subset C, namely the set-theoretical complement of U. Therefore, the sets $\mathfrak{a}a$ are generally called *basic open* and the sets $\mathfrak{c}a$ *basic closed*; in the case of frames/locales, the $\mathfrak{c}a$'s form a closure system, and the prefix "basic" is omitted. The letters α and $\mathfrak{a}$ remind of the Greek $\alpha\nu o\iota\kappa\tau o\varsigma$ and the Latin $\mathfrak{apertus}$ for *open*, while $\gamma\varepsilon\nu\varepsilon\sigma\iota\varsigma$ is Greek for *generation*, and $\mathfrak{clusus}$ is Latin for *closed*, whence we chose the letters γ and $\mathfrak{c}$. (In [4,33,43], v stands for α and u for γ, while in [27] u has the meaning of α, and c is our γ; in [38] $\mathfrak{o}$ stands for $\mathfrak{a}$).

The complementarity between open and closed sets in spaces is reflected by the next proposition (for weaker statements see [4,27,33,38,40]).

Proposition 5.4. *$\mathfrak{c}a$ is the complement and so the pseudocomplement of $\mathfrak{a}a$ in $\mathcal{T}A$, hence also in $\mathcal{S}lA$ and in $\mathcal{N}A$ if A is a lattice.*

Proof. For the least l-ideal $\{\top\}$, one obtains $\mathfrak{a}a \cap \mathfrak{c}a = \{\top\}$, because x lies in $\mathfrak{a}a \cap \mathfrak{c}a$ iff $a \leq x = a^x$, which implies $x = a^x = \top$. If $\mathfrak{a}a \cup \mathfrak{c}a \subseteq C$ for some l-ideal C then each $x \in A$ satisfies $a^x \in \mathfrak{a}a \subseteq C$, $a^{xx} \in \mathfrak{c}a \subseteq C$, and therefore $x = a^x \wedge a^{xx} \in C$. Thus, A is the only upper bound of $\{\mathfrak{a}a, \mathfrak{c}a\}$ in $\mathcal{T}A$. □

Proposition 5.5. *For any $a \in A$, the following conditions are equivalent:*

(a) *$a \vee x$ exists for all $x \in A$.*
(b) *$\mathfrak{c}a$ is a nuclear range.*
(c) *$\mathfrak{c}a$ is the complement of $\mathfrak{a}a$ in $\mathcal{S}lA$ resp. in $\mathcal{N}A$.*
(d) *$\mathfrak{a}a$ has a complement in $\mathcal{S}lA$ resp. in $\mathcal{N}A$.*

Thus, A is a lattice iff for all $a \in A$, $\mathfrak{a}a$ and $\mathfrak{c}a$ are complementary elements of $\mathcal{S}lA$, or equivalently, α_a and γ_a are complementary elements of NA.

Proof. (a) $\Leftrightarrow$ (b): y is the join of a and x iff y is the least element of $\mathfrak{c}a$ above x.
(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d): Use Corollary 5.3 and Proposition 5.4. □

Macnab [33] showed that for any nucleus g on a Heyting algebra the composite map $\beta_a g$ is a boolean nucleus determined by the value ga, and established a series of interesting equations for the boolean nuclei β_a via γ. In the case of semilattices, the nuclei γ_a need not exist. Instead, one has the identities displayed in the next two lemmas.

Lemma 5.6. *For $a, c \in A$, the nucleus $\beta_{\overline{c}}$ with $\overline{c} = \beta_a c$ fulfils $\beta_{\overline{c}} x = (x^a \wedge c^a)^a$.*

Proof. Put $b = c^a$. Then $b^a = \overline{c}$, $\beta_{\overline{c}} x = (x^{b^a})^{b^a} = ((x \wedge b)^a \wedge b)^a = (x^a \wedge b)^a$, since $(x \wedge b)^a \wedge x \wedge b \leq a$ implies $(x \wedge b)^a \wedge b \leq x^a \leq (x \wedge b)^a$. □

Lemma 5.7. *Let a be an element of A and g a prenucleus on A. Then:*
1. *$(gx)^a = x^a \wedge (ga)^a$ for $x \geq a$.*
2. *$\beta_a g \beta_a = \beta_b$ for $b = \beta_a ga$. Hence, $\beta_b = \beta_a \vee g$ in $\mathrm{N}A$ if $g \in \mathrm{N}A$.*

Proof. (1) $a \leq x \leq gx$ yields $ga \leq gx$, $(gx)^a \leq x^a \wedge (ga)^a$. On the other hand, $gx \wedge x^a \leq g(x \wedge x^a) \leq ga$ implies $gx \wedge x^a \wedge (ga)^a \leq a$, i.e. $x^a \wedge (ga)^a \leq (gx)^a$.
(2) By (1) for x^{aa} instead of x and Lemma 5.6 for $c = ga$, $x^{aa} \geq a$ yields $\beta_a g \beta_a x = (g(x^{aa}))^{aa} = (x^{aaa} \wedge (ga)^a)^a = (x^a \wedge (ga)^a)^a = \beta_{\overline{c}} x = \beta_b x$. □

Lemma 5.7 together with Proposition 3.2 leads to a result that was obtained by Macnab [33] for the case of modal operators on Heyting algebras:

Theorem 5.8. *Let $a \in A$, $g \in \mathrm{N}A$ and $b = \beta_a ga$. Then $g \geq \beta_a \Leftrightarrow g = \beta_b$. The boolean nuclear ranges of A form an upset in $\mathcal{N}A$, i.e. a downset in $\mathcal{S}lA$.*

We finish this part with diverse characterizations of boolean lattices in terms of basic open, boolean, and basic closed nuclear ranges (cf. [27,39]).

Theorem 5.9. *For a Heyting semilattice A, the following are equivalent:*
(b_0) *A is a boolean lattice.*
(a_1) *The nuclear ranges are the basic open l-ideals.*
(b_1) *The nuclear ranges are the boolean l-ideals.*
(c_1) *The nuclear ranges are the basic closed l-ideals.*
(a_2) *$\mathfrak{a}_A : A \longrightarrow \mathcal{N}A$ is a dual isomorphism.*
(b_2) *$\mathfrak{b}_A : A \longrightarrow \mathcal{N}A$ is an isomorphism.*
(c_2) *$\mathfrak{c}_A : A \longrightarrow \mathcal{N}A$ is an isomorphism.*
(a_3) *$\mathfrak{a}_A$ is complementary to $\mathfrak{b}_A$ in $(\mathcal{N}A)^A$.*
(b_3) *$\mathfrak{b}_A$ is isotone.*
(c_3) *$\mathfrak{c}_A$ coincides with $\mathfrak{b}_A$.*

Proof. (b_0) $\Rightarrow$ (c_2) $\Rightarrow$ (c_1): If A is a boolean lattice then so is each $\mathfrak{c}a$; and conversely, each $C \in \mathcal{N}A$ satisfies $C = \mathfrak{c}b$ for $b = \bot C$ (indeed, for $b \leq a$, one obtains $a = a \vee b = \neg\neg a \vee b = \neg a \rightarrow b \in C$).
(c_1) $\Rightarrow$ (a_1): By Proposition 5.5, (c_1) entails that A is a lattice, each C in $\mathcal{N}A$ is of the form $\mathfrak{c}b$, and its complement $\mathfrak{a}b$ equals some $\mathfrak{c}c$; by uniqueness of complements, it follows that $C = \mathfrak{a}c$. Thus, each $C \in \mathcal{N}A$ is basic open.

$(a_1) \Rightarrow (a_2)$: By Theorem 5.1, $\mathfrak{a}_A$ is an embedding of A in $\mathcal{S}lA = (\mathcal{N}A)^{op}$.

$(a_2) \Rightarrow (c_1)$ is shown by a similar argument as for $(c_1) \Rightarrow (a_1)$, using Proposition 5.5 and the fact that (a_2) forces A to be a lattice, as A and $\mathcal{N}A$ are dually isomorphic semilattices.

$(c_1) \Rightarrow (c_3)$: $\mathfrak{b}a = \mathfrak{c}b$ implies $a = \bot\mathfrak{b}a = \bot\mathfrak{c}b = b$, hence $\mathfrak{b}a = \mathfrak{c}a$.

$(c_3) \Rightarrow (a_3)$ is clear by Proposition 5.4, and $(a_3) \Rightarrow (b_3)$ is obvious.

$(b_3) \Rightarrow (b_2)$: $\mathfrak{b}a \subseteq \mathfrak{c}a$ holds anyway, and if $\mathfrak{b}_A : A \longrightarrow \mathcal{N}A$ is isotone then $a \leq b$ implies $b \in \mathfrak{b}b \subseteq \mathfrak{b}a$ ($\mathcal{N}A$ carries the reverse inclusion order!), whence $\mathfrak{b}_A$ agrees with the embedding $\mathfrak{c}_A$. Now, $\bot \leq a$ implies $a \in \mathfrak{b}a \subseteq \mathfrak{b}\bot$. Thus, $A = \mathfrak{b}\bot$, and by Theorem 5.8 each nuclear range is boolean, so $\mathfrak{b}_A$ is an isomorphism.

The trivial implications $(b_2) \Rightarrow (b_1) \Rightarrow (b_0)$ close the circuit
$(b_0) \Rightarrow (c_2) \Rightarrow (c_1) \Rightarrow (a_1) \Rightarrow (a_2) \Rightarrow (c_3) \Rightarrow (a_3) \Rightarrow (b_3) \Rightarrow (b_2) \Rightarrow (b_1) \Rightarrow (b_0)$. □

It is quite surprising that isotonicity of the map $\mathfrak{b}_A$ resp. β_A is already sufficient (and necessary) for A to be a boolean lattice. Many of the above implications and equivalences remain valid for arbitrary implicative semilattices (possibly missing a least element).

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