Algebra Univers. (2018) 79:25

1420-8911/18/020001-7
published online April 18, 2018
https://doi.org/10.1007/s00012-018-0517-9

Algebra Universalis

CrossMark

A note on homomorphisms between products of algebras

Ivan Chajda, Martin Goldstern and Helmut Länger

Abstract. Let $\mathcal{K}$ be a congruence distributive variety and call an algebra hereditarily directly irreducible (HDI) if every of its subalgebras is directly irreducible. It is shown that every homomorphism from a finite direct product of arbitrary algebras from $\mathcal{K}$ to an HDI algebra from $\mathcal{K}$ is essentially unary. Hence, every homomorphism from a finite direct product of algebras $\mathbf{A}_i$ $(i \in I)$ from $\mathcal{K}$ to an arbitrary direct product of HDI algebras $\mathbf{C}_j$ $(j \in J)$ from $\mathcal{K}$ can be expressed as a product of homomorphisms from $\mathbf{A}_{\sigma(j)}$ to $\mathbf{C}_j$ for a certain mapping σ from J to I. A homomorphism from an infinite direct product of elements of $\mathcal{K}$ to an HDI algebra will in general not be essentially unary, but will always factor through a suitable ultraproduct.

Mathematics Subject Classification. 06B05.

Keywords. Direct product of chains, Homomorphism, Essentially unary mapping, Ultrafilter.

1. Introduction

Let $\mathbf{A}_i, i \in I$, and $\mathbf{B}_j, j \in J$, be algebras of the same type and f a homomorphism from $\prod_{i\in I} \mathbf{A}_i$ to $\prod_{j\in J} \mathbf{B}_j$. For every $k \in J$ let p_k denote the projection from $\prod_{j\in J} \mathbf{B}_j$ onto $\mathbf{B}_k$ and $f_k := p_k \circ f$. More generally, for any $J_0 \subseteq J$ we let $p_{J_0} \colon \prod_{j\in J} \mathbf{B}_j \to \prod_{j\in J_0} \mathbf{B}_j$ be the canonical projection map. It is evident that $f = (f_j : j \in J)$. Hence, the task of describing f is reduced to the task of describing the homomorphisms f_k from $\prod_{i\in I} \mathbf{A}_i$ to $\mathbf{B}_k$.

Presented by G. Czédli.
I. Chajda and H. Länger gratefully acknowledge support of this research by ÖAD, project CZ 04/2017, as well as by IGA, project PřF 2018 012. M. Goldstern gratefully acknowledges support by the Austrian Science Fund (FWF), project I 3081-N35. H. Länger gratefully acknowledges support of this research by the Austrian Science Fund (FWF), project I 1923-N25.

Birkhäuser

In [3] the authors solve this problem for the case that the algebras $\mathbf{A}_i$ and $\mathbf{B}_j$ are conservative median algebras and the index sets are finite. More generally, Couceiro et al. [2] considers the case that the $\mathbf{A}_i$ are median algebras and the $\mathbf{B}_j$ are tree-median algebras. It turns out that the method developed in [2,3] can be further generalized to lattices. Let us note that every distributive lattice is a median algebra (but not conversely). We are even able to extend this result to arbitrary lattices $\mathbf{A}_i$ provided the $\mathbf{B}_j$ are chains. For lattice concepts used in the rest of the paper the reader is referred to the monographs [1,5].

We call a mapping $f\colon \prod_{i\in I} A_i \to C$ *essentially unary* if there exists an $i_0 \in I$ and a mapping $g\colon A_{i_0} \to C$ with $g \circ p_{i_0} = f$. In this case we say that "f depends only on the i_0-th coordinate", or that "f factors through p_{i_0}".

From $f = g \circ p_{i_0}$ it easily follows that g is a homomorphism if and only if f is.

2. Fraser–Horn property and HDI algebras

Definition 2.1. A class $\mathcal{K}$ of algebras has the *Fraser–Horn property* if there are no skew congruences on any product $\mathbf{A}_1 \times \mathbf{A}_2$ with $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{K}$, or more explicity:

For all $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{K}$, for every congruence $\theta \in \mathrm{Con}(\mathbf{A}_1 \times \mathbf{A}_2)$ there are congruences $\theta_1 \in \mathrm{Con}(\mathbf{A}_1)$, $\theta_2 \in \mathrm{Con}(\mathbf{A}_2)$ such that $\theta = \theta_1 \times \theta_2$, i.e. $\theta = \{((x_1, x_2), (y_1, y_2)) \mid x_1\theta_1 y_1, x_2\theta_2 y_2\}$.

The following lemma is known from [4].

Lemma 2.2. *Let $\mathcal{K}$ be a congruence distributive (CD) variety. Then $\mathcal{K}$ has the Fraser–Horn property.*

For the rest of the paper we fix a variety $\mathcal{K}$ with the Fraser–Horn property.

We call an algebra non-trivial if its universe contains at least two elements.

Definition 2.3. We call an algebra $\mathbf{A}$ *hereditarily directly irreducible* (HDI) if every subalgebra $\mathbf{B} \leq \mathbf{A}$ is directly irreducible, i.e., is not isomorphic to a direct product of two non-trivial factors.

Fact 2.4. (1) The variety of lattices is congruence distributive.
(2) A lattice is HDI if and only if it is a chain.

Theorem 2.5. *Let $\mathcal{K}$ be a variety with the Fraser–Horn property. If n is a positive integer, $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are in $\mathcal{K}$ and $\mathbf{C} \in \mathcal{K}$ is HDI, then every homomorphism f from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ to $\mathbf{C}$ is essentially unary, i.e., factors through one of the projections p_i.*

Proof. Let $\theta = \ker(f)$. By a straightforward generalization of the Fraser–Horn property we know that $\theta = \theta_1 \times \cdots \times \theta_n$, where each θ_i is a congruence on $\mathbf{A}_i$.

The homomorphism theorem tells us that $\mathbf{B}' := f(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$ is isomorphic to the direct product $(\mathbf{A}_1/\theta_1) \times \cdots \times (\mathbf{A}_n/\theta_n)$. By our assumption, $\mathbf{B}'$ is directly irreducible, so at most one of these factors can be non-trivial, so there is at most one i such that $\mathbf{A}_i/\theta_i$ has more than one element. So f depends only on the i-th coordinate. □

Remark 2.6. If $f\colon \mathbf{A}_1 \times \mathbf{A}_2 \to \mathbf{C}$ is not constant, then there is at most one $i \in \{1,2\}$ such that f factors through p_i.

As a consequence of the above theorem we obtain the following statement.

Theorem 2.7. *If n is a positive integer, $\mathbf{A}_1,\dots,\mathbf{A}_n \in \mathcal{K}$, where $\mathcal{K}$ has the Fraser–Horn property, $(\mathbf{C}_j; j \in J)$ is a non-empty family of HDI algebras in $\mathcal{K}$, and f is a homomorphism from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ to $\prod_{j\in J} \mathbf{C}_j$ then there exists a mapping $\sigma\colon J \to \{1,\dots,n\}$ and for every $j \in J$ a homomorphism g_j from $\mathbf{A}_{\sigma(j)}$ to $\mathbf{C}_j$ such that*

$$f(x_1,\dots,x_n) = (g_j(x_{\sigma(j)}); j \in J)$$

for all $(x_1,\dots,x_n) \in A_1 \times \cdots \times A_n$.

Proof. Apply Theorem 2.5 to the mappings $f_j := p_j \circ f$, $j \in J$. □

Theorem 2.8. *Let $\mathcal{K}$ be a variety with the Fraser–Horn property. Let n, k be positive integers and let $\mathbf{A}_1,\dots,\mathbf{A}_n, \mathbf{C}_1,\dots,\mathbf{C}_k$ be non-trivial HDI algebras in $\mathcal{K}$ and assume $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n \cong \mathbf{C}_1 \times \cdots \times \mathbf{C}_k$. Then $n = k$ and there exists a permutation $\sigma \in \mathrm{S}_n$ such that $\mathbf{C}_i \cong \mathbf{A}_{\sigma(i)}$ for all $i = 1,\dots,n$.*

Proof. Let f denote an isomorphism from $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ to $\mathbf{C}_1 \times \cdots \times \mathbf{C}_k$. According to Theorem 2.7, there exist mappings σ from $\{1,\dots,k\}$ to $\{1,\dots,n\}$ and τ from $\{1,\dots,n\}$ to $\{1,\dots,k\}$, for every $j \in \{1,\dots,k\}$ a homomorphism g_j from $\mathbf{A}_{\sigma(j)}$ to $\mathbf{C}_j$ and for every $i \in \{1,\dots,n\}$ a homomorphism h_i from $\mathbf{C}_{\tau(i)}$ to $\mathbf{A}_i$ such that

$$f(x_1,\dots,x_n) = (g_1(x_{\sigma(1)}),\dots,g_k(x_{\sigma(k)}))$$

for all $(x_1,\dots,x_n) \in A_1 \times \cdots \times A_n$ and

$$f^{-1}(y_1,\dots,y_k) = (h_1(y_{\tau(1)}),\dots,h_n(y_{\tau(n)}))$$

for all $(y_1,\dots,y_k) \in C_1 \times \cdots \times C_k$. The injectivity of f implies $k \geq n$ and the injectivity of f^{-1} implies $n \geq k$. This shows $n = k$. Moreover, again since f is injective we have $\sigma \in \mathrm{S_n}$. Finally, the injectivity of f implies the injectivity of $g_1,\dots,g_n$ and the surjectivity of f implies the surjectivity of $g_1,\dots,g_n$. This shows that $g_1,\dots,g_n$ are isomorphisms, i.e. $\mathbf{C}_i \cong \mathbf{A}_{\sigma(i)}$ for all $i = 1,\dots,n$. □

Corollary 2.9. *If an algebra in $\mathcal{K}$ is isomorphic to a finite product of non-trivial HDI algebras, then these factors are uniquely determined up to order and isomorphisms.*

Proof. This follows from Theorem 2.8. □

We can generalize this to infinite direct products as follows. Recall that an ultrafilter on a set I is a family U of subsets of I which is upwards closed and also closed under intersections such that for all $I_0 \subseteq I$ exactly one of I_0,

$I \backslash I_0$ is in U. For any family $(A_i : i \in I)$ of sets and any ultrafilter U on I we define the equivalence relation $\sim_U$ on $\prod_i A_i$ by

$$(x_i : i \in I) \sim_U (y_i : i \in I) \;\Leftrightarrow\; \{i \in I \mid x_i = y_i\} \in U,$$

and we write $\prod_i A_i/U$ for the set of equivalence classes, the "*ultraproduct* of the A_i modulo U". The canonical map from $\prod_i A_i$ to $\prod A_i/U$ is denoted by κ_U. If $(\mathbf{A}_i)_{i\in I}$ is a family of algebras of the same type, then the relation $\sim_U$ is a congruence relation on the product $\prod_i \mathbf{A}_i$.

Theorem 2.10. *Let $\mathcal{K}$ be a variety with the Fraser–Horn property. Let I be a non-empty set, and for each $i \in I$ let $\mathbf{A}_i$ be an algebra in $\mathcal{K}$. Let $\mathbf{C}$ be an HDI algebra in $\mathcal{K}$, and let $h\colon \prod_{i\in I} \mathbf{A}_i \to \mathbf{C}$ be a homomorphism which is not constant. Then there is a unique ultrafilter U on I such that h factors through κ_U, i.e., there is a homomorphism $h'\colon \prod_{i\in I} \mathbf{A}_i/{\sim_U} \to \mathbf{C}$ such that $h = h' \circ \kappa_U$.*

In particular: If there is no $i \in I$ such that h factors through p_i, then U will be a non-principal ultrafilter.

Proof. Let U be defined as the set of all $M \subseteq I$ such that h factors through p_M, i.e., such that there exists $f_M\colon \prod_{i\in M} \mathbf{A}_i \to \mathbf{C}$ with $h = f_M \circ p_M$.

It is clear that U is upwards closed, and from Theorem 2.5 and Remark 2.6 we get: If $M_1 \subseteq I$ and $M_2 := I \backslash M_1$, then $M_1 \in U$ and $M_2 \notin U$ or conversely. As h is not constant, we have $\emptyset \notin U$.

We now show that U is closed under intersections: Given $M_1, M_2 \in U$, then we can write $\prod_{i\in I} \mathbf{A}_i$ as the direct product of four factors:

$$B_{11} = \prod_{i\in M_1\cap M_2} \mathbf{A}_i, B_{10} = \prod_{i\in M_1\backslash M_2} \mathbf{A}_i, B_{01} = \prod_{i\in M_2\backslash M_1} \mathbf{A}_i, B_{00} = \prod_{i\notin M_1\cup M_2} \mathbf{A}_i,$$

with corresponding projections p_{11}, p_{10}, p_{01}, p_{00}.

Since none of the sets $M_1\backslash M_2$, $M_2\backslash M_1$, and $I\backslash(M_1 \cup M_2)$ are in U, h cannot factor through any of p_{10}, p_{01}, or p_{00}. Hence (by Theorem 2.5), h must factor through p_{11}, so $M_1 \cap M_2 \in U$. So we have shown that U is a filter, and even an ultrafilter.

We now check that h factors through the canonical map $\kappa_U : \prod_i \mathbf{A}_i \to \prod_i \mathbf{A}_i/U$. All we have to show is that for all $x \sim_U y \in \prod_i \mathbf{A}_i$ we have $h(x) = h(y)$. Now $x \sim_U y$ implies that the set $M := \{i \mid x(i) = y(i)\}$ is in U; by definition of U, there is some f_M with $h = f_M \circ p_M$, so we get $h(x) = f(p_M(x)) = f(p_M(y)) = h(y)$.

Finally, we show that U is unique. So let U' be an ultrafilter such that h factors through $\kappa_{U'}$. It is enough to show $U' \subseteq U$:

Let $M \in U'$, and let $U'{\restriction}M := \{N \cap M \mid N \in U'\}$ be the restriction of U' to M. The map $\kappa_{U'}$ can be written as $\kappa_{U'} = \kappa_{U'\restriction M} \circ p_M$; as h factors through $\kappa_{U'}$, h also factors through p_M, so $M \in U$. □

Remark 2.11. Theorem 2.5 was used in the proof of Theorem 2.10; but we can also view Theorem 2.5 as a special case of Theorem 2.10, as any ultrafilter on a finite index set must be principal.

3. Lattices

Theorem 2.10 is in some sense best possible, in the sense that homomorphisms from an infinite product $\prod_i \mathbf{A}_i$ into an HDI algebra will in general not factor through any single projection p_j, as the following example shows.

Example 3.1. Let U be an ultrafilter on the infinite index set I, and for each $i \in I$ let $\mathbf{A}_i$ be the 2-element lattice $\{0, 1\}$. Then the ultraproduct $\prod_{i \in I} \mathbf{A}_i/U$ is again the 2-element lattice.

Identifying $\prod_i \mathbf{A}_i$ with the power set lattice $(P(I), \cup, \cap)$, the canonical map $\kappa_U \colon P(I) \to \{0, 1\}$ maps each element of U to 1 and everything else to 0. If U is a non-principal ultrafilter, then h_U does not factor through any projection.

This example can be generalized to any Fraser–Horn variety where the class of HDI algebras is described by a set of first order formulas: If $\prod_i \mathbf{A}_i$ is a product of algebras, and $(h_i : i \in I)$ is a family of homomorphisms $h_i \colon \mathbf{A}_i \to \mathbf{C}_i$, where each $\mathbf{C}_i$ is HDI, then the family $(h_i : i \in I)$ naturally defines a homomorphism $h \colon \prod_i \mathbf{A}_i \to \prod_i \mathbf{C}_i$.

If U is an ultrafilter on I, then the algebra $\mathbf{C} := \prod_i \mathbf{C}_i/U$ is again HDI (as $\mathbf{C}$ satisfies all first order statements that are true in each $\mathbf{C}_i$). Let $\kappa_U^C \colon \prod_i \mathbf{C}_i \to \prod_i \mathbf{C}_i/U$ and $\kappa_U^A \colon \prod_i \mathbf{A}_i \to \prod_i \mathbf{A}_i/U$ be the canonical maps. Then the map $\bar{h} := \kappa_U^C \circ h \colon \prod_i \mathbf{A}_i \to \mathbf{C}$ trivially factors through κ_U^A, i.e., there is $h' \colon \prod_i \mathbf{A}_i/U \to \mathbf{C}$ with $\bar{h} = h' \circ \kappa_U^A$. By the uniqueness claim in Theorem 2.10, we see that U is the set of all $M \subseteq I$ such that $\bar{h}$ factors through p_M. So if U is non-principal, then $\bar{h}$ does not factor through any p_i.

Fact 3.2. Let $\mathbf{A}$ be a lattice. Then the following are equivalent:

- There is a non-constant homomorphism from $\mathbf{A}$ into a chain.
- There is a non-constant homomorphism from $\mathbf{A}$ into the 2-element chain.
- The lattice $\mathbf{A}$ has a prime ideal.

The following corollary can be seen as a weak version of Theorem 2.5.

Corollary 3.3. *The class of lattices without a prime ideal is closed under finite direct products.*

The following example shows that even this weak version cannot be generalized to infinite products, not even if all factors are equal.

Example 3.4. (a) There are non-trivial lattices $\mathbf{M}$ such that no (finite or infinite) direct power of $\mathbf{M}$ has a prime ideal.

(b) On the other hand, there are lattices $\mathbf{A}$ without a prime ideal such that any infinite direct power $\mathbf{A}^I$ will contain a prime ideal.

Proof of (a). Let $\mathcal{M}$ be the class of all lattices of height 3 with at least 5 elements, i.e., the class of all bounded lattices $\mathbf{M}$ in which all elements except for $\sup \mathbf{M}$ and $\inf \mathbf{M}$ are incomparable. It is clear that no lattice in $\mathcal{M}$ has a prime ideal. The class $\mathcal{M}$ is closed under ultraproducts, since the property of being in $\mathcal{M}$ can be expressed by a first order statement.

If $\mathbf{M} = \prod_{i\in I} \mathbf{M}_i$ is an arbitrary direct product with factors $\mathbf{M}_i \in \mathcal{M}$, and $h\colon \mathbf{M} \to \mathbf{C}$ is a homomorphism into a chain, then h factors through an ultraproduct $\prod_{i\in I} \mathbf{M}_i \to \prod_{i\in I} \mathbf{M}_I/U \to \mathbf{C}$, $h = h' \circ \kappa_U$. The map h' and therefore also h must be constant. □

Proof of (b). Let $\mathbf{A}$ be the lattice obtained from $\mathbb{N} = \{0, 1, 2, \dots\}$ by replacing each odd number $2k + 1$ by a 3-element antichain a_k, b_k, c_k, and each even number $2k$ by a new element d_k. It is easy to see that $\mathbf{A}$ has no prime ideal.

We will show that every infinite power $\mathbf{A}^I$ contains a prime ideal. Clearly it is enough to show this for the case of countable I, say $I = \mathbb{N}$.

For any ultrafilter U on $\mathbb{N}$ the following set J_U is an ideal on $\prod_{i\in I} \mathbf{A}_i$:

$$J_U := \{(x_n : n \in \mathbb{N}) \mid \exists k\ \exists C \in U\ \forall n \in C : x_n \leq d_k\}$$

We now show that J_U is a prime ideal. If

$$\bar{x} = (x_i : i \in \mathbb{N}),\ \bar{y} = (y_i : i \in \mathbb{N}),\ \bar{z} = (z_i : i \in \mathbb{N}),\quad \bar{x} \wedge \bar{y} = \bar{z} \in J_U,$$

then there is some set $C \in U$ and some natural number $k \in \mathbb{N}$ such that $z_n \leq d_k$ holds for all $n \in C$. Now the two sets

$$\{n : x_n \geq d_{k+1}\},\ \{n : y_n \geq d_{k+1}\}$$

cannot both belong to U, as their intersection D is disjoint to C. (Since $n \in D$ implies $x_n \wedge y_n \geq d_{k+1}$.)

Without loss of generality we have $\{n : x_n \leq d_{k+1}\} \in U$, so $\bar{x} \in J_U$. □

Acknowledgements

Open access funding provided by TU Wien (TUW). We are grateful to the referee of a previous version of this paper for alerting us to [4], and to Gábor Czédli for suggesting the definition of HDI algebras.

References

[1] Birkhoff, G.: Lattice Theory. Corr. Repr. of the 1967 3rd Edn. American Mathematical Society Colloquium Publications, vol. 25. American Mathematical Society (AMS), Providence (1979)

[2] Couceiro, M., Foldes, S., Meletiou, G.C.: On homomorphisms between products of median algebras. Algebra Universalis **78**, 545–553 (2017)

[3] Couceiro, M., Marichal, J.-L., Teheux, B.: Conservative median algebras and semilattices. Order **33**, 121–132 (2016)

[4] Fraser, G.A., Horn, A.: Congruence relations in direct products. Proc. Am. Math. Soc. **26**, 390–394 (1970)

[5] Grätzer, G.: Lattice Theory: Foundation. Birkhäuser, Basel (2011)

Ivan Chajda and Helmut Länger
Department of Algebra and Geometry
Palacký University Olomouc
17. listopadu 12
771 46 Olomouc
Czech Republic
e-mail: ivan.chajda@upol.cz

Martin Goldstern and Helmut Länger
Institute of Discrete Mathematics and Geometry
TU Wien
Wiedner Hauptstraße 8-10
1040 Vienna
Austria
e-mail [M. Goldstern]: martin.goldstern@tuwien.ac.at
e-mail [H. Länger]: helmut.laenger@tuwien.ac.at

Received: 3 April 2017.
Accepted: 15 January 2018.