Aequat. Math. 94 (2020), 651–677

0001-9054/20/040651-27
published online August 7, 2019
https://doi.org/10.1007/s00010-019-00670-9

Aequationes Mathematicae

On the equality problem of generalized Bajraktarević means

Richárd Grünwald and Zsolt Páles

Dedicated to the 95th birthday of Professor János Aczél.

Abstract. The purpose of this paper is to investigate the equality problem of generalized Bajraktarević means, i.e., to solve the functional equation

$$f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\cdots+p_n(x_n)f(x_n)}{p_1(x_1)+\cdots+p_n(x_n)}\right)=g^{(-1)}\left(\frac{q_1(x_1)g(x_1)+\cdots+q_n(x_n)g(x_n)}{q_1(x_1)+\cdots+q_n(x_n)}\right), \tag{*}$$

which holds for all $(x_1,\ldots,x_n)\in I^n$, where $n\geq 2$, I is a nonempty open real interval, the unknown functions $f,g:I\to\mathbb{R}$ are strictly monotone, $f^{(-1)}$ and $g^{(-1)}$ denote their generalized left inverses, respectively, and $p=(p_1,\ldots,p_n):I\to\mathbb{R}_+^n$ and $q=(q_1,\ldots,q_n):I\to\mathbb{R}_+^n$ are also unknown functions. This equality problem in the symmetric two-variable (i.e., when $n=2$) case was already investigated and solved under sixth-order regularity assumptions by Losonczi (Aequationes Math 58(3):223–241, 1999). In the nonsymmetric two-variable case, assuming the three times differentiability of f, g and the existence of $i\in\{1,2\}$ such that either p_i is twice continuously differentiable and p_{3-i} is continuous on I, or p_i is twice differentiable and p_{3-i} is once differentiable on I, we prove that (*) holds if and only if there exist four constants $a,b,c,d\in\mathbb{R}$ with $ad\neq bc$ such that

$$cf+d>0, \qquad g=\frac{af+b}{cf+d}, \qquad \text{and} \qquad q_\ell=(cf+d)p_\ell \qquad (\ell\in\{1,\ldots,n\}).$$

In the case $n\geq 3$, we obtain the same conclusion with weaker regularity assumptions. Namely, we suppose that f and g are three times differentiable, p is continuous and there exist $i,j,k\in\{1,\ldots,n\}$ with $i\neq j\neq k\neq i$ such that p_i,p_j,p_k are differentiable.

Mathematics Subject Classification. 39B30, 39B40, 26E60.

Keywords. Equality of means, Quasi-arithmetic mean, Bajraktarević mean, Generalized inverse.

The research of the first author was supported by the ÚNKP-18-2 New National Excellence Program of the Ministry of Human Capacities. The research of the second author was supported by the Hungarian Scientific Research Fund (OTKA) Grant K-111651 and by the EFOP-3.6.1-16-2016-00022 project. This project is co-financed by the European Union and the European Social Fund.

The authors are indebted to the anonymous referees for careful reading of the manuscript and for their useful suggestions.

1. Introduction

Throughout this paper, the symbols $\mathbb{R}$ and $\mathbb{R}_+$ will stand for the sets of real and positive real numbers, respectively, and I will always denote a nonempty open real interval. In the theory of quasi-arithmetic means the characterization of the equality of means with different generators is a basic problem which was completely solved in the book [7]. Using this characterization, the homogeneous quasi-arithmetic means can also be found: they are exactly the power means and the geometric mean. In [2] (cf. also [3]) Bajraktarević introduced a new generalization of quasi-arithmetic means by adding a weight function to the formula of quasi-arithmetic means. He also described the equality of such means (called Bajraktarević means since then) in the at least 3-variable setting assuming three times differentiability. Daróczy and Losonczi [4], later Daróczy and Páles [5] arrived at the same conlusion with first-order differentiability and without differentiability, respectively, but assuming equality for all $n \in \mathbb{N}$. As an application of the characterization of the equality, Aczél and Daróczy [1] determined the homogeneous Bajraktarević means that include Gini means which were introduced by Gini [6]. Losonczi [9] described the equality of two-variable Bajraktarević means under sixth-order regularity assumptions and an algebraic condition which was later removed in [10]. Using these results, the homogeneous two-variable means were also determined by Losonczi [11,12].

The purpose of this paper is to extend the definition of Bajraktarević means in a nonsymmetric way by replacing each appearance of the weight function by a possibly different one. We also take strictly monotone functions instead of strictly monotone and continuous ones.

Given a subset $S \subseteq \mathbb{R}$, the smallest convex set containing S, which is identical to the smallest interval containing S, will be denoted by $\operatorname{conv}(S)$. For our definition of generalized Bajraktarević means, we shall need the following lemma about the existence and properties of the left inverse of strictly monotone (but not necessarily continuous) functions.

Lemma 1. *Let $f : I \to \mathbb{R}$ be a strictly monotone function. Then there exists a uniquely determined monotone function $g : \operatorname{conv}(f(I)) \to I$ such that g is the left inverse of f, i.e.,*

$$(g \circ f)(x) = x \qquad (x \in I). \tag{1}$$

Furthermore, g is monotone in the same sense as f, continuous,

$$(f \circ g)(y) = y \qquad (y \in f(I)), \tag{2}$$

and

$$\liminf_{x \to g(y)} f(x) \le y \le \limsup_{x \to g(y)} f(x) \qquad (y \in \operatorname{conv}(f(I))). \tag{3}$$

Thus, if f is lower (resp. upper) semicontinuous at $g(y)$, then $(f \circ g)(y) \leq y$ (resp. $y \leq (f \circ g)(y)$).

Proof. Without loss of generality, we may assume that $f : I \to \mathbb{R}$ is a strictly increasing function. Then $f : I \to f(I)$ is a bijection. The interval I is open, therefore, f has a left and a right limit at every point $x \in I$, which will be denoted by $f_-(x)$ and $f_+(x)$, respectively. We introduce the notation $J_x := [f_-(x), f_+(x)]$, where $x \in I$. Then, for all elements $u < x < v \in I$, we have that

$$f_+(u) < f_-(x) \leq f(x) \leq f_+(x) < f_-(v).$$

From these inequalities, it follows that $f(x) \in J_x$ holds for all $x \in I$ and $J_x \cap J_u = \emptyset$ whenever u is distinct from x.

The convex hull of $f(I)$ is the smallest interval $J \subseteq \mathbb{R}$ containing $f(I)$. The opennes of I implies that $\inf f(I), \sup f(I) \notin J$, hence $J :=]\inf f(I), \sup f(I)[$. We show that

$$J = \bigcup_{x \in I} J_x. \tag{4}$$

If $x \in I$, then, for all $u < x$, we have $f_-(x) > f_+(u) = \inf_{u<t} f(t) \geq \inf f(I)$. Similarly, $f_+(x) < \sup f(I)$, therefore, $J_x \subseteq J$. This proves the inclusion $\supseteq$ in (4). To prove the reverse inclusion in (4), let $y \in J$. Define

$$x := \sup\{u \in I \mid f(u) \leq y\}.$$

Then, for all $n \in \mathbb{N}$, there exists $u_n \in I$ such that $x - \frac{1}{n} < u_n$ and $f(u_n) \leq y$. Thus, $u_n \leq x$ and hence u_n tends to x as $n \to \infty$. Therefore,

$$f_-(x) \leq \limsup_{n \to \infty} f(u_n) \leq y.$$

On the other hand, let $u_n \in I$ be an arbitrary sequence converging to x such that $x < u_n$. Then $y < f(u_n)$, whence we obtain

$$y \leq \lim_{n \to \infty} f(u_n) = f_+(x).$$

The above inequalities imply that $y \in J_x$, which completes the proof of the inclusion $\subseteq$ in (4).

Let $y \in J = \mathrm{conv}(f(I))$ be an arbitrarily fixed element. Then there exists a uniquely determined element $x \in I$ such that $y \in J_x$, hence we define the function $g : J \to I$ by the prescription $g(y) := x$.

Therefore, if $x \in I$ is an arbitrary element, then it is obvious that $f(x) \in J_x$ and hence $g(f(x)) = x$. Thus, Eq. (1) is valid for all $x \in I$.

To see that g is nondecreasing, let $y_1 < y_2$ be arbitrary elements of J. Then there exist elements $x_1, x_2 \in I$ such that $y_i \in J_{x_i}$. If x_2 were strictly smaller than x_1, then we would have

$$y_2 \leq f_+(x_2) < f_-(x_1) \leq y_1.$$

This contradiction shows that $g(y_1) = x_1 \leq x_2 = g(y_2)$.

To prove that g is continuous, let $y \in J$ and choose $\varepsilon > 0$ so that $g(y) \pm \varepsilon$ be in I. Define $W_\varepsilon :=]f_-(g(y)-\varepsilon), f_+(g(y)+\varepsilon)[$. Then

$$f_-(g(y)-\varepsilon) < f_-(g(y)) \leq y \leq f_+(g(y)) < f_+(g(y)+\varepsilon),$$

hence W_ε is a neighborhood of y. By the monotonicity of g, for $w \in W_\varepsilon$, we have that

$$g(y)-\varepsilon = g\big(f_-(g(y)-\varepsilon)\big) \leq g(w) \leq g\big(f_+(g(y)+\varepsilon)\big) = g(y)+\varepsilon,$$

which yields that g is continuous at y.

If $y \in f(I)$, then there exists a uniquely determined element $x \in I$ such that $f(x) = y$ and hence, using (1), we get that

$$(f \circ g)(y) = f((g \circ f)(x)) = f(x) = y,$$

which shows that (2) holds for all $y \in f(I)$.

To see that (3) is valid, let $y \in J$. By the definition of $g(y)$, there exists a unique element $v \in I$ such that $y \in J_v$ and $g(y) = v$. Then, for all $x < v = g(y)$, we have

$$f(x) \leq f_+(x) < f_-(v) \leq y.$$

Therefore, upon taking the left limit $x \to v-0$, we get

$$\liminf_{x \to g(y)} f(x) = \lim_{x \to g(y)-0} f(x) \leq y,$$

which proves the left hand side inequality in (3). The verification of the right hand side inequality is completely analogous, therefore it is omitted.

Finally, we prove the uniqueness of g. Assume that $h : J \to I$ is a nondecreasing function which is the left inverse of f. We are going to show that h coincides with g on J. Let $y \in J$ be arbitrary. Then there exists $x \in I$ such that $f_-(x) \leq y \leq f_+(x)$ and $g(y) = x$. Let (x_n) be a strictly increasing and (x'_n) be a strictly decreasing sequence converging to x. Then, for all $n \in \mathbb{N}$, we have

$$f(x_n) < f_-(x) \leq y \leq f_+(x) < f(x'_n).$$

By the monotonicity of h, it follows that

$$x_n = (h \circ f)(x_n) \leq h(y) \leq (h \circ f)(x'_n) = x'_n.$$

Taking the limit $n \to \infty$, we arrive at

$$x \leq h(y) \leq x,$$

which proves that $h(y) = x = g(y)$. □

The function g described in the above lemma is called the *generalized left inverse of the strictly monotone function* $f : I \to \mathbb{R}$ and is denoted by $f^{(-1)}$. It is clear from (1) and (2) that the restriction of $f^{(-1)}$ to $f(I)$ is the inverse of f in standard sense. Therefore, $f^{(-1)}$ is the continuous and monotone extension of the inverse of f to the smallest interval containing the range of f.

Given a strictly monotone function $f : I \to \mathbb{R}$ and an n-tuple of positive valued functions $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$, we introduce the *$n$-variable generalized Bajraktarević mean* $A_{f,p} : I^n \to I$ by the following formula:

$$A_{f,p}(\mathbf{x}):=f^{(-1)}\left(\frac{p_1(x_1)f(x_1)+\ldots+p_n(x_n)f(x_n)}{p_1(x_1)+\ldots+p_n(x_n)}\right) \qquad (\mathbf{x}=(x_1,\ldots,x_n) \in I^n), \tag{5}$$

and, to simplify the notations, we will use the following definition:

$$R_{f,p}(\mathbf{x}) := \frac{p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n)}{p_1(x_1) + \cdots + p_n(x_n)}. \tag{6}$$

Theorem 2. *Let $n \geq 2$, $f : I \to \mathbb{R}$ be strictly monotone and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$. Then the function $A_{f,p} : I^n \to I$ given by (5) is well-defined and it is a mean, that is,*

$$\min(\mathbf{x}) \leq A_{f,p}(\mathbf{x}) \leq \max(\mathbf{x}) \qquad (\mathbf{x} = (x_1, \dots, x_n) \in I^n). \tag{7}$$

Proof. We may assume that f is strictly increasing (in the decreasing case the proof is very similar). To show that, for all $\mathbf{x} = (x_1, \dots, x_n) \in I^n$, the formula for $A_{f,p}(\mathbf{x})$ is well-defined and (7) holds, consider the ratio $R_{f,p}(\mathbf{x})$.

Due to the positivity of the values of $p_i(x_i)$, we can see that $R_{f,p}(\mathbf{x})$ is a convex combination of the values $f(x_1), \dots, f(x_n)$, therefore,

$$\begin{aligned} f(\min(\mathbf{x})) = \min(f(x_1), \dots, f(x_n)) &\leq R_{f,p}(\mathbf{x}) \\ &\leq \max(f(x_1), \dots, f(x_n)) = f(\max(\mathbf{x})). \end{aligned} \tag{8}$$

This shows that $R_{f,p}(\mathbf{x})$ is an element of $\operatorname{conv}(f(I))$, which is the domain of $f^{(-1)}$ and hence $A_{f,p}(\mathbf{x}) = f^{(-1)}(R_{f,p}(\mathbf{x}))$ is well-defined. Furthermore, using that $f^{(-1)}$ is nondecreasing and is the left inverse of f, the inequalities in (8) yield

$$\min(\mathbf{x}) = f^{(-1)}(f(\min(\mathbf{x}))) \leq f^{(-1)}(R_{f,p}(\mathbf{x})) \leq f^{(-1)}(f(\max(\mathbf{x}))) = \max(\mathbf{x}).$$

This finally proves the mean value inequalities stated in (7). □

Theorem 3. *Let $n \geq 2$, $f : I \to \mathbb{R}$ be strictly increasing and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$. Then, for all $\mathbf{x} = (x_1, \dots, x_n) \in I^n$, the equality $y = A_{f,p}(\mathbf{x})$ holds if and only if*

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) \begin{cases} < 0 & \text{for } z \in I,\ z < y, \\ > 0 & \text{for } z \in I,\ z > y. \end{cases} \tag{9}$$

If f is strictly decreasing, then the inequalities (9) hold with reversed inequality signs.

Proof. Assume that $f : I \to \mathbb{R}$ is strictly increasing, let $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ and $y := A_{f,p}(\mathbf{x})$. If $z < y$, then $f(z) < R_{f,p}(\mathbf{x})$, because in the opposite case we would have $f(z) \geq R_{f,p}(\mathbf{x})$ which implies $z = f^{(-1)}(f(z)) \geq$

$f^{(-1)}(R_{f,p}(\mathbf{x})) = A_{f,p}(\mathbf{x}) = y$, contradicting the choice of z. Rearranging the inequality $f(z) < R_{f,p}(\mathbf{x})$, it easily follows that

$$\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i)) < 0.$$

In the case $z > y$, we get $f(z) > R_{f,p}(\mathbf{x})$, which implies the second inequality in (9).

Observe that the function

$$z \mapsto \varphi(z) := \sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i))$$

is strictly increasing. Therefore, it changes sign at most one point in I. If (9) holds for y, then φ changes sign at y. On the other hand, as we have seen it above, φ also changes sign at $A_{f,p}(\mathbf{x})$. Hence $y = A_{f,p}(\mathbf{x})$ must hold. □

Corollary 4. *Let $n \geq 2$, $f : I \to \mathbb{R}$ be continuous, strictly monotone, and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$. Then, for all $\mathbf{x} = (x_1, \dots, x_n) \in I^n$, the value $y = A_{f,p}(\mathbf{x})$ is the unique solution of the equation*

$$\sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i)) = 0. \tag{10}$$

Proof. The function

$$y \mapsto \varphi(y) := \sum_{i=1}^{n} p_i(x_i)(f(y) - f(x_i))$$

is strictly monotone and continuous. Therefore, it vanishes at most one point in I. Applying Theorem 3, we obtain that φ changes sign at $y = A_{f,p}(\mathbf{x})$. Thus, using that φ is continuous, φ vanishes at $y = A_{f,p}(\mathbf{x})$. □

The next result establishes a sufficient condition for the equality of the n-variable generalized Bajraktarević means. We will call this situation the canonical case of the equality.

Theorem 5. *Let $n \geq 2$, $f, g : I \to \mathbb{R}$ be strictly monotone and $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$, $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. If there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that*

$$cf + d > 0, \qquad g = \frac{af + b}{cf + d}, \qquad \text{and} \qquad q_i = (cf + d)p_i \qquad (i \in \{1, \dots, n\}) \tag{11}$$

hold on I, then the n-variable generalized Bajraktarević means $A_{f,p}$ and $A_{g,q}$ are identical on I^n.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ be arbitrary. Using the formulas (11), for $z \in I$, we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} q_i(x_i)(g(z) - g(x_i)) \\
&\quad = \sum_{i=1}^{n} (cf(x_i)+d)p_i(x_i) \left(\frac{(af(z)+b)(cf(x_i)+d) - (af(x_i)+b)(cf(z)+d)}{(cf(x_i)+d)(cf(z)+d)} \right) \\
&\quad = \frac{ad-bc}{cf(z)+d} \Big(\sum_{i=1}^{n} p_i(x_i)(f(z)-f(x_i)) \Big).
\end{aligned}$$

It shows that $\sum_{i=1}^{n} q_i(x_i)(g(z) - g(x_i))$ changes sign at y if and only if $\sum_{i=1}^{n} p_i(x_i)(f(z) - f(x_i))$ changes sign at y. Hence, applying Theorem 3, $A_{f,p}(\mathrm{x}) = A_{g,q}(\mathrm{x})$ holds. The element x being arbitrary in I^n, we get the statement of the theorem. □

With the aid of the following lemma, we can reduce the regularity assumptions in our statements. For the formulation of this and subsequent results, we define the *diagonal* $\mathrm{diag}(I^n)$ of I^n and the map $\Delta_n : I \to \mathrm{diag}(I^n)$ by

$$\mathrm{diag}(I^n) := \{(x, \ldots, x) \in \mathbb{R}^n \mid x \in I\} \qquad \text{and} \qquad \Delta_n(x) := (x, \ldots, x) \qquad (x \in I).$$

For all $i \in \{1, \ldots, n\}$, let $e_i \in \mathbb{R}^n$ denote the ith vector of the standard base of $\mathbb{R}^n$, i.e., let $e_i := (\delta_{ij})_{j=1}^{n}$, where δ stands for the Kronecker symbol.

Given $p = (p_1, \ldots, p_n) : I \to \mathbb{R}_+^n$ and $q = (q_1, \ldots, q_n) : I \to \mathbb{R}_+^n$, we will also use the following notations:

$$p_0 := p_1 + \cdots + p_n, \qquad q_0 := q_1 + \cdots + q_n, \qquad \text{and} \qquad r_0 := \frac{q_0}{p_0}.$$

Lemma 6. *Let $n \geq 2$, $f, g : I \to \mathbb{R}$ be continuous strictly monotone functions, and $p = (p_1, \ldots, p_n) : I \to \mathbb{R}_+^n$, $q = (q_1, \ldots, q_n) : I \to \mathbb{R}_+^n$. Assume that there exists an open set $U \subseteq I^n$ containing* $\mathrm{diag}(I^n)$ *such that $A_{f,p} = A_{g,q}$ holds on U. Then the following two assertions hold.*

- (i) *For all $i \in \{1, \ldots, n\}$, the function p_i is continuous on I if and only if the function q_i is continuous on I.*
- (ii) *Let $k \in \mathbb{N}$. Assume that $f, g : I \to \mathbb{R}$ are k times differentiable (resp. k times continuously differentiable) functions on I with nonvanishing first derivatives. Then, for all $i \in \{1, \ldots, n\}$, the function p_i is k times differentiable (resp. k times continuously differentiable) on I if and only if q_i is k times differentiable (resp. k times continuously differentiable) on I.*

Proof. In what follows, we will prove that the regularity properties possessed by p_i are transferred to the corresponding q_i. The reverse statements can by similarly verified.

For $i \in \{1, \dots, n\}$, denote

$$U_i := \{(x, y) \in I^2 \mid \Delta_n(x) + (y - x)e_i \in U\}.$$

Then U_i is an open set containing $\operatorname{diag}(I^2)$. By our assumption, we have that, for all $(x, y) \in U_i$,

$$A_{g,q}(\Delta_n(x) + (y - x)e_i) = A_{f,p}(\Delta_n(x) + (y - x)e_i).$$

This is equivalent to the following equality

$$\begin{aligned} &\frac{(q_0(x) - q_i(x))g(x) + q_i(y)g(y)}{q_0(x) - q_i(x) + q_i(y)} \\ &\quad = \big(g \circ f^{-1}\big)\left(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\right) \qquad ((x, y) \in U_i). \end{aligned} \tag{12}$$

Observe that, for $x, y \in I$ with $x \neq y$, the inequalities $p_i(x) < p_0(x)$ and $f(x) \neq f(y)$ imply that

$$\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)} \neq f(y).$$

Therefore,

$$\big(g \circ f^{-1}\big)\left(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\right) \neq g(y).$$

Thus, solving Eq. (12) with respect to $q_i(y)$, we get

$$q_i(y) = (q_0(x) - q_i(x)) \frac{(g \circ f^{-1})\big(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\big) - g(x)}{g(y) - (g \circ f^{-1})\big(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\big)} \qquad ((x, y) \in U_i,\ x \neq y). \tag{13}$$

Let $x_0 \in I$ be an arbitrarily fixed point. The pair (x_0, x_0) is an interior point of U_i, therefore, there exists $x \in I \setminus \{x_0\}$ such that $(x, x_0) \in U_i$. Then the set

$$V_i := \{y \in I \mid (x, y) \in U_i,\ x \neq y\}$$

is a neighborhood of x_0 on which we have equality (13) for q_i. Provided that f and g are continuous on I and p_i is continuous at x_0, it follows that $g \circ f^{-1}$ is continuous on $f(I)$ and hence the mapping

$$y \mapsto \big(g \circ f^{-1}\big)\left(\frac{(p_0(x) - p_i(x))f(x) + p_i(y)f(y)}{p_0(x) - p_i(x) + p_i(y)}\right) \tag{14}$$

is continuous at x_0. This shows that the right hand side of (13) is a continuous function of y at x_0 and hence q_i is continuous at x_0. This proves the first assertion.

Provided that, for some $k \in \mathbb{N}$, the functions $f, g : I \to \mathbb{R}$ are k times differentiable (resp. k times continuously differentiable) on I with nonvanishing first derivatives and that p_i is k times differentiable (resp. k times continuously differentiable) at x_0, it follows, by standard calculus rules, that $g \circ f^{-1}$ is k times differentiable (resp. k times continuously differentiable) and hence the mapping (14) is also k times differentiable (resp. k times continuously differentiable) at x_0. This implies that the right hand side of (13) is a k times differentiable (resp. k times continuously differentiable) function of y at x_0 and hence q_i is k times differentiable (resp. k times continuously differentiable) at x_0. This proves the second statement. □

The following theorem is of basic importance for our investigations.

Theorem 7. *Let $n \geq 2$, $f, g : I \to \mathbb{R}$ be continuous, strictly monotone and $p = (p_1, \ldots, p_n) : I \to \mathbb{R}_+^n$ be a continuous function on I. Let further $q = (q_1, \ldots, q_n) : I \to \mathbb{R}_+^n$. Assume that there exists an open set $U \subseteq I^n$ containing* $\operatorname{diag}(I^n)$ *so that $A_{f,p} = A_{g,q}$ holds on U and there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ and a nonempty open subinterval J of I such that* (11) *holds on J. Then q is continuous on I and* (11) *is also valid on I.*

Proof. First of all, using Lemma 6 and the continuity of f, g and p, it is clear that q is continuous on I.

Assume that $A_{f,p} = A_{g,q}$ holds on some open set U containing $\operatorname{diag}(I^n)$ and for some constants $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ there exists a nonempty open subinterval J of I such that (11) holds on J. We may assume that J is a maximal subinterval of I with this property. To complete the proof, we have to show that $J = I$. To the contrary, suppose that $J \neq I$. Then one of the strict inequalities

$$\inf I < \inf J =: \alpha \qquad \text{or} \qquad \sup J < \sup I \tag{15}$$

must be valid. We may suppose that the first inequality in (15) holds. Hence, due to the continuity of f, p_1, and q_1 at α, it follows from (11) that $q_1(\alpha) = (cf(\alpha) + d)p_1(\alpha)$. Therefore, $q_1(\alpha) > 0$ implies that $cf(\alpha) + d > 0$. Consequently, using the continuity of all the functions, for all $x \in \bar{J} := J \cup \{\alpha\}$, we get that

$$cf(x)+d>0, \qquad g(x) = \frac{af(x)+b}{cf(x)+d} \qquad \text{and} \qquad q_i(x) = (cf(x)+d)p_i(x) \qquad (i \in \{1, \ldots, n\})$$

are valid. By the continuity of f, there is an element $\bar{\alpha} \in I$ with $\bar{\alpha} < \alpha$ such that $cf(x) + d > 0$ for all $x \in \bar{I} :=]\bar{\alpha}, \alpha] \cup J$. Define the functions $\bar{g} : \bar{I} \to \mathbb{R}$ and $\bar{q} : \bar{I} \to \mathbb{R}_+^n$ by

$$\bar{g}(x) := \frac{af(x)+b}{cf(x)+d} \qquad \text{and} \qquad \bar{q}_i(x) := (cf(x)+d)p_i(x) \qquad (x \in \bar{I},\, i \in \{1, \ldots, n\}). \tag{16}$$

Thus, for all $x \in \bar{J}$, the equations

$$g(x) = \bar{g}(x) \qquad \text{and} \qquad q_i(x) = \bar{q}_i(x) \qquad (i \in \{1, \dots, n\}) \tag{17}$$

hold. On the other hand, the maximality property of J implies that there is no $\beta < \alpha$ such that (17) is valid for all $x \in]\beta, \alpha] \cup J$. Furthermore, the equality $A_{f,p} = A_{g,q}$ on U and Theorem 5 applied to the conditions (16) yield that

$$A_{g,q}(\mathbf{x}) = A_{f,p}(\mathbf{x}) = A_{\bar{g},\bar{q}}(\mathbf{x}) \tag{18}$$

is also valid for all $\mathbf{x} \in \bar{U} := (\bar{I})^n \cap U$. The point $(\alpha, \dots, \alpha)$ is an interior point of $\bar{U}$, therefore, there exists $r > 0$ such that $]\alpha - r, \alpha + r[^n \subseteq \bar{U}$ and hence (18) holds for all $\mathbf{x} \in]\alpha - r, \alpha + r[^n$.

In what follows, we assume that g is strictly increasing and hence $\bar{g}$ must be also strictly increasing. The functions g and $\bar{g}$ are identical on $[\alpha, \alpha + r[$, therefore, their inverses are also equal on $[g(\alpha), g(\alpha + r)[$.

The following claim will be useful for the rest of the proof.

Claim. *If* $\mathbf{x} = (x_1, \dots, x_n) \in]\alpha - r, \alpha + r[^n$ *such that* $\alpha \le A_{g,q}(\mathbf{x})$, *then*

$$\frac{q_1(x_1)g(x_1) + \cdots + q_n(x_n)g(x_n)}{q_1(x_1) + \cdots + q_n(x_n)} = \frac{\bar{q}_1(x_1)\bar{g}(x_1) + \cdots + \bar{q}_n(x_n)\bar{g}(x_n)}{\bar{q}_1(x_1) + \cdots + \bar{q}_n(x_n)}. \tag{19}$$

Indeed, the condition on $\mathbf{x}$ implies that $\alpha \le A_{g,q}(\mathbf{x}) \le \max(x) < \alpha + r$ also holds, hence $g(A_{g,q}(\mathbf{x})) = \bar{g}(A_{g,q}(\mathbf{x}))$. On the other hand, in view of (18), we have the equality $A_{g,q}(\mathbf{x}) = A_{\bar{g},\bar{q}}(\mathbf{x})$. Therefore, $g(A_{g,q}(\mathbf{x})) = \bar{g}(A_{\bar{g},\bar{q}}(\mathbf{x}))$, which implies Eq. (19).

Let $y_0 \in]\alpha, \alpha + r[$ be fixed. Then the inequality $g(\alpha) < g(y_0)$ implies that

$$g(\alpha) < \frac{q_i(\alpha)g(\alpha) + (q_0 - q_i)(y_0)g(y_0)}{q_i(\alpha) + (q_0 - q_i)(y_0)} \qquad (i \in \{1, \dots, n\}).$$

Now, by the continuity of the functions $g, q_1, \dots, q_n$, we can find a positive number $\delta_0 := \delta(y_0) < \min(y_0 - \alpha, \alpha + r - y_0) < r$ such that, for all $x \in]\alpha - \delta_0, \alpha]$ and $y \in]y_0 - \delta_0, y_0 + \delta_0[$,

$$g(\alpha) \le \frac{q_i(x)g(x) + (q_0 - q_i)(y)g(y)}{q_i(x) + (q_0 - q_i)(y)} \qquad (i \in \{1, \dots, n\}). \tag{20}$$

Applying the inverse of g side by side to this inequality, it follows that $\alpha \le A_{g,q}(x_1, \dots, x_n)$, where $x_i := x$ and $x_j := y$ for all $j \in \{1, \dots, n\} \setminus \{i\}$. Therefore, in view of the Claim above and equality (17), for all $x \in]\alpha - \delta_0, \alpha]$ and $y \in]y_0 - \delta_0, y_0 + \delta_0[$, we have that

$$\frac{q_i(x)g(x) + (q_0 - q_i)(y)g(y)}{q_i(x) + (q_0 - q_i)(y)} = \frac{\bar{q}_i(x)\bar{g}(x) + (q_0 - q_i)(y)g(y)}{\bar{q}_i(x) + (q_0 - q_i)(y)} \qquad (i \in \{1, \dots, n\}).$$

This equality can be rewritten as

$$\begin{aligned} q_i(x)\bar{q}_i(x)(g(x) - \bar{g}(x)) + (q_0 - q_i)(y)(q_i(x)g(x) \\ - \bar{q}_i(x)\bar{g}(x)) + (q_0 - q_i)(y)g(y)(\bar{q}_i(x) - q_i(x)) = 0. \end{aligned} \tag{21}$$

Consider the sets

$$S := \{x \in]\alpha - r, \alpha[\, : g(x) \neq \bar{g}(x)\}, \quad S_i := \{x \in]\alpha - r, \alpha[\, : q_i(x) \neq \bar{q}_i(x)\}, \quad (i \in \{1, \dots, n\}).$$

In the next step we show that

$$S \cap]\alpha - \delta_0, \alpha[= S_i \cap]\alpha - \delta_0, \alpha[\qquad (i \in \{1, \dots, n\}). \tag{22}$$

If $x \in]\alpha - \delta_0, \alpha[\setminus S$, then $g(x) = \bar{g}(x)$. Using this, (21) simplifies to the product equality

$$(q_0 - q_i)(y) \cdot (g(x) - g(y)) \cdot (q_i(x) - \bar{q}_i(x)) = 0.$$

The first factor is not zero, because it is the sum of positive terms. Using that $x < \alpha < y_0 - \delta_0 < y$, the strict monotonicity of g implies that $g(x) < g(y)$, proving that the second factor is also not zero. Therefore, we must have $q_i(x) = \bar{q}_i(x)$, which shows that $x \in]\alpha - \delta_0, \alpha[\setminus S_i$. Conversely, if $x \in]\alpha - \delta_0, \alpha[\setminus S_i$, then $q_i(x) = \bar{q}_i(x)$. In this case (21) reduces to the product equality

$$q_i(x) \cdot (q_i(x) + (q_0 - q_i)(y)) \cdot (g(x) - \bar{g}(x)) = 0.$$

The first two factors are positive, hence we must have $g(x) = \bar{g}(x)$, which proves that $x \in]\alpha - \delta_0, \alpha[\setminus S$ and completes the proof of equality (22). The maximality of the interval J, in view of (22), implies that

$$\sup S \cap]\alpha - \delta_0, \alpha[= \sup S_i \cap]\alpha - \delta_0, \alpha[= \alpha \qquad (i \in \{1, \dots, n\}). \tag{23}$$

Let $i \in \{1, \dots, n\}$ be fixed and $y_1, y_2 \in]y_0 - \delta_0, y_0 + \delta_0[$ be arbitrary such that $y_1 \neq y_2$. Replacing y by y_1 and y_2 in (21), and then subtracting the two equations so obtained side by side, we get that

$$\begin{aligned} &((q_0 - q_i)(y_1) - (q_0 - q_i)(y_2)) \cdot (q_i(x)g(x) - \bar{q}_i(x)\bar{g}(x)) \\ &\quad + ((q_0 - q_i)(y_1)g(y_1) - (q_0 - q_i)(y_2)g(y_2)) \cdot (\bar{q}_i(x) - q_i(x)) = 0. \end{aligned} \tag{24}$$

Let $x_1, x_2 \in]\alpha - \delta_0, \alpha[$ be arbitrary. Substituting x by x_1 and then by x_2 in (24), we get a homogeneous linear system of two equations of the form

$$\xi \cdot (q_i(x_k)g(x_k) - \bar{q}_i(x_k)\bar{g}(x_k)) + \eta \cdot (\bar{q}_i(x_k) - q_i(x_k)) = 0 \qquad (k \in \{1, 2\}) \tag{25}$$

which is nontrivially solvable with respect to (ξ, η), because the equalities

$$\xi := (q_0 - q_i)(y_1) - (q_0 - q_i)(y_2) = 0 \quad \text{and} \quad \eta := (q_0 - q_i)(y_1)g(y_1) - (q_0 - q_i)(y_2)g(y_2) = 0$$

cannot be satisfied simultaneously. Indeed, if $\xi = 0$, then $(q_0 - q_i)(y_1) = (q_0 - q_i)(y_2) > 0$. This equality together with $\eta = 0$ imply that $g(y_1) = g(y_2)$. The strict monotonicity of g then yields $y_1 = y_2$, which contradicts the choice of y_1 and y_2. Hence the determinant of the system (25) must be equal to zero, that is,

$$\begin{vmatrix} q_i(x_1)g(x_1) - \bar{q}_i(x_1)\bar{g}(x_1) & \bar{q}_i(x_1) - q_i(x_1) \\ q_i(x_2)g(x_2) - \bar{q}_i(x_2)\bar{g}(x_2) & \bar{q}_i(x_2) - q_i(x_2) \end{vmatrix} = 0.$$

If $x_1, x_2 \in S\cap]\alpha-\delta_0, \alpha[= S_i \cap]\alpha-\delta_0, \alpha[$ are arbitrary, then $\bar{q}_i(x_1) \neq q_i(x_1)$ and $\bar{q}_i(x_2) \neq q_i(x_2)$, therefore, the above determinantal equality can be rewritten as

$$\frac{q_i(x_1)g(x_1) - \bar{q}_i(x_1)\bar{g}(x_1)}{\bar{q}_i(x_1) - q_i(x_1)} = \frac{q_i(x_2)g(x_2) - \bar{q}_i(x_2)\bar{g}(x_2)}{\bar{q}_i(x_2) - q_i(x_2)}.$$

Therefore, there exists a real constant c_i such that

$$c_i = \frac{q_i(x)g(x) - \bar{q}_i(x)\bar{g}(x)}{\bar{q}_i(x) - q_i(x)}$$

holds for all $x \in S\cap]\alpha-\delta_0, \alpha[$. Solving this equation with respect to $\bar{g}(x)$, we obtain that

$$\bar{g}(x) = \frac{q_i(x)}{\bar{q}_i(x)}(g(x) + c_i) - c_i \tag{26}$$

is valid for all $x \in S\cap]\alpha-\delta_0, \alpha[$. Subsituting formula (26) into (21), for all $x \in S\cap]\alpha-\delta_0, \alpha[$ and $y \in]y_0-\delta_0, y_0+\delta_0[$, we arrive at the equation

$$(\bar{q}_i(x) - q_i(x)) \cdot \big(q_i(x)(g(x) + c_i) + (q_0 - q_i)(y)(g(y) + c_i)\big) = 0,$$

which simplifies to the identity

$$q_i(x)(g(x) + c_i) = -(q_0 - q_i)(y)(g(y) + c_i) \qquad (x \in S\cap]\alpha-\delta_0, \alpha[,\ y \in]y_0-\delta_0, y_0+\delta_0[).$$

Therefore, there exists a real constant d_i such that

$$q_i(x)(g(x)+c_i) = d_i = -(q_0-q_i)(y)(g(y) + c_i) \qquad (x \in S\cap]\alpha-\delta_0, \alpha[,\ y \in]y_0-\delta_0, y_0+\delta_0[).$$

Using these equalities on the domain indicated, inequality (20) implies that

$$g(\alpha) \leq \frac{q_i(x)g(x)+(q_0 - q_i)(y)g(y)}{q_i(x)+(q_0 - q_i)(y)} = \frac{d_i - c_i q_i(x) - d_i - c_i(q_0 - q_i)(y)}{q_i(x) + (q_0 - q_i)(y)} = -\,c_i. \tag{27}$$

Therefore, for all $x \in S\cap]\alpha-\delta_0, \alpha[$, we have that $g(x) < g(\alpha) \leq -c_i$, which yields that $d_i < 0$ and $q_i(x) = \frac{d_i}{g(x)+c_i}$. This shows that q_i is strictly increasing on $S\cap]\alpha-\delta_0, \alpha[$. As a consequence of this property, it follows that the equality $q_i(x)(g(x) + c_i) = d_i$ uniquely determines the constants c_i and d_i. Indeed, if $q_i(x)(g(x) + c_i') = d_i'$ were also true for all $x \in S\cap]\alpha-\delta_0, \alpha[$ and for some constants c_i' and d_i', then subtracting the two equations side by side, we get $q_i(x)(c_i - c_i') = d_i - d_i'$. If $c_i \neq c_i'$, then this last equality yields that q_i is constant, which contradicts its strict monotonicity. Therefore, $c_i = c_i'$ implying that $d_i = d_i'$ is also valid.

In the final step, instead of a fixed element $y_0 \in]\alpha, \alpha+r[$, we take another arbitrary element $y' \in]\alpha, \alpha+r[$. Repeating the same argument as above, there exists a positive number $\delta' := \delta(y')$ and real constants c_i', d_i' such that

$$q_i(x)(g(x)+c_i') = d_i' = -(q_0 - q_i)(y)(g(y) + c_i') \qquad (x \in S\cap]\alpha-\delta', \alpha[,\ y \in]y'-\delta', y'+\delta'[).$$

On the set $S \cap]\alpha - \min(\delta', \delta_0), \alpha[$, we have both $q_i(x)(g(x) + c_i) = d_i$ and $q_i(x)(g(x) + c_i') = d_i'$. Due to the uniqueness property, it follows that $c_i' = c_i$ and $d_i' = d_i$. Therefore,

$$d_i = -(q_0 - q_i)(y)(g(y) + c_i) \tag{28}$$

is valid for all $y \in]y' - \delta', y' + \delta'[$, in particular, for $y = y'$. The point y' being arbitrary, we can see that (28) holds for all $y \in]\alpha, \alpha + r[$. Comparing the signs of both sides, we obtain that $g(y) + c_i > 0$ for all $y \in]\alpha, \alpha + r[$. Upon taking the limit $y \to \alpha + 0$, it follows that $g(\alpha) + c_i \geq 0$. On the other hand, by (27), we also have that $g(\alpha) + c_i \leq 0$, whence $g(\alpha) + c_i = 0$ follows. Using that (23) holds, we may also take the limit $x \to \alpha - 0$ in the equality

$$q_i(x)(g(x) + c_i) = d_i \qquad (x \in S \cap]\alpha - \delta_0, \alpha[),$$

whence we arrive at the equality $d_i = 0$, which is the desired contradiction. □

2. Partial derivatives of Bajraktarević means

In the next result we determine the partial derivatives of the Bajraktarević means up to third order at diagonal points of I^n under tight regularity assumptions. For instance, as stated below in assertions (1), (2b), (3c), we prove the existence of partial derivatives of the form ∂_i^m only assuming the $(m-1)$ times continuous differentiability of p_i.

Theorem 8. *Let $n \geq 2$, $\ell \in \{1, 2, 3\}$, let $f : I \to \mathbb{R}$ be an ℓ times differentiable function on I with a nonvanishing first derivative, and let $p = (p_1, \ldots, p_n) : I \to \mathbb{R}_+^n$. Then we have the following assertions.*

(1) *If $\ell = 1$, $i \in \{1, \ldots, n\}$, and p_i is continuous on I, then the first-order partial derivative $\partial_i A_{f,p}$ exists on* $\operatorname{diag}(I^n)$ *and*

$$\partial_i A_{f,p} \circ \Delta_n = \frac{p_i}{p_0}.$$

(2a) *If $\ell = 2$, $i, j \in \{1, \ldots, n\}$ with $i \neq j$, furthermore, p_i and p_j are differentiable on I, then the second-order partial derivative $\partial_i \partial_j A_{f,p}$ exists on* $\operatorname{diag}(I^n)$ *and*

$$\partial_i \partial_j A_{f,p} \circ \Delta_n = -\frac{(p_i p_j)'}{p_0^2} - \frac{p_i p_j}{p_0^2} \cdot \frac{f''}{f'}.$$

(2b) *If $\ell = 2$, $i \in \{1, \ldots, n\}$, and p_i is continuously differentiable on I, then the second-order partial derivative $\partial_i^2 A_{f,p}$ exists on* $\operatorname{diag}(I^n)$ *and*

$$\partial_i^2 A_{f,p} \circ \Delta_n = 2\frac{p_i'(p_0 - p_i)}{p_0^2} + \frac{p_i(p_0 - p_i)}{p_0^2} \cdot \frac{f''}{f'}.$$

(3a) *If* $\ell = 3$, $i, j, k \in \{1, \ldots, n\}$ *with* $i \neq j \neq k \neq i$, *furthermore,* p_i, p_j, *and* p_k *are differentiable on* I, *then the third-order partial derivative* $\partial_i \partial_j \partial_k A_{f,p}$ *exists on* I^n *and*

$$\partial_i \partial_j \partial_k A_{f,p} \circ \Delta_n = 2\frac{p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k}{p_0^3} + 2\frac{(p_i p_j p_k)'}{p_0^3} \cdot \frac{f''}{f'} + \frac{p_i p_j p_k}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right).$$

(3b) *If* $\ell = 3$, $i, j \in \{1, \ldots, n\}$ *with* $i \neq j$, *furthermore,* p_i *is twice differentiable and* p_j *is differentiable on* I, *then the third-order partial derivative* $\partial_i^2 \partial_j A_{f,p}$ *exists on* I^n *and*

$$\partial_i^2 \partial_j A_{f,p} \circ \Delta_n = \frac{2p_i' p_j' (2p_i - p_0) + p_j(2(p_i')^2 - p_i'' p_0)}{p_0^3} + \frac{(2p_i' p_j + p_i p_j')(2p_i - p_0)}{p_0^3} \cdot \frac{f''}{f'} + \frac{p_i p_j}{p_0^3}\left((3p_i - p_0)\left(\frac{f''}{f'}\right)^2 - p_i \frac{f'''}{f'}\right).$$

(3c) *If* $\ell = 3$, $i \in \{1, \ldots, n\}$ *and* p_i *is twice continuously differentiable on* I, *then the third-order partial derivative* $\partial_i^3 A_{f,p}$ *exists on* $\operatorname{diag}(I^n)$ *and*

$$\partial_i^3 A_{f,p} \circ \Delta_n = \frac{3(p_0 - p_i)\big(p_0 p_i'' - 2(p_i')^2\big)}{p_0^3} + 3\frac{p_i'(p_0 - 2p_i)(p_0 - p_i)}{p_0^3} \cdot \frac{f''}{f'} - \frac{p_i(p_0 - p_i)}{p_0^3}\left(3p_i\left(\frac{f''}{f'}\right)^2 - (p_0 + p_i)\frac{f'''}{f'}\right).$$

Proof. Let $\ell \in \{1, 2, 3\}$. Assume that $f : I \to \mathbb{R}$ is an ℓ times differentiable function on I with a nonvanishing first derivative. We have the following formulas for the derivatives of f^{-1}:

$$\big(f^{-1}\big)' = \frac{1}{f'} \circ f^{-1}, \quad \big(f^{-1}\big)'' = -\frac{f''}{(f')^3} \circ f^{-1}, \quad \big(f^{-1}\big)''' = \frac{3(f'')^2 - f' f'''}{(f')^5} \circ f^{-1}. \tag{29}$$

In this proof, let δ denote the extended Kronecker symbol, which, for $i, j, k \in \mathbb{N}$, is defined by:

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta_{ijk} := \begin{cases} 1 & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, in order to make the calculations shorter, we use the notation $R := R_{f,p}$, where $R_{f,p}$ was defined in (6). Then $A_{f,p} = f^{-1} \circ R_{f,p} = f^{-1} \circ R$.

To compute the partial derivatives of R, we introduce the notations

$$P(x_1, \ldots, x_n) := p_1(x_1) + \cdots + p_n(x_n),$$

$$Q(x_1,\dots,x_n) := p_1(x_1)f(x_1) + \cdots + p_n(x_n)f(x_n).$$

Then $R\cdot P = Q$ and we have that

$$P\circ\Delta_n = p_0, \qquad Q\circ\Delta_n = p_0 f, \qquad R\circ\Delta_n = f, \qquad \text{and} \qquad f^{-1}\circ R\circ\Delta_n = \mathrm{id}\,. \tag{30}$$

To prove the first assertion of the theorem, let $x\in I$ be fixed. Then, using the continuity of p_i and the differentiability of f at x, we get

$$\begin{aligned}(\partial_i R\circ\Delta_n)(x) &= \lim_{y\to x}\frac{R(\Delta_n(x)+(y-x)e_i)-R(\Delta_n(x))}{y-x}\\ &= \lim_{y\to x}\frac{1}{y-x}\bigg(\frac{(p_0(x)-p_i(x))f(x)+p_i(y)f(y)}{p_0(x)-p_i(x)+p_i(y)} - f(x)\bigg)\\ &= \lim_{y\to x}\frac{p_i(y)}{p_0(x)-p_i(x)+p_i(y)}\cdot\frac{f(y)-f(x)}{y-x} = \frac{p_i f'}{p_0}(x). \end{aligned}\tag{31}$$

Therefore, using standard differentiation rules, the last identity in (30) and (31), we obtain

$$\partial_i A_{f,p}\circ\Delta_n = \partial_i\big(f^{-1}\circ R\big)\circ\Delta_n = \bigg(\frac{\partial_i R}{f'\circ f^{-1}\circ R}\bigg)\circ\Delta_n = \frac{\frac{p_i}{p_0}f'}{f'} = \frac{p_i}{p_0}.$$

This completes the proof of assertion (1).

For the proof of statement (2a), let $i,j\in\{1,\dots,n\}$ with $i\neq j$ be fixed and assume that p_i and p_j are differentiable and f is twice differrentiable on I. Then, for all $\alpha,\beta\in\{i,j\}$ with $\alpha\neq\beta$, the partial derivatives ∂_α and $\partial_\alpha\partial_\beta$ of P and Q and hence of R exist at every point in I^n. Furthermore, for all $(x_1,\dots,x_n)\in I^n$, we have that

$$\begin{aligned}\partial_\alpha P(x_1,\dots,x_n) &= p'_\alpha(x_\alpha), \qquad \partial_\alpha Q(x_1,\dots,x_n) = (p_\alpha f)'(x_\alpha),\\ \partial_\alpha\partial_\beta P(x_1,\dots,x_n) &= 0, \qquad \partial_\alpha\partial_\beta Q(x_1,\dots,x_n) = 0.\end{aligned}\tag{32}$$

Differentiating the identity $R\cdot P = Q$ with respect to the jth and then with respect to the ith variable, in view of the equalities in the second line in (32), it follows that

$$\partial_i\partial_j R\cdot P + \partial_j R\cdot\partial_i P + \partial_i R\cdot\partial_j P = 0$$

holds on I^n, whence, using (31) and (32), we arrive at

$$\begin{aligned}\partial_i\partial_j R\circ\Delta_n &= \bigg(-\frac{\partial_j R\cdot\partial_i P + \partial_i R\cdot\partial_j P}{P}\bigg)\circ\Delta_n\\ &= -\frac{p_j f'}{p_0^2}\cdot p'_i - \frac{p_i f'}{p_0^2}\cdot p'_j = -\frac{(p_i p_j)' f'}{p_0^2}.\end{aligned}\tag{33}$$

Applying the chain rule, the first two formulas in (29) and then (30), (31), (33), it follows that

$$\partial_i\partial_j A_{f,p}\circ\Delta_n = \Big(\big((f^{-1})''\circ R\big)\cdot\partial_i R\cdot\partial_j R + \big((f^{-1})'\circ R\big)\cdot\partial_i\partial_j R\Big)\circ\Delta_n$$
$$= -\frac{f''}{(f')^3}\cdot\frac{p_i f'}{p_0}\cdot\frac{p_j f'}{p_0}+\frac{1}{f'}\cdot\frac{-(p_ip_j)'f'}{p_0^2} = -\frac{(p_ip_j)'}{p_0^2}-\frac{p_ip_j}{p_0^2}\cdot\frac{f''}{f'}.$$

To justify assertion (2b), let $x\in I$ be fixed. Let $i\in\{1,\dots,n\}$ and assume that p_i is continuously differentiable and f is twice differentiable on I. Then the partial derivative ∂_i of P and Q and hence of R exist at every point in I^n. Differentiating the identity $R\cdot P=Q$ with respect to the ith variable, we have that $\partial_i R\cdot P+R\cdot\partial_i P=\partial_i Q$, whence

$$\partial_i R=\frac{\partial_i Q-R\cdot\partial_i P}{P}.$$

Using this, we obtain

$$\begin{aligned}
&(\partial_i^2 R\circ\Delta_n)(x)\\
&\quad=\lim_{y\to x}\frac{\partial_i R(\Delta_n(x)+(y-x)e_i)-\partial_i R(\Delta_n(x))}{y-x}\\
&\quad=\lim_{y\to x}\frac{1}{y-x}\left(\frac{(p_if)'(y)-\frac{(p_0(x)-p_i(x))f(x)+p_i(y)f(y)}{p_0(x)-p_i(x)+p_i(y)}p_i'(y)}{p_0(x)-p_i(x)+p_i(y)}-\frac{p_i(x)f'(x)}{p_0(x)}\right)\\
&\quad=\lim_{y\to x}\left(\frac{(p_0(x)-p_i(x))p_i'(y)}{(p_0(x)-p_i(x)+p_i(y))^2}\cdot\frac{f(y)-f(x)}{y-x}\right.\\
&\qquad\left.+\frac{1}{y-x}\left(\frac{(p_if')(y)}{p_0(x)-p_i(x)+p_i(y)}-\frac{(p_if')(x)}{p_0(x)}\right)\right)\\
&\quad=\frac{(p_0-p_i)p_i'f'}{p_0^2}(x)+\lim_{y\to x}\left(\frac{1}{p_0(x)-p_i(x)+p_i(y)}\cdot\frac{(p_if')(y)-(p_if')(x)}{y-x}\right.\\
&\qquad\left.-\frac{(p_if')(x)}{(p_0(x)-p_i(x)+p_i(y))p_0(x)}\cdot\frac{p_i(y)-p_i(x)}{y-x}\right)\\
&\quad=\left(2\frac{p_i'(p_0-p_i)f'}{p_0^2}+\frac{p_if''}{p_0}\right)(x).
\end{aligned}\tag{34}$$

Applying standard calculus rules, the first two formulas in (29) and then (30), (31), (34), we conclude

$$\begin{aligned}
\partial_i^2 A_{f,p}\circ\Delta_n&=\partial_i^2\big(f^{-1}\circ R\big)\circ\Delta_n\\
&=\Big(\big((f^{-1})''\circ R\big)\cdot(\partial_i R)^2+\big((f^{-1})'\circ R\big)\cdot\partial_i^2 R\Big)\circ\Delta_n\\
&=-\frac{f''}{(f')^3}\left(\frac{p_if'}{p_0}\right)^2+\frac{1}{f'}\left(2\frac{p_i'(p_0-p_i)f'}{p_0^2}+\frac{p_if''}{p_0}\right)\\
&=2\frac{p_i'(p_0-p_i)}{p_0^2}+\frac{p_i(p_0-p_i)}{p_0^2}\cdot\frac{f''}{f'}.
\end{aligned}$$

To prove assertion (3a), let $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ and assume that p_i, p_j and p_k are differentiable on I. Then, for all $\alpha, \beta, \gamma \in \{i, j, k\}$ with $\alpha \neq \beta \neq \gamma \neq \alpha$, the partial derivatives ∂_α, $\partial_\alpha\partial_\beta$ and $\partial_\alpha\partial_\beta\partial_\gamma$ of P, Q and hence of R exist at every point in I^n. Furthermore, for all $(x_1, \dots, x_n) \in I^n$, we have the equalities in (32) and in addition

$$\partial_\alpha\partial_\beta\partial_\gamma P(x_1, \dots, x_n) = 0, \qquad \partial_\alpha\partial_\beta\partial_\gamma Q(x_1, \dots, x_n) = 0. \tag{35}$$

Differentiating the identity $R \cdot P = Q$ with respect to the kth variable, then with respect to the jth variable and then with respect to the ith variable, in view of the last two formulas in (32) and (35), we get $\partial_i\partial_j\partial_k R \cdot P + \partial_i\partial_j R \cdot \partial_k P + \partial_i\partial_k R \cdot \partial_j P + \partial_j\partial_k R \cdot \partial_i P = 0$. Thus, applying the first formula in (33) and (32), we arrive at

$$\begin{aligned}\partial_i\partial_j\partial_k R \circ \Delta_n &= \left(-\frac{\partial_i\partial_j R \cdot \partial_k P + \partial_i\partial_k R \cdot \partial_j P + \partial_j\partial_k R \cdot \partial_i P}{P} \right) \circ \Delta_n \\ &= \frac{(p_i p_j)' p_k' f' + (p_i p_k)' p_j' f' + (p_j p_k)' p_i' f'}{p_0^3} \\ &= \frac{2(p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k) f'}{p_0^3}. \end{aligned} \tag{36}$$

Hence, using (29) and then (30), (31), (36), (33), we obtain

$$\begin{aligned} &\partial_i\partial_j\partial_k A_{f,p} \circ \Delta_n \\ &\quad = \partial_i\partial_j\partial_k \big(f^{-1} \circ R\big) \circ \Delta_n \\ &\quad = \Big(\big((f^{-1})''' \circ R \big) \cdot \partial_i R \cdot \partial_j R \cdot \partial_k R + \big((f^{-1})' \circ R \big) \cdot \partial_i\partial_j\partial_k R \\ &\qquad + \big((f^{-1})'' \circ R \big) \cdot (\partial_i R \cdot \partial_j\partial_k R + \partial_j R \cdot \partial_i\partial_k R + \partial_k R \cdot \partial_i\partial_j R) \Big) \circ \Delta_n \\ &\quad = \frac{3(f'')^2 - f' f'''}{(f')^5} \cdot \frac{p_i f'}{p_0} \cdot \frac{p_j f'}{p_0} \cdot \frac{p_k f'}{p_0} + \frac{1}{f'} \cdot 2 \frac{(p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k) f'}{p_0^3} \\ &\qquad - \frac{f''}{(f')^3} \left(\frac{p_i f'}{p_0} \cdot \frac{-(p_j p_k)' f'}{p_0^2} + \frac{p_j f'}{p_0} \cdot \frac{-(p_i p_k)' f'}{p_0^2} + \frac{p_k f'}{p_0} \cdot \frac{-(p_i p_j)' f'}{p_0^2} \right) \\ &\quad = 2 \frac{p_i p_j' p_k' + p_i' p_j p_k' + p_i' p_j' p_k}{p_0^3} + 2 \frac{(p_i p_j p_k)'}{p_0^3} \cdot \frac{f''}{f'} + \frac{p_i p_j p_k}{p_0^3} \left(3 \left(\frac{f''}{f'} \right)^2 - \frac{f'''}{f'} \right). \end{aligned}$$

To verify assertion (3b), let $i, j \in \{1, \dots, n\}$ with $i \neq j$ and assume that p_i is twice and p_j is once differentiable on I. Then, for all $\alpha, \beta \in \{i, j\}$ with the assumption α and β are not equal to j simultaneously, the partial derivatives ∂_α, $\partial_\alpha\partial_\beta$ and $\partial_i^2\partial_j$ of P, Q and hence of R exist at every point in I^n. Furthermore, for all $(x_1, \dots, x_n) \in I^n$, we have (32), (35), and in addition

$$\begin{aligned} \partial_i^2 P(x_1, \dots, x_n) &= p_i''(x_i), \qquad \partial_i^2 Q(x_1, \dots, x_n) = (p_i f)''(x_i), \\ \partial_i^2\partial_j P(x_1, \dots, x_n) &= 0, \qquad \partial_i^2\partial_j Q(x_1, \dots, x_n) = 0. \end{aligned} \tag{37}$$

Differentiating the equality $R \cdot P = Q$ with respect to the jth variable, and then with respect to the ith variable twice, using (32) and (37), we get

$$\partial_i^2\partial_j R \cdot P + \partial_i^2 R \cdot \partial_j P + 2\partial_i\partial_j R \cdot \partial_i P + \partial_j R \cdot \partial_i^2 P = 0.$$

Thus, applying (34), (32), (33), (31), and (37), we arrive at

$$\begin{aligned}\partial_i^2\partial_j R \circ \Delta_n &= \bigg(-\frac{\partial_i^2 R \cdot \partial_j P + 2\partial_i\partial_j R \cdot \partial_i P + \partial_j R \cdot \partial_i^2 P}{P}\bigg) \circ \Delta_n \\ &= -\bigg(2\frac{(p_0 - p_i)p_i' f'}{p_0^2} + \frac{p_i f''}{p_0}\bigg) \cdot \frac{p_j'}{p_0} + 2\frac{(p_i p_j)' f'}{p_0^2} \cdot \frac{p_i'}{p_0} - \frac{p_j f'}{p_0} \cdot \frac{p_i''}{p_0} \\ &= \frac{(2p_i' p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i'' p_0))f' - p_0 p_i p_j' f''}{p_0^3}. \qquad (38)\end{aligned}$$

Therefore, using (29) and then (30), (31), (38), (33), (34), we get

$$\begin{aligned}\partial_i^2\partial_j A_{f,p} \circ \Delta_n &= \Big(\big((f^{-1})''' \circ R\big) \cdot (\partial_i R)^2 \cdot \partial_j R + \big((f^{-1})' \circ R\big) \cdot \partial_i^2\partial_j R \\ &\qquad + \big((f^{-1})'' \circ R\big) \cdot \big(2\partial_i R \cdot \partial_i\partial_j R + \partial_j R \cdot \partial_i^2 R\big)\Big) \circ \Delta_n \\ &= \frac{3(f'')^2 - f' f'''}{(f')^5} \cdot \frac{(p_i f')^2}{p_0^2} \cdot \frac{p_j f'}{p_0} \\ &\quad + \frac{1}{f'} \cdot \frac{(2p_i' p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i'' p_0))f' - p_0 p_i p_j' f''}{p_0^3} \\ &\quad - \frac{f''}{(f')^3}\bigg(-2\frac{p_i f'}{p_0} \cdot \frac{(p_i p_j)' f'}{p_0^2} + \frac{p_j f'}{p_0} \cdot \frac{p_0(2p_i' f' + p_i f'') - (p_i p_i)' f'}{p_0^2}\bigg) \\ &= \frac{2p_i' p_j'(2p_i - p_0) + p_j(2(p_i')^2 - p_i'' p_0)}{p_0^3} + \frac{(2p_i' p_j + p_i p_j')(2p_i - p_0)}{p_0^3} \cdot \frac{f''}{f'} \\ &\quad + \frac{p_i p_j}{p_0^3}\bigg((3p_i - p_0)\Big(\frac{f''}{f'}\Big)^2 - p_i \frac{f'''}{f'}\bigg),\end{aligned}$$

which completes the proof of case (3b).

To prove assertion (3c), let $i \in \{1, \dots, n\}$ and assume that p_i is twice continuously differentiable on I. Then the partial derivatives ∂_i, ∂_i^2 of P, Q and hence of R exist at every point in I^n. We have that

$$\partial_i^2 R = \partial_i\bigg(\frac{\partial_i Q - R \cdot \partial_i P}{P}\bigg) = \frac{\partial_i^2 Q \cdot P - Q \cdot \partial_i^2 P - 2\partial_i Q \cdot \partial_i P + 2R \cdot (\partial_i P)^2}{P^2}.$$

Then, for all $x, y \in I$, we get

$$\begin{aligned}&\partial_i^2 R(\Delta_n(x) + (y - x)e_i) \\ &\quad = \frac{(p_i f)''(y)((p_0 - p_i)(x) + p_i(y)) - \big(((p_0 - p_i)f)(x) + (p_i f)(y)\big)p_i''(y)}{((p_0 - p_i)(x) + p_i(y))^2}\end{aligned}$$

$$
-\frac{2(p_if)'(y)p_i'(y)-2\frac{((p_0-p_i)f)(x)+(p_if)(y)}{(p_0-p_i)(x)+p_i(y)}(p_i')^2(y)}{((p_0-p_i)(x)+p_i(y))^2}
$$

$$
=\frac{(p_0-p_i)(x)\big(((p_0-p_i)(x)+p_i(y))p_i''(y)-2(p_i')^2(y)\big)}{((p_0-p_i)(x)+p_i(y))^3}(f(y)-f(x))
$$

$$
+\frac{2(p_0-p_i)(x)(p_i'f')(y)}{((p_0-p_i)(x)+p_i(y))^2}+\frac{(p_if'')(y)}{(p_0-p_i)(x)+p_i(y)}. \tag{39}
$$

Therefore, using (39) and (34), the twice continuous differentiability of p_i, we obtain that

$$
(\partial_i^3R\circ\Delta_n)(x)=\lim_{y\to x}\frac{\partial_i^2R(\Delta_n(x)+(y-x)e_i)-\partial_i^2R(\Delta_n(x))}{y-x}
$$

$$
=\lim_{y\to x}\frac{1}{y-x}\Bigg(\frac{(p_0-p_i)(x)\big(((p_0-p_i)(x)+p_i(y))p_i''(y)-2(p_i')^2(y)\big)}{((p_0-p_i)(x)+p_i(y))^3}(f(y)-f(x))
$$

$$
+\frac{2(p_0-p_i)(x)(p_i'f')(y)}{((p_0-p_i)(x)+p_i(y))^2}+\frac{(p_if'')(y)}{(p_0-p_i)(x)+p_i(y)}
$$

$$
-\bigg(2\frac{p_i'(p_0-p_i)f'}{p_0^2}+\frac{p_if''}{p_0}\bigg)(x)\Bigg)
$$

$$
=\bigg(\frac{3(p_0-p_i)\big(p_0p_i''-2(p_i')^2\big)f'}{p_0^3}+\frac{3(p_0-p_i)p_i'f''}{p_0^2}+\frac{p_if'''}{p_0}\bigg)(x). \tag{40}
$$

Hence, applying (29), (31), (40), and (34), we conclude

$$
\partial_i^3A_{f,p}\circ\Delta_n=\partial_i^3\big(f^{-1}\circ R\big)\circ\Delta_n
$$

$$
=\Big(((f^{-1})'''\circ R)(\partial_iR)^3+((f^{-1})'\circ R)\cdot\partial_i^3R+((f^{-1})''\circ R)(3\partial_iR\cdot\partial_i^2R)\Big)\circ\Delta_n
$$

$$
=\frac{3(f'')^2-f'f'''}{(f')^5}\bigg(\frac{p_if'}{p_0}\bigg)^3
$$

$$
+\frac{1}{f'}\bigg(\frac{3(p_0-p_i)\big(p_0p_i''-2(p_i')^2\big)f'}{p_0^3}+\frac{3(p_0-p_i)p_i'f''}{p_0^2}+\frac{p_if'''}{p_0}\bigg)
$$

$$
-\frac{f''}{(f')^3}\bigg(3\frac{p_if'}{p_0}\bigg(2\frac{p_i'(p_0-p_i)f'}{p_0^2}+\frac{p_if''}{p_0}\bigg)\bigg)
$$

$$
=\frac{3(p_0-p_i)\big(p_0p_i''-2(p_i')^2\big)}{p_0^3}+3\frac{p_i'(p_0-2p_i)(p_0-p_i)}{p_0^3}\cdot\frac{f''}{f'}
$$

$$
-\frac{p_i(p_0-p_i)}{p_0^3}\bigg(3p_i\bigg(\frac{f''}{f'}\bigg)^2-(p_0+p_i)\frac{f'''}{f'}\bigg),
$$

which completes the proof of assertion (3c). □

Lemma 9. *Let $n\geq 2$ and $f,g:I\to\mathbb{R}$ be differentiable functions on I with nonvanishing first derivatives and $i\in\{1,\dots,n\}$. Let $p=(p_1,\dots,p_n):I\to\mathbb{R}_+^n$ and $q=(q_1,\dots,q_n):I\to\mathbb{R}_+^n$ such that p_i and q_i are continuous on I. If $\partial_iA_{f,p}=\partial_iA_{g,q}$ holds on $\operatorname{diag}(I^n)$, then*

$$\frac{q_i}{q_0} = \frac{p_i}{p_0} \tag{41}$$

holds on I.

Proof. In view of Theorem 8, we have

$$\frac{q_i}{q_0} = \partial_i A_{g,q} \circ \Delta_n = \partial_i A_{f,p} \circ \Delta_n = \frac{p_i}{p_0}.$$

□

Lemma 10. *Let $n \geq 2$ and $f, g : I \to \mathbb{R}$ be twice differentiable functions on I with nonvanishing first derivatives. Let $p = (p_1, \dots, p_n) : I \to \mathbb{R}^n_+$ and $q = (q_1, \dots, q_n) : I \to \mathbb{R}^n_+$ be continuous functions on I and assume that, for all $i \in \{1, \dots, n\}$, (41) holds on I. Let $j, k \in \{1, \dots, n\}$. Then the following two assertions hold.*

(i) *Provided that $j \neq k$ and p_j, p_k, q_j, q_k are differentiable functions on I, if $\partial_j \partial_k A_{f,p} = \partial_j \partial_k A_{g,q}$ holds on $\operatorname{diag}(I^n)$, then there exists a nonzero constant γ such that, for all $i \in \{1, \dots, n\}$,*

$$q_i^2 g' = \gamma p_i^2 f' \tag{42}$$

is valid on I.

(ii) *Provided that $j = k$ and p_j, q_j are continuously differentiable functions on I, if $\partial_j^2 A_{f,p} = \partial_j^2 A_{g,q}$ holds on $\operatorname{diag}(I^n)$, then there exists a nonzero constant γ such that, for all $i \in \{1, \dots, n\}$, (42) is valid on I.*

Proof. From Lemma 9 we obtain that $q_i = r_0 p_i$ holds for all $i \in \{0, \dots, n\}$. Assume that $j \neq k$. Then, using assertion (2a) of Theorem 8, we have that

$$\begin{aligned} -\frac{(p_j p_k)'}{p_0^2} - \frac{p_j p_k}{p_0^2} \cdot \frac{f''}{f'} &= \partial_j \partial_k A_{f,p} \circ \Delta_n = \partial_j \partial_k A_{g,q} \circ \Delta_n \\ &= -\frac{(r_0^2 p_j p_k)'}{r_0^2 p_0^2} - \frac{r_0^2 p_j p_k}{r_0^2 p_0^2} \cdot \frac{g''}{g'}. \end{aligned}$$

Thus, after reduction, we get that

$$\frac{1}{2}\left(\frac{f''}{f'} - \frac{g''}{g'}\right) = \frac{r_0'}{r_0} \tag{43}$$

is valid on I. Hence, there exists $\gamma \in \mathbb{R} \setminus \{0\}$ such that

$$r_0 = \sqrt{\gamma \cdot \frac{f'}{g'}} \tag{44}$$

holds on I, whence, by Lemma 9 again, it follows that, for all $i \in \{1, \dots, n\}$, (42) is valid.

If $j = k$, then, applying assertion (2b) of Theorem 8, with a similar calculation we arrive at the same differential equation for r_0 and conclude that (42) is also valid in this case. □

For a three times differentiable function $f : I \to \mathbb{R}$ with a nonvanishing first derivative, we introduce its Schwarzian derivative $S(f) : I \to \mathbb{R}$ by the following formula:

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{45}$$

The following lemma plays a basic role in our proofs.

Lemma 11. *Let $f, g : I \to \mathbb{R}$ be three times differentiable functions on I with nonvanishing first derivatives. Then $S(f) = S(g)$ is valid on I if and only if there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive on I and*

$$g = \frac{af + b}{cf + d} \tag{46}$$

holds on I.

Proof. Denote the function $g \circ f^{-1}$ by φ. Then, by the well-known chain rule for the Schwarzian derivative (see [8, Chapter 10]),

$$S(g) = S(\varphi \circ f) = (S(\varphi) \circ f) \cdot (f')^2 + S(f).$$

Therefore, $S(f) = S(g)$ holds on I if and only if $S(\varphi) = 0$ is satisfied on $J = f(I)$. This latter equality however can be valid (again by [8]) if and only if φ is a Möbius transform on J, i.e., there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cx + d > 0$ and $\varphi(x) = \frac{ax+b}{cx+d}$ for $x \in J$. Substituting x by $f(u)$, these properties are equivalent to the positivity of the function $cf + d$ and the equality (46) on the interval I, respectively. □

Our first main result is contained in the following theorem. It completely characterizes the equality of two generalized Bajraktarević means with at least three variables.

Theorem 12. *Let $n \geq 3$ and $f, g : I \to \mathbb{R}$ be three times differentiable functions on I with nonvanishing first derivatives. Let $p = (p_1, \dots, p_n) : I \to \mathbb{R}_+^n$ be a continuous function on I and $q = (q_1, \dots, q_n) : I \to \mathbb{R}_+^n$. Assume that there exist $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$ such that p_i, p_j, p_k are differentiable functions on I. Then the following assertions are equivalent.*

(i) *The n-variable generalized Bajraktarević means $A_{f,p}$ and $A_{g,q}$ are identical on I^n.*
(ii) *There is an open subset U of I^n containing $\operatorname{diag}(I^n)$ such that the n-variable generalized Bajraktarević means $A_{f,p}$ and $A_{g,q}$ are identical on U.*
(iii) *The function q is continuous, the functions q_i, q_j, q_k are differentiable on I, and the equalities*

$$\begin{aligned} \partial_\ell A_{f,p} &= \partial_\ell A_{g,q} \qquad (\ell \in \{1, \dots, n-1\}), \\ \partial_i \partial_j A_{f,p} &= \partial_i \partial_j A_{g,q}, \end{aligned}$$

$$\partial_i\partial_j\partial_k A_{f,p} = \partial_i\partial_j\partial_k A_{g,q}$$

hold on $\operatorname{diag}(I^n)$.

(iv) *There exist* $a, b, c, d \in \mathbb{R}$ *with* $ad \neq bc$ *such that* $cf + d$ *is positive on* I,

$$g = \frac{af + b}{cf + d} \qquad \textit{and} \qquad q_\ell = (cf + d)p_\ell \qquad (\ell \in \{1, \ldots, n\})$$

hold on I.

Proof. The implication (i)⇒(ii) is obvious. Applying Lemma 6, it is also easy to see that assertion (iii) follows from statement (ii). The implication (iv)⇒(i) is a consequence of Theorem 5. It remains to prove that assertion (iii) implies statement (iv).

Without loss of generality, we can assume that $i = 1$, $j = 2$, and $k = 3$. One can easily see that, if $\partial_\ell A_{f,p} = \partial_\ell A_{g,q}$ holds for all $\ell \in \{1, \ldots, n-1\}$, then it is also valid for $\ell = n$. Using Lemma 9, we have that $q_\ell = r_0 p_\ell$ holds for all $\ell \in \{0, \ldots, n\}$. Hence, using the equality $q'_\ell = r'_0 p_\ell + r_0 p'_\ell$, we get that

$$\begin{aligned}
&2\frac{p_1p'_2p'_3 + p'_1p_2p'_3 + p'_1p'_2p_3}{p_0^3} + 2\frac{p_1p_2p'_3 + p_1p'_2p_3 + p'_1p_2p_3}{p_0^3}\cdot\frac{f''}{f'}\\
&\quad + \frac{p_1p_2p_3}{p_0^3}\left(3\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right)\\
&\quad = \partial_1\partial_2\partial_3 A_{f,p}\circ\Delta_n = \partial_1\partial_2\partial_3 A_{g,q}\circ\Delta_n\\
&\quad = 2\frac{r_0^3(p_1p'_2p'_3 + p'_1p_2p'_3 + p'_1p'_2p_3)}{r_0^3p_0^3} + 4\frac{r'_0r_0^2(p_1p_2p'_3 + p_1p'_2p_3 + p'_1p_2p_3)}{r_0^3p_0^3}\\
&\qquad + 6\frac{r_0(r'_0)^2p_1p_2p_3}{r_0^3p_0^3}\\
&\qquad + 6\frac{r_0^2r'_0p_1p_2p_3}{r_0^3p_0^3}\cdot\frac{g''}{g'} + 2\frac{r_0^3(p_1p_2p'_3 + p_1p'_2p_2 + p'_1p_2p_3)}{r_0^3p_0^3}\cdot\frac{g''}{g'}\\
&\qquad + \frac{r_0^3p_1p_2p_3}{r_0^3p_0^3}\left(3\left(\frac{g''}{g'}\right)^2 - \frac{g'''}{g'}\right).
\end{aligned}$$

Thus, applying (43) three times, after reduction, it follows that

$$\left(\frac{f''}{f'}\right)^2 - \frac{1}{3}\cdot\frac{f'''}{f'} = \frac{1}{2}\left(\frac{f''}{f'} - \frac{g''}{g'}\right)^2 + \left(\frac{f''}{f'} - \frac{g''}{g'}\right)\frac{g''}{g'} + \left(\frac{g''}{g'}\right)^2 - \frac{1}{3}\cdot\frac{g'''}{g'}$$

is valid on I. Whence we obtain that $S(f) = S(g)$ holds on I. Therefore, using Lemma 11, there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive and (46) holds on I. Substituting (46) into (44), we get that $r_0 = \delta(cf + d)$ holds on I, where $\delta := \sqrt{\frac{\gamma}{ad-bc}} > 0$. Therefore,

$$q_\ell = r_0p_\ell = (\delta cf + \delta d)p_\ell \qquad (\ell \in \{1, \ldots, n\}),$$

and

$$g = \frac{af+b}{cf+d} = \frac{\delta af + \delta b}{\delta cf + \delta d},$$

which proves that assertion (iv) holds with the constant vector $(\bar{a}, \bar{b}, \bar{c}, \bar{d}) := \delta \cdot (a, b, c, d)$. □

Our second main theorem has two variants concerning the regularity assumptions and characterizes the equality of generalized two-variable non-symmetric Bajraktarević means.

Theorem 13. *Let $f, g : I \to \mathbb{R}$ be three times differentiable functions on I with nonvanishing first derivatives. Let $p = (p_1, p_2) : I \to \mathbb{R}_+^2$ and $q = (q_1, q_2) : I \to \mathbb{R}_+^2$ such that $p_1 \neq p_2$. Assume that there exists $i \in \{1, 2\}$ such that one of the following regularity conditions is satisfied.*

(a) *p_i is twice continuously differentiable and p_{3-i} is continuous on I.*
(b) *p_i is twice differentiable and p_{3-i} is once differentiable on I.*

Then the following assertions are pairwise equivalent.

(i) *The two-variable generalized Bajraktarević means $A_{f,p}$ and $A_{g,q}$ are identical on I^2.*
(ii) *There is an open subset U of I^2 containing $\operatorname{diag}(I^2)$ such that the two-variable generalized Bajraktarević means $A_{f,p}$ and $A_{g,q}$ are identical on U.*
(iv) *There exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive on I,*

$$g = \frac{af+b}{cf+d}, \qquad q_1 = (cf+d)p_1, \qquad \text{and} \qquad q_2 = (cf+d)p_2$$

hold on I.

Proof. The implication (i)⇒(ii) is obvious. The implication (iv)⇒(i) is a consequence of Theorem 5. It remains to prove that (ii) implies statement (iv) in both regularity settings.

Applying Lemma 6, one can see that we have the following assertions, from statement (ii), under the regularity assumptions (a) and (b) of Theorem 13, respectively.

(iii) The function q_i is twice continuously differentiable, q_{3-i} is continuous on I, furthermore

$$\partial_i A_{f,p} = \partial_i A_{g,q}, \qquad \partial_i^2 A_{f,p} = \partial_i^2 A_{g,q}, \qquad \text{and} \qquad \partial_i^3 A_{f,p} = \partial_i^3 A_{g,q}$$

hold on $\operatorname{diag}(I^2)$.

(iii)' The function q_i is twice differentiable, q_{3-i} is once differentiable on I, furthermore

$$\partial_i A_{f,p} = \partial_i A_{g,q}, \qquad \partial_i^2 A_{f,p} = \partial_i^2 A_{g,q}, \qquad \text{and} \qquad \partial_i^2 \partial_{3-i} A_{f,p} = \partial_i^2 \partial_{3-i} A_{g,q}$$

hold on $\operatorname{diag}(I^2)$.

Without loss of generality, we can assume that $i = 1$. Then, using the first equation of (iii) or (iii)' and Lemma 9, we have $q_j = r_0 p_j$ for all $j \in \{0, 1, 2\}$. Due to the equality $r_0 = q_i/p_i$, it follows that r_0 is twice differentiable. Furthermore, by the second equation of assertion (iii) or (iii)' we have that (43) also holds by the second statement of Lemma 10. Observe that, differentiating (43), we can obtain that

$$\frac{r_0''}{r_0} = \frac{1}{4}\left(2\frac{f'''}{f'} - 2\frac{g'''}{g'} - \left(\frac{f''}{f'}\right)^2 + 3\left(\frac{g''}{g'}\right)^2 - 2\frac{f''}{f'}\cdot\frac{g''}{g'}\right). \tag{47}$$

Under the regularity assumption (a) of Theorem 13, the third equality in condition (iii) and formula (3c) of Theorem 8 yield that

$$\begin{aligned}
&-\frac{p_2\big(6(p_1')^2 - 3p_1''(p_1+p_2)\big)}{p_0^3} - 3\frac{p_1'p_2(p_1-p_2)}{p_0^3}\cdot\frac{f''}{f'} - 3\frac{p_1^2p_2}{p_0^3}\left(\frac{f''}{f'}\right)^2 \\
&\quad + \frac{p_1p_2(2p_1+p_2)}{p_0^3}\cdot\frac{f'''}{f'} \\
&= \partial_1^3 A_{f,p}\circ\Delta_2 = \partial_1^3 A_{g,q}\circ\Delta_2 \\
&= -\frac{r_0p_2\big(6r_0^2(p_1')^2 + 12r_0r_0'p_1p_1' + 6(r_0')^2p_1^2 - 3r_0(p_1+p_2)(r_0p_1'' + 2r_0'p_1' + r_0''p_1)\big)}{r_0^3p_0^3} \\
&\quad -3\frac{r_0^2p_2(r_0p_1' + r_0'p_1)(p_1-p_2)}{r_0^3p_0^3}\cdot\frac{g''}{g'} - 3\frac{r_0^3p_1^2p_2}{r_0^3p_0^3}\cdot\left(\frac{g''}{g'}\right)^2 \\
&\quad + \frac{r_0^3p_1p_2(2p_1+p_2)}{r_0^3p_0^3}\cdot\frac{g'''}{g'}.
\end{aligned} \tag{48}$$

Hence, from (48), using (43) and (47), it follows that

$$\begin{aligned}
&-3\frac{p_1'p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'} - \frac{g''}{g'}\right) - 3\frac{p_1^2p_2}{p_0^3}\left(\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right) \\
&+ \frac{p_1p_2(2p_1+p_2)}{p_0^3}\left(\frac{f'''}{f'} - \frac{g'''}{g'}\right) \\
&+3\frac{p_1'p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'} - \frac{g''}{g'}\right) + \frac{3}{2}\cdot\frac{p_1^2p_2}{p_0^3}\left(\frac{f''}{f'} - \frac{g''}{g'}\right)^2 \\
&-\frac{3}{4}\cdot\frac{p_1p_2(p_1+p_2)}{p_0^3}\left(2\frac{f'''}{f'} - 2\frac{g'''}{g'} - \left(\frac{f''}{f'}\right)^2 + 3\left(\frac{g''}{g'}\right)^2 - 2\frac{f''}{f'}\cdot\frac{g''}{g'}\right) \\
&+\frac{3}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'} - \frac{g''}{g'}\right)\frac{g''}{g'} = 0,
\end{aligned}$$

whence we get

$$\frac{1}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\frac{f'''}{f'} - \frac{g'''}{g'}\right) - \frac{3}{4}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right) = 0,$$

which simplifies to

$$\frac{1}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\big(S(f)-S(g)\big)=0. \tag{49}$$

Using that $p_1\neq p_2$, by continuity, it follows that there exists an open nonempty subinterval $J\subseteq I$ such that $p_1(x)\neq p_2(x)$ holds for $x\in J$. Therefore, the above equation implies that $S(f)=S(g)$ holds on J. Thus, by Lemma 11, there exist $a,b,c,d\in\mathbb{R}$ with $ad\neq bc$ such that $cf+d$ is positive and (46) holds on J and hence, by Theorem 7, this is also valid on I. Substituting (46) into (44), we get that $r_0=\delta(cf+d)$ holds on I, where $\delta:=\sqrt{\frac{\gamma}{ad-bc}}>0$. Therefore, with the same argument as at the end of the proof of Theorem 12, we can see that assertion (iv) holds with the constant vector $(\bar{a},\bar{b},\bar{c},\bar{d}):=\delta\cdot(a,b,c,d)$.

Under the assumption (b) of Theorem 13, the third equality of condition (iii)' and formula (3b) of Theorem 8 imply that

$$\begin{aligned}
&\frac{2p_1'p_2'(p_1-p_2)+p_2(2(p_1')^2-p_1''(p_1+p_2))}{p_0^3}+\frac{(2p_1'p_2+p_1p_2')(p_1-p_2)}{p_0^3}\cdot\frac{f''}{f'}\\
&\quad+\frac{p_1p_2(2p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}\right)^2\\
&\quad-\frac{p_1^2p_2}{p_0^3}\cdot\frac{f'''}{f'}=\partial_1^2\partial_2A_{f,p}\circ\Delta_n\\
&=\partial_1^2\partial_2A_{g,q}\circ\Delta_n=\frac{2r_0(r_0p_1'+r_0'p_1)(r_0p_2'+r_0'p_2)(p_1-p_2)}{r_0^3p_0^3}\\
&\quad+\frac{r_0p_2\big(2\big(r_0^2(p_1')^2+2r_0r_0'p_1p_1'+(r_0')^2p_1^2\big)-r_0(r_0p_1''+2r_0'p_1'+r_0''p_1)(p_1+p_2)\big)}{r_0^3p_0^3}\\
&\quad+\frac{r_0^2(2p_2(r_0p_1'+r_0'p_1)+p_1(r_0p_2'+r_0'p_2))(p_1-p_2)}{r_0^3p_0^3}\cdot\frac{g''}{g'}\\
&\quad+\frac{r_0^3p_1p_2(2p_1-p_2)}{r_0^3p_0^3}\left(\frac{g''}{g'}\right)^2-\frac{r_0^3p_1^2p_2}{r_0^3p_0^3}\cdot\frac{g'''}{g'}.
\end{aligned} \tag{50}$$

Hence, from (50), using (43) and (47), we arrive at

$$\begin{aligned}
&\frac{2p_1p_1'p_2-2p_1'p_2^2+p_1^2p_2'-p_1p_2p_2'}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)\\
&\quad+\frac{p_1p_2(2p_1-p_2)}{p_0^3}\left(\left(\frac{f''}{f'}\right)^2-\left(\frac{g''}{g'}\right)^2\right)\\
&\quad-\frac{p_1^2p_2}{p_0^3}\left(\frac{f'''}{f'}-\frac{g'''}{g'}\right)-\frac{2p_1p_1'p_2-2p_1'p_2^2+p_1^2p_2'-p_1p_2p_2'}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)\\
&\quad+\frac{1}{4}\cdot\frac{p_1p_2(p_1+p_2)}{p_0^3}\left(2\frac{f'''}{f'}-2\frac{g'''}{g'}-\left(\frac{f''}{f'}\right)^2+3\left(\frac{g''}{g'}\right)^2-2\frac{f''}{f'}\cdot\frac{g''}{g'}\right)\\
&\quad-\frac{1}{2}\cdot\frac{p_1p_2(2p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)^2-\frac{3}{2}\cdot\frac{p_1p_2(p_1-p_2)}{p_0^3}\left(\frac{f''}{f'}-\frac{g''}{g'}\right)\frac{g''}{g'}=0.
\end{aligned}$$

Therefore, we have that (49) holds, thus following a similar train of thought as above, we get assertion (iv). □

Theorem 14. *Let $f, g : I \to \mathbb{R}$ be six times differentiable functions on I with nonvanishing first derivatives. Let $p : I \to \mathbb{R}_+$ and $q : I \to \mathbb{R}_+$ be continuous functions on I and assume that p is three times differentiable on I. Then the following assertions are equivalent.*

(i) *The 2-variable generalized Bajraktarević means $A_{f,(p,p)}$ and $A_{g,(q,q)}$ are identical on I^2.*
(ii) *There is an open subset U of I^2 containing $\operatorname{diag}(I^2)$ such that the 2-variable generalized Bajraktarević means $A_{f,(p,p)}$ and $A_{g,(q,q)}$ are identical on U.*
(iii) *The function q is three times differentiable and the equalities*

$$\partial_1^j \partial_2^j A_{f,(p,p)} = \partial_1^j \partial_2^j A_{g,(q,q)} \qquad (j \in \{1, 2, 3\})$$

hold on $\operatorname{diag}(I^2)$.
(iv) *Either there exist $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$ such that $cf + d$ is positive on I,*

$$g = \frac{af + b}{cf + d} \qquad \text{and} \qquad q = (cf + d)p$$

hold on I or there exist two polynomials P and Q of at most second degree such that P and Q are positive on $f(I)$ and $g(I)$, respectively, and there exist two constants $\alpha, \beta \in \mathbb{R}$ such that

$$g = G^{-1} \circ (\alpha F \circ f + \beta), \qquad p = P^{-\frac{1}{2}} \circ f, \qquad \text{and} \qquad q = Q^{-\frac{1}{2}} \circ g$$

hold on I, where F and G denote a primitive function of $1/P$ and $1/Q$, respectively.

Proof. The implication (i)⇒(ii) is obvious. Applying Lemma 6, it is also easy to see that assertion (iii) follows from statement (ii). The proof of the implication (iii)⇒(iv) is based on a result of Losonczi [9] (who classified the solutions into 1+32 classes) and a recent characterization of the equality of two-variable (symmetric) Bajraktarević means with two-variable quasi-aritmetic means by Páles and Zakaria [13]. The proof of the implication (iv)⇒(i) is also described in the paper [13]. □

Acknowledgements

Open access funding provided by University of Debrecen (DE).

References

[1] Aczél, J., Daróczy, Z.: Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind. Publ. Math. Debrecen **10**, 171–190 (1963)

[2] Bajraktarević, M.: Sur une équation fonctionnelle aux valeurs moyennes. Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II **13**, 243–248 (1958)

[3] Bajraktarević, M.: Sur une généralisation des moyennes quasilinéaires. Publ. Inst. Math. **3**(17), 69–76 (1963)

[4] Daróczy, Z., Losonczi, L.: Über den Vergleich von Mittelwerten. Publ. Math. Debrecen **17**(1970), 289–297 (1971)

[5] Daróczy, Z., Páles, Zs: On comparison of mean values. Publ. Math. Debrecen **29**(1–2), 107–115 (1982)

[6] Gini, C.: Di una formula compressiva delle medie. Metron **13**, 3–22 (1938)

[7] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1934), (first edition), (1952) (second edition)

[8] Hille, E.: Ordinary Differential Equations in the Complex Domain. Wiley, New York (1976)

[9] Losonczi, L.: Equality of two variable weighted means: reduction to differential equations. Aequationes Math. **58**(3), 223–241 (1999)

[10] Losonczi, L.: Equality of two variable means revisited. Aequationes Math. **71**(3), 228–245 (2006)

[11] Losonczi, L.: Homogeneous non-symmetric means of two variables. Demonstratio Math. **40**(1), 169–180 (2007)

[12] Losonczi, L.: Homogeneous symmetric means of two variables. Aequationes Math. **74**(3), 262–281 (2007)

[13] Páles, Zs., Zakaria, A.: On the equality of Bajraktarević means to quasi-arithmetic means (2019) (submitted)

Richárd Grünwald and Zsolt Páles
Institute of Mathematics
University of Debrecen
Debrecen Pf. 400
4002
Hungary
e-mail: richard.grunwald96@gmail.com

Zsolt Páles
e-mail: pales@science.unideb.hu

Received: April 12, 2019
Revised: July 25, 2019