



Iterated joining means

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Dedicated to Professor Karol Baron on the occasion of his 70th birthday.

Abstract. Using a simple dynamical system generated by means M and N which are considered on adjacent intervals, we show how to find their joints, that is means extending both M and N . The procedure of joining is a local version of that presented in Jarczyk (Publ. Math. Debr. 91:235–246, 2017). Among joints are those semiconjugating some functions defined by the use of the so-called marginal functions of M and N .

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1. Introduction

Given any interval I a function $L: I \times I \rightarrow I$ is called a *mean on I* if

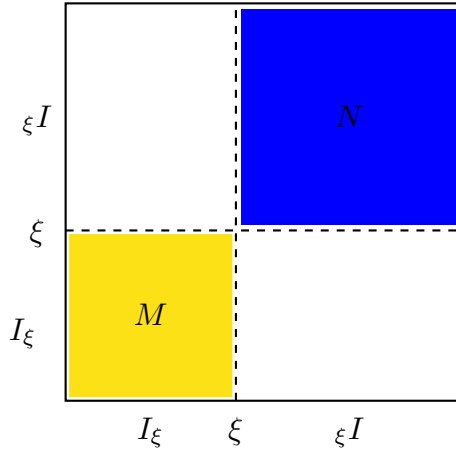
$$\min\{x, y\} \leq L(x, y) \leq \max\{x, y\}, \quad x, y \in I.$$

It is said to be *strict* if both the inequalities are sharp whenever $x \neq y$. Having an interior point ξ of I we split I into two intervals:

$$I_\xi = \{x \in I: x \leq \xi\} \quad \text{and} \quad {}_\xi I = \{x \in I: x \geq \xi\}. \quad (1.1)$$

The present paper is, in a sense, a continuation of [1, 2] where the following problem was studied:

For any means M and N on I_ξ and ${}_\xi I$, respectively, find their common extension L to a mean on I .



In other words we are looking for means L on I satisfying the condition

$$L|_{I_\xi \times I_\xi} = M \quad \text{and} \quad L|_{\xi I \times \xi I} = N.$$

Any such L , denoted also by $M \oplus N$, is called a *joint of M and N* .

A first trial to answer that question was given in the paper [1], whereas a more comprehensive description of possible solutions of the problem was presented in [2]. In both papers the authors were interested only in joints of M and N , which have all the values strictly depending on the values of M and N . This is done there by using the *marginal functions* $h_1, h_2 : I \rightarrow I$ defined by

$$h_1(x) = \begin{cases} M(x, \xi), & \text{if } x \in I_\xi, \\ N(x, \xi), & \text{if } x \in \xi I, \end{cases}$$

and

$$h_2(y) = \begin{cases} M(\xi, y), & \text{if } y \in I_\xi, \\ N(\xi, y), & \text{if } y \in \xi I. \end{cases}$$

In the next section we summarize some results of [2] underlying the global approach to the problem of joining means. The remaining Sects. 3–5 contain main outcomes of the present paper, where more general local notions and iteration give a wider possibility of joining means. As a conclusion we observe that some special limit joints of means M and N semiconjugate functions naturally coming from their marginal functions h_1 and h_2 .

2. Joiners and marginal joints

In what follows, besides intervals (1.1), we use others:

$$I_\xi^\circ := \{x \in I : x < \xi\} \quad \text{and} \quad \xi I^\circ := \{x \in I : \xi < x\}.$$

Moreover, given any functions $f: I_\xi \rightarrow I_\xi$ and $g: {}_\xi I \rightarrow {}_\xi I$, satisfying $f(\xi) = g(\xi)$, we define the sum $f \cup g: I \rightarrow I$ by

$$(f \cup g)(x) := \begin{cases} f(x), & \text{if } x \in I_\xi, \\ g(x), & \text{if } x \in {}_\xi I. \end{cases}$$

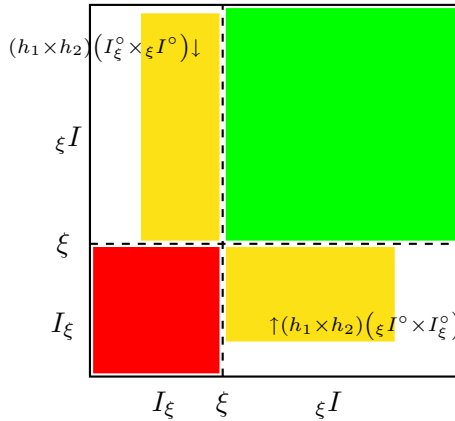
Fix means M and N on I_ξ and ${}_\xi I$, respectively, consider their marginal functions h_1 and h_2 , and denote by $h_1 \times h_2$ their product. Thus $h_1 \times h_2 : I \times I \rightarrow I \times I$ is defined by the equality

$$(h_1 \times h_2)(x, y) = (h_1(x), h_2(y)).$$

The notion of a joiner, recalled below, is basic for the method of joining means presented in [2]. We say that a multifunction

$$\mathbf{K} : (h_1 \times h_2)(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ) \rightarrow 2^I$$

is a *joiner of the pair* (M, N) if



$$(h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^{-1}(\mathbf{K}((h_1 \times h_2)(x, y))) \cap [x, y] \neq \emptyset, \quad (x, y) \in I_\xi^\circ \times {}_\xi I^\circ,$$

and

$$(h_2|_{{}_\xi I} \cup h_1|_{I_\xi})^{-1}(\mathbf{K}((h_1 \times h_2)(x, y))) \cap [y, x] \neq \emptyset, \quad (x, y) \in {}_\xi I^\circ \times I_\xi^\circ.$$

Notice that each joiner has non-empty values.

There are a lot of joiners. Some of them are trivial, for instance all those \mathbf{K} with values containing ξ . Two examples of less trivial but single-valued joiners originate in the paper [1] (for details see [2, Examples 1 and 2]). Example 4.1 below provides a description of a joiner with values which are not singletons. A simple condition sufficient for a multifunction to be a joiner can be found in Theorem 4.1.

Another notion, fundamental to the idea of joining means, is that of a marginal joint of means. The following result (see [2, Theorem 1]) serves as a starting point for its definition.

Theorem 2.1. *If \mathbf{K} is a joiner of the pair (M, N) , then the values of the multifunction $M \oplus_{\mathbf{K}} N$ defined as*

$$\left\{ \begin{array}{ll} \{M(x, y)\}, & \text{if } (x, y) \in I_{\xi} \times I_{\xi}, \\ (h_1|_{I_{\xi}} \cup h_2|_{\xi I})^{-1}(\mathbf{K}((h_1 \times h_2)(x, y))) \cap [x, y], & \text{if } (x, y) \in I_{\xi}^{\circ} \times \xi I^{\circ}, \\ (h_2|_{I_{\xi}} \cup h_1|_{\xi I})^{-1}(\mathbf{K}((h_1 \times h_2)(x, y))) \cap [y, x], & \text{if } (x, y) \in \xi I^{\circ} \times I_{\xi}^{\circ}, \\ \{N(x, y)\}, & \text{if } (x, y) \in \xi I \times \xi I, \end{array} \right.$$

are non-empty and its every selection is a mean on the interval I , extending both the means M and N .

For an extension of Theorem 2.1 to local joiners see Theorem 4.2. Any mean $M \oplus_{\mathbf{K}} N$, being a selection of $M \oplus_{\mathbf{K}} N$ constructed above, is called a *marginal \mathbf{K} -joint of the means M and N* .

3. A dynamical system induced by a pair of means

As previously let h_1 and h_2 be marginal functions for means M and N given on I_{ξ} and ξI , respectively. Joiners considered in the previous section are of global type: they have to be defined on the whole set $(h_1 \times h_2)(I_{\xi}^{\circ} \times \xi I^{\circ} \cup \xi I^{\circ} \times I_{\xi}^{\circ})$. However, it would be desirable and nice to join means making use of some local joiners, defined only in some small sets, for instance in neighbourhoods of the point (ξ, ξ) . One possible idea to fulfil that requirement is to use a simple dynamical system presented below, connected with the means M and N .

The product $h_1 \times h_2$ maps the square $I \times I$ into itself, so it can be iterated. To study properties of sequences of its iterates we begin with the following observation. Clearly,

$$h_1(\xi) = M(\xi, \xi) = \xi = N(\xi, \xi) = h_2(\xi),$$

hence

$$(h_1 \times h_2)(\xi, \xi) = (h_1(\xi), h_2(\xi)) = (\xi, \xi).$$

If $x \in I_{\xi}$, that is $x \in I$ and $x \leq \xi$, then $x \leq M(x, \xi) \leq \xi$, so $x - \xi \leq h_1(x) - \xi \leq 0$. Similarly if $x \in \xi I$ then $\xi \leq N(x, \xi) \leq x$, and thus $0 \leq h_1(x) - \xi \leq x - \xi$. Consequently,

$$0 \leq \frac{h_1(x) - \xi}{x - \xi} \leq 1, \quad x \in I \setminus \{\xi\}. \tag{3.1}$$

Analogously we get

$$0 \leq \frac{h_2(y) - \xi}{y - \xi} \leq 1, \quad y \in I \setminus \{\xi\}.$$

To obtain further properties of the sequence $((h_1 \times h_2)^n)_{n \in \mathbb{N}}$ of iterates of $h_1 \times h_2$ we need a little bit stronger assumptions.

Theorem 3.1. *Assume that the interval I is compact and the functions h_1, h_2 are continuous. Then*

- (i) $(h_1^n(I))_{n \in \mathbb{N}}$ and $(h_2^n(I))_{n \in \mathbb{N}}$ are decreasing (in the inclusion sense) sequences of compact intervals containing ξ ;
- (ii) $\bigcap_{n=1}^\infty h_1^n(I) = [x_1^-, x_1^+]$ and $\bigcap_{n=1}^\infty h_2^n(I) = [x_2^-, x_2^+]$, where

$$x_i^- = \inf \{x \in I : h_i(x) = x\} \text{ and } x_i^+ = \sup \{x \in I : h_i(x) = x\}, i = 1, 2;$$

- (iii) $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ is an attractor of the system $((h_1 \times h_2)^n)_{n \in \mathbb{N}}$: for every open set $G \subset \mathbb{R}^2$ containing the rectangle $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ there is an integer $n_0 \in \mathbb{N}$ such that

$$(h_1 \times h_2)^n(I \times I) \subset G, \quad n \geq n_0.$$

Proof. Assertion (i) follows directly from the compactness of I , the continuity of h_1, h_2 , and the obvious inclusions $h_1(I) \subset I$ and $h_2(I) \subset I$.

To prove (ii) we focus on the set $\bigcap_{n=1}^\infty h_1^n(I)$. Observe that it is a compact interval containing all fixed points of h_1 . It follows that $\xi \in [x_1^-, x_1^+] \subset \bigcap_{n=1}^\infty h_1^n(I)$. Suppose that $[x_1^-, x_1^+] \subsetneq \bigcap_{n=1}^\infty h_1^n(I)$, take any point $x_0 \in \bigcap_{n=1}^\infty h_1^n(I) \setminus [x_1^-, x_1^+]$ and assume, for instance, that $x_0 > x_1^+$. Define the function $H: I_\xi \rightarrow I_\xi$ by

$$H(x) = \begin{cases} h_1(x), & \text{if } \xi \leq x \leq x_1^+, \\ \max \{h_1(x), x_1^+\}, & \text{if } x_1^+ < x. \end{cases}$$

Clearly, H is continuous and, by (3.1) and the definition of x_1^+ ,

$$x_1^+ \leq H(x) < x, \quad x \in I \cap (x_1^+, +\infty).$$

Thus, by [3, Theorem 1.24], the sequence $(H^n(x))_{n \in \mathbb{N}}$ converges to x_1^+ uniformly on $I \cap (x_1^+, +\infty)$. Since $h_1(\xi I) \subset \xi I$ and $x_0 > \xi$ it follows that

$$x_0 \in \bigcap_{n=1}^\infty h_1^n(I) = \bigcap_{n=1}^\infty h_1^n(I_\xi) \subset \bigcap_{n=1}^\infty H^n(I_\xi).$$

Take any sequence $(x_n)_{n \in \mathbb{N}}$ of points of I_ξ° such that $x_0 = H^n(x_n)$ for all $n \in \mathbb{N}$. Then $x_1^+ < x_0 \leq x_n, n \in \mathbb{N}$, and, in view of the uniform convergence of $(H^n|_{I \cap (x_1^+, +\infty)})_{n \in \mathbb{N}}$ to x_1^+ , we see that

$$x_1^+ = \lim_{n \rightarrow \infty} H^n(x_n) = x_0,$$

a contradiction. Therefore $\bigcap_{n=1}^\infty h_1^n(I) = [x_1^-, x_1^+]$. Analogously we show the second equality and complete the proof of (ii).

To see (iii) we argue in a standard way. Take any open set $G \subset \mathbb{R}^2$ containing $[x_1^-, x_1^+] \times [x_2^-, x_2^+]$ and suppose to the contrary that $C_n := (h_1 \times h_2)^n(I \times I) \setminus G \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\{C_n\}_{n \in \mathbb{N}}$ is a family of non-empty

closed subsets of the compact space $I \times I$, with the finite intersection property, and thus $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. But this is impossible as

$$\begin{aligned} \bigcap_{n=1}^{\infty} C_n &= \bigcap_{n=1}^{\infty} (h_1 \times h_2)^n (I \times I) \setminus G = \bigcap_{n=1}^{\infty} (h_1^n \times h_2^n) (I \times I) \setminus G \\ &= \bigcap_{n=1}^{\infty} (h_1^n(I) \times h_2^n(I)) \setminus G = \left(\bigcap_{n=1}^{\infty} h_1^n(I) \times \bigcap_{n=1}^{\infty} h_2^n(I) \right) \setminus G \\ &= ([x_1^-, x_1^+] \times [x_2^-, x_2^+]) \setminus G = \emptyset. \end{aligned}$$

□

Remark 3.2. It is possible that the attractor $\bigcap_{n=1}^{\infty} h_1^n(I) \times \bigcap_{n=1}^{\infty} h_2^n(I)$ is really big, even the whole square $I \times I$. For example, if the means M and N are defined by

$$M(x, y) = \min \{x, y\} \quad \text{and} \quad N(x, y) = \max \{x, y\},$$

respectively, then $h_1(x) = h_2(x) = x$ for all $x \in I$, and thus $\bigcap_{n=1}^{\infty} h_1^n(I) = \bigcap_{n=1}^{\infty} h_2^n(I) = I$.

Such a situation cannot occur if we impose a slightly stronger condition on the means M and N .

Theorem 3.3. *Assume that the interval I is compact, the functions h_1, h_2 are continuous and ξ is their unique fixed point. Then $\bigcap_{n=1}^{\infty} h_1^n(I) = \{\xi\}$ and $\bigcap_{n=1}^{\infty} h_2^n(I) = \{\xi\}$, that is*

$$\lim_{n \rightarrow \infty} (h_1 \times h_2)^n (x, y) = (\xi, \xi)$$

uniformly in $I \times I$.

Proof. It is enough to observe that now $x_1^- = x_1^+ = x_2^- = x_2^+ = \xi$ and to apply Theorem 3.1. □

Remark 3.4. If the means M and N are strict then ξ is the unique fixed point of the functions h_1 and h_2 .

4. Local joiners and iterated joints

Now we are in a position to define a local joiner. Let h_1 and h_2 be the marginal functions for means M and N given on I_ξ and ${}_\xi I$, respectively, and let $p \in \mathbb{N}$. We say that a multifunction \mathbf{K} , defined on a subset of $I \times I$ which contains the set $(h_1 \times h_2)^p (I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$ and takes the values in 2^I , is a *p-joiner of the pair (M, N)* if

$$\left((h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^p \right)^{-1} (\mathbf{K} ((h_1 \times h_2)^p (x, y))) \cap [x, y] \neq \emptyset, \quad (x, y) \in I_\xi^\circ \times {}_\xi I^\circ,$$

and

$$\left((h_2|_{I_\xi} \cup h_1|_{\xi I})^p \right)^{-1} (\mathbf{K}((h_1 \times h_2)^p(x, y))) \cap [y, x] \neq \emptyset, \quad (x, y) \in {}_\xi I^\circ \times I_\xi^\circ.$$

Notice that the notion of 1-joiner coincides, in fact, with that of a joiner introduced in Sect. 2. Any p -joiner, where $p \in \mathbb{N}$, is called a *local joiner of the pair* (M, N) .

There is a lot of local joiners for the pair (M, N) . Some of them are trivial. This is the case, for instance, if all the values of \mathbf{K} contain ξ . A less trivial local joiner is provided by the following example, where $K(x, y)$ contains both x and y for all (x, y) from the domain of \mathbf{K} .

Example 4.1. If $x \in I_\xi$ then $x \in [\inf(h_1^p)^{-1}(\{h_1^p(x)\}), \xi]$. Similarly, if $y \in {}_\xi I$ then $y \in [\xi, \sup(h_2^p)^{-1}(\{h_2^p(y)\})]$. Therefore

$$h_1^p(x), h_2^p(y) \in h_1^p \left(\left[\inf(h_1^p)^{-1}(\{h_1^p(x)\}), \xi \right] \right) \cup h_2^p \left(\left[\xi, \sup(h_2^p)^{-1}(\{h_2^p(y)\}) \right] \right)$$

for all $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$. Analogously one can show that

$$h_2^p(y), h_1^p(x) \in h_2^p \left(\left[\inf(h_2^p)^{-1}(\{h_2^p(y)\}), \xi \right] \right) \cup h_1^p \left(\left[\xi, \sup(h_1^p)^{-1}(\{h_1^p(x)\}) \right] \right)$$

for all $(x, y) \in {}_\xi I^\circ \times I_\xi^\circ$. It follows that the formulas

$$\mathbf{K}(u, v) = h_1^p \left(\left[\inf(h_1^p)^{-1}(\{u\}), \xi \right] \right) \cup h_2^p \left(\left[\xi, \sup(h_2^p)^{-1}(\{v\}) \right] \right)$$

if $(u, v) \in (h_1 \times h_2)^p(I_\xi^\circ \times {}_\xi I^\circ)$ and

$$\mathbf{K}(u, v) = h_2^p \left(\left[\inf(h_2^p)^{-1}(\{v\}), \xi \right] \right) \cup h_1^p \left(\left[\xi, \sup(h_1^p)^{-1}(\{u\}) \right] \right)$$

if $(u, v) \in (h_1 \times h_2)^p({}_\xi I^\circ \times I_\xi^\circ)$, define a p -joiner of the pair (M, N) . Observe that $u, v \in \mathbf{K}(u, v)$ for all $(u, v) \in (h_1 \times h_2)^p(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$.

Assuming additionally that h_1 and h_2 are continuous and increasing (not necessarily strictly) we have

$$\mathbf{K}(u, v) = [u, h_1^p(\xi)] \cup [h_2^p(\xi), v] = [u, \xi] \cup [\xi, v] = [u, v]$$

if $(u, v) \in (h_1 \times h_2)^p(I_\xi^\circ \times {}_\xi I^\circ)$ and

$$\mathbf{K}(u, v) = [v, h_2^p(\xi)] \cup [h_1^p(\xi), u] = [v, \xi] \cup [\xi, u] = [v, u]$$

if $(u, v) \in (h_1 \times h_2)^p({}_\xi I^\circ \times I_\xi^\circ)$, that is

$$\mathbf{K}(u, v) = [\min\{u, v\}, \max\{u, v\}]$$

for all $(u, v) \in (h_1 \times h_2)^p(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$.

The following result gives a simple condition sufficient for a multifunction to be a local joiner for a given pair of means.

Theorem 4.1. *If the functions h_1, h_2 are continuous then every multifunction $\mathbf{K} : (h_1 \times h_2)^p(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ) \rightarrow 2^I$, satisfying the condition*

$$\mathbf{K}(u, v) \cap [\min\{u, v\}, \max\{u, v\}] \neq \emptyset$$

for all $(u, v) \in (h_1 \times h_2)^p(I_\xi^\circ \times {}_\xi I^\circ \cup {}_\xi I^\circ \times I_\xi^\circ)$, is a p -joiner of the pair (M, N) .

Proof. Take any pair $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$ and put $u = h_1^p(x)$ and $v = h_2^p(y)$. Then $u \leq x \leq v$, so we can find a $w \in [u, v]$ such that $w \in \mathbf{K}(u, v)$. Since h_1, h_2 are continuous and

$$h_1^p(x) = u \leq w \leq v = h_2^p(y)$$

it follows that $w \in (h_1|_{I_\xi} \cup h_2|_{I_\xi})^p(z)$ for a $z \in [x, y]$. Consequently, the set

$$((h_1|_{I_\xi} \cup h_2|_{I_\xi})^p)^{-1}(\mathbf{K}((h_1 \times h_2)^p(x, y))) \cap [x, y]$$

contains z . Similarly one can show that the set

$$((h_2|_{I_\xi} \cup h_1|_{I_\xi})^p)^{-1}(\mathbf{K}((h_1 \times h_2)^p(x, y))) \cap [y, x]$$

is non-empty for all $(x, y) \in {}_\xi I^\circ \times I_\xi^\circ$. □

Now we generalize Theorem 2.1 showing how, making use of a given local joiner, one can find a family of means joining two means given on adjacent intervals.

Theorem 4.2. *If \mathbf{K} is a p -joiner of the pair (M, N) , then the values of the multifunction $(M \oplus_{\mathbf{K}} N)_p$ defined as*

$$\begin{cases} \{M(x, y)\}, & \text{if } (x, y) \in I_\xi \times I_\xi, \\ ((h_1|_{I_\xi} \cup h_2|_{{}_\xi I})^p)^{-1}(\mathbf{K}((h_1 \times h_2)^p(x, y))) \cap [x, y], & \text{if } (x, y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ ((h_2|_{I_\xi} \cup h_1|_{{}_\xi I})^p)^{-1}(\mathbf{K}((h_1 \times h_2)^p(x, y))) \cap [y, x], & \text{if } (x, y) \in {}_\xi I^\circ \times I_\xi^\circ, \\ \{N(x, y)\}, & \text{if } (x, y) \in {}_\xi I \times {}_\xi I, \end{cases}$$

are non-empty and its every selection is a mean on the interval I , extending both the means M and N .

Proof. The multifunction $(M \oplus_{\mathbf{K}} N)_p$ defined on $I \times I$ and, by the definition of joint, all its values are non-empty. Taking any of its selections $(M \oplus_{\mathbf{K}} N)_p$ we see that

$$(M \oplus_{\mathbf{K}} N)_p \begin{cases} = \{M(x, y)\}, & \text{if } (x, y) \in I_\xi \times I_\xi, \\ \in [x, y], & \text{if } (x, y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ \in [y, x], & \text{if } (x, y) \in {}_\xi I^\circ \times I_\xi^\circ, \\ = \{N(x, y)\}, & \text{if } (x, y) \in {}_\xi I \times {}_\xi I, \end{cases}$$

and thus $(M \oplus_{\mathbf{K}} N)_p$ is a mean extending M and N . □

Any mean $(M \oplus_{\mathbf{K}} N)_p$ constructed as above is called an *iterated marginal \mathbf{K} -joint* of the pair (M, N) of degree p .

The next result indicates an important connection between \mathbf{K} -joints of degrees i and $i + 1$.

Theorem 4.3. *Assume that the functions h_1 and h_2 take the value ξ only at ξ . If \mathbf{K} is a p -joiner of the pair (M, N) , then*

$$(h_1|_{I_\xi} \cup h_2|_{\xi I})^{-1} ((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))) \subset (M \oplus_{\mathbf{K}} N)_{i+1}(x, y),$$

for all $(x, y) \in I_\xi^\circ \times \xi I^\circ$ and $i \geq p$, and

$$(h_2|_{I_\xi} \cup h_1|_{\xi I})^{-1} ((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))) \subset (M \oplus_{\mathbf{K}} N)_{i+1}(x, y),$$

for all $(x, y) \in \xi I^\circ \times I_\xi^\circ$ and $i \geq p$.

Proof. Fix any integer $i \geq p$. Then \mathbf{K} is also an i -joiner of the pair (M, N) . Take an arbitrary point $(x, y) \in I_\xi^\circ \times \xi I^\circ$. Then $(h_1 \times h_2)(x, y) = (h_1(x), h_2(y)) \in I_\xi \times \xi I$ and, since h_1 and h_2 do not take the value ξ besides ξ , we see that actually $(h_1 \times h_2)(x, y) \in I_\xi^\circ \times \xi I^\circ$. Let z be any element of the set

$$(h_1|_{I_\xi} \cup h_2|_{\xi I})^{-1} ((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))).$$

Then, by the definition of $(M \oplus_{\mathbf{K}} N)_i$, we get

$$(h_1|_{I_\xi} \cup h_2|_{\xi I})(z) \in \left((h_1|_{I_\xi} \cup h_2|_{\xi I})^i \right)^{-1} \left(K \left((h_1 \times h_2)^i (h_1(x), h_2(y)) \right) \right) \cap [h_1(x), h_2(y)]$$

that is

$$z \in \left((h_1|_{I_\xi} \cup h_2|_{\xi I})^{i+1} \right)^{-1} \left(K \left((h_1 \times h_2)^{i+1} (h_1(x), h_2(y)) \right) \right) \cap [h_1(x), h_2(y)] \subset (M \oplus_{\mathbf{K}} N)_{i+1}(x, y),$$

which completes the proof of the first inclusion. The second one can be verified analogously. □

Remark 4.4. If the means M and N are strict then the functions h_1 and h_2 take the value ξ only at ξ .

Now, using Theorem 4.3, we define a particular sequence $((M \oplus_{\mathbf{K}} N)_i)_{i \geq p}$ of \mathbf{K} -joints of the pair (M, N) . Assume that h_1 and h_2 are continuous, increasing, and take the value ξ only at ξ . Take any selection $(M \oplus_{\mathbf{K}} N)_p$ for the multifunction $(M \oplus_{\mathbf{K}} N)_p$. Fix an integer $i \geq p$ and assume that we have chosen a selection $(M \oplus_{\mathbf{K}} N)_i$ of $(M \oplus_{\mathbf{K}} N)_i$. Observe that

$$\min \{h_1(x), h_2(y)\} \leq (M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y)) \leq \max \{h_1(x), h_2(y)\}$$

for all $x, y \in I$, so, by the continuity of h_1 and h_2 ,

$$(M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y)) \in h_1(I_\xi) \cup h_2(\xi I), \quad (x, y) \in I_\xi \times \xi I,$$

and

$$(M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y)) \in h_2(I_\xi) \cup h_1(\xi I), \quad (x, y) \in \xi I \times I_\xi.$$

Consequently, the formula

$$\left\{ \begin{array}{ll} \{M(x, y)\}, & \text{if } (x, y) \in I_\xi \times I_\xi, \\ (h_1|_{I_\xi} \cup h_2|_{\xi I})^{-1}((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))), & \text{if } (x, y) \in I_\xi^\circ \times \xi I^\circ, \\ (h_2|_{I_\xi} \cup h_1|_{\xi I})^{-1}((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))), & \text{if } (x, y) \in \xi I^\circ \times I_\xi^\circ, \\ \{N(x, y)\}, & \text{if } (x, y) \in \xi I \times \xi I, \end{array} \right. \quad (4.1)$$

defines a multifunction with non-empty values. Moreover, the increase of h_1 and h_2 implies that

$$(h_1|_{I_\xi} \cup h_2|_{\xi I})^{-1}((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))) \subset [x, y], \quad (x, y) \in I_\xi^\circ \times \xi I^\circ,$$

and

$$(h_2|_{I_\xi} \cup h_1|_{\xi I})^{-1}((M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y))) \subset [y, x], \quad (x, y) \in \xi I^\circ \times I_\xi^\circ.$$

Therefore, any selection of the multifunction (4.1) is a mean on I , extending M and N . According to Theorem 4.3 it is also a selection of $(M \oplus_{\mathbf{K}} N)_{i+1}$. Denote it by $(M \oplus_{\mathbf{K}} N)_{i+1}$. In such a way we obtain the announced sequence $((M \oplus_{\mathbf{K}} N)_{i \geq p})$ of single-valued \mathbf{K} -joints of the pair (M, N) . Observe that we have

$$(h_1|_{I_\xi} \cup h_2|_{\xi I})((M \oplus_{\mathbf{K}} N)_{i+1}(x, y)) = (M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y)) \quad (4.2)$$

for all $(x, y) \in I_\xi^\circ \times \xi I^\circ$ and

$$(h_2|_{I_\xi} \cup h_1|_{\xi I})((M \oplus_{\mathbf{K}} N)_{i+1}(x, y)) = (M \oplus_{\mathbf{K}} N)_i((h_1 \times h_2)(x, y)) \quad (4.3)$$

for all $(x, y) \in \xi I^\circ \times I_\xi^\circ$. Moreover, putting $L = (M \oplus_{\mathbf{K}} N)_p$, we get

$$(M \oplus_{\mathbf{K}} N)_{p+i}(x, y) \in \left\{ \begin{array}{ll} \left((h_1|_{I_\xi} \cup h_2|_{\xi I})^i \right)^{-1} \left(L \left((h_1 \times h_2)^i(x, y) \right) \right), & \text{if } (x, y) \in I_\xi^\circ \times \xi I^\circ, \\ \left((h_2|_{I_\xi} \cup h_1|_{\xi I})^i \right)^{-1} \left(L \left((h_1 \times h_2)^i(x, y) \right) \right), & \text{if } (x, y) \in \xi I^\circ \times I_\xi^\circ, \end{array} \right.$$

for all $i \in \mathbb{N}$.

5. Limit joints

In the final section, applying Theorem 4.3 and the procedure described afterwards, we introduce joints satisfying some functional equations of semiconjugacy. To this aim we accept all assumptions of Theorem 4.3, that is we let h_1 and h_2 be the marginal functions for a pair (M, N) of means given on I_ξ and

ξI , respectively, and \mathbf{K} be a p -joiner of (M, N) for some $p \in \mathbb{N}$, and we assume that h_1 and h_2 take the value ξ only at ξ .

Fix any sequence $((M \oplus_{\mathbf{K}} N)_i)_{i \geq p}$ of selections of multifunctions given by (4.1). The formulas

$$\phi_{\mathbf{K}}^-(x, y) = \liminf_{i \rightarrow \infty} (M \oplus_{\mathbf{K}} N)_i(x, y) \tag{5.1}$$

and

$$\phi_{\mathbf{K}}^+(x, y) = \limsup_{i \rightarrow \infty} (M \oplus_{\mathbf{K}} N)_i(x, y) \tag{5.2}$$

define means on $I \times I$ extending both M and N .

Theorem 5.1. *Assume that the functions h_1 and h_2 are continuous and take the value ξ only at ξ . Let \mathbf{K} be a p -joiner of the pair (M, N) and define means $\phi_{\mathbf{K}}^-$ and $\phi_{\mathbf{K}}^+$ by (5.1) and (5.2), respectively. Then the functions $\phi_{\mathbf{K}}^-|_{I_\xi^\circ \times_\xi I^\circ}$ and $\phi_{\mathbf{K}}^+|_{I_\xi^\circ \times_\xi I^\circ}$ satisfy the functional equation*

$$\phi(h_1(x), h_2(y)) = h_{12}(\phi(x, y)), \tag{5.3}$$

where $h_{12} = h_1|_{I_\xi} \cup h_2|_{\xi I}$, whereas $\phi_{\mathbf{K}}^-|_{\xi I^\circ \times I_\xi^\circ}$ and $\phi_{\mathbf{K}}^+|_{\xi I^\circ \times I_\xi^\circ}$ are solutions of the equation

$$\phi(h_1(x), h_2(y)) = h_{21}(\phi(x, y)), \tag{5.4}$$

where $h_{21} = h_2|_{I_\xi} \cup h_1|_{\xi I}$.

Proof. It is enough to observe that, since h_1 and h_2 take the value ξ only at ξ , we have

$$\text{if } (x, y) \in I_\xi^\circ \times_\xi I^\circ \text{ then } (h_1 \times h_2)(x, y) \in I_\xi^\circ \times_\xi I^\circ$$

and

$$\text{if } (x, y) \in \xi I^\circ \times I_\xi^\circ \text{ then } (h_1 \times h_2)(x, y) \in \xi I^\circ \times I_\xi^\circ,$$

and tend with i to infinity in (4.2) and (4.3) taking into account the assumed continuity of h_1 and h_2 . □

It seems to be interesting to describe the solutions $\phi: I_\xi^\circ \times_\xi I^\circ \rightarrow I$ of functional equation (5.3) and solutions $\phi: \xi I^\circ \times I_\xi^\circ$ of (5.4) in the class of functions satisfying the condition

$$x \leq \phi(x, y) \leq y, \quad (x, y) \in I_\xi^\circ \times_\xi I^\circ,$$

and

$$y \leq \phi(x, y) \leq x, \quad (x, y) \in \xi I^\circ \times I_\xi^\circ,$$

respectively. Notice that both of the equations express the semiconjugacy of $h_1 \times h_2$ and either h_{12} or h_{21} on suitable rectangles.

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