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Category-measure duality: convexity, midpoint convexity and Berz sublinearity

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Abstract. Category-measure duality concerns applications of Baire-category methods that have measure-theoretic analogues. The set-theoretic axiom needed in connection with the Baire category theorem is the Axiom of Dependent Choice, DC rather than the Axiom of Choice, AC. Berz used the Hahn–Banach theorem over $\mathbb{Q}$ to prove that the graph of a measurable sublinear function that is $\mathbb{Q}_+$-homogeneous consists of two half-lines through the origin. We give a category form of the Berz theorem. Our proof is simpler than that of the classical measure-theoretic Berz theorem, our result contains Berz's theorem rather than simply being an analogue of it, and we use only DC rather than AC. Furthermore, the category form easily generalizes: the graph of a Baire sublinear function defined on a Banach space is a cone. The results are seen to be of automatic-continuity type. We use Christensen Haar null sets to extend the category approach beyond the locally compact setting where Haar measure exists. We extend Berz's result from Euclidean to Banach spaces, and beyond. Passing from sublinearity to convexity, we extend the Bernstein–Doetsch theorem and related continuity results, allowing our conditions to be 'local'—holding off some exceptional set.

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1. Introduction

The Berz theorem of our title is his characterization of a function $S : \mathbb{R} \to \mathbb{R}$ which is *sublinear,* that is—it is *subadditive* ([50, Ch. 3], [87]):

$$S(u + v) \leqslant S(u) + S(v),$$

and *homogeneous* with respect to non-negative integer scaling. Following Berz [5], we call S *sublinear on a set* Σ if S is subadditive and

$$S(nx) = nS(x) \text{ for } x \in \Sigma, n = 0, 1, 2, \dots$$

(here we do not require $nx \in \Sigma$), equivalently, if Σ is closed under non-negative rational scaling,

$$S(qx) = qS(x) \text{ for } x \in \Sigma, q \in \mathbb{Q}_+ = \mathbb{Q} \cap [0, \infty);$$

in words, S is positively $\mathbb{Q}$-homogeneous on Σ and $S(0) = 0$. An important class of functions with these two properties but with a more general domain occurs in mathematical finance—the *coherent risk measures* introduced by Artzner et al. [2] (cf. §6.5); for textbook treatments see [67], [36, 4.1]. In Sect. 4 we characterize such functions in the category setting when the domain is a Banach space. Working in a locally convex Fréchet space and under various axiomatic assumptions Ajtai [1], Wright [102], and Garnir [41], motivated by semi-norm considerations, study the continuity of a subadditive function S with the property $S(2x) = 2S(x)$.

Berz used the Hahn–Banach theorem over $\mathbb{Q}$ to prove that the graph of a (Lebesgue) measurable sublinear function consists of two half-lines through the origin ([62, §16.4, 5]; cf. [8]). Recall that in a topological space X, a subset H is *Baire* (has the *Baire property, BP*) if $H = (V \backslash M_V) \cup M_H$ for some open set V and meagre sets M_V, M_H in the sense of the topology on X; similarly a function $f : X \to \mathbb{R}$ is *Baire* if preimages of (Euclidean) open subsets of $\mathbb{R}$ are Baire subsets in the topology of X. Our first result is the Baire version of Berz's theorem on the line. Below $\mathbb{R}_\pm$ denotes the non-negative and non-positive half-lines.

Theorem 1B (ZF + DC). *For $S : \mathbb{R} \to \mathbb{R}$ sublinear and Baire, there are $c^\pm \in \mathbb{R}$ such that*

$$S(x) = c^\pm x, \text{ for } x \in \mathbb{R}_\pm.$$

As we shall see in Sect. 3, Theorem 1B implies the classical Berz theorem as a corollary:

Theorem 1M (ZF + DC, containing Berz [5] with AC). *For $S : \mathbb{R} \to \mathbb{R}$ sublinear and measurable, there are $c^\pm \in \mathbb{R}$ such that*

$$S(x) = c^\pm x, \text{ for } x \in \mathbb{R}_\pm.$$

For the relative strengths of the usual Hahn–Banach theorem, HB and the Axiom of Choice, AC, see [81,82]; Pincus and Solovay [83] provide a model of set theory in which the Axiom of Dependent Choice(s), DC holds but HB fails. HB is derivable from the Prime Ideal Theorem, PI, an axiom weaker than AC: for literature see again [81,82]; moreover, HB for separable normed spaces is not provable from DC [32, Cor. 4]. For more on this (with references), see Appendix 1 of the fuller arXiv version of this paper.

Theorems 1B and 1M may be combined, into 'Theorem 1(B+M)', say. Following necessary topological preliminaries (Lemma S, Theorem BL; Steinhaus–Weil property) in Sect. 2, the two cases are proved together in Sect. 3 bi-topologically, by switching between the two relevant density topologies of Sect. 2 ([13,17,22,75]). Here we also prove Theorem 2 (local boundedness for subadditive functions) and Theorem 3, the corresponding continuity result.

We introduce universal measurability (used in Sect. 4 in defining Christensen's notion of Haar null sets—in contexts where there may be no Haar measure—[28,29]), and use this to note a variant on Theorem 2, Theorem 2H ('H for Haar').

The sector between the lines $c^{\pm}x$ in the upper half-plane is a two-dimensional cone. This suggests the generalization to Banach spaces that we prove in Sect. 4 (Theorems 4B, 4M, 4F—'F for F-space').

The results above for the Baire/measurable functions on $\mathbb{R}$ are to be expected: they follow from the classical Bernstein–Doetsch continuity theorem for locally bounded, midpoint convex functions on normed vector spaces, to which we turn in Sect. 5 (see e.g. [62, 6.4.2] quoted for $\mathbb{R}^d$, but its third proof there applies more generally, as does Theorem B below, also originally for $\mathbb{R}^d$; see also [47, Ch. III]), once one proves their local boundedness (Sect. 3, Th. 2), since a sublinear function is necessarily midpoint convex. Indeed, by $\mathbb{Q}$-homogeneity and subadditivity,

$$f\left(\frac{1}{2}(x+y)\right) = \frac{1}{2}\left(f(x+y)\right) \leqslant \frac{1}{2}\left(f(x)+f(y)\right)$$

(we remark that the inequality here generalizes to Jensen's for *dyadic* rational convex combinations). We handle the Berz sublinear case first (in Sect. 3), as the arguments are simpler, and turn to mid-convexity matters in Sect. 5, where we prove the following two results (for topological and convexity terminology see respectively Sects. 2 and 5).

Theorem M (Mehdi's Theorem, [68, Th. 4]; cf. [102]). *For a Banach space X, if $S : X \to \mathbb{R}$ is midpoint convex and Baire, then S is continuous.*

Theorem FS (cf. [35]). *For a Banach space X, if $S : X \to \mathbb{R}$ is midpoint convex and universally measurable (or even $\mathcal{H}$-measurable), then S is continuous.*

For the Banach context both there and in Sect. 4, we rely on the following dichotomy result, Theorem B, especially on its second assertion, which together with an associated Corollary B in Sect. 4 (on boundedness), enables passage from a general Banach space to a separable one (wherein the Christensen theory of Haar null sets is available). See [24] and Appendix 2 of the fuller arXiv version of this paper.

Theorem B (Blumberg's Dichotomy Theorem, [24, Th. 1]; cf. [92]). *For any normed vector space X and $S : X \to \mathbb{R}$ midpoint convex: for any $x_0 \in X$, either S is not continuous at $x_0 \in X$, or there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_0 with $\{S(x_n)\}_{n\in\mathbb{N}}$ unbounded above.*

In particular, for a Banch space X, if for any closed separable subspace $B \subseteq X$ the restriction $S|B$ is continuous (for instance $S|B$ is locally bounded above on B), then S is continuous.

In Sect. 5 we switch to a form of mid-convexity that is assumed to hold only on a co-meagre or co-null set (so on an open set of a *density topology*—see Sect. 2); we term this *weak midpoint convexity* (on a set Σ, say), and use it to show in particular that a Baire/measurable midpoint convex function is continuous and convex. We close in Sect. 6 with some complements.

Theorem 1B (under the usual tacit assumption ZF + AC) was given in [20, Th. 5]. The results imply the classical results that Baire/measurable additive functions are linear (see [16] for historical background); indeed, an additive function $A(.)$ is sublinear and $A(-x) = -A(x)$, so $c^+ = -c^-$.

The primacy of category within *category-measure* duality is one of our two main themes here. This is something we have emphasised before [13,16, 17]; with the key words transposed, Oxtoby [79] calls this *measure-category* duality, so from a different viewpoint—he has next to no need of Steinhaus's theorem ([79, Ch. 4], cf. [75]), which is crucial for us. Our second main theme, new here, is AC versus DC. As so much of the extensive relevant background is still somewhat scattered, we have summarized what we need in detail in an appendix (which has its own separate references); this may be omitted by the expert (or uninterested) reader, and so is included only in the fuller arXiv version of this paper (as Appendix 1).

Without further comment, we work with ZF + DC, rather than ZF + AC, throughout the paper. It is natural that DC should dominate here. DC suffices for the *common parts* of the Baire category and Lebesgue measure cases: for the first, see Blair [23], and for the second, see Solovay (Appendix 1.3; [95, p. 25]). For the *contrasts*—or 'wedges'—between them, see Appendix 1.5. It is here that further set-theoretic assumptions become crucial; in brief, *measure theory needs stronger assumptions.*

2. Topological preliminaries: Steinhaus–Weil property

Fundamental for our purposes is the *Steinhaus–Weil property*[1] [21,22]—that the difference set $A - A$ has a non-empty interior for any non-negligible set A with the Baire property, briefly: *Baire set*—as opposed to Baire *topology*. We focus on *Baire* topological spaces on which the Steinhaus–Weil theorem holds. (See [95, Remark to Th. 6.1] for failure of the Steinhaus–Weil property in a group; cf. [61,88] for extensions of this property.) This is just what is needed to make the infinite combinatorics used in our proofs work.

Call a an (outer) *Lebesgue-density* point of a set A if $\lim_{\delta\downarrow 0} |A \cap (a - \delta, a + \delta)|/2\delta = 1$, where $|S|$ is the outer measure of S; the *Lebesgue density theorem* asserts that almost all points of a set are density points. (On this point the

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[1] Initially, as in the Steinhaus–Piccard–Pettis context, this concerns $\mathbb{R}$; the wider context is due to Weil and concerns (Haar) measurability in locally compact groups [99, p. 50], cf. [45]. These distinctions blur in our bitopological context.

survey [27] is a classic. For further background see [100] and literature cited there; cf. the recent [21].) By analogy, say that a is a *Baire-density* point of A if $V \setminus A$ is meagre, for some open neighbourhood V of a; if A is Baire, then it is immediate from BP that, except for a meagre set, all points of A are Baire density points. Each of the category and measure notions of density defines a *density topology* (denoted respectively $\mathcal{D}_{\mathcal{B}}/\mathcal{D}_{\mathcal{L}}$—with $\mathcal{L}$ denoting Lebesgue measurable sets), in which a set W is *density-open* if all its points are category/measure density points of W, the latter case was introduced by Goffman and his collaborators—see [42,43]. Both refine the usual Euclidean topology, $\mathcal{E}$; see [21] for properties common to both topologies. We call meagre/null sets *negligible*, and say that *quasi all* points of a set have a property if, but for a negligible subset, all have the property. These negligible sets form a σ-ideal; see Fremlin [38], Lukes et al. [64], Wilczyński [21,100] for background, and also [12,74]. Below (for use in Sect. 4) we consider a further σ-ideal: the *(left) Haar null sets* (Solecki [94]–[96]) of a Banach space and by extension use the same language of negligibles there. The corresponding density topologies may also be studied via *Hashimoto* topologies (cf. [3,48], [64, 1C], [30]), obtained by declaring as basic open the sets of the form $U \setminus N$ with $U \in \mathcal{E}$ and N the appropriate negligible. (That these sets, even under DC, form a topology follows from $\mathcal{E}$ being second countable—cf. [55, 4.2] and [21].)

The definition above of a Baire-density point may of course be repeated verbatim in the context of any topology $\mathcal{T}$ on any set X by referring to $\mathcal{B}(\mathcal{T})$, the Baire sets of $\mathcal{T}$. In particular, working with $\mathcal{T} = \mathcal{D}_{\mathcal{L}}$ in place of $\mathcal{E}$ we obtain a topology $\mathcal{D}_{\mathcal{B}}(\mathcal{D}_{\mathcal{L}})$. Since $\mathcal{B}(\mathcal{D}_{\mathcal{L}}) = \mathcal{L}$ (see [58, 17.47] and [13]),

$$\mathcal{D}_{\mathcal{B}} = \mathcal{D}_{\mathcal{B}}(\mathcal{E}), \qquad \mathcal{D}_{\mathcal{L}} = \mathcal{D}_{\mathcal{B}}(\mathcal{D}_{\mathcal{L}}).$$

Lemma S (Multiplicative Sierpiński Lemma; [12, Lemma S], cf. [91]). *For A, B Baire/measurable in $(0, \infty)$ with respective density points (in the category/measure sense) a, b, and for $n = 1, 2, \ldots$ there exist positive rationals q_n and points a_n, b_n converging (metrically) to a, b through A, B respectively so that $b_n = q_n a_n$.*

Proof. For $n = 1, 2, \ldots$ and the consecutive values $\varepsilon = 1/n$, the sets $B_\varepsilon(a) \cap A$ and $B_\varepsilon(b) \cap B$ are Baire/measurable non-negligible. So by Steinhaus's theorem (see e.g. [62, §3.7], [7, Th. 1.1.1]; cf. [16]), the set $[B \cap B_\varepsilon(b)] \cdot [A \cap B_\varepsilon(a)]^{-1}$ contains interior points, and so in particular a rational point q_n. Thus for some $a_n \in B_\varepsilon(a) \cap A$ and $b_n \in B_\varepsilon(b) \cap B$ we have $q_n = b_n a_n^{-1} > 0$, and as $|a - a_n| < 1/n$ and $|b - b_n| < 1/n$, $a_n \to a, b_n \to b$. $\square$

Remark. The result above is a consequence of the Steinhaus–Weil property regarded as a corollary of the Category Interior Theorem ([14, Th. 4.4]; cf. [44,45]). The latter, applied to the topology $\mathcal{D}$ that is either of the above two topologies $\mathcal{D}_{\mathcal{B}}/\mathcal{D}_{\mathcal{L}}$, asserts that $U - V$ or UV^{-1} is an $\mathcal{E}$-open nhd (of the relevant neutral element) for U, V open under $\mathcal{D}$, since $\mathcal{D}$ is a shift-invariant

Baire topology satisfying the Weak Category Convergence condition of [13] for
either of the shift actions $x \mapsto x + a$, $x \mapsto xa$. The Category Interior Theorem
in turn follows from the Category Embedding Theorem ([13]; cf. [72]). Now
$a \in A^o := \mathrm{int}_{\mathcal{D}}(A)$, $b \in B^o$, as a and b are respectively density points of A and
B, and $B_\varepsilon(a) \cap A^o$ and $B_\varepsilon(b) \cap B^o$ are in $\mathcal{D}$, as $\mathcal{D}$ refines $\mathcal{E}$.

Our approach below is, as in Lemma S, via the *Steinhaus–Weil property*
of certain non-negligible sets Σ: 0 is a (usual) interior point of $\Sigma - \Sigma$. Our
motivation comes from some infinite combinatorics going back to Kestelman
[60] in 1947 that has later resurfaced in the work of several authors: Kemper-
man [59] in 1957, Borwein and Ditor [25] in 1978, Trautner [98] in 1987, Harry
Miller [71] in 1989, Grosse-Erdmann [45] in 1989, and [8–10,14] from 2008.
The *Kestelman–Borwein–Ditor Theorem* (KBD below) asserts (in particular)
that *for any Baire/measurable non-negligible Σ and any null sequence $z_n \to 0$,
there are $t \in \Sigma$ and an infinite $\mathbb{M}$ such that $t + z_m \in \Sigma$ for $m \in \mathbb{M}$.*
On $\mathbb{R}$, KBD is both a consequence and a sharpening of the Baire Category
Theorem (BC below), for BC implies KBD, and conversely—the proof of KBD
requires a sequence of applications of BC [72]. The power of these ideas is
shown in the proof of the Uniform Convergence Theorem of regular variation
([7, Ch. 1], [9]).
None of this is special to $\mathbb{R}$: one can work in a Polish abelian group. Then
KBD in this setting implies as an almost immediate consequence the Effros
Theorem ([76], cf. [69]), and so the Open Mapping Theorem [78], as well as
other classical results, for instance the Banach–Steinhaus Theorem—see the
survey [77] and the more recent developments in [21] and [22, Th. 2].
The significance of KBD is three-fold.
Firstly, if KBD applies for the non-negligible sets Σ of some family of sets,
then these sets have the Steinhaus–Weil property. If not, choose $z_n \notin \Sigma - \Sigma$
with $z_n \to 0$ (henceforth termed a 'null' sequence); now there are $t \in \Sigma$ and
an infinite $\mathbb{M}$ such that $t + z_m \in \Sigma$ for $m \in \mathbb{M}$, so $z_m = (t + z_m) - t \in \Sigma - \Sigma$,
a contradiction. A variant argument, relevant to Sect. 5, may be used for the
difference of two such sets $\Sigma_1 - \Sigma_2$: for x_1, x_2 respective density points, work
as above with $\Sigma := (\Sigma_1 - x_1) \cap (\Sigma_2 - x_2)$ to obtain $0 \in \mathrm{int}(\Sigma - \Sigma)$, i.e.
$x_1 - x_2 \in \mathrm{int}(\Sigma_1 - \Sigma_2)$; cf. [14].
Secondly, several proofs of KBD rely on elementary induction, i.e. recursion
through the natural numbers via DC (see Sect. 1). As a result our Berz-type
theorems depend only on DC rather than on the full strength of AC used by
Berz.
Finally, any application of KBD in a topological vector space context may
be deemed to take place in the separable subspace generated by the null se-
quence, so enabling restriction to the separable case.

Definition. For X a separable Banach space, say that the increasing sequence
of sets $\{\Sigma_m\}_{m \in \mathbb{N}}$ *quasi covers* X (is a *quasi cover* of X) if $\bigcup_{m=0}^{\infty} \Sigma_m$ is dense

in X, and that $f : X \to \mathbb{R}$ is *quasi σ-continuous* under the quasi cover if $f|\Sigma_m$ is continuous for each $m = 0, 1, 2, \ldots$.

In applications $X \backslash \bigcup_{m=0}^{\infty} \Sigma_m$ will be negligible.

Separability is a natural condition in the next result—see the closing comments in [103].

Theorem BL (Baire Continuity Theorem [15, Th. 11.8]; Baire–Luzin Theorem; cf. [46, end of Section 55], [103, Th. II]). *For a separable Banach space, if $f : X \to \mathbb{R}$ is Baire, or measurable with respect to a regular σ-finite measure μ, then f is quasi σ-continuous with respect to some quasi cover $\{\Sigma_m\}_{m \in \mathbb{N}}$, with the sets Σ_m being non-empty and respectively in $\mathcal{D}_{\mathcal{B}}$ or μ-non null μ-measurable. Thus, for $X = \mathbb{R}$ under the Lebesgue measure, the sets Σ_m may be taken non-empty density open, i.e. in $\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{\mathcal{B}}(\mathcal{D}_{\mathcal{L}})$.*

Remarks. 1. In the category case, with $\Sigma_m = \Sigma_0$ for all m and Σ_0 co-meagre, this is Baire's Theorem ([79, Th. 8.1]). In the Lebesgue measure case this is a useful form of Luzin's Theorem formulated in [11]. The extension to a regular (i.e. $\mathcal{G}$-outer regular) σ-finite measure may be made via Egoroff's Theorem (cf. [46, §21 Th. A]).

2. Below, and especially in Sect. 5 (see for instance, the introductory paragraph to Th. 6), it is helpful if the sets Σ_m are not only in $\mathcal{D}_{\mathcal{B}}$ but also norm dense. So, in particular, sets that are locally co-meagre come to mind; however, any Baire set that is locally co-meagre is co-meagre. (For Σ Baire, its quasi-interior—the largest (regular) open set equal to Σ modulo a meagre set—is then locally dense, so everywhere dense and so co-meagre.)

3. For f Baire, $f|V$ is continuous in the usual sense (i.e. $\mathcal{E} \to \mathcal{E}$) on a $\mathcal{D}_{\mathcal{B}}$-open set V [79, Th. 8.1].

In an infinite-dimensional separable Banach space: we cannot rely on the Haar measure, as here it does not exist; but we can nevertheless rely on a σ-ideal of sets whose 'negligibility' is predicated on the Borel probability measures of that space[2]. We recall below their definition and two key properties, the first of which relies on *separability* (hence the frequent recourse below to separable Banach subspaces): the Steinhaus–Weil property and the *weak extension of the Fubini theorem* (WFT; see below) due to Christensen [28], which may be applied here. For this we need to recall some definitions.

Firstly, for G a complete metric group (e.g. a Banach space viewed as an additive group), $B \subseteq G$ is *universally measurable* if B is measurable with respect to every Borel measure on G—for background, see e.g. cf. [38, 434D, 432]. Examples are analytic subsets (see e.g. [86, Part 1 §2.9], or [58, Th. 21.10], [38, 434Dc]) and the σ-algebra that they generate. Beyond these are the *provably*

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[2] Also relevant here is their regularity: for their outer regularity (approximation by open sets) see [80, Th. II.1.2], and their inner regularity (approximation by compacts—the Radon property) see [80, Ths. II.3.1 and 3.2], the latter relying on completeness.

$\mathbf{\Delta}_2^1$ sets of [34]; for their definition see Appendix 1.1 in the extended arXiv version of this paper.

Consistently with this, say that a function $S : G \to \mathbb{R}$ is *universally measurable* if S^{-1} takes open sets to universally measurable sets in G. Actually we just need these inverse images to have universally measurable intersections with closed separable subgroups of G.

Secondly, we need the σ-ideal of *Haar null* sets (defined below). This is a generalization of Christensen [28,29] to a non-locally compact group of the notion of a Haar measure-zero set: see again Hoffmann-Jørgensen [86, Part 3, Th. 2.4.5] and Solecki [94–96] (and [51] in the function space setting).

Christensen [28] shows that in an abelian Polish group $(G, \cdot)$ the family $\mathcal{H}(G, \cdot)$ of Haar null sets forms a σ-ideal. This was extended for 'left Haar null' sets (see below) by Solecki [96, Th. 1] in the more general setting of (not necessarily abelian) Polish groups $(G, \cdot)$ *amenable at 1*, the scope of which he studies, in particular proving that any abelian Polish group is amenable at 1 [96, Prop 3.3]; this includes, as additive groups, separable F- (and hence Banach) spaces.

A subset of a Polish group G is *left Haar null* [96] if it is contained in a universally measurable set B such that for some Borel probability measure μ on G

$$\mu(gB) = 0 \qquad (\forall g \in G).$$

Solecki also considers the (in general) narrower family of Haar null sets (as above, but now $\mu(gBh) = 0$ for all $g, h \in G$). Below we work in vector spaces and so the non-abelian distinctions vanish.

We can clarify the terminology in Theorem FS above: for X a separable Banach space, $Y \subseteq X$ is $\mathcal{H}$-*measurable* ('H' for Haar) if $Y = B \cup H$ for some universally measurable B and Haar-null H, and $f : X \to \mathbb{R}$ is $\mathcal{H}$-*measurable* if $f^{-1}(U)$ is $\mathcal{H}$-measurable for each open $U \subseteq \mathbb{R}$.

With these concepts at hand, we turn to the two properties mentioned above.

First, the Steinhaus–Weil Theorem holds for universally measurable sets that are not Haar null; this was proved by Solecki (actually for left Haar null [96, Th. 1(ii)]; cf. Hoffmann-Jørgensen [86, Part 3, Th. 2.4.6]) by implicitly proving KBD. It may be checked that his proof uses only DC. One may also show that the KBD theorem follows from amenability at 1: see [22].

Next, Christensen's WFT [28] (for a detailed proof, see [26]) concerns the product $H \times T$ of a locally compact group T, equipped with the Haar measure η, and an arbitrary abelian Polish group H, and is a 'one-way round' theorem (only for T-sections): if $A \subseteq H \times T$ is universally measurable, then A is Haar null iff the sections $A(h) := \{t \in T : (h, t) \in A\}$ are Haar measure-zero except possibly for a Haar null set (in the sense above) of $h \in H$ (i.e. for 'quasi all'

$h \in H$). The 'other way round' (for H-sections) may fail, as was shown by Christensen [28, Th. 6]. For another counter-example, see Jabłońska [52].

3. Sublinearity and Berz's Theorem

We begin with Theorem 2 on subadditive functions. We deal with the category and measure versions together via the Steinhaus–Weil property, and use DC rather than AC. The Lebesgue measurable case (with AC) is classical [50, Ch. 7]. See also [8, Prop. 1], [62, Th. 16.2.2], [20, Prop. 7′].

Recalling that $\mathcal{T}$ is a Baire space topology if Baire's theorem holds under $\mathcal{T}$, we say that a vector space X has a *Steinhaus–Weil topology* $\mathcal{T}$ if the *non-meagre Baire sets* of $\mathcal{T}$ *have the Steinhaus–Weil property*. Thus $\mathcal{E}$ and $\mathcal{D}_{\mathcal{L}}$ are Steinhaus–Weil topologies for $\mathbb{R}$ that are Baire topologies, by the classical Steinhaus–Piccard–Pettis Theorems (see e.g. [16]).

As above, we say that $S : X \to \mathbb{R}$ is $\mathcal{T}$-*Baire* if S^{-1} takes (Euclidean) open sets of $\mathbb{R}$ to $\mathcal{T}$-Baire sets in X. Thus $\mathcal{E}$-Baire means Baire in the usual sense, and $\mathcal{D}_{\mathcal{L}}$-Baire means Lebesgue measurable.

Theorem 2. *For X a vector space with a Steinhaus–Weil, Baire topology $\mathcal{T}$ and $S : X \to \mathbb{R}$ subadditive: if S is $\mathcal{T}$-Baire, then it is locally bounded.*

Proof. Suppose $|S(u + z_n)| \to \infty$ for some $u \in X$ and null sequence $z_n \to 0$. As the level sets $H_n^{\pm} := \{t : |S(\pm t)| \leqslant n\}$ are $\mathcal{T}$-Baire and $\mathcal{T}$ is Baire, for some k the set $H_k^{\pm}$ is non-meagre. As $\mathcal{T}$ is Steinhaus–Weil, $H_k^{\pm} - H_k^{\pm}$ has 0 in its interior. So there is $n \in \mathbb{N}$ such that $z_m \in H_k^{\pm} - H_k^{\pm}$ for all $m \geqslant n$. For $m \geqslant n$, choose $a_m, b_m \in H_k^{\pm}$ with

$$z_m = a_m - b_m$$

for $m \geqslant n$. Then for all $m \geqslant n$

$$S(u) - 2k \leqslant S(u) - S(-a_m) - S(b_m) \leqslant S(u + a_m - b_m)$$
$$= S(u + z_m) \leqslant S(u) + S(a_m) + S(-b_m) \leqslant S(u) + 2k,$$

contradicting unboundedness.   □

The key to Theorem 1B is Theorem 3 below. It may be regarded as a subadditive analogue of Ostrowski's Theorem for additive functions (cf. [16], [66]). The result extends its counterpart in [17, Prop. 13], with a simpler proof, and uses only DC (via KBD). As it depends on the Steinhaus–Weil property, it handles the Baire and measurable cases together.

Theorem 3. *If $S : \mathbb{R} \to \mathbb{R}$ is subadditive, locally bounded with $S(0) = 0$, and:*

(i) *there is a symmetric set Σ (i.e. $\Sigma = -\Sigma$) containing 0 with $S|\Sigma$ continuous at 0;*

(ii) *for each $\delta > 0$, $\Sigma_\delta^+ := \Sigma \cap (0, \delta)$ has the Steinhaus–Weil property*
     —*then S is continuous at 0 and so everywhere.*
          In particular, this conclusion holds if there is a symmetric set Σ containing 0 on which

$$S(u) = c^{\pm}u \text{ for some } c^{\pm} \in \mathbb{R} \text{ and all } u \in \Sigma \cap \mathbb{R}_+, \text{ or all } u \in \Sigma \cap \mathbb{R}_- \text{ resp.,}$$

and Σ is Baire/measurable, non-negligible in each $(0, \delta)$ for $\delta > 0$.

Proof. If S is not continuous at 0, then (see e.g. [50, 7.4.3], cf. [20, Prop. 7])

$$\infty > \lambda_+ := \limsup_{t \to 0} S(t) > \liminf_{t \to 0} S(t) \geqslant 0,$$

the first inequality by local boundedness at 0, the last by subadditivity. Choose $z_n \to 0$ with $S(z_n) \to \lambda_+ > 0$. Let $\varepsilon = \lambda_+/3$. By continuity on Σ at 0, there is $\delta > 0$ with $|S(t)| < \varepsilon$ for $t \in \Sigma \cap (-\delta, \delta)$. By the Steinhaus–Weil property of Σ_δ there is n such that $z_m \in \Sigma_\delta^+ - \Sigma_\delta^+$ for all $m \geqslant n$. Choose $a_m, b_m \in \Sigma_\delta^+$ with $z_m = a_m - b_m$; so by subadditivity

$$S(z_m) \leqslant S(a_m) + S(-b_m) \leqslant 2\varepsilon.$$

Taking limits,

$$\lambda_+ \leqslant 2\varepsilon < \lambda_+.$$

This contradiction shows that S is continuous at 0. As in [50, Th. 2.5.2], continuity at all points follows by noting that

$$S(x) - S(-h) \leqslant S(x + h) \leqslant S(x) + S(h).$$

The remaining assertion (implying $S(0) = 0$) follows from the Steinhaus–Piccard–Pettis Theorem via Theorem 2, since continuity at 0 on Σ is implied by

$$|c^{\pm}u| \leqslant |u| \cdot \max\{|c^+|, |c^-|\}. \qquad \square$$

Proofs of Theorems 1B and 1M. Let $S : \mathbb{R} \to \mathbb{R}$ be sublinear and either Baire or measurable, that is, Baire in one of the two topologies $\mathcal{D}_\mathcal{B}(\mathcal{E})$ or $\mathcal{D}_\mathcal{B}(\mathcal{D}_\mathcal{L})$. By Theorem BL S is quasi σ-continuous. Taking Σ_m as in Th. BL of Sect. 2 (with m fixed), apply Lemma S to $A = B = \Sigma_m \subseteq \mathbb{R}_+$. Fix (non-zero!) $a, b \in \Sigma_m$; as these are density points, there are $a_n, b_n \in \Sigma_m$ and $q_n \in \mathbb{Q}_+$ so that

$$b_n = q_n a_n; \quad a = \lim_n a_n, \ b = \lim_n b_n.$$

As S is sublinear,

$$S(q_n a_n)/S(a_n) = q_n = b_n/a_n \to b/a.$$

But $S|\Sigma_m$ is continuous at a and b, so

$$S(b)/S(a) = \lim_n S(b_n)/S(a_n) = b/a.$$

So on Σ_m, $S(u) = c_m u$. But $\Sigma_m \supseteq \Sigma_0 \neq \emptyset$, so $c_m = c_0$ for all m. So $S(x) = c_0 x$ for $x \in \Sigma^+ := \bigcup_{m \in \mathbb{N}} \Sigma_m$, i.e. for almost all $x > 0$. Repeat for $\mathbb{R}_-$ with an

analogous set Σ^- and put $\Sigma := \{0\} \cup \Sigma^+ \cup \Sigma^-$. We may assume $-\Sigma = \Sigma$ (otherwise pass to the subset $\Sigma \cap (-\Sigma)$, which is quasi all of $\mathbb{R}$). By Th. 3, S is continuous at 0 and so everywhere. In summary: S is linear on the *dense* subset $\Sigma^+ \subseteq \mathbb{R}_+$ and continuous, and likewise on the *dense* subset $\Sigma^- \subseteq \mathbb{R}_-$. So S is linear on the whole of $\mathbb{R}_+$, and on the whole of $\mathbb{R}_-$. $\square$

For later use (in Sect. 4 below) we close this section with a variant on Theorem 2, Theorem 2H ('H for Haar'). The proof is the same as that of Theorem 2 above, but needs a little introduction. Recall (from Sect. 2) that a function $S : X \to \mathbb{R}$ with X a Banach space is *universally measurable* if S^{-1} takes open sets to universally measurable sets (as in Sect. 2) in X; further we say that a σ-ideal $\mathcal{H}$ of subsets of a topological vector space X (the 'negligible sets') is *proper* if $X \notin \mathcal{H}$, and that $\mathcal{H}$ has the *Steinhaus–Weil property* if universally measurable sets that are not in $\mathcal{H}$ have the interior point property.

Theorem 2H. *Let X be a topological vector space carrying a proper σ-ideal with the Steinhaus–Weil property. If $S : X \to \mathbb{R}$ is subadditive and universally measurable, then S is locally bounded.*

4. Banach-space versions

In Theorem 4B and 4M below we extend the category and measure results in Theorems 1B and 1M to the setting of a Banach space X. Since the conclusions are derived from continuity (and local boundedness), our results are first established for separable (sub-) spaces, which then extend to the non-separable context, by Theorem B (Sect. 1). The key in each case is an appropriate application of Theorem 2 (or Theorem 2H). The category case here is covered by the Piccard-Pettis Theorem, true for non-meagre Baire sets in X; in fact more is true, as variants of KBD hold in analytic groups with translation-invariant metric—see [15, Ths. 1.2, 5.1] or [75, Th. 2], which also covers F-spaces, so include Fréchet spaces (see the end of this section). In the absence of Haar measure, the analogous 'measure case' arising from universal measurability is technically more intricate, but nevertheless true —see below. It is here that our methodology requires us to *pass down* to separable subspaces of a Banach space X. That it suffices to reduce the case of a general Banach space X to the separable case follows from the result below, a corollary of Theorem B of Sect. 1. Henceforth we write B_δ, Σ_δ respectively for the closed unit ball $\{x : ||x|| \leqslant \delta\}$ and the δ-sphere $\{x : ||x|| = \delta\}$, and use the following notation for *lines* and *rays*:

$$R(u) := \{\lambda u \in X : \lambda \in \mathbb{R}\}, \quad R_\pm(u) := \{\lambda u \in X : \lambda \in \mathbb{R}_\pm \cup \{0\}\}.$$

Corollary B. *For X a Banach space and $S : X \to \mathbb{R}$ sublinear, if $S|B$ is locally bounded above for each closed separable subspace B, then $\{|S(x)|/||x|| : x \neq 0\}$ is bounded.*

Proof. By Theorem B, S is continuous on X, so there is $\delta > 0$ with

$$||S(v)|| \leqslant 1 \qquad (||v|| \leqslant \delta).$$

Furthermore, for any $x \neq 0$ taking $u := x/||x||$, the restriction of S to the ray $R_+(u)$ is positively homogeneous by Theorem 1B, and so

$$|S(x)| = |S(||x||u)| = |S(\delta u)||x||/\delta| \leqslant ||x||/\delta. \qquad \square$$

The proof of the category case in Theorem 4B below would have been easier had we used AC to construct the function $c(x)$; but, as we wish to rely only on DC, more care is needed.

We offer two proofs. The first uses Theorems 1B and 2 (and is laid out so as to extend easily to the more demanding F-space setting of Theorem 4F below); the second is more direct, but uses a classical selection (uniformization) theorem, together with a Fubini-type theorem for negligible sets in a product space. Both proofs have Banach-space 'measure' analogues.

Theorem 4B. *For X a Banach space, and $S : X \to \mathbb{R}$ Baire, if S is subadditive and $\mathbb{Q}_+$-homogeneous, then*

(i) *S is continuous and convex with epigraph a convex cone pointed at 0, and*
(ii) *there is a bounded function $c : X \to \mathbb{R}$ such that*

$$S(x) = c(x)||x||.$$

First Proof. Since S is midpoint convex, and we first seek to establish continuity, we begin by establishing it for any separable subspace; we then use Theorem and Corollary B above to draw the same conclusion about X itself. Consequently, we may w.l.o.g. assume X is itself *separable*. By Theorem 2 applied to the usual meagre sets, S is locally bounded at 0, so there are M and $\delta > 0$ such that

$$|S(x)| \leqslant M \qquad (x \in B_\delta).$$

In particular, for $v \in \Sigma_\delta$, $|S(v)| \leqslant M$. For $u \in X$ define a ray-restriction of S by

$$f_u(x) := S(x) \quad (x \in R(u)).$$

For fixed u, as the mapping $\lambda \longmapsto \lambda u$ from $\mathbb{R}$ into X is continuous, the set $R(u)$ is σ-compact. So for any fixed u, f_u is Baire; indeed, $f_u(x) \in (a,b)$ iff $S(x) \in (a,b)$ and $x \in R(u)$, i.e.

$$\{x : f_u(x) \in (a,b)\} = \{x : S(x) \in (a,b)\} \cap R(u),$$

and $R(u)$ has the Baire property (being σ-compact). So by Th. 1B, for any fixed u the function f_u is continuous and there exist $c^{\pm} \in \mathbb{R}$ with $S(\lambda u) = c^{\pm}\lambda$ according to the sign of λ. This justifies the *definitions* below for $u \in X$:

$$c^+(u) := S(u), \qquad c^-(u) := -S(-u).$$

Then, for fixed u, by the continuity of f_u, $S(\lambda u) = \lambda S(u) = c^+(u)\lambda$ for $\lambda \geqslant 0$, so that S is *positively homogeneous* on $R_+(u)$; likewise, $S(\lambda u) = S((-\lambda)(-u)) = (-\lambda)S(-u) = c^-(u)\lambda$ for $\lambda \leqslant 0$. Then for $u = x/||x||$ with $x \neq 0$, as $v := \delta u \in \Sigma_\delta$,

$$|S(x)| = |S(||x||u)| = |S(v)| \cdot ||x||/\delta \leqslant (M/\delta)||x||.$$

So S is continuous at 0, and so by subadditivity everywhere, as in the proof of Theorem 3. By continuity (as S is positively homogeneous) S is fully convex [85, Th. 4.7], as opposed to just midpoint convex (cf. Sect. 5); so its epigraph is a convex cone pointed at the origin [85, Th. 13.2].

Finally, for $x \neq 0$, take $c(x) := S(x/||x||)$, which as above is bounded by M/δ; then

$$S(x) = c(x)||x||. \qquad \qquad \square$$

Second Proof. As above, we again assume that X is separable. By Theorem BL (Sect. 2) there is a co-meagre subset Σ with $S|\Sigma$ continuous. By passage to $\Sigma \cap (-\Sigma)$ we may assume Σ is symmetric; we may also assume that Σ is a $\mathcal{G}_\delta$. By the Kuratowski-Ulam Theorem [79, Th. 15.1], for quasi all $x \neq 0$, say for $x \in D$ with D a $\mathcal{G}_\delta$-set, the ray $R_+(x) \cap \Sigma$ is co-meagre on Σ. By the Steinhaus Theorem, Sierpiński's Lemma S applies. By Theorem 1B for $s \in \Sigma \cap R_+(x)$ there is c with $S(s) = c||s||$ (as $s = x||s||/||x||$). Now $S|\Sigma$ is continuous so a Borel function, as Σ is a $\mathcal{G}_\delta$, and for fixed $x \in D$, $S(a)/||a||$ is constant for (density) points a of $\Sigma \cap R_+(x)$. (This uses the isometry of $R_+(x)$ and $\mathbb{R}_+$.) So by Novikov's Theorem (see e.g. [56, p. x],—cf. [58, 36.14]) there is a Borel function $c : D \to \mathbb{R}$ such that $S(x) = c(x)||x||$ for $x \in D$. By Theorem 1B, since S is bounded near the origin, $c(x)$ is also bounded on D near 0 (as in the previous proof). From this boundedness near 0, as in the proof of Theorem 3, $S(x)$ is continuous for all x. By continuity, S is positively homogeneous, so again convex with epigraph a convex cone pointed at the origin. $\qquad \square$

Remarks. 1. In the first proof, one may show that S is continuous at 0 by considering a (null) non-vanishing sequence $z_n \to 0$. Put $u_n := z_n/||z_n||$; by DC select $c_n^{\pm}$ such that $S(\lambda u_n) = c_n^{\pm}\lambda$, according to the sign of λ. As S is locally bounded at 0, there are M and $\delta > 0$ such that

$$|S(x)| \leqslant M \qquad (x \in B_\delta).$$

W.l.o.g. $\delta \in \mathbb{Q}_+$, so for $x = \delta u_n \in B_\delta$, $|S(x)| = |c_n^+\delta| \leqslant M$; then $|c_n^+| \leqslant M/\delta$. So

$$S(z_n) = S(||z_n||u_n) = c_n^+||z_n|| \leqslant (M/\delta)||z_n|| \to 0.$$

2. The second proof uses the Fubini-like Kuratowski-Ulam Theorem [79, 15.1] (cf. [28]). This can fail in a non-separable metric context, as shown in [84] (cf. [70]), but see [39, 97].

Either argument for Theorem 4B above has an immediate *Lebesgue measure* analogue for $X = \mathbb{R}^d$, and beyond that a *Haar measure* analogue for X a locally compact group with Haar measure η, by the classical Fubini Theorem (see e.g. [79, Th. 14.2]). But we may reach out further still for a measure analogue, Theorem 4M below, by employing the σ-ideal of Haar null sets (Sect. 2). Whilst our argument is simpler (through not involving radial open-ness), there is a close relation to the result of [35], which is concerned with *convex* functions S that are measurable in the following sense: S^{-1} takes open sets to sets that, modulo Haar null sets, are universally measurable sets in X (we turn to convexity in Sect. 5: see especially Th. 7 and 8). Below (recalling from Sect. 2) a function $S : G \to \mathbb{R}$ is *universally measurable* if S^{-1} takes open sets to universally measurable sets in G; this means that, as in Th. 2 Sect. 3, the level sets $H_n^{\pm}$ are universally measurable, so if G is amenable at 1, in particular if G is an abelian Polish group, for some $k \in \mathbb{N}$ the level set $H_k^{\pm}$ is not Haar null (since $X = \bigcup_{n \in \mathbb{N}} H_n^{\pm}$ is not Haar null). This aspect would remain unchanged if the level sets were universally measurable modulo Haar null sets.

Theorem 4M. *For X a Banach space, and $S : X \to \mathbb{R}$ universally measurable: if S is subadditive and $\mathbb{Q}_+$-homogeneous, then*

(i) *S is continuous and convex with epigraph a convex cone pointed at 0, and*
(ii) *there is a bounded function $c : X \to \mathbb{R}$ such that*

$$S(x) = c(x)\|x\|.$$

First Proof. Proceed as in the first proof of Theorem 4B (reducing as there to separability), but in lieu of Theorem 2 apply Theorem 2H here to the σ-ideal of Haar null sets $\mathcal{H}(X, +)$. $\qquad \square$

Second Proof. Reduce as before to the separable case. With WFT above, as a replacement for the Kuratowski-Ulam theorem, we may follow the proof strategy in the second proof of Theorem 4B, largely verbatim. Regarding the line $R(x)$ (for $x \neq 0$) as a locally compact group isomorphic to $\mathbb{R}$, take $\mu := \mu_\Sigma \times \eta_1$ to be a probability measure with atomless spherical component μ_Σ (a probability on the unit sphere of X; this can be done since atomless measures form a dense $\mathcal{G}_\delta$ under the weak topology—cf. [80, Th. 8.1]) and radial component η_1, a *probability* on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue (Haar) measure. We claim that, for quasi all x, $S|R_+(x)$ is quasi σ-continuous on a (Haar/Lebesgue) co-null set (via the Haar-absolutely continuous measure η_1 above). For if not, there is a set C that is not Haar null with $S|R_+(x)$ not quasi σ-continuous for $x \in C$. So there is $u \in X$ with $\mu(u + C) > 0$, and so $u + C$ is not radial. Put $m(B) := \mu(u + B)$ for Borel sets B, again a probability measure. By Theorem BL and WFT, $S|R_+(x)$ is quasi σ-continuous for

m-almost all x, except on some set E with $m(E) = 0$. This is a contradiction for points in $C \backslash E$. Now continue as in Theorem 4B.   □

F-spaces. Recall that an *F-space* is a topological vector space with topology generated by a complete translation-invariant metric d_X ([57, Ch. 1], [89, Ch. 1]). Thus the topology is generated by the *F-norm* $||x|| := d_X(0, x)$, which satisfies the triangle inequality with $||\alpha x|| \leqslant ||x||$ for $|\alpha| \leqslant 1$, and under it scalar multiplication is jointly continuous. This continuity implies that a vector x can be scaled down to an arbitrarily small size. Consequently, the proofs above may be re-worked to yield F-space versions of Theorems 4B and 4M. However, in the absence of a norm (see §6.5 for normability), there is no isometry between the rays $R(x)$ below and $\mathbb{R}_+$, only an injection $\Delta : R(x) \to \mathbb{R}_+$. We are thus left with a result that has a somewhat weaker representation of S.

We need the F-norm to be *unstarlike*, in the sense that the 'norm-length' (i.e. the range of the norm) of all rays be the same, say unbounded for convenience. This last property holds for the L^p spaces for $0 < p < 1$ with the familiar F-norm $||f|| := \left(\int |f(t)|^p dt \right)^{1/p}$.

Unstarlikeness is an F-norm, rather than a topological, property; it will hold after re-norming, albeit with $(0, 1)$ as the common range, when taking the F-norm to be $||x|| := \sup_n 2^{-n}(||x||_n/(1 + ||x||_n))$, for $|| \cdot ||_n$ a *distinguishing* sequence of semi-norms, since $\varphi_{x,n}(t) := t||x||_n/(1 + t||x||_n)$ maps $[0, \infty)$ onto $[0, 1)$. Examples here are provided by spaces of continuous functions such as $C(\Omega)$, for $\Omega := \bigcup_n K_n$ with $K_n \subseteq \text{int}(K_{n+1})$ a chain of compact subsets of a Euclidean space, and with $||f||_n := ||f|_{K_n}||_\infty$ for $|| \cdot ||_\infty$ the supremum norm. Likewise this holds in the subspace $H(\Omega)$ of holomorphic functions, and in $C^\infty(\Omega)$ when $||f||_n := \max\{||D^\alpha f||_\infty : |\alpha| < n\}$ for multi-indices α—see [89, §1.44–47]). Being infinite-dimensional, none of them are normable as they are either locally bounded or Heine-Borel (or both)—cf. [89, Th. 1.23].

Theorem 4F. *For X an F-space and $S : X \to \mathbb{R}$ Baire, if S is subadditive and $\mathbb{Q}_+$-homogeneous, then*

(i) *S is continuous and convex with epigraph a convex cone pointed at 0, and*
(ii) *for any unstarlike F-norm $|| \cdot ||$ (with $||tx|| \to \infty$ $(t \to \infty)$ for all $x \neq 0$), there are a bounded function $c : X \to \mathbb{R}$, a constant δ, and a map $\Delta : X \to \mathbb{R}_+$ with each $\Delta|R(x)$ an injection such that*

$$S(x) = c(x)\Delta(x), \text{ where } ||x/\Delta(x)|| = \delta.$$

In particular, if X is normable with norm $|| \cdot ||_X$, then $\Delta(x) = ||x||_X/\delta$.

Proof. Let $||.||$ be an unstarlike F-norm. For any $x \neq 0$, the map $\varphi_x : t \mapsto ||tx||$ is a continuous injection with $\varphi_x(0) = 0$ and $\varphi_x(1) = ||x||$; so for all $\delta > 0$ and $||x|| \geqslant \delta$ we may define $\delta(x) := \min\{t : ||tx|| = \delta\}$, the infimum being attained. So $||\delta(x)x|| = \delta$. The unstarlike property implies that $\delta(x)$ is likewise well defined for all $x \neq 0$.

We now assume w.l.o.g. that X is separable, as in the earlier variants of Th. 4, for the same reasons (though we need the F-norm analogue of Corollary B, also valid—see below for the relevant positive homogeneity). Proceed as in the first proofs of Theorems 4B and 4M, with a few changes, which we now indicate. Of course we refer respectively to the σ-ideals of meagre sets and Haar null sets.

With this in mind one deduces again positive homogeneity, and for some M, $\delta > 0$ thence, for $x \neq 0$ and with $v = \delta(x)x \in \Sigma_\delta$, that as $\delta(x) > 0$

$$|S(x)| = |S(v/\delta(x))| = |S(v)|/\delta(x) \leqslant M/\delta(x).$$

Now $\delta(x) \to \infty$ as $x \to 0$, and so S is continuous at 0; indeed, for each $n \in \mathbb{N}$ the function $x \mapsto ||nx||$ is continuous at $x = 0$, so by DC there is a positive sequence $\{\eta(n)\}_{n \in \mathbb{N}}$ such that $||nx|| < \delta$ for all $x \in B_{\eta(n)}$. So $\delta(x) > n$ for $x \in B_{\eta(n)}$ and $n \in \mathbb{N}$, and so

$$|S(x)| \leqslant M/\delta(x) < M/n \qquad (x \in B_{\eta(n)}).$$

Thereafter, taking $c(x) := c^+(\delta(x)x) = S(\delta(x)x)$, which is bounded by M,

$$S(x) = S(\delta(x)x/\delta(x)) = c(x)\Delta(x), \text{ where } \Delta(x) := 1/\delta(x).$$

So $||x/\Delta(x)|| = \delta$. If the F-norm is a norm, $\delta(x) := \delta/||x||$; then $||x\delta(x)|| = \delta$, so that $\Delta(x) := ||x||/\delta$. □

Theorem 4F implies Theorem 4B and 4M by taking $c(x)/\delta$ in place of $c(x)$.

5. Convexity

We begin by recalling a classical result, Theorem BD below, which motivates the themes of this section. These focus on the two properties of a function S of *midpoint convexity*

$$S\left(\frac{1}{2}(x+y)\right) \leqslant \frac{1}{2}\left(S(x) + S(y)\right) \qquad (x, y \in X)$$

(for X a vector space), and of *convexity,* which, for purposes of emphasis, we also refer to (as in [68]) as *full* (or $\mathbb{R}$-) *convexity*:

$$S((1-t)x + ty) \leqslant (1-t)S(y) + tS(y) \qquad (x, y \in X, \ t \in (0,1)),$$

by considering the weaker property of *weak midpoint convexity on a set* Σ (Sect. 1):

$$S\left(\frac{1}{2}(x+y)\right) \leqslant \frac{1}{2}\left(S(x) + S(y)\right) \qquad (x, y \in \Sigma);$$

here we do not require $(x+y)/2 \in \Sigma$, in contrast to the still weaker property of the restriction map $S|\Sigma$ being midpoint convex. The latter does require that $(x+y)/2$ lie in the domain Σ of the restriction map.

It is instructive to observe that the indicator function $\mathbf{1}_{\mathbb{P}}$ of the irrationals $\mathbb{P}$ is weakly midpoint convex on $\mathbb{P}$ but not midpoint convex; we return to this in Theorem 6 below.

In Theorems 5–7 below, we give *local* results, with the hypotheses holding on a set Σ. The smaller Σ is, the more powerful (and novel) the conclusions are. For instance, Σ might be locally co-meagre (and so co-meagre, as noted in the remarks to the definition of quasi σ-continuity in Sect. 2).

Theorem BD (Bernstein–Doetsch Continuity Theorem, [62, §6.4])**.** *For X a normed vector space, if $S : X \to \mathbb{R}$ is midpoint convex and locally bounded above somewhere (equivalently everywhere), then S is continuous and so fully convex.*

Proof. This is immediate from Theorem B (see Sect. 1). See also the 'third proof' in [62, §6.4], as the other two apply only in $\mathbb{R}^d$. $\qquad\square$

The theorem gives rise to a sharp dichotomy for midpoint convex functions, similar to that for additive functions: they are either continuous everywhere or discontinuous everywhere ('totally discontinuous'), since local boundedness (from above) is 'transferable' between points [62, Th. 6.2.1]. So on the one hand, a Hamel basis yields discontinuous additive examples (the 'Hamel pathology' of [7, §1.1.4]) and, on the other, a smidgen's worth of regularity prevents this—see Corollary 1 below —and midpoint convex functions are then continuous.

A closely related result (for which see e.g. [93, Prop. 1.18]) we give as Theorem BD* below, whose proof we include, as it is so simple.

Theorem BD*. *For X a normed vector space, if $S : X \to \mathbb{R}$ is fully convex and locally bounded above, then S is continuous.*

Proof. W.l.o.g. assume that S is bounded above in the unit ball, by K say (otherwise translate to the origin and rescale the norm). For $x \neq 0$ in the unit ball, setting $u = x/||x||$, and first writing x as a convex combination of 0 and u, then 0 as a convex combination of $-u$ and x,

$$S(x) \leqslant (1 - ||x||)S(0) + ||x||S(u), \qquad (1 + ||x||)S(0) \leqslant ||x||S(-u) + S(x).$$

From here, for *all* x in the unit ball, $|S(x) - S(0)| \leqslant (K + |S(0)|)||x||$, since

$$||x||[S(0) - S(-u)] \leqslant S(x) - S(0) \leqslant ||x||[S(u) - S(0)]. \qquad\square$$

Thus the emphasis in convexity theory is on generic differentiability; for background see again [93]. In Theorem 6 below we derive continuity and full (i.e. $\mathbb{R}$-) convexity, as in Theorem BD [62, §6.4], for functions possessing the weaker property of weak midpoint convexity on certain subsets Σ of their domain with negligible complement, for instance co-meagre or co-null sets. The results below vary their contexts between $\mathbb{R}$ and a general Banach space, and refer to sets with the following Steinhaus–Weil property.

Definition. We say that Σ is *locally Steinhaus–Weil*, or has the *Steinhaus–Weil property locally*, if for $x, y \in \Sigma$ and, for all $\delta > 0$ sufficiently small, the sets $\Sigma_z^+ := \Sigma \cap B_\delta(z)$, for $z = x, y$, have the interior point property that $\Sigma_x^+ - \Sigma_y^+$ has $x - y$ in its interior. (As in Sect. 4 $B_\delta(x)$ is the closed ball about x of radius δ.) See [14] for conditions under which this property is implied by the interior point property of the sets $\Sigma_x^+ - \Sigma_x^+$ (cf. [4]). Note that if Σ has the local Steinhaus–Weil property, then Σ and so $\mathrm{cl}(\Sigma)$ is *dense in itself* , which will be relevant.

Examples of locally Steinhaus–Weil sets relevant here are the following:

(i) Σ density-open in the case $X := \mathbb{R}^n$ (by Steinhaus's Theorem);

(ii) Σ locally non-meagre at all points $x \in \Sigma$ (by the Piccard-Pettis Theorem—such sets can be extracted as subsets of a second-category set, using separability or by reference to the Banach Category Theorem [75, Ch.16]);

(iii) Σ universally measurable and not Haar null at any point (by the Christensen-Solecki Interior-points Theorem of Sect. 2—again such sets can be extracted using separability);

(iv) Σ a Borel subset of a Polish abelian group and not Haar-meagre in the sense of Darji [31] at any point (by Jabłońska's generalization of the Piccard Theorem, [53, Th.2], cf. [54], and since Haar-meagre sets form a σ-ideal [31, Th. 2.9]); see also §6.6.

In (ii) recall from Sect. 2 that if Σ has the Baire property and is locally non-meagre, then it is co-meagre (since its quasi interior is everywhere dense).

For contrast with Corollary 2 below, we first note that local boundedness of midpoint convex functions follows from regularity almost exactly as in the subadditive case of Theorem 2 of Sect. 3.

Theorem 2′. *For X a vector space with a Steinhaus–Weil, Baire topology $\mathcal{T}$ and $S : X \to \mathbb{R}$ midpoint convex: if S is $\mathcal{T}$-Baire, then it is locally bounded.*

Proof. Suppose $|S(u + z_n)| \to \infty$ for some $u \in X$ and null sequence $z_n \to 0$. As the level sets $H_n^\pm := \{t : |S(\pm t)| \leqslant n\}$ are $\mathcal{T}$-Baire and $\mathcal{T}$ is Baire, for some k the set $H_k^\pm$ is non-meagre. As $\mathcal{T}$ is Steinhaus–Weil, $H_k^\pm - H_k^\pm$ has 0 in its interior.

First suppose that $S(u + z_n) \to +\infty$. Then there is $n \in \mathbb{N}$ such that $4z_m \in H_k^\pm - H_k^\pm$ for all $m \geqslant n$. For $m \geqslant n$, choose $a_m, b_m \in H_k^\pm$ with

$$4z_m = a_m - b_m$$

for $m \geqslant n$. Then, as

$$u + z_m = \frac{1}{2}2u + \frac{1}{4}a_m + \frac{1}{4}(-b_m),$$

for all $m \geqslant n$

$$S(u + z_m) \leqslant \frac{1}{2}S(2u) + \frac{1}{4}S(a_m) + \frac{1}{4}S(-b_m) \leqslant \frac{1}{2}S(2u) + \frac{1}{2}k,$$

contradicting upper unboundedness.

If on the other hand $S(u+z_n) \to -\infty$, then argue similarly, but now choose k, n and $a_m, b_m \in H_k^\pm$ so that

$$-2z_m = a_m - b_m,$$

for all $m \geqslant n$. Then

$$S(u/2) - \frac{1}{4}S(a_m) - \frac{1}{4}S(-b_m) \leqslant \frac{1}{2}S(u+z_m),$$

contradicting lower unboundedness. $\square$

As with Theorem 2H (at the end of Sect. 3) here as well, Theorem $2'$ has a 'Haar'-type variant with the same proof, which we need below in Cor. 1 and Theorem FS (universally measurable case).

Theorem 2H$'$. *Let X be a topological vector space carrying a proper σ-ideal with the Steinhaus–Weil property. If $S : X \to \mathbb{R}$ is midpoint convex and universally measurable, then S is locally bounded.*

These results immediately yield a Banach-space version of Theorem BD in the separable context. The non-separable variant must wait.

Corollary 1. *For a separable Banach space X, if $S : X \to \mathbb{R}$ midpoint convex is Baire or universally measurable, then it is locally bounded and so continuous.*

Proof. Apply Theorem $2'$ or 2H$'$ respectively to the σ-ideal of meagre or Haar null sets. $\square$

Our aim now is to identify in Theorem 5 below, for any function weakly midpoint convex on a subset Σ of $\mathbb{R}$, a canonical continuous convex function, using continuity on sets Σ with the local Steinhaus–Weil property. This may be compared to the *upper-hull* construction in [62, §6.3], which, however, refers to the full domain of a function. Thereafter in Theorem 6, a variant of Theorem BD, we will deduce *continuity* of a function on $\mathbb{R}$ that is midpoint convex on just such a subset Σ, extending the result thereafter to the separable Banach context of Theorem 7. As corollaries we then deduce Theorems M and FS of Sect. 1.

Below we write $\limsup_{y \to x}^{\Sigma}$, $\lim_{y \to x}^{\Sigma} S(y)$ for the upper limit or limit of $S(y)$ as y tends to x through Σ. We recall that if Σ has the local Steinhaus–Weil property, then Σ and so $\mathrm{cl}(\Sigma)$ is *dense in itself*. For ease of exposition, we state Theorem 5 below in the format '(i)–(iv), twice'. The second (ii) is actually part of Theorem 6, and is proved there.

Theorem 5 (Canonical Extension Theorem). *For $\Sigma \subseteq \mathbb{R}$ locally Steinhaus–Weil and $I := \mathrm{cl}(\Sigma)$, if $S : \mathbb{R} \to \mathbb{R}$ is both continuous and weakly midpoint convex on Σ, and*

$$\bar{S}(x) = \bar{S}^{\Sigma}(x) := \limsup_{y \to x}^{\Sigma} S(y) \qquad (x \in I)$$

– *then*

(i) *the limit exists for* $x \in I$: $\bar{S}(x) := \lim_{y \to x}^{\Sigma} S(y)$;

(ii) $\bar{S} = S$ *on* Σ;

(iii) *for each* $x \in I$ *with* $x = \sup\{t \in \Sigma : t < x\} = \inf\{t \in \Sigma : t > x\}$

$$S(x) \leqslant \bar{S}(x);$$

(iv) *the restriction map* $\bar{S}|I$ *is fully convex: for* $x, y \in I$, $t \in (0,1)$ *with* $tx + (1-t)y \in I$,

$$\bar{S}(tx + (1-t)y) \leqslant t\bar{S}(x) + (1-t)\bar{S}(y).$$

In particular, $\bar{S}$ and so also S is locally bounded above on I.

   Furthermore, the conclusions (i) (with finite limit),(ii), (iii) and (iv) above hold with $I = \mathbb{R}$ for a quasi cover $\{\Sigma_m\}_{m \in \mathbb{N}}$ of $\mathbb{R}$ with each Σ_m locally Steinhaus–Weil and $S : \mathbb{R} \to \mathbb{R}$ quasi σ-continuous under the quasi cover and weakly midpoint convex on $\Sigma = \bigcup_{m \in \mathbb{N}} \Sigma_m$.

   For the proof we need three lemmas. Unlike in Theorem 5(iv) above, in Lemma 1 below 'fully convex on Σ' does not require the convexity of Σ, but rather that the convexity inequality holds for all relevant arguments. Under the weaker assumption that the restriction map $S|\Sigma$ is midpoint convex, the conclusion of Lemma 1 would require further that $(1-t)a + tb \in \Sigma$. Below conv(Σ) denotes the *convex hull* of Σ.

Lemma 1 (Full convexity on Σ). *For $\Sigma \subseteq \mathbb{R}$ locally Steinhaus–Weil, if $S : \mathbb{R} \to \mathbb{R}$ is both continuous and weakly midpoint convex on Σ, then S is fully convex on Σ :*

$$S((1-t)a + tb) \leqslant (1-t)S(a) + tS(b) \qquad (a, b \in \Sigma, t \in (0,1)).$$

In particular, as the convex combination on the right is bounded by $\max(S(a), S(b))$:

$$S((1-t)a + tb) \leqslant \max(S(a), S(b)) \qquad (a, b \in \Sigma, t \in (0,1)), \qquad (\dagger)$$

so S is locally bounded above on int(conv(Σ)).

Proof. For any T, as above, write $B_\varepsilon^T(x) := B_\varepsilon(x) \cap T$. Fix $t \in (0,1)$ and $a, b \in \Sigma$ with $a < b$. Put

$$u := (1-t)a + tb : \qquad t = (u-a)/(b-a).$$

Now $\Sigma - u$ has the Steinhaus–Weil property locally, and exponentiation is a homeomorphism, so since $(b-u)(u-a)^{-1} + 1 > 1$, for small enough ε the set

$$B_\varepsilon^{\Sigma-u}(b-u)[B_\varepsilon^{\Sigma-u}(u-a)]^{-1} + 1 = -B_\varepsilon^{\Sigma-u}(b-u)B_\varepsilon^{\Sigma-u}(a-u)^{-1} + 1$$

has $(b-u)(u-a)^{-1} + 1 > 1$ in its interior, and so has a rational element $r > 1$.

Taking successively $\varepsilon = 1/n$ for $n \in \mathbb{N}$, select as above rational $r_n > 1$ and a_n, b_n in Σ such that

$$a_n \to a, b_n \to b, \ r_n = 1 + \frac{b_n - u}{u - a_n} = \frac{b_n - a_n}{u - a_n} \to \frac{b - a}{u - a} = 1/t.$$

So with $q_n = 1/r_n \in \mathbb{Q}_+$,

$$u = a_n + q_n(b_n - a_n) = (1 - q_n)a_n + q_n b_n, \text{ and } 0 < q_n < 1.$$

As a, b are relative-continuity points and q_n is rational with $q_n \to t$,

$$\begin{aligned} S(u) &= S((1 - q_n)a_n + q_n b_n) \\ &\leqslant (1 - q_n)S(a_n) + q_n S(b_n) \to (1 - t)S(a) + tS(b). \end{aligned}$$

So for any $a, b \in \Sigma$ and $0 < t < 1$,

$$S((1 - t)a + tb) \leqslant (1 - t)S(a) + tS(b).$$

That is: S is fully convex on Σ, and, in particular, (†) follows.

As for the last assertion, for $x \in \mathrm{int}(\mathrm{conv}(\Sigma))$, choose $a, b \in \Sigma, 0 < t < 1$ with $x = (1 - t)a + tb$; then (a, b) is a neighbourhood of x on which, by (†), S is bounded above by $\max(S(a), S(b))$. □

Corollary 2 (Boundedness on Σ). *For $\Sigma \subseteq \mathbb{R}$ locally Steinhaus–Weil, if $S : \mathbb{R} \to \mathbb{R}$ is both continuous and weakly midpoint convex on Σ, then, for each $x \in \mathrm{int}(\mathrm{conv}(\Sigma)) \cap \mathrm{cl}(\Sigma)$ and each sequence $\{u_n\}_{n \in \mathbb{N}}$ in Σ converging to x, the sequence $\{S(u_n)\}_{n \in \mathbb{N}}$ is bounded.*

Proof. If not, then, by Lemma 1, for some sequence $\{u_n\}_{n \in \mathbb{N}}$ of points in Σ with limit $x \in \mathrm{int}(\mathrm{conv}(\Sigma), S(u_n) \to -\infty$. W.l.o.g. the sequence $\{u_n\}$ is strictly monotonic, and moreover $u_n \downarrow x$ (otherwise, mutatis mutandis, replace u_n and x by their negatives). Put $v := u_1, w := u_2$ and write $w = tx + (1 - t)v$; then $0 < t < 1$ as $x < w < v$. For $n \geqslant 3$ write $w = t_n u_n + (1 - t_n)v$ for some $0 < t_n < 1$, possible as $u_n < u_2 < u_1$. Then t_n is convergent, to $s \in [0, 1]$ say, with $w = sx + (1 - s)v = tx + (1 - t)v$, and so $s = t$, as $x < v$, i.e. $0 < t < 1$. By Lemma 1, as $u_n, v \in \Sigma$ (and $w \in \Sigma$),

$$S(w) = S(t_n u_n + (1 - t_n)v) \leqslant t_n S(u_n) + (1 - t_n)S(v),$$

giving in the limit $S(w) \leqslant -\infty$, as $t_n \to t > 0$, a contradiction. □

The following result is stated as we need it—for the line; we raise, and leave open here, the question of whether it holds in an infinite-dimensional Banach space. It does, however, hold under a stronger $\mathbb{Q}$-affine assumption on the set Σ—see Lemma 2' below. The latter assumption gives an easy way of capturing the key idea in Lemma 2: one given 'direction of access' $\{v_n\}_{n \in \mathbb{N}}$ (taken through a set Σ) to a fixed location ($v_n \to x$ below) is made accessible

from a second given direction $\{u_n\}_{n\in\mathbb{N}}$ (again through Σ), if not exactly then at least approximately—up to an error taken from Σ :

$$v_{m(n)} = (1 - t_n)u_{m(n)} + t_n w_n : \quad v_{m(n)} - u_{m(n)} = t_n(w_n - u_{m(n)})$$
$$(w_n \in \Sigma, t_n \searrow 0).$$

Framed this way, the condition demands that for $u \in \Sigma$ the set $\Sigma - u$ is capable of replicating any small difference $v - u$ with $v \in \Sigma$ after arbitrary downscaling (by t), i.e. $\Sigma - u$ contains a dense set of 'long' vectors $w - u$.

Lemma 2 (Unique limits on $\mathbb{R}$)**.** *For $\Sigma \subseteq \mathbb{R}$ locally Steinhaus–Weil and $S : \mathbb{R} \to \mathbb{R}$, if $S|\Sigma$ is both continuous and weakly midpoint convex on Σ, then for any $x \in \mathbb{R}$ and for any sequences in Σ with $u_n \uparrow x$ and $v_n \downarrow x$,*

$$\lim S(u_n) = \lim S(v_n),$$

when both limits exist.

Furthermore, this conclusion holds for a quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$ of $\mathbb{R}$ with each Σ_m locally Steinhaus–Weil and $S : \mathbb{R} \to \mathbb{R}$ quasi σ-continuous under the quasi cover and weakly midpoint convex on $\Sigma = \bigcup_{m\in\mathbb{N}} \Sigma_m$.

Proof. Write $\mathbf{u} := \{u_n\}_{n\in\mathbb{N}}$, $\mathbf{v} := \{v_n\}_{n\in\mathbb{N}}$ and $L := L(\mathbf{u}) = \lim S(u_n), R := L(\mathbf{v}) = \lim S(v_n)$; we show that $L = R$ in two steps—by deriving two inequalities. Let $\varepsilon > 0$; then there is N so that for $n > N$,

$$R - \varepsilon \leqslant S(v_n) \leqslant R + \varepsilon \text{ and } L - \varepsilon \leqslant S(u_n) \leqslant L + \varepsilon.$$

For any $n > N$ choose $t_n \in (0, 1)$ with $t_n|R - L| < \varepsilon$, and $m = m(n) > N$ in order to express the 'right-sided' sequence $\mathbf{v}$ in terms of the 'left-sided' sequence $\mathbf{u}$:

$$v_{m(n)} = (1 - t_n)u_{m(n)} + t_n v_n.$$

This is possible as $u_n \uparrow x$ and $v_n \downarrow x$. Now fix any $n > N$. As $u_{m(n)}, v_{m(n)}, v_n \in \Sigma$, by Lemma 1,

$$R - \varepsilon \leqslant S(v_{m(n)}) \leqslant (1 - t_n)S(u_{m(n)}) + t_n S(v_n)$$
$$\leqslant (1 - t_n)(L + \varepsilon) + t_n(R + \varepsilon) = L + t_n(R - L) + \varepsilon \leqslant L + 2\varepsilon.$$

So $R - L \leqslant 3\varepsilon$; letting $\varepsilon \downarrow 0$ yields $R \leqslant L$.

For the reverse inequality, proceed similarly by exchanging the roles of the sequences $\mathbf{u}$ and $\mathbf{v}$: $u_{m(n)} = (1 - t_n)v_{m(n)} + t_n u_n$ with $t_n \in (0, 1)$ and $t_n|R - L| < \varepsilon$, to obtain

$$L - \varepsilon \leqslant S(u_{m(n)}) \leqslant (1 - t_n)S(v_{m(n)}) + t_n S(u_n) \leqslant R + 2\varepsilon.$$

This time $L - R \leqslant 3\varepsilon$ and so $L \leqslant R$.

For the final assertion, again proceed similarly in two steps. Define ε and N as above, and refer to $v_{m(n)} = (1-t_n)u_{m(n)}+t_n v_n$ and $u_{m(n)} = (1-t_n)v_{m(n)} + t_n u_n$, according to the desired inequality. Then fixing $n > N$, find Σ_ℓ with ℓ

so large that $v_{m(n)}, u_{m(n)}, u_n, v_n$ are all in Σ_ℓ. Then, using the continuity of S on Σ_ℓ, and letting $\varepsilon \downarrow 0$ conclude $L \leqslant R$, $R \leqslant L$, as before. $\qquad\square$

Lemma 2′ (Banach-space variant of unique limits). *For a Banach space X and $x \in X$, and $\mathbb{Q}$-affine (closed under rational affine combinations) Σ, if $S : X \to \mathbb{R}$ is both weakly midpoint convex on Σ and locally bounded on Σ at w, then for any sequences in Σ with $u_n \to x$ and $v_n \to x$ with $\{S(u_n)\}_{n\in\mathbb{N}}$ and $\{S(v_n)\}_{n\in\mathbb{N}}$ convergent*

$$\lim S(u_n) = \lim S(v_n).$$

Proof. Put $A := \lim S(u_n)$, $B := \lim S(v_n)$. By the symmetry of the assumptions we may suppose that $A < B$. Noting that the translate $x + \Sigma$ is $\mathbb{Q}$-affine and the translate $S_x(t) = S(x + t)$ is midpoint convex on $x + \Sigma$, w.l.o.g suppose that $x = 0$. Choose $\delta > 0$ and K such that $|S(y)| \leqslant K$ for all $y \in \Sigma$ with $|y| \leqslant \delta$. For $\varepsilon := (B - A)/3 > 0$, there is $m(0)$ so that for $n > m(0)$,

$$B - \varepsilon \leqslant S(v_n).$$

Let $t_n \downarrow 0$ be dyadic rational, e.g. $t_n = 2^{-n}$. Then $s_n := 1/t_n \to \infty$. For each n choose $m(n) > n$ such that $||u_{m(n)}|| < \delta/3$ and $||s_n u_{m(n)}|| < \delta/3, ||s_n v_{m(n)}|| < \delta/3$. Put

$$w_n := s_n v_{m(n)} + (1 - s_n) u_{m(n)} \in \Sigma$$

(as Σ is $\mathbb{Q}$-affine). Then

$$||w_n|| = ||s_n v_{m(n)}|| + ||s_n u_{m(n)}|| + ||u_{m(n)}|| \leqslant \delta,$$

so that $|S(w_n)| \leqslant K$ and

$$v_{m(n)} = (1 - t_n) u_{m(n)} + t_n w_n.$$

Here S is midpoint convex on Σ, so

$$B - \varepsilon \leqslant S(v_{m(n)}) \leqslant (1 - t_n) S(u_{m(n)}) + t_n S(w_n) \to A.$$

But $B - \varepsilon \leqslant A$ gives the contradiction $B - A \leqslant \varepsilon \leqslant (B - A)/3$. $\qquad\square$

Lemma 3 (Regularization). *For $\Sigma \subseteq \mathbb{R}$ locally Steinhaus–Weil and $I \subseteq \operatorname{int}(\operatorname{conv}(\Sigma)) \cap \operatorname{cl}(\Sigma)$, if $S : \mathbb{R} \to \mathbb{R}$ with $S|\Sigma$ both weakly midpoint convex and continuous, write*

$$\bar{S}(x) := \limsup\nolimits_{y \to x}^{\Sigma} S(y) \qquad (x \in I).$$

   Then

(i) *the limit exists for $x \in I$: $\bar{S}(x) := \lim_{y \to x}^{\Sigma} S(y)$;*
(ii) *the function $\bar{S} : I \to \bar{\mathbb{R}}$ is continuous on I.*

   Both conclusions hold with $I = \mathbb{R}$ for a quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$ of $\mathbb{R}$ with each Σ_m locally Steinhaus–Weil and $S : \mathbb{R} \to \mathbb{R}$ quasi σ-continuous under the quasi cover and weakly midpoint convex on $\Sigma = \bigcup_{m\in\mathbb{N}} \Sigma_m$.

Proof. (i) By Lemma 2, condition (i) holds.

(ii) Here we use only that $I \subseteq \mathrm{cl}(\Sigma)$. Suppose that $\bar{S}(x_n) \to L \neq \bar{S}(x)$ for some sequence $x_n \to x$ in I, with L possibly infinite. If $\bar{S}(x_n)$ is finite for an infinite set of n, then for such n we may choose $y_n \in B_{1/n}(x_n) \cap \Sigma$, as Σ is dense in itself, with $|S(y_n) - \bar{S}(x_n)| < 2^{-n}$. Then, down a subsequence, $y_n \to x$ and $S(y_n) \to L$, contradicting $S(y_n) \to \bar{S}(x)$, by (i).

Otherwise, without loss of generality, each term of the sequence $\{\bar{S}(x_n)\}_{n \in \mathbb{N}}$ is infinite and so, being a convergent sequence, all the terms are equal to $L \neq \bar{S}(x)$. So we may choose $y_n \in B_{1/n}(x_n) \cap \Sigma$ with $S(y_n) \to L$. Then again $y_n \to x$, again contradicting $S(y_n) \to \bar{S}(x)$.

As for the final assertion, (i) holds by the last part of Lemma 2, and in (ii) we noted that only $I \subseteq \mathrm{cl}(\Sigma)$ is required. $\qquad\square$

Proof of Theorem 5. (i) and (ii) follow by Lemma 3(i) and by the finiteness result in Corollary 2.

By the continuity of S on Σ, $S|\Sigma = \bar{S}|\Sigma$.

(iii) As in Lemma 1, for any $x \in I$ with $x = \sup\{t \in \Sigma : t < x\} = \inf\{t \in \Sigma : t > x\}$, write $x = t_x u_x + (1 - t_x) v_x$ with $u_x < x < v_x$, $u_x, v_x \in \Sigma \cap I$, and $t_x \in (0, 1)$; then

$$S(x) \leqslant t_x S(u_x) + (1 - t_x) S(v_x).$$

Taking limits as $u_x \uparrow x, v_x \downarrow x$, and w.l.o.g. assuming $t_x \to \tau_x$ (by the boundedness of the points t_x),

$$S(x) \leqslant \tau_x \bar{S}(x) + (1 - \tau_x) \bar{S}(x) = \bar{S}(x).$$

(iv) For distinct $x, y, z \in I \subseteq \mathrm{cl}(\Sigma)$, with $z = \alpha x + (1 - \alpha)y$ and $\alpha \in (0, 1)$, put $\beta = 1 - \alpha$. In Σ choose $x_n \to x, y_n \to y$ and $z_n \to \alpha x + \beta y \in I$; then choose α_n so that with $\beta_n = 1 - \alpha_n$

$$z_n := \alpha_n x_n + \beta_n y_n : \qquad \alpha_n := (y_n - z_n)/(y_n - x_n) \to [y - \alpha x + \beta y]/(y - x) = \alpha.$$

Then, as $x_n, y_n, z_n \in \Sigma$ and $\alpha_n \in (0, 1)$, by Lemma 1

$$S(z_n) = S(\alpha_n x_n + \beta_n y_n) \leqslant \alpha_n S(x_n) + \beta_n S(y_n);$$

so, since $z_n \to \alpha x + \beta y$ through Σ, Lemma 3(i) yields

$$\bar{S}(\alpha x + \beta y) \leqslant \alpha \bar{S}(x) + \beta \bar{S}(y).$$

For $t \in \mathrm{cl}(\Sigma) \cap \mathrm{int}(\mathrm{conv}(\mathrm{cl}(\Sigma)))$ choose $a, b \in \Sigma$ with $t \in (a, b)$ and $u_n \in \Sigma \cap (a, b)$ with $u_n \to t$. Then as in (†)

$$\bar{S}(t) = \lim S(u_n) \leqslant \max(S(a), S(b)) = \max(\bar{S}(a), \bar{S}(b)).$$

The local boundedness of S on $\mathrm{cl}(\Sigma)$ follows from (iii).

As for the final assertion concerning quasi covers, we defer property (ii) to Theorem 6 below, where we use only the other properties (i), (iii) and (iv), to be proved next. As (i) here holds by Lemma 3, we turn to (iii). For any $x \in \mathbb{R}$,

since Σ is dense and the sequence $\{\Sigma_m\}_{m\in\mathbb{N}}$ is increasing, there is m such that $x \in \text{int}(\text{conv}(\Sigma_m))$; say $x \in (a,b)$ with $a,b \in \Sigma_m$. Then for $y \in (a,b)$

$$S(y) \leqslant \max\{S(a), S(b)\},$$

and so, now with $\Sigma = \bigcup_{m=0}^{\infty} \Sigma_m$,

$$\bar{S}(x) := \bar{S}^{\Sigma}(x) = \limsup_{y\to x}^{\Sigma} S(y) \leqslant \max\{S(a), S(b)\} < \infty.$$

First we note that, for $u \in \Sigma_m$, by (iii) above and as $\Sigma_m \subseteq \Sigma$,

$$S(u) \leqslant \bar{S}^{\Sigma_m}(u) \leqslant \bar{S}^{\Sigma}(u).$$

More generally, for $x \in \mathbb{R}$, choose $u_n, v_n \in \Sigma_{m(n)}$ with $u_n \uparrow x, v_n \downarrow x$, and $t_n \in [0,1]$ with $x = t_n u_n + (1-t_n)v_n$; then, by Lemma 1 applied to $\Sigma_m = \Sigma_{m(n)}$ use Th. BLH and apply Theorem 8.   □

$$S(x) \leqslant t_n S(u_n) + (1-t_n)S(v_n) \leqslant t_n \bar{S}^{\Sigma_m}(u_n) + (1-t_n)\bar{S}^{\Sigma_m}(v_n)$$
$$\leqslant t_n \bar{S}(u_n) + (1-t_n)\bar{S}(v_n).$$

By Lemma 3, $\bar{S}^{\Sigma}$ is continuous on $\text{cl}(\Sigma) = \mathbb{R}$, so, assuming w.l.o.g. that $t_n \to t$, pass to the limit:

$$S(x) \leqslant t\bar{S}(x) + (1-t)\bar{S}(x) = \bar{S}(x).$$

Finally, condition (iv) is proved similarly, but more simply, using Lemma 3(ii), i.e. the continuity of $\bar{S}$. If $x < z < y$ are in $\mathbb{R}$, and $\alpha \in (0,1)$ with $z = \alpha x + (1-\alpha)y$, choose $u_n, v_n, w_n \in \Sigma_{m(n)}$ with $u_n \to x, v_n \to y, w_n \to z$, and $t_n \in [0,1]$ with $w_n = t_n u_n + (1-t_n)v_n$. Then, by Lemma 1 and the continuity of $\bar{S}$,

$$S(w_n) \leqslant t_n S(u_n) + (1-t_n)S(v_n) \leqslant t_n \bar{S}(u_n) + (1-t_n)\bar{S}(v_n),$$
$$\bar{S}(z) = \lim S(w_n) \leqslant \alpha\bar{S}(x) + (1-\alpha)\bar{S}(y)$$

as $t_n \to \alpha$.   □

Recall that for $S : \mathbb{R} \to \mathbb{R}$ Baire/measurable, by Th. BL (Sect. 2) S is quasi σ-continuous under a quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$ of $\mathbb{R}$ with Σ_m an *increasing* sequence of sets respectively in $\mathcal{D}_B$ or $\mathcal{D}_L$, hence *each having the Steinhaus–Weil property locally*. Before returning to a Banach space setting we prove (using this 'quasi' apparatus) a result in $\mathbb{R}$, which completes the tasks of Theorem 5 and which thereafter we shall apply (twice) in the context of a 'typical' line segment. Our variant of the Continuity Theorem BD with an alternative condition to local boundedness is this.

Theorem 6. *For a quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$ of $\mathbb{R}$ with each Σ_m locally Steinhaus–Weil, if $S : \mathbb{R} \to \mathbb{R}$ is weakly midpoint convex on $\Sigma = \bigcup_{m=0}^{\infty} \Sigma_m$ and quasi σ-continuous under the quasi cover, then S is continuous on Σ. Furthermore, if S is midpoint convex, then it is continuous (and so convex).*

Note that, in the final assertion, we really need the stronger hypothesis of midpoint convexity. Indeed, the example of the indicator function $\mathbf{1}_{\mathbb{P}}$ of

the irrationals, mentioned earlier, is weakly midpoint convex on $\mathbb{P}$, but only continuous on $\mathbb{P}$.

Proof. Assume S is quasi σ-continuous under the quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$ and is either midpoint convex, or weakly midpoint convex on $\Sigma := \bigcup_{m=0}^{\infty} \Sigma_m$. We use Theorem 5(i),(iii),(iv), as it applies to quasi covers. To prove the continuity of S, by Lemma 3(ii) and Th. 5(iii) and (iv) with $\bar{S} := \bar{S}^\Sigma$, it suffices to consider any x with $S(x) < \bar{S}(x) < \infty$ and to deduce a contradiction. Fix such an x. We consider only $x \in \Sigma$ when S is weakly midpoint convex.

We may take $x = 0$, since translation preserves the quasi covering property (replace S by $S_x(t) := S(x+t)$, which on $\Sigma - x$ is weakly midpoint convex and σ-quasi continuous if S is).

Put $\varepsilon := [\bar{S}(0) - S(0)]/4$. By the continuity of $\bar{S}$ at 0, for some $\Delta > 0$,

$$|\bar{S}(0) - \bar{S}(y)| \leqslant \varepsilon \qquad (y \in (-\Delta, \Delta)). \tag{*}$$

For the purposes of the proof, call $\alpha > 2$ *commensurate* with a set T if there exists $t \in T$ with $\alpha t \in T$. Take any interval $(a, b) \subseteq (\Delta/2, \Delta)$ with

$$(\Delta/4) < (a/2) < (b/2) < (\Delta/2).$$

Being a quasi cover, Σ is dense; so since $2 < 4a/\Delta$ and Σ is dense in $(\Delta/4, \Delta/2)$, choose $t \in \Sigma \cap (\Delta/4, \Delta/2)$ and $\alpha \in (2, 4a/\Delta)$, as close to 2 as desired, subject to

$$\alpha t \in (a, b) \cap \Sigma;$$

this is possible as $\alpha(\Sigma \cap (\Delta/4, \Delta/2))$ is dense in $\alpha(\Delta/4, \Delta/2)$, and $\alpha\Delta/4 < a$, whereas $b < \Delta < \alpha\Delta/2$. Choose $m = m(\alpha)$ with $t, \alpha t \in T_m := \Sigma_m \cap [0, \Delta]$. W.l.o.g. $0 \in \Sigma_m$, if $0 \in \Sigma$. This proves the *claim*: α is commensurate with T_m and is as close to 2 as desired.

For any $\alpha > 2$, put

$$q = q(\alpha) := (\alpha - 1)/\alpha = 1 - (1/\alpha) > 1/2;$$

then, for α commensurate with $T_{m(\alpha)}$ as above, t is the convex combination

$$t = q.0 + \frac{1}{\alpha}.(\alpha t), \qquad t, \alpha t \in [0, \Delta] \cap \Sigma_{m(\alpha)}.$$

By (*) with $y = t$, Lemma 3(ii) and Th. 5(ii) applied at $t \in \Sigma_m$ to $\bar{S}^{\Sigma_m}$,

$$\bar{S}(0) - \varepsilon \leqslant \bar{S}(t) = \bar{S}^{\Sigma_m}(t) = S(t). \tag{**}$$

First, suppose S is *weakly midpoint convex*, so $0 \in \Sigma_m$. As $t_\alpha := \alpha t \in \Sigma_m$,

$$S(t) \leqslant qS(0) + (1/\alpha)S(t_\alpha),$$

since S on Σ_m is continuous so has full convexity as in Lemma 1. But

$$S(t_\alpha) \leqslant \bar{S}(t_\alpha) \leqslant [\bar{S}(0) + \varepsilon],$$

by Th. 5(iii) and (*) with t_α for y. In sum, for sufficiently small $\alpha > 2$ commensurate with $T_{m(\alpha)}$,

$$\bar{S}(0) - \varepsilon \leqslant qS(0) + (1/\alpha)[\bar{S}(0) + \varepsilon],$$

independently of $m(\alpha)$. Let $\alpha \downarrow 2$ through values commensurate with $T_{m(\alpha)} \subseteq \Sigma$, possible by the *claim* above. Then, as $\lim 1/\alpha = 1/2 = \lim q(\alpha)$,

$$\bar{S}(0) - \varepsilon \leqslant (1/2)S(0) + (1/2)[\bar{S}(0) + \varepsilon].$$

This gives the contradiction

$$0 < \bar{S}(0) - S(0) \leqslant 3\varepsilon = (3/4)[\bar{S}(0) - S(0)].$$

Now suppose S is *midpoint convex*. The proof is similar but simpler: interchange q and $(1/\alpha)$ with α dyadic. Choose $t \in \Sigma'$ and a *dyadic* $\alpha > 2$, as close to 2 as desired, subject to the simpler condition

$$\alpha t \in (a, b).$$

With $q(\alpha)$ as before, write $t \in \Sigma_m$ as the *dyadic* convex combination

$$t = (1/\alpha).0 + q.\alpha t/(\alpha - 1), \qquad t, t\alpha/(\alpha - 1) \in [0, \Delta]$$

(as $\alpha/(\alpha - 1) < 2$ and $t < \Delta/2$). This time put $t_\alpha := \alpha t/(\alpha - 1)$. As before (**) holds, by (*) with t and t_α for y. As S is midpoint convex,

$$S(t) \leqslant (1/\alpha)S(0) + qS(t_\alpha),$$

since α is dyadic. Continue as before to conclude that, for any sufficiently small dyadic $\alpha > 2$,

$$\bar{S}(0) - \varepsilon \leqslant (1/\alpha)S(0) + q[\bar{S}(0) + \varepsilon].$$

Letting $\alpha \downarrow 2$, we again deduce the contradiction

$$\bar{S}(0) - \varepsilon \leqslant (1/2)S(0) + (1/2)[\bar{S}(0) + \varepsilon]. \qquad \square$$

As a corollary we now have a result on separable Banach spaces, which by Theorem B will enable us to prove in their more general setting Theorems M and FS, stated in Sect. 1. Note the local character of the key assumption.

Theorem 7. *For a separable Banach space X, a quasi cover $\{\Sigma_m\}_{m \in \mathbb{N}}$, with each Σ_m Baire, locally non-meagre and so locally Steinhaus–Weil, if $S : X \to \mathbb{R}$ is midpoint convex, Baire, and quasi σ-continuous under $\{\Sigma_m\}_{m \in \mathbb{N}}$, then S is continuous.*

Proof. Since each Σ_m has the Steinhaus–Weil property locally, we may proceed as in Theorem 6 above to consider $x \neq 0$ with $S(x) < \bar{S}(x)$; define $\varepsilon > 0$ as there and choose $\Delta > 0$ similarly so that (*) holds for $y \in B_\Delta(x)$. Take $\delta < \Delta/2$ and $\Sigma' := \Sigma \cap B_\delta(x)$, which is non-meagre. By the Kuratowski-Ulam Theorem (as in the second proof of Th. 4B, Sect. 4), choose $\sigma \in \Sigma'$ such that the ray

$$R_x(\sigma) := \{x + \lambda(\sigma - x) : \lambda \geqslant 0\}$$

meets Σ' in a non-meagre set: otherwise $\Sigma' \cap R_x(\sigma)$ is meagre for all $\sigma \in \Sigma'$, and so Σ' is meagre. As $\Sigma' \cap R_x(\sigma)$ is Baire, there is an interval $I := [s, s']$ along $R_x(\sigma)$ for which $\Sigma' \cap I$ is co-meagre in I. Continue as in Theorem 6, working in $R_x(\sigma)$ rather than $\mathbb{R}_+$, to obtain a contradiction to $S(x) < \bar{S}(x)$, so deducing the continuity of S. $\qquad\square$

As an immediate corollary we are now able to prove Theorem M due to Mehdi (albeit for a general topological vector space) and Theorem FS in a form that is slightly weaker than the result of Fischer and Słodkowski [35] (where universal measurability is modulo Haar null sets, i.e. $\mathcal{H}$-measurability and in Sect. 2, as that requires Theorem 8 below).

Proof of Theorem M. By Theorem B we may assume w.l.o.g. that X is separable. By Theorem BL, S is continuous relative to a co-meagre (so dense) set Σ. Since Σ has the Steinhaus–Weil property locally, we may apply Theorem 7 above with $\Sigma_m \equiv \Sigma$, as S is midpoint convex, so deducing the continuity of S. $\qquad\square$

Proof of Theorem FS (universally measurable case). As above, we may again assume that X is separable. For any distinct points a, b, consider the line L through a and b, and let λ be the Lebesgue measure on L. Then $S|L : L \to \mathbb{R}$ is universally measurable, so λ-measurable and so quasi σ-continuous (under a quasi cover by 'density open' sets Σ_m), by Luzin's Theorem. As S is midpoint convex by Theorem 6, $S|L$ is continuous on L and so fully convex on L. So S is fully convex. By Theorem 2H$'$, S is locally bounded, so continuous by Theorem BD*. $\qquad\square$

We close with an analogue of Theorem 7. We will need to argue as in Theorem 6 but twice over: once, in the 'measure-case' mode of Theorem 6 (using quasi σ-continuity), to establish that the continuity points form a big set (as in Luzin's Theorem), and then again, but now in the 'category mode' of Theorem 6 as in Theorem 7 (where Σ is dense and locally Steinhaus–Weil). This reflects the hybrid nature of Christensen's definition of Haar null sets.

Theorem 8. *For a separable Banach space X, a quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$, with each Σ_m locally non-Haar null and so locally Steinhaus–Weil, under which $S : X \to \mathbb{R}$ is quasi σ-continuous : if S is midpoint convex and universally measurable, then S is continuous.*

Proof. Put $\Gamma := \{x \in X : S \text{ is continuous at } x\}$; then Γ is universally measurable. Indeed, by Lemma 3, $\bar{S}$ is well-defined and continuous (from the given quasi cover $\{\Sigma_m\}_{m\in\mathbb{N}}$). Thus S is discontinuous at x iff $S(x) \neq \bar{S}(x)$, and so, since $\bar{S}$ is continuous and S universally measurable, the complement of Γ is

$$\bigcup_{q\in\mathbb{Q}} \{x \,:\, S(x) < q < \bar{S}(x)\} \cup \{x : \bar{S}(x) < q < S(x)\}$$
$$= \bigcup_{q\in\mathbb{Q}} S^{-1}(-\infty, q) \cap \bar{S}^{-1}(q, \infty) \cup \bar{S}^{-1}(-\infty, q) \cap S^{-1}(q, \infty),$$

so universally measurable.

We claim first that $\Gamma \cap U$ is non-Haar null for all non-empty open U. If not, $U \cap \Gamma$ is Haar null for some non-empty open U; then, by the definition of Haar nullity (see Sect. 3), there exist a Borel set $H \supseteq U \cap \Gamma$ and a Borel probability measure μ such that $\mu(g + H) = 0$ for all $g \in X$. W.l.o.g. $U = u + B_\delta$; as X is separable, a countable number of translates $t_i + U$ of U, and so also of B_δ, cover X. So $\mu(u + v + B_\delta) > 0$ for some $v := t_i$. Put $\mu_v(E) := \mu(v + E)$ for Borel $E \subseteq X$; then μ_v is finite with $\mu_v(U) > 0$, and S is quasi σ-continuous under an appropriate quasi cover $\{\Sigma'_m\}_{m \in \mathbb{N}}$ comprising sets of locally positive μ_v-measure, by Luzin's Theorem. Proceed as in Theorem 7, but this time applying Christensen's WFT in place of the Kuratowski-Ulam Theorem (using $\{\Sigma'_m\}_{m \in \mathbb{N}}$), to deduce that S is continuous at x for each $x \in U$, so contradicting the assumption that H is Haar null (and so not the whole of B_δ).

Being universally measurable and locally non-Haar null, Γ has the Steinhaus–Weil property locally, by a theorem of Christensen [28, Th. 2] (cf. Solecki [96, Th. 1(ii) via Prop. 3.3(i)]). With $\Sigma = \Gamma$, as $X = \bar{\Gamma}$, proceed once more as in Theorem 7, again applying Christensen's WFT in place of the Kuratowski-Ulam Theorem. This gives that S is continuous on X. $\square$

Theorem FS in the $\mathcal{H}$-measurable case is an immediate corollary of Theorem 8 and Theorem BLH, the following variant form of Theorem BL concerned with $\mathcal{H}$-measurablity.

Theorem BLH (cf. Th. BL). *For a separable Banach space X and $\mathcal{H}$-measurable $f : X \to \mathbb{R}$, f is quasi σ-continuous.*

Proof. Take a basis $\{U_i\}$ in $\mathbb{R}$ and choose B_i universally measurable and H_i Haar-null such that $f^{-1}(U_i) = B_i \cup H_i$, and put $H = \bigcup_{i \in \mathbb{N}} H_i$, which is Haar-null. Choose a probability measure μ witnessing the Haar-nullity of H, that is: choose $N \supseteq H$ universally measurable and a probability measure μ with $\mu(x + N) = 0$, for all $x \in X$. Then for $g := f|_{X \setminus N}$, since the pre-image

$$g^{-1}(U_i) = B_i \setminus N$$

is universally measurable for each i, g is μ-measurable. Now apply Theorem BL to obtain an increasing sequence of μ-measurable non-μ-null sets $\Sigma_m \subseteq X \setminus N$ with g continuous on Σ_m (for each m) and with $\Sigma = \bigcup_{m \in \mathbb{N}} \Sigma_m$ co-μ-null on $X \setminus N$, and so co-μ-null on X. As $f = g$ on Σ_m, this sequence yields a quasi cover relative to which f is quasi σ-continuous. $\square$

This yields an immediate consequence:

Proof of Theorem FS (universally $\mathcal{H}$-measurable case). By Th. BLH w.l.o.g. S is universally measurable and σ-continuous. Now apply BLH and apply Theorem 8. $\square$

6. Complements

1. Berz's other theorems. A sublinear function S has $\mathbb{Q}_+$-convex epigraph C. This observation allows Berz to deduce from the $\mathbb{Q}$-version of the Hahn–Banach theorem that S is the supremum of all the additive functions f which it majorizes; the proof refers to the $\mathbb{Q}$-hyperplanes defined by f that support the epigraph. Since a Baire/measurable S is locally bounded (Th. 2 above), all of the additive minorants of S supporting C are bounded above and so linear by *Darboux's Theorem* (see e.g. [16] and the references cited there). This allows Berz to deduce that their upper envelope comprises the two half-lines defining S (equivalently, this is the upper envelope of the supremum and infimum of the additive minorants of S). Hence Berz deduces a third result: when S is symmetric about the origin it may be represented as a *norm*. Indeed, embed $x \mapsto \{f(x)\}_f$ so that $f(x)$ is the projection of x onto the f co-ordinate space; then a norm is defined by identifying x with its imaging and setting

$$||x|| := \sup_f |f(x)| = S(x).$$

2. Automatic continuity. The proof of Theorem 1 is inspired by an idea due to Goldie appearing in [6, I, Th. 5.7] (cf. [7, Th. 3.2.5]), and more fully exploited in a recent series of papers including [17–20, Prop. 3]. The theme here is the interplay between *functional inequalities* (as with subadditivity, convexity etc.) and *functional equations* (as with additivity and the Cauchy functional equation). Here, minimal regularity implies continuity—whence the term automatic continuity—and linearity; see e.g. [15] and the references cited there.

3. Automatic continuity and group action. An automatic continuity theorem of Hoffmann-Jørgensen is particularly relevant here for the discussion of the Baire-Berz Theorem. Hoffmann-Jørgensen proves in [86, Part 3: Th. 2.2.12] the (sequential) continuity of a Baire function $f : X \to Y$ when a single non-meagre group T acts on the two (Hausdorff) spaces X and Y with $f(tx) = tf(x)$, by appealing to a KBD argument (under T rather than under addition) in X. In the Baire-Berz Theorem it is a meagre group, namely $\mathbb{Q}_+$, that acts multiplicatively on the Banach spaces X and $Y = \mathbb{R}$; but it is the additive structure of a Banach space which permits the use of KBD to obtain global continuity from continuity on a smaller set.

4. Convex and coherent risk measures. As remarked in Sect. 1, Berz's sublinearity theorem is connected with the theory of *coherent* risk measures [36, §4.1]. The key properties are convexity and positive homogeneity ($\rho(\lambda x) = \lambda \rho(x)$ for $\lambda \geqslant 0$). Under positive homogeneity, convexity is equivalent to subadditivity. This paper thus extends to sublinearity studies of the related areas of convexity, subadditivity and additivity, for which see e.g. [8,10].

In the economic/financial context, positive homogeneity—a form of scale-invariance—means that large and small firms (or agents) have similar preferences; see e.g. Lindley [63, Ch. 5]. This is far from the case in practice, which is why *convex* risk measures (in which positive homogeneity is dropped) are often preferred; again, see e.g. [36, §4.1]. Sensitivity to scale here is related to the curvature of utility functions, and the 'law of diminishing returns'. This incidentally underpins the viability of the insurance industry; again see e.g. Lindley [63, Ch. 5].

The two half-lines in Berz's theorem correspond to taking long and short positions in one dimension. One can extend to many dimensions, as in [36], where the 'broken line' becomes a cone, and as we do in Sect. 4. Berz himself worked in one dimension, as his motivation was normability (below).

5. Normability. As norms are necessarily sublinear, Berz's third result (6.1) addresses the question of which sublinear functions are realized as norms. In this connection, the criterion for normability of a topological vector space was established by Kolmogorov, see e.g. [89, Th. 1.39]; for recent metric characterizations of normability—in terms of translation-invariant metrics—see the Oikhberg-Rosenthal result [73] demanding continuity of scaling and *isometry* of all one-dimensional subspaces $R(x)$ with $\mathbb{R}$, a theme present in Thms 4B/M. Šemrl's relaxation [90] drops this continuity when spaces are of dimension at least 2. (As for relaxation of homogeneity see [65].) Invariant metrics are provided by the Birkhoff-Kakutani normability theorem—see e.g. [89, Th. 1.24], [49, Th. 8.3], or for recent accounts [40, Ch. 1–4], [75, §2.1].

6. Beyond local compactness: Haar category-measure duality. In the absence of the Haar measure, the definition (in Sect. 2) of left Haar null subsets of a topological group G required $\mathcal{U}(G)$, the universally measurable sets—by dint of the role of the totality of (probability) measures on G. The natural dual of $\mathcal{U}(G)$ is the class $\mathcal{U}_{\mathcal{B}}(G)$ of *universally Baire sets*, defined, for G with a Baire topology, as those sets B whose preimages $f^{-1}(B)$ are Baire (have the Baire property) in any compact Hausdorff space K for any continuous $f : K \to G$. Initially considered in [33] for $G = \mathbb{R}$, these have attracted continued attention for their role in the investigation of axioms of determinacy and large cardinals—see especially [101].

Analogously to the left Haar null sets, we have in G the family of *left Haar meagre* sets, $\mathcal{HM}(G)$, to comprise the sets M coverable by a universally Baire set B for which there are a compact Hausdorff space K and a continuous $f : K \to G$ with $f^{-1}(gB)$ meagre in K for all $g \in G$. These were introduced, in the *abelian* Polish group setting and with K metrizable, by Darji [31], cf. [53], and shown there to form a σ-ideal of meagre sets (co-extensive with the meagre sets for G locally compact). Analogously to $\mathcal{H}$-measurablity, Jabłońska [53] defines $Y \subseteq X$ to be $\mathcal{D}$-*measurable* ('D' for Darji) if $Y = B \cup H$ for some Borel B and Haar-meagre H, and $f : X \to \mathbb{R}$ to be $\mathcal{D}$-*measurable* if $f^{-1}(U)$ is $\mathcal{D}$-measurable for each open $U \subseteq \mathbb{R}$. Since Haar-meagre sets are meagre, this

implies that f is Baire. Thus Theorem BD for $\mathcal{D}$-measurable S is subsumed in Theorem M. For a survey of similarities between the dual concepts see [52].

Acknowledgements

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