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Aequationes Mathematicae

On convex iterative roots of non-monotonic mappings

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Abstract. Let I be an interval. We consider the non-monotonic convex self-mappings $f : I \to I$ such that f^2 is convex. They have the property that all iterates f^n are convex. In the class of these mappings we study three families of functions possessing convex iterative roots. A function f is said to be *iteratively convex* if f possesses convex iterative roots of all orders. A mapping f is said to be *dyadically convex* if for every $n \geq 2$ there exists a convex iterative root $f^{1/2^n}$ of order 2^n and the sequence $\{f^{1/2^n}\}$ satisfies the condition of compatibility, that is $f^{1/2^n} \circ f^{1/2^n} = f^{1/2^{n-1}}$. A function f is said to be *flowly convex* if it possesses a convex semi-flow of f, that is a family of convex functions $\{f^t, t > 0\}$ such that $f^t \circ f^s = f^{t+s}$, $t, s > 0$ and $f^1 = f$. We show the relations among these three types of convexity and we determine all convex iterative roots of non-monotonic functions.

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1. Introduction

Let I be an interval and let $f : I \to I$ be a convex continuous function. If f is increasing then the composition $f^2 = f \circ f$ is also convex, however if f is not increasing then f^2 need not be convex. We will show that if f^2 is convex then all iterates f^n of the function f are convex. The natural question is: when does the converse property hold Namely, which convex functions have the property that they are the iterates of convex functions. In general even increasing smooth convex functions do not have to be iterates of convex functions. Ger in paper [1] discovered a special family of C^1 increasing convex functions which are not the second iterates of convex functions. Up to now we do not have an exact characterization of such functions, though we can explicitly determine the family of all increasing C^1 functions which posses convex iterative roots. In this note we concentrate on the problem of characterizing non-monotonic functions possessing convex iterative roots and their determination. A lot of information on convex iterative roots of strictly increasing functions can be found in [2,6–9,12].

If I is an interval bounded above then every convex function $f : I \to I$ is continuous in $\operatorname{Int} I$ and there exist the limits of f at the ends of I (see [3]). Thus in the case of closed interval we will assume that f is continuous at the ends of I. Moreover, if the interval I is unbounded above then we will assume that f is continuous.

2. Preliminary results

Let $I = [a, b]$, where $a < b \leq \infty$, and f be a continuous function which maps I into itself.

Lemma 1. *If f and f^2 are convex then there exists $x_0 \in I$, where f attends the local minimum, so that $f(x_0) \geq x_0$ or f is strictly decreasing.*

Proof. It is well known that for every convex function f there exist $a \leq x_1 \leq x_2 \leq b$ such that f attends its minimum at x_1 and x_2, $f_{|[a,x_1]}$ is strictly decreasing (if $x_1 \neq a$), $f_{|[x_2,b]}$ is strictly increasing (if $x_2 \neq b$) and $f(x) = f(x_1)$ for $x \in [x_1, x_2]$ (see e.g. [3]). We will show that $f(x_1) \geq x_1$. Suppose on the contrary that $f(x_1) < x_1$. By the continuity of f at x_1 there exists a neighbourhood U_{x_1} of x_1 such that $f[U_{x_1}] \subset [f(x_1), x_1)$. If $x_1 = b$ then f is strictly decreasing. Let $x_1 \neq b$. Put

$$U_{x_1}^- := U_{x_1} \cap [a, x_1], \quad U_{x_1}^+ := U_{x_1} \cap [x_1, b].$$

Since $f_1 := f|_{U_{x_1}^-}$ is strictly decreasing, $f_2 := f|_{U_{x_1}^+}$ is increasing and $f_{|[f(x_1),x_1]}$ is strictly decreasing, we get that $f^2|_{U_{x_1}^-} = f|_{[f(x_1),x_1)} \circ f_1$ is strictly increasing and $f^2|_{U_{x_1}^+} = f|_{[f(x_1),x_1)} \circ f_2$ is decreasing, so f^2 has a local maximum at x_1 but this contradicts the convexity of f^2. □

By the way we have proved more

Remark 1. If f and f^2 are convex and $[x_1, x_2]$ is an interval of constancy of f then $f(x_1) \geq x_1$.

Applying an obvious fact, that for every convex and increasing h and convex f the composition $h \circ f$ is convex, we prove the following

Lemma 2. *If f is convex and has a local minimum at $x_0 \in (a, b)$ so that $f(x_0) \geq x_0$ then all iterates f^n are convex.*

Proof. We have $f(x) \geq f(x_0) \geq x_0$, so $f[I] \subset [x_0, b]$. Putting $h := f|_{[x_0,b]}$ we get $f^n(x) = h^n(x)$ for $x \in [x_0, b]$. Since h is convex and increasing, the functions $f^{n+1} = f^n|_{[x_0,b]} \circ f = h^n \circ f$ for $n \geq 1$ are convex as a composition of the convex function f and the convex increasing mappings h^n. □

Theorem 1. *If f and f^2 are convex then all iterates f^n are convex.*

Proof. For increasing functions this assertion is trivial. If f is decreasing then f^2 is increasing and convex. Hence $f^{2n} = (f^2)^n$ are convex. On the other hand $f^{2n+1} = f^{2n} \circ f$ is convex as a composition of a convex function and a convex increasing function. If f is not monotonic then the assertion is a simple consequence of Lemmas 1 and 2. $\square$

As a direct consequence of Lemmas 1 and 2 we get the following.

Theorem 2. *If f is convex then f^2 is convex if and only if there exists $x_0 \in I$, where f attends the local minimum so that $f(x_0) \geq x_0$ or f is strictly decreasing.*

Similarly, applying the fact that the composition $h \circ f$ of concave functions f with increasing h is concave, we prove the analogous properties for concave functions.

Remark 2. Let f be concave. Then f^2 is concave if and only if there exists $x_0 \in I$, where f attends the local maximum so that $f(x_0) \geq x_0$ or f is strictly decreasing. Then all iterates f^n are concave.

If f is convex and f^2 is not convex then it follows by Theorem 2, that the iterates f^n are piecewise monotonic functions with increasing number of oscillations.

Let us note the following obvious fact.

Lemma 3. *If f and f^2 are convex and f has a strict local minimum at $x_0 < b$ then all iterates f^n have a strict local minimum at x_0.*

Every convex function f has one or two fixed points. (If $b = \infty$ then $\lim_{x \to \infty} f(x) = \infty$ and we admit ∞ as a fixed point of f.) If f has two fixed points then one of them equals b. In this case denote by p_f this fixed point which is different from b. However, if f has one fixed point then we denote it also by p_f.

Lemma 4. *If f and f^2 are convex and $[x_1, x_2]$ is the interval of constancy of f and $a < x_1 < x_2 \leq p_f$ then $[x_1, x_2]$ is the interval of constancy of all f^n.*

Proof. We have $p_f > f(x) > x$ for $x_2 \leq x < p_f$ and $p_f < f(x) < x$ for $x > p_f$. Hence f maps $[x_2, b]$ into itself and all iterates $f^n|_{[x_2,b]}$ are strictly increasing, since $f|_{[x_2,b]}$ is strictly increasing. The constancy of f in $[x_1, x_2]$ implies that $f^n|_{[x_1,x_2]}$ is constant. Put $g(x) := f(x)$ for $x \in [a, x_1]$. We have

$$g([a, x_1]) = f([a, x_1]) \subset [f(x_1), b] = [f(x_2), b] \subset [x_2, b],$$

because $x_2 \leq p_f$. In consequence, $f^{n+1}(x) = f^n \circ g(x) = f^n|_{[x_2,b]} \circ g(x)$ for $x \in [a, x_1]$. Thus all f^{n+1} are strictly decreasing on $[a, x_1]$ since g is strictly decreasing. $\square$

Lemma 5. *If f and f^2 are convex and $[x_1, x_2]$ is the interval of constancy of f and $p_f \in [x_1, x_2)$ then for every $n \geq 2$ there exist $y_1 \leq x_1$ and $y_2 > x_2$ such that $[y_1, y_2]$ is the interval of constancy of f^n. If $x_1 = p_f$ then $y_1 = p_f$. If $x_1 < p_f$ then $y_1 < x_1$. If $y_1 \neq a$ then $f^n|_{[a,y_1]}$ is strictly decreasing. If $y_2 \neq b$ then $f^n|_{[y_2,b]}$ is strictly decreasing.*

Proof. Define $\{y_1, y_2\} := f^{-1}[\{x_2\}]$ and $y_1 < p_f < y_2$. If y_1 with such a property does not exist we put $y_1 := a$. If such y_2 does not exist we put $y_2 := b$. The rest of the proof is a simple verification analogous to that in Lemma 4. □

Let f and f^2 be convex. The following three cases may occur:

(A) f has a strict minimum at x_0,
(B) f has an interval of constancy $[x_1, x_2]$ such that $x_2 \leq p_f$,
(C) f has an interval of constancy $[x_1, x_2]$ such that $x_1 \leq p_f < x_2$.

Lemma 6. *If f satisfies (A) or (B) then*
(i) $p_f > f^{n+1}(x) > f^n(x)$, $n \geq 0$, *for* $x < p_f$ *and* $f(x) < p_f$*;*
(ii) $p_f \leq f^{n+1}(x) < f^n(x)$, $n \geq 0$ *for* $x < p_f$ *and* $f(x) > p_f$*;*
(iii) $p_f \leq f^{n+1}(x) < f^n(x)$ *for* $x > p_f$.
If f satisfies (C) then $p_f \leq f^{n+1}(x) \leq f^n(x)$ for $x \in [a, b]$.

The proof is a simple verification based on the inequalities $f(x) > x$ for $x < p_f$ and $p_f \leq f(x) < x$ for $x > p_f$.

Definition 1. Let $f : X \to X$. Every function $g : X \to X$ such that $g^n = f$ is said to be an *iterative root of n-th order of f*.

If we do not assume any conditions on functions g and f then iterative roots are not determined uniquely, nevertheless we denote each of them by $f^{\frac{1}{n}}$. This symbol is ambiguous.

It follows by Theorem 1 that every convex function f which possesses a convex square iterative root $f^{\frac{1}{2}}$ has the property that f^2 is also convex. Thus in all theorems concerning convex iterative roots we will assume that f and f^2 are convex.

As a simple consequence of Lemma 6 we get

Corollary 1. *Assume f has a convex iterative root of n-th order $f^{\frac{1}{n}}$.*

If f satisfies (A) then $f^{\frac{1}{n}}$ satisfies (A) at the same point x_0.

If f satisfies (B) then $f^{\frac{1}{n}}$ satisfies (B) in the same interval $[x_1, x_2]$.

If f satisfies (C) then $f^{\frac{1}{n}}$ satisfies (C) in an interval $[a_n, b_n] \subset [x_1, x_2]$ for some $a_n < p_f < b_n$.

Moreover, in cases (A), (B),

$$x < f^{\frac{1}{n}}(x) < f(x) \text{ for } x < p_f \text{ and } f^{\frac{1}{n}}(x) > f(x) \text{ for } x > p_f.$$

However, in cases (C),

$$f^{\frac{1}{n}}(x) \geq f(x) \geq p_f \quad \text{for } x \in [a, b].$$

Example 1. Let $p \in (a,b)$ and $\mathcal{G}$ be a family of all convex functions $g : [a,b] \to [p,b]$ such that $g(x) = p$ for $x \in [p,b]$. It is easy to see that every function $g \in \mathcal{G}$ is an n-th convex iterative root of the constant function $f(x) = p$, $x \in [a,b]$.

Let f and f^2 be convex. Define two intervals $I_a := [a, y_0]$ and $I_b := [y_0, b]$, where

$$y_0 := \begin{cases} x_0, & \textit{in cases } (A) \\ x_2, & \textit{in cases } (B) \\ p_f, & \textit{in cases } (C) \end{cases}.$$

Lemma 7. *Let f and f^2 be convex and f have a convex iterative root $f^{\frac{1}{n}}$. Put $S := f^{\frac{1}{n}}|_{I_b}$ and $K := f^{\frac{1}{n}}|_{I_a}$. Then $S : I_b \to I_b$ is increasing, $K : I_a \to I_b$ is decreasing and*

$$S^{n-1} \circ K = f|_{I_a}. \tag{1}$$

Proof. By Corollary 1, $f^{\frac{1}{n}}[I_b] \subset I_b$. Hence $S : I_b \to I_b$ and $S^n = f|_{I_b}$. By the same Corollary we have $K(x) \geq K(y_0) = f^{\frac{1}{n}}(y_0) \geq y_0$ for $x \in I_a$. Thus $K : I_a \to I_b$.

Now, let $x \in I_a$, then $K(x) \in I_b$ and $f^{\frac{1}{n}} \circ K(x) = S \circ K(x) \in I_b$. Further, $f^{\frac{1}{n}} \circ (f^{\frac{1}{n}} \circ K)(x) = f^{\frac{1}{n}} \circ (S \circ K)(x) = S \circ S \circ K(x)$ and so on. Repeating this operation $n-1$ times we get $(f^{\frac{1}{n}})^{n-1} \circ K(x) = S^{n-1} \circ K(x)$, $x \in I_a$. On the other hand $(f^{\frac{1}{n}})^{n-1} \circ K(x) = (f^{\frac{1}{n}})^{n-1} \circ f^{\frac{1}{n}}(x) = f(x)$ for $x \in I_a$. Thus we get (1). □

Let us introduce the following formal notation

$$f^{\frac{k}{n}} := (f^{\frac{1}{n}})^k = \underbrace{f^{\frac{1}{n}} \circ \ldots \circ f^{\frac{1}{n}}}_{k \textit{ times}}.$$

Note that the iterative roots are not unambiguous, thus the equality $\frac{k}{n} = \frac{p}{q}$ does not imply that $f^{\frac{k}{n}} = f^{\frac{p}{q}}$.

Lemma 8. *Let f and f^2 be convex and f have a convex iterative root $f^{\frac{1}{n}}$. Putting $g := f|_{I_a}$ and $K := f^{\frac{1}{n}}|_{I_a}$ we have*

$$f^{\frac{n-1}{n}} \circ K = g \tag{2}$$

and

$$f(a) \leq f^{\frac{n-1}{n}}(b). \tag{3}$$

Proof. Note that (2) is only a modified notation of (1). It follows, by Lemma 7, that $K(a) \in I_b$ and the mapping $f^{\frac{n-1}{n}}|_{I_b}$ is increasing. Hence, by (1), we obtain $f(a) = g(a) = f^{\frac{n-1}{n}}(K(a)) \leq f^{\frac{n-1}{n}}(b)$. □

Remark 3. In cases (A) and (B) inequality (3) is equivalent to

$$f^{\frac{1}{n}}(f(a)) \leq f(b), \tag{4}$$

however, in cases (C) inequality (3) implies (4).

Proof. We have $f(a) \geq f(y_0) \geq y_0$, because $f|_{I_a}$ is non-increasing. Hence, by (3), $f(a), f^{\frac{n-1}{n}}(b) \in [y_0, b]$. Since $f^{\frac{n-1}{n}}|_{I_b}$ is increasing, (3) implies that

$$f^{\frac{1}{n}}(f(a)) \leq f^{\frac{1}{n}}(f^{\frac{n-1}{n}}(b)) = f(a).$$

Conversely, in cases (A) and (B), $f^{\frac{n-1}{n}}|_{I_b}$ is strictly increasing, thus the last inequality implies (3). □

Theorem 3. *Let f and f^2 be convex and satisfy (A) or (B) and let $g := f|_{I_a}$ and $h := f|_{I_b}$. If f has a convex iterative root $f^{\frac{1}{n}}$ then h has a convex iterative root $h^{\frac{1}{n}}$,*

$$f^{\frac{1}{n}}(x) = \begin{cases} h^{\frac{1}{n}}(x), & x \in I_b \\ (h^{\frac{n-1}{n}})^{-1} \circ g(x), & x \in I_a, \end{cases} \tag{5}$$

and

$$f(a) \leq h^{\frac{n-1}{n}}(b). \tag{6}$$

Conversely, if h possesses a convex iterative root $h^{\frac{1}{n}}$ satisfying (6) and such that $(h^{\frac{n-1}{n}})^{-1} \circ g$ is convex then (5) defines a convex iterative root of f.

Proof. If $f^{\frac{1}{n}}$ is a convex iterative root of f then, by Lemma 7 and Corollary 1, $f^{\frac{1}{n}}[I_b] \subset I_b$ and the mapping $h^{\frac{1}{n}} := f^{\frac{1}{n}}|_{I_a}$ is a convex, strictly increasing iterative root of h. Furthermore Lemma 7 implies, that $K : I_a \to I_b$, so by (2) in Lemma 8,

$$h^{\frac{n-1}{n}}(K(x)) = f^{\frac{n-1}{n}}(K(x)) = g(x) \text{ for } x \in I_a,$$

where $K := f^{\frac{1}{n}}|_{I_a}$. Hence $f^{\frac{1}{n}}(x) = K(x) = (h^{\frac{n-1}{n}})^{-1} \circ g(x)$ for $x \in I_a$. Thus we get (5). Obviously (3) gives (6).

Conversely, let $h^{\frac{1}{n}}$ be a convex iterative root of h satisfying (6). Note that (A) and (B) imply that $p_f \in \mathrm{Int\,I_b}$, $h(x) > x$ for $x < p_f$ and $h(x) < x$ for $x > p_f$. By Corollary 1, $x < h^{\frac{1}{n}}(x)$ for $x < p_f$, and $h^{\frac{1}{n}}(x) < x$ for $x < p_f$. We will show that $h^{\frac{n-1}{n}}(x) < h(x)$ for $x < p_f$. Suppose on the contrary, that there exists an $\overline{x} < p_f$ such that $h^{\frac{n-1}{n}}(\overline{x}) \geq h(\overline{x})$. Then $h(\overline{x}) = h^{\frac{1}{n}}(h^{\frac{n-1}{n}}(\overline{x})) \geq h^{\frac{1}{n}}(h(\overline{x})) = h(h^{\frac{1}{n}}(\overline{x}))$. Since $h^{\frac{1}{n}}(\overline{x}) \in I_b$ and $h^{\frac{1}{n}}$ is strictly increasing we have $\overline{x} \geq h^{\frac{n-1}{n}}(\overline{x})$. A contradiction.

In particular we get

$$h^{\frac{n-1}{n}}(y_0) < h(y_0). \tag{7}$$

In view of (6) and (7)

$$g[I_a] = [g(y_0), g(a)] = [h(y_0), f(a)] \subset [h^{\frac{n-1}{n}}(y_0), h^{\frac{n-1}{n}}(b)] = h^{\frac{n-1}{n}}[I_b].$$

Thus we may define the functions

$$R(x) := (h^{\frac{n-1}{n}})^{-1} \circ g(x), \quad x \in I_a$$

and

$$Q(x) := \begin{cases} h^{\frac{1}{n}}(x), & x \in I_a \\ R(x), & x \in I_b. \end{cases}$$

We have $Q^n = f$. In fact, since $Q : I \to I_b$ we have $Q^n = Q \circ Q^{n-1} = h^{\frac{1}{n}} \circ Q^{n-1} = h^{\frac{1}{n}} \circ Q \circ Q^{n-2} = h^{\frac{2}{n}} \circ Q^{n-2} = \cdots = h^{\frac{n-1}{n}} \circ Q$. If $x \in I_b$ then

$$Q^n(x) = h^{\frac{n-1}{n}} \circ g(x) = h^{\frac{n-1}{n}} \circ h^{\frac{1}{n}} \circ h(x) = f(x).$$

If $x \in I_a$ then

$$Q^n(x) = h^{\frac{n-1}{n}} \circ R(x) = h^{\frac{n-1}{n}} \circ (h^{\frac{n-1}{n}})^{-1} \circ g(x) = g(x) = f(x).$$

Put $z_0 := R(y_0) = (h^{\frac{n-1}{n}})^{-1} \circ g(y_0)$. We have $h^{\frac{n-1}{n}}(z_0) = g(y_0) = h(y_0) = h^{\frac{n-1}{n}} \circ h^{\frac{1}{n}}(y_0)$. Hence $z_0 = h^{\frac{1}{n}}(y_0)$, so the function Q is correctly defined and continuous. The functions R and $h^{\frac{1}{n}}$ are convex and Q has a minimum at y_0, so Q is convex. $\square$

An analogous result holds also in case (C) but the formula (5) should be modified. It has a more complicated form. The method of the proof is also simple but slightly burdensome.

3. Main results

Definition 2. We will say that a function f is *iteratively convex* if for every $n \geq 2$ there exists a convex iterative root of n-th order of f.

Definition 3. We will say that a function f is *dyadically convex* if for every $n \geq 1$ there exists a convex iterative root $f^{\frac{1}{2^n}}$ of 2^n-th order and the sequence $\{f^{\frac{1}{2^n}}\}$ satisfies the condition of compatibility, that is

$$f^{\frac{1}{2^n}} \circ f^{\frac{1}{2^n}} = f^{\frac{1}{2^{n-1}}}, \quad n \geq 1. \tag{8}$$

Definition 4. We will say that a function f is *flowly convex* if there exists a convex semi-flow of f also called an iteration semi-group of f, i.e. a family of convex functions $\{f^t, t > 0\}$ such that $f^t \circ f^s = f^{t+s}$ for $t, s > 0$ and $f^1 = f$.

Remark 4. Note that if f is dyadically convex then f and f^2 are convex and every flowly convex function is iteratively and dyadically convex.

Let us recall

$$I_b := \begin{cases} [x_0, b] & \textit{in case } (A) \\ [x_2, b] & \textit{in case } (B). \end{cases}$$

Theorem 4. *If f is dyadically convex then $f(a) \le f(b)$. If moreover, f is not constant in a right side of its fixed point p_f and has a convex iterative root $f^{\frac{1}{n}}$ then*

$$f^{\frac{1}{n}} = h^{\frac{1}{n}} \circ e, \tag{9}$$

where $h^{\frac{1}{n}}$ is a convex strictly increasing iterative root of $h := f|_{I_b}$. Moreover, $f[I_a] \subset h[I_b]$ and

$$e(x) := \begin{cases} x, & x \in I_b \\ h^{-1} \circ f(x), & x \in I_a \end{cases}. \tag{10}$$

Proof. Put $g_n := f^{\frac{1}{2^n}}|_{I_b}$ and $I_b^- := I_b \cap (-\infty, p_f)$, $I_b^+ := I_b \cap (p_f, \infty)$. We have that $g_n^2 = g_{n-1}$. In cases (A) and (B), Corollary 1 and Lemma 6 imply, that for $x \in I_b^-$ the sequence $\{g_n(x)\}$ is decreasing and for $x \in I_b^+$ it is increasing. However, in case (C) this sequence is increasing for all $x \in I_b$. Thus the limit

$$\varepsilon(x) := \lim_{n \to \infty} g_n(x) \text{ for } x \in I_b$$

exists. Since g_n are convex and increasing, ε is also convex and increasing. Thus ε is continuous in $\mathrm{Int\, I_b}$. We will show that ε is continuous at the ends of the interval I_b.

First we consider the left end of I_b. Let (A) or (B) hold. Suppose that ε is discontinuous at $\overline{a} := \mathrm{inf\, I_b}$. Then $\varepsilon(\overline{a}) < \varepsilon(\overline{a}+) =: q$ because ε is decreasing. Hence there exists $n_0 \in \mathbb{N}$ such that $g_{n_0}(\overline{a}) < q$. By the continuity of g_{n_0} there exists a neighbourhood $U_{\overline{a}}$ of $\overline{a}$ such that $g_{n_0}(x) < q$ for $x \in U_{\overline{a}}$. On the other hand $q \le \varepsilon(x) \le g_{n_0}(x)$ for $x \in I_b^-$, since $\{g_n(x)\}$ is decreasing in I_b^-. This is a contradiction.

In case (C) $I_b = [p_f, b]$, so $\overline{a} = p_f$, the sequence $\{g_n(x)\}$ is increasing and $g_n(x) < x$ in the whole I_b. This gives $\varepsilon(x) \le x$ in I_b and $\varepsilon(p_f+) \le p_f$. Since ε is increasing, $p_f = \varepsilon(p_f) \le \varepsilon(p_f+)$, so $\varepsilon(p_f+) = p_f = \varepsilon(p_f)$.

Now we prove that ε is continuous at b. Suppose that $r := \varepsilon(b-) < \varepsilon(b)$. By Corollary 1 and Lemma 6 we get that for every $x \in I_b^+$ $g_n(x) \le g_{n+1}(x) \le \varepsilon(x)$. Thus there exists n_0 such that $g_{n_0}(b) > r$. By the continuity of g_{n_0} at b there exists a neighbourhood U_b of b such that $g_{n_0}(x) > r$ for $x \in U_b$. On the other hand $g_{n_0}(x) \le \varepsilon(x) \le \varepsilon(b-) = r$. This is a contradiction.

Since ε is continuous in I_b and the sequences $g_n|_{I_b^-}$ and $g_n|_{I_b^+}$ are monotonic, they converge uniformly to $\varepsilon|_{I_b^+}$ and $\varepsilon|_{I_b^-}$, respectively. Hence g_n converges to ε uniformly.

It is easy to verify the following property:

(P) If I is a compact interval and the continuous mappings $u_n, v_n : I \to I$ converge uniformly on I to u and v, respectively then $u_n \circ v_n$ converges uniformly to $u \circ v$.

Since g_n converges uniformly to ε in I_b and $g_n \circ g_n = g_{n-1}$ we get by (P) the equality

$$\varepsilon(\varepsilon(x)) = \varepsilon(x) \text{ for } x \in I_b.$$

Hence $\varepsilon(x) = x$ for $x \in J := \varepsilon[I_b]$ and J is a closed interval, since ε is continuous. Obviously $p_f \in J$. In view of the inequalities $g_n(x) \geq f(x) > p_f$ for $x \in I_b^+$ we obtain that $\varepsilon(x) > p_f$ for $x \in I_b^+$, which implies that $p_f \notin \varepsilon[I_b^+]$, so $J \neq \{p_f\}$. Thus $\alpha := \inf \mathrm{J} < \sup \mathrm{J} =: \beta$ and we have $\alpha \leq \varepsilon(x) \leq \beta$ for $x \in I_b$ and $\varepsilon(x) = x$ for $x \in [\alpha, \beta]$. Since ε is increasing

$$\varepsilon(x) = \begin{cases} \alpha, & x \leq \alpha \\ x, & x \in [\alpha, \beta] \\ \beta, & x \geq \beta \end{cases} .$$

The convexity of ε implies the equality $\beta = b$.

Now we prove that $f(a) \leq f(b)$. Suppose on the contrary that $f(a) > f(b)$ and $f(b) < b$. By the inequality $p_f = f(p_f) \leq f(b)$ we obtain that $f(a) > p_f$, so $f(a) > \alpha$ and $\varepsilon(f(a)) = f(a)$. Hence, by Lemma 8 and Remark 3,

$$f(a) = \lim_{n \to \infty} g_n(f(a)) = \lim_{n \to \infty} f^{\frac{1}{2^n}}(f(a)) \leq f(b).$$

If $f(b) = b$ then the inequality is obvious.

Now, assume that (A) or (B) holds and f has a convex iterative root $f^{\frac{1}{n}}$. Then $h^{\frac{1}{n}} := f^{\frac{1}{n}}|I_b$ is injective. We have $h = h^{\frac{n-1}{n}} \circ h^{\frac{1}{n}}$. Hence $(h^{\frac{n-1}{n}})^{-1} \circ h = h^{\frac{1}{n}}$ and $(h^{\frac{n-1}{n}})^{-1}(x) = h^{\frac{1}{n}} \circ h^{-1}(x)$ for $x \in h[I_b]$.

The inequality $f(a) \leq f(b)$ implies that $g[I_a] \subset h[I_b]$, where $g = f|_{I_a}$, whence

$$(h^{\frac{n-1}{n}})^{-1}(g(x)) = h^{\frac{1}{n}} \circ h^{-1}(g(x)) = h^{\frac{1}{n}} \circ h^{-1}(f(x)) \quad \text{for } x \in I_a,$$

so by Theorem 3

$$f^{\frac{1}{n}}(x) = h^{\frac{1}{n}} \circ h^{-1} \circ f(x) = h^{\frac{1}{n}} \circ e(x) \quad \text{for } x \in I_a.$$

This proves (9). $\square$

Theorem 5. *A non-decreasing function is dyadically convex if and only if it is flowly convex.*

Proof. Let $f^{\frac{1}{2^n}}$, $n \geq 1$ be convex iterative roots of f satisfying (8). It follows, by Corollary 1, that $f^{\frac{1}{2^n}}$ are non-decreasing. Define

$$f^{\frac{k}{2^n}} := (f^{\frac{1}{2^n}})^k, \quad n, k \in \mathbb{N}.$$

The following property holds:

(P1) If $\frac{l}{2^n} = \frac{k}{2^m}$ then $f^{\frac{l}{2^n}} = f^{\frac{k}{2^m}}$.

In fact, assume that $n \geq m$. Then in view of (8) we get by induction that $f^{\frac{1}{2^m}} = (f^{\frac{1}{2^n}})^{2^{n-m}}$ which gives that $f^{\frac{k}{2^m}} = (f^{\frac{1}{2^m}})^k = (f^{\frac{1}{2^n}})^{2^{n-m}k} = (f^{\frac{1}{2^m}})^l = f^{\frac{l}{2^n}}$.

Denote $D := \{\frac{k}{2^n},\ k, n \in N\}$. Property (P1) allows us to uniquely define the functions f^w for $w \in D$. We will show more.

(P2) If $w_1, w_2 \in D$ and $x < p_f$ then $f^{w_1}(x) \leq f^{w_2}(x)$, however for $x > p_f$ $f^{w_1}(x) \geq f^{w_2}(x)$.

In fact, if $\frac{l}{2^n} < \frac{k}{2^m}$ and $x < p_f$ then it follows, by Corollary 1, that $x < f^{\frac{1}{2^n}}(x) \leq f(x) \leq p_f$, so $f^{\frac{k_1}{2^n}}(x) \leq f^{\frac{k_2}{2^n}}(x)$ for $k_1 < k_2$. Since $\frac{l}{2^n} < \frac{k2^{n-m}}{2^m}$, we have $f^{\frac{l}{2^n}}(x) < f^{\frac{k2^{n-m}}{2^n}}(x) = f^{\frac{k}{2^m}}(x)$.

If $p_f < x$ and $f(b) \neq b$ we have $p_f \leq f(x) \leq f^{\frac{1}{2^n}}(x) < x$, so $f^{\frac{k_1}{2^n}}(x) \geq f^{\frac{k_2}{2^n}}(x)$ for $k_1 < k_2$. Then as previously we get that $f^{\frac{l}{2^n}}(x) \geq f^{\frac{k}{2^m}}(x)$. If $f(b) = b$ then $f^{\frac{l}{2^n}}(b) = f^{\frac{k}{2^m}}(b)$.

For $n \geq m$ we have

$$f^{\frac{l}{2^n}} \circ f^{\frac{k}{2^m}} = f^{\frac{l}{2^n}} \circ f^{\frac{k2^{n-m}}{2^n}} = f^{\frac{l+k2^{n-m}}{2^n}} = f^{\frac{l}{2^n}+\frac{k}{2^m}},$$

which can be written as follows

$$f^u \circ f^u = f^{u+v} \quad \text{for } u, v \in D.$$

Define

$$g^t(x) := \begin{cases} inf\, f^w(x), & w > t,\, w \in D,\, x < p_f \\ sup\, f^w(x), & w > t,\, w \in D,\, x > p_f. \end{cases} \quad t > 0 \tag{11}$$

Since the set of dyadic numbers D is dense in $\mathbb{R}^+$, for every $t, s > 0$ there exist the monotonic sequences $\{u_n\}, \{v_n\} \subset D$ such that $u_n \to t$ and $v_n \to s$. In view of the convexity of g^t we infer, similarly as in Theorem 4, that $\{f^{u_n}\}$ and $\{f^{w_n}\}$ converge uniformly on I, respectively, to g^t and g^s. Thus, by property (P),

$$g^t \circ g^s = \lim_{n\to\infty} f^{u_n} \circ \lim_{n\to\infty} f^{v_n} = \lim_{n\to\infty} f^{u_n} \circ f^{v_n} = \lim_{n\to\infty} f^{u_n+v_n} = g^{t+s}.$$

We will show that $f = g^1$. Let $x < p_f$, $w > 1$ and $w \in D$. Then by (P2) $f^w(x) \geq f^1(x) = f(x)$, so $g^1(x) \geq f(x)$. Similarly, for $x > p_f$ we get the inequality $g^1(x) \leq f(x)$. Every semi-flow of convex functions is continuous and monotonic (see Th. 19.2 and Lemma 4.1 in [12]). This means that the mappings $t \to g^t(x)$ are continuous and increasing for $x < p_f$ they, however for $x > p_f$ are decreasing. Suppose that $g^1(x_0) > f(x_0)$ for an $x_0 < p_f$. By the continuity of $t \to g^t(x_0)$ at 1 there exists and $0 < s < 1$ such that $g^s(x_0) > f(x_0)$. Fix a $w \in D$ such that $0 < s < w < 1$. In view of (11) $g^s(x_0) \leq f^w(x_0)$. On the other hand, by (P2), $f^w(x_0) \leq f(x_0)$, which gives that $g^s(x_0) \leq f(x_0)$, but this contradicts the previous inequality. Thus $g^1(x) = f(x)$ for $x < p_f$. Similarly, we prove for $x > p_f$. Thus f is flowly convex.

Conversely, if f has a convex semi-flow $\{f^t : t > 0\}$ then $f^{\frac{1}{2^n}}$ are convex iterative roots of 2^n-th order satisfying (8), thus f is diadically convex. □

Theorem 6. *Let f and f^2 be convex and f be non-constant in a right side of its fixed point p_f. Then f is dyadically convex if and only if*

1° $f(a) \leq f(b)$,

2° *the function* $h := f|_{I_b}$ *is flowly convex,*
3° *the function* e *defined by* (10) *is convex.*

Proof. Let f be dyadically convex. Note that f satisfies (A) or (B), so $h := f|_{I_b}$ is strictly increasing and, by Theorem 5, it is flowly convex. Next, by Theorem 4, we get assertion 1° and $f[I_a] \subset h[I_b]$. This relation allows us to correctly define the function e by formula (10). Let $f^{\frac{1}{2^n}}$ be convex iterative roots satisfying (8) and $\{h^t : t > 0\}$ be a convex semi-flow of h constructed similarly as in the proof of Theorem 5. Moreover, by Theorem 4 we have

$$f^{\frac{1}{2^n}}(x) = h^{\frac{1}{2^n}} \circ e(x), \quad n \in \mathbb{N}, \quad x \in I. \tag{12}$$

Since $\{h^t : t > 0\}$ is a continuous semi-flow, we have $\lim_{n\to\infty} h^{\frac{1}{2^n}}(x) = x$ for $x \in I_b$. Hence (12) implies the existence of the limit

$$\lim_{n\to\infty} f^{\frac{1}{2^n}}(x) = e(x), \quad x \in I,$$

because $e[I] = I_b$. The convexity of all $f^{\frac{1}{2^n}}$ yields the convexity of e.

Conversely, if $f|_{I_b}$ is flowly convex then obviously $f|_{I_b}$ is dyadically convex. The inequality $f(a) \leq f(b)$ allows us to define a function e by (10). If $h^{\frac{1}{2^n}}$ are convex iterative roots of $f|_{I_b}$ satisfying (8) and e is convex then in view of (12) $f^{\frac{1}{2^n}} := h^{\frac{1}{2^n}} \circ e$ are convex iterative roots of f and

$$f^{\frac{1}{2^n}} \circ f^{\frac{1}{2^n}} = h^{\frac{1}{2^n}} \circ e \circ h^{\frac{1}{2^n}} \circ e = h^{\frac{1}{2^n}} \circ h^{\frac{1}{2^n}} \circ e = h^{\frac{1}{2^{n-1}}} \circ e = f^{\frac{1}{2^{n-1}}}.$$

□

Theorem 7. *A convex function* f *non-constant in a right side of its fixed point* p_f *is dyadically convex if and only if* f *is flowly convex.*

Proof. If f is embeddable in a convex semi-flow $\{f^t : t > 0\}$ then $f^{\frac{1}{2^n}}$ are convex iterative roots of 2^n-th order satisfying (8), thus f is dyadically convex. Conversely, let f be dyadically convex. It follows by Theorem 1 that f^2 is convex. By Theorem 6 $f(a) \leq f(b)$, (10) correctly defines a convex function e. Moreover, $f|_{I_b}$ is embeddable in a convex semi-flow $\{h^t : t > 0\}$. Then the mappings $f^t := h^t \circ e$ for $t > 0$ are convex semi-flows of f. □

By the last Theorem and Remark 4 we get

Corollary 2. *If* f *is dyadically convex and non-constant in a right neighbourhood of its fixed point, then* f *is iteratively convex.*

Theorem 8. *Every strictly increasing dyadically convex function is of class* C^1 *except a fixed point* p_f.

Proof. By Theorem 5 f possesses a convex semi-flow. Hence $f|_{[a,p_f]}$ (if $p_f \neq a$) and $f|_{[p_f,b]}$ (if $p_f \neq b$) are embeddable in convex semi-flows. It has been proved in [7], that every convex semi-flow $\{g^t : t > 0\}$ defined on an open interval of a strictly increasing function g without fixed points is differentiable, that is all functions g^t are differentiable. Hence we infer that $f|_{(a,p_f)}$ and $f|_{(p_f,b)}$ are differentiable. The convexity of f implies the existence of one sided derivatives at the ends of the intervals (we admit here that $f'(b-) = \infty$). Hence by Theorem 2 in book [3] (see p.156) we infer that $f|_{[a,p_f]}$ and $f|_{[p_f,b]}$ are of class C^1. □

To prove the next theorem we apply the following result.

Lemma 9. ([3,11]) *If $h : I \to I$ is strictly increasing, convex, of class C^1 in I and $h'(p_f) > 0$ then for every $n \geq 2$ the equation $\varphi^n = h$ has a unique C^1 solution.*

Theorem 9. *A convex increasing function f of class C^1 such that $f'(p_f) \neq 0$ is iteratively convex if and only if it is flowly convex.*

Proof. If f is flowly convex then obviously is iteratively convex.

Assume, now, that f is iteratively convex. We may also assume that f is injective, since otherwise case (B) holds and f is constant in an interval $[a, x_2]$, where $x_2 \leq p_f$. By Corollary 1 all $f^{\frac{1}{n}}$ are constant in $[a, x_2]$. Thus we can consider only iterative roots of $f|_{[x_2,b]}$ which are strictly increasing.

It follows by the iterative convexity of f that for every $n \in N$ there exits a convex iterative root $f^{\frac{1}{2^n}}$. We shall show that $f^{\frac{1}{2^n}}$ is of class C^1.

Let $J := (\alpha, \beta)$ and $\varphi : J \to J$ be a convex strictly increasing function such that $\varphi(x) \neq x$ for $x \in J$. Denote by $Z_\varphi(J)$ the set of all points of non-differentiability of φ. In [6] A. Smajdor showed that $Z_\varphi(J) \subsetneqq Z_{\varphi^2}(J)$. Hence

$$Z_{f^{\frac{1}{2^n}}}(J) \subsetneqq Z_f(J),$$

for $J = (a, p_f)$ and $J = (p_f, b)$ if $a < p_f < b$ and otherwise for $J = (a, b)$. Since $Z_f(J) = \emptyset$, we get $Z_{f^{\frac{1}{2^n}}}(J) = \emptyset$ which means that $f^{\frac{1}{2^n}}$ is differentiable in J.

If $a < p_f < b$ then $f^{\frac{1}{2^n}}$ as a convex function is one sided differentiable at p_f, and $f^{\frac{1}{2^n}}|_{[a,p_f]}$ and $f^{\frac{1}{2^n}}|_{[p_f,b]}$ are of class C^1 (see [3] p. 156). Moreover we have

$$[(f^{\frac{1}{2^n}})'_-(p_f)]^{2^n} = f'_-(p_f) \quad \text{and} \quad [(f^{\frac{1}{2^n}})'_+(p_f)]^{2^n} = f'_+(p_f).$$

Since $f'_-(p_f) = f'_+(p_f)$ we get $(f^{\frac{1}{2^n}})'_-(p_f) = (f^{\frac{1}{2^n}})'_+(p_f)$, which shows that $f^{\frac{1}{2^n}}$ is of class C^1 in $[a, b]$. If $p_f = a$ or $p_f = b$ then, by analogous argumentation, we get the same property.

By Lemma 9 for every $n \geq 1$ f has a unique C^1 iterative root h_n of 2^n order. We have ${h_n}^{2^n} = f$, so

$$({h_n}^2)^{2^{n-1}} = (h_n \circ h_n)^{2^{n-1}} = {h_n}^{2^n} = f.$$

Hence by the uniqueness we get

$$h_n^{\,2} = h_{n-1}. \tag{13}$$

On the other hand $f^{\frac{1}{2^n}}$ is also a C^1 iterative root of f of 2^n order. Thus, by the uniqueness, $f^{\frac{1}{2^n}} = h_n$ and consequently, by (13), we get (8), so f is dyadically convex and by Theorem 7 it is flowly convex. □

Lemma 10. ([4,5] p. 436) *Every convex strictly increasing C^1 function with positive derivative has at most one convex iterative root of every order.*

Theorem 10. *Let f be convex, of class C^1 and $f'(p_f) \neq 0$. If f is iteratively convex then $f(a) \leq f(b)$.*

Proof. We may assume that $f(b) < b$. Note that f^2 is convex and the assumption $f'(p_f) \neq 0$ implies that f satisfies condition (A) or (B). Let $f^{\frac{1}{n}}$, $n \geq 2$ be convex iterative roots of f. It follows by Theorem 9 that the function $h := f|_{I_b}$ possesses a convex semi-flow $\{h^t : t > 0\}$. Lemma 10 yields the equality $f^{\frac{1}{n}}|_{I_b} = h^{\frac{1}{n}}$. Thus $f^{\frac{1}{n}}|_{I_b}$ is uniquely determined and consequently $f^{\frac{n-1}{n}}|_{I_b} = h^{\frac{n-1}{n}}$ for $n \geq 2$. Every convex semi-flow is continuous (see [12]). This means that for every $x \in I_b$ the mapping $t \to h^t(x)$ is continuous. Hence $\lim_{n\to\infty} f^{\frac{n-1}{n}}(b) = \lim_{n\to\infty} h^{\frac{n-1}{n}}(b) = h^1(b) = h(b) = f(b)$. By Lemma 8 $f(a) \leq f^{\frac{n-1}{n}}(b)$ for $n \geq 2$, so $f(a) \leq f(b)$. □

Theorem 11. *Let f be convex, of class C^1 and $f'(p_f) \neq 0$. Then f is iteratively convex if and only if it is flowly convex.*

Proof. Let f be iteratively convex. Note that f^2 is convex and f satisfies (A) or (B). By Theorem 9 the function $h := f|_{I_b}$ is flowly convex. We will show that the function e defined by (10) is convex.

Let $h^{\frac{1}{n}}$ be a convex iterative root of h. Let $\{\overline{h^t} : t \geq 0\}$ be a convex semi-flow of h. Convex iterative roots are uniquely determined, so $\overline{h^{\frac{1}{n}}} = h^{\frac{1}{n}}$. We have $h^{\frac{n-1}{n}} \circ h^{\frac{1}{n}} = h$, so $(h^{\frac{n-1}{n}})^{-1}(x) = h^{\frac{1}{n}} \circ h^{-1}(x)$ for $x \in h[I_b]$. By Theorem 3 $f^{\frac{1}{n}}(x) = (h^{\frac{n-1}{n}})^{-1} \circ f(x)$ for $x \in I_a$. However, by Theorem 10, we have $f[I_a] \subset h[I_b]$. Combining these conditions we obtain $f^{\frac{1}{n}}(x) = h^{\frac{1}{n}} \circ h^{-1} \circ f(x)$ for $x \in I_a$, so $f^{\frac{1}{n}}(x) = h^{\frac{1}{n}} \circ e(x)$ for $x \in I_a$, where e is defined by (10), which gives that

$$f^{\frac{1}{n}}(x) = \overline{h^{\frac{1}{n}}} \circ e(x), \quad \text{for} \quad x \in I. \tag{14}$$

The semi-flow $\{\overline{h^t} : t \geq 0\}$ is continuous as it is convex, so that there exists the limit $\lim_{t\to 0} \overline{h^t}(x) = \overline{h^0}(x)$ for $x \in I_b$. Since $\overline{h^1} = h$ is injective, $\overline{h^0}(x) = x$ for $x \in I_b$. Hence, by (14), there exists the limit $\lim_{n\to\infty} f^{\frac{1}{n}}(x) = e(x)$ for $x \in I$, so the convexity of the functions $f^{\frac{1}{n}}$ implies the convexity of e.

By Theorem 9 $f(a) \leq f(b)$. Thus the assumptions of Theorem 6 are fulfilled, so f is flowly convex and in consequence, by Theorem 7, f is flowly convex.

□

Example 2. Let $I = [a, b]$ and $x_0 \in (a, b)$. Define

$$f(x) := \begin{cases} g(x), & x < x_0 \\ h(x), & x \geq x_0, \end{cases}$$

where $h(x_0) = g(x_0)$ and $f(a) \leq f(b)$. Consider two cases:

1^0 $g : [a, x_0] \to (x_0, b]$ is decreasing and affine,
$h : [x_0, b] \to (x_0, b]$ is increasing, convex, not affine and

2^0 $g : [a, x_0] \to (x_0, b]$ is decreasing and convex,
$h : [x_0, b] \to (x_0, b]$ is increasing and affine.

At first glance the graphs are similar to each other, but in case 1^0, the mapping f is never iteratively convex. However, in case 2^0 f is iteratively convex.

In fact, in case 1^0 the function e defined by (10) is not convex, since $g \circ h^{-1}$ is concave. In case 2^0 e is convex, since $g \circ h^{-1}$ is convex. Obviously, an affine function is flowly convex. Hence by Theorems 6 and 9 we get our assertions.

The criteria on the iterative convexity of a given convex function are still unknown, however, we can determine all strictly increasing flowly convex functions with one fixed point. To give a complete construction it suffices to restrict the construction to the case of the functions which have one fixed point on the left end of their domains. In fact, we can consider independently constructions for functions restricted to the intervals $[a, p]$ and $[p, b]$, where p is their fixed point. If g is a function defined on $[a, p]$ such that $x < g(x) < p$ for $x \in [a, p)$ and $f := \gamma \circ g \circ \gamma^{-1}$, where $\gamma(x) := -x + 2p$, then $p < f(x) < x$ for $[p, 2p - a]$. Note that f is convex if and only if g is concave and $f^t := \gamma \circ g^t \circ \gamma^{-1}$, $t > 0$ is a convex semi-flow if and only if g^t is a concave semi-flow.

To construct flowly iterative functions we apply the following result of Smajdor.

Lemma 11. ([7]) *Let f be convex (concave) and of class C^1 in $[p, b)$ and $p < f(x) < x$ for $x \in (p, b)$ and $f'(p) \neq 0$. Then f is flowly convex (concave) if and only if the functional equation*

$$G(f(x)) = f'(x)G(x), \quad t \geq 0 \tag{15}$$

has a convex (concave) solution.

Theorem 12. *Let G be a convex (concave) function defined on $[p, b]$ such that $G < 0$ in $(p, b]$, $G(p) = 0$, $G'(p+) \neq -\infty$ and $\int_a^b \frac{du}{G(u)} = -\infty$. Then $f(x) := \alpha^{-1}(1 + \alpha(x))$, $x \in (p, b]$, where $\alpha(x) := -\int_x^b \frac{du}{G(x)}$ is flowly convex (concave), of class C^1, $f(p) = p$ and $f'(p) \neq 0$.*

Conversely, every flowly convex (concave) strictly increasing function f defined on $[p, a]$, of class C^1, with fixed point p and $f'(p) \neq 0$ is of the above form.

Proof. Define $f^t(x) := \alpha^{-1}(t + \alpha(x)), \quad t \geq 0$. It is easy to see that f^t are of class C^1 in (p, b),

$$G(x) = \frac{\partial f^t(x)}{\partial x}$$

and

$$G(f^t(x)) = (f^t)'(x)G(x), \quad t \geq 0.$$

Putting $t = 1$ we see that G satisfies (15). It was shown in [10] (Th.4) that f^t are differentiable at p and $(f^t)'(p) = \exp \mathrm{t\, G'(p)}$, so $0 < (f^1)'(p) \leq 1$. Now, it follows, by Lemma 11, that $f := f^1$ is flowly convex.

Conversely, let f be flowly convex. By Lemma 11 there exists a convex solution $G_0 \leq 0$ of (15). Obviously, $G_0(p) = 0$ and $G_0(x) < 0$ for $x \in (p, b)$. Put $\beta(x) := -\int_x^b \frac{du}{G_0(x)}$. It is easy to see that $\beta(f(x)) = \beta(x) + c$ for a $c > 0$ and $\lim_{x \to p} \beta(x) = \infty$. Define $\alpha(x) := \frac{1}{c}\beta(x)$ and $G(x) = c\, G_0(x)$. Hence $f(x) := \alpha^{-1}(1 + \alpha(x))$ and $\alpha(x) := -\int_x^b \frac{du}{G(x)}$, which proves the theorem. $\square$

Compliance with ethical standards

Conflict of interest The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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