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Gołąb–Schinzel equation on cylinders

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Abstract. We determine continuous solutions of the Gołąb–Schinzel functional equation on cylinders.

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1. Introduction

The Gołąb–Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y) \tag{1}$$

has its origins in various algebraic problems. For more details concerning applications of Eq. (1) and its generalized versions in determining substructures of algebraic structures we refer to the monograph [1], a survey paper [5] and references therein.

It turns out (cf. [11]) that Eq. (1) on a restricted domain expresses the symmetries of nonlinear differential equations corresponding to some problems in mathematical meteorology and fluid mechanics (e.g. evaporation of cloud droplets and water discharging from a reservoir). Therefore, it is natural to ask on the form of solutions of (1) on various types of restricted domains. Several results concerning this problem can be found e.g. in [2], [4–7], [11,12], [14,15] and [17–19]. All of them concern the case where an unknown function f is a real function of a real variable. In a recent paper [8] equation (1) has been investigated in a more general setting, namely on a convex cone in a real linear space.

In this paper we study the Gołąb–Schinzel functional equation on a special type of restricted domain, called *a cylinder*. More precisely we determine continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equations

$$f(x + f(x)y) = f(x)f(y) \quad \text{for} \quad (x, y) \in \mathbb{R} \times A \tag{2}$$

and

$$f(x + f(x)y) = f(x)f(y) \quad \text{for} \quad (x, y) \in A \times \mathbb{R}, \tag{3}$$

where A is a non-empty subset of $\mathbb{R}$. In [14] equation (3) was studied in the case where A consists of two elements and f satisfies some additional assumptions. Further functional equations on cylinders have also been already investigated. In particular, several results concerning the solutions of the Cauchy equations on cylinders can be found in [9] and [10]. The analogous problem for the d'Alembert equation has been recently studied in [3].

In the sequel, for every non-empty subset A of $\mathbb{R}\backslash\{0\}$, by $\langle A\rangle$ we denote the subgroup of the multiplicative group of non-zero reals generated by A. Furthermore, for $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, let $\phi_{(f,y)} : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi_{(f,y)}(x) = x + f(x)y \quad \text{for} \quad x \in \mathbb{R} \tag{4}$$

and let

$$Z_f := \{x \in \mathbb{R} : f(x) = 0\}.$$

Obviously, if f is continuous then, for every $y \in \mathbb{R}$, so is $\phi_{(f,y)}$. Let us also recall that f is said to be *a projection* provided $f \circ f = f$, and it is said to be *an involution* provided $f \circ f = \mathrm{id}_{\mathbb{R}}$. Note that each involution is bijective and the only increasing involution on $\mathbb{R}$ is the identity.

2. Auxiliary results

Remark 2.1. It is well known (cf. e.g. [20, Theorem 1.4, p. 3]) that every subgroup of the additive group of reals is either cyclic, or dense in $\mathbb{R}$. Since the multiplicative group of positive reals is isomorphic to the additive group of reals (by the logarithm), every subgroup of the multiplicative group of positive reals is either dense in $(0,\infty)$, or cyclic. Moreover, if the second possibility holds, the subgroup is of the form $\{d^n : n \in \mathbb{Z}\}$ for some $d \in (0,\infty)$.

Remark 2.2. Assume that $C \subset \mathbb{R}\backslash\{0\}$ and $C\backslash\{-1, 1\} \neq \emptyset$. Then $\langle C\rangle \cap (0,\infty)$ is a non-trivial subgroup of the multiplicative group of positive reals. Therefore, according to Remark 2.1, $\langle C\rangle \cap (0,\infty)$ is either dense in $(0,\infty)$, or $\langle C\rangle \cap (0,\infty) = \{d^n : n \in \mathbb{Z}\}$ for some $d \in (0,\infty)\backslash\{1\}$. In the second case, for every $c \in C \cap (-\infty, 0)$ there is an $n \in \mathbb{Z}$ such that $c^2 = d^n$. Thus

$$C \subset \{-d^{n/2} : n \in \mathbb{Z}\} \cup \{d^n : n \in \mathbb{Z}\}. \tag{5}$$

Proposition 2.3. *Let $C \subset \mathbb{R}\backslash\{0\}$ be such that $C\backslash\{-1, 1\} \neq \emptyset$ and let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function with $F(1) = 1$.*

(A) If $\langle C\rangle \cap (0,\infty)$ is dense in $(0,\infty)$ then:

(i) in the case where $C \cap (-\infty, 0) \neq \emptyset$*, F satisfies the equation*

$$F(cx) = cF(x) \quad \textit{for} \quad x \in \mathbb{R}, c \in C \tag{6}$$

if and only if $F(x) = x$ *for* $x \in \mathbb{R}$*;*

(ii) in the case where $C \subset (0, \infty)$*, F satisfies equation* (6) *if and only if there exists* $\alpha \in \mathbb{R}$ *such that*

$$F(x) = \begin{cases} \alpha x & \textit{for} \quad x \in (-\infty, 0), \\ x & \textit{for} \quad x \in [0, \infty). \end{cases} \tag{7}$$

(B) If $\langle C \rangle \cap (0, \infty) = \{d^n : n \in \mathbb{Z}\}$ *for some* $d \in (0, \infty) \backslash \{1\}$ *then F satisfies equation* (6) *if and only if there exist continuous 1-periodic functions* $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ *such that* $p_1(0) = 1$*,*

$$F(x) = \begin{cases} -xp_2(\log_d(-x)) & \textit{for} \quad x \in (-\infty, 0), \\ 0 & \textit{for} \quad x = 0, \\ xp_1(\log_d x) & \textit{for} \quad x \in (0, \infty), \end{cases} \tag{8}$$

$$p_2 = -p_1 \quad \textit{whenever} \quad -d^n \in C \textit{ for some } n \in \mathbb{Z}$$

and

$$p_2(x + 1/2) = -p_1(x) \quad \textit{for} \quad x \in \mathbb{R} \tag{9}$$

whenever $-d^{n/2} \in C$ *for some odd* $n \in \mathbb{Z}$*.*

Proof. Setting $x = 1$ in (6), we get $F(c) = c$ for $c \in C$. Furthermore, as $C \backslash \{-1, 1\} \neq \emptyset$, applying (6) with $x = 0$, we obtain $F(0) = 0$. Hence, in view of (6), we get

$$\begin{aligned} \emptyset \neq C \subset C_F &:= \{c \in \mathbb{R}\backslash\{0\} : F(cx) = cF(x) \ \text{ for } \ x \in \mathbb{R}\} \\ &= \{c \in \mathbb{R}\backslash\{0\} : F(cx) = F(c)F(x) \ \text{ for } \ x \in \mathbb{R}\backslash\{0\}\}. \end{aligned}$$

Therefore, according to [13, Lemma 18.5.1, p. 552], C_F is a subgroup of the multiplicative group of non-zero reals. Thus, $\langle C \rangle \subset C_F$, which gives

$$F(cx) = cF(x) \quad \text{for} \quad x \in \mathbb{R}, c \in \langle C \rangle. \tag{10}$$

Since $F(1) = 1$, this implies that

$$F(c) = c \quad \text{for} \quad c \in \langle C \rangle. \tag{11}$$

Suppose that $\langle C \rangle \cap (0, \infty)$ is dense in $(0, \infty)$. Then, as F is continuous, from (11) we derive that

$$F(x) = x \quad \text{for} \quad x \in [0, \infty). \tag{12}$$

Furthermore, making use of (6) and (12), we obtain $cx = F(cx) = cF(x)$ for $x \in (-\infty, 0)$, $c \in C \cap (-\infty, 0)$. Thus, if $C \cap (-\infty, 0) \neq \emptyset$, we have $F(x) = x$ for $x \in (-\infty, 0)$. Hence, taking (12) into account, we conclude that $F(x) = x$ for $x \in \mathbb{R}$. Obviously, F of this form satisfies (6) and so (i) is proved. If $C \subset (0, \infty)$ then, setting $x = -1$ in (6), we get $F(-c) = F(-1)c$ for $c \in \langle C \rangle$. Hence $F(x) = -F(-1)x$ for $x \in -\langle C \rangle$. Therefore, as F is continuous and $\langle C \rangle \cap (0, \infty)$ is dense in $(0, \infty)$, taking (12) into account, we obtain (7) with

$\alpha := -F(-1)$. It it easy to check that if $C \subset (0,\infty)$ then, for every $\alpha \in \mathbb{R}$, F of the form (7) satisfies (6). This proves (ii).

Now, assume that $\langle C\rangle \cap (0,\infty) = \{d^n : n \in \mathbb{Z}\}$ for some $d \in (0,\infty)\backslash\{1\}$. In view of (10), we get

$$F(dx) = dF(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{13}$$

Since $d \neq 1$, this gives $F(0) = 0$. Furthermore, the function $p_1 \colon \mathbb{R} \to \mathbb{R}$ given by $p_1(x) = F(d^x)d^{-x}$ for $x \in \mathbb{R}$, is continuous and 1-periodic. Obviously, we also have

$$F(x) = xp_1(\log_d x) \quad \text{for} \quad x \in (0,\infty). \tag{14}$$

Moreover, the function $F_- : \mathbb{R} \to \mathbb{R}$ given by $F_-(x) = F(-x)$ for $x \in \mathbb{R}$, also satisfies equation (13). So, as previously we get that $F_-(x) = xp_2(\log_d x)$ for $x \in (0,\infty)$ with some continuous 1-periodic function $p_2 \colon \mathbb{R} \to \mathbb{R}$. Thus

$$F(x) = F_-(-x) = -xp_2(\log_d(-x)) \quad \text{for} \quad x \in (-\infty,0).$$

Therefore, as $F(0) = 0$, taking (14) into account, we obtain (8). Note also that, making use of (13) and (14), we get $d = dF(1) = F(d) = dp_1(1) = dp_1(0)$. Hence $p_1(0) = 1$.

Suppose that $-d^n \in C$ for some $n \in \mathbb{Z}$. Then, in view of (6) and (8), we obtain

$$\begin{aligned} -xp_2(\log_d(-x)) &= F(x) = F(-d^n(-x/d^n)) = -d^nF(-x/d^n) \\ = -d^n(-x/d^n)p_1(\log_d(-x/d^n)) &= xp_1(\log_d(-x)) \quad \text{for} \quad x \in (-\infty,0). \end{aligned}$$

Therefore, $p_2 = -p_1$.

If $-d^{n/2} \in C$ for some odd $n \in \mathbb{Z}$ then, applying (6) and (8), we get

$$\begin{aligned} d^{n/2}xp_2(\log_d x + 1/2) &= d^{n/2}xp_2(\log_d(d^{n/2}x)) \\ &= F(-d^{n/2}x) = -d^{n/2}F(x) = -d^{n/2}xp_1(\log_d x) \quad \text{for} \quad x \in (0,\infty), \end{aligned}$$

which implies (9). The converse is easy to check. □

Corollary 2.4. *Assume that* $C \subset \mathbb{R}\backslash\{0\}$ *is such that* $\langle C\rangle \cap (0,\infty) = \{d^n : n \in \mathbb{Z}\}$ *for some* $d \in (0,\infty)\backslash\{1\}$ *and* C *has non-empty intersections with the sets* $\{-d^n : n \in \mathbb{Z}\}$ *and* $\{-d^{n/2} : n \in \mathbb{Z}$ *is odd*$\}$. *Then a continuous function* $F \colon \mathbb{R} \to \mathbb{R}$, *with* $F(1) = 1$, *satisfies Eq.* (6) *if and only if there exists a continuous* $1/2$*-periodic function* $p : \mathbb{R} \to \mathbb{R}$ *with* $p(0) = 1$ *such that*

$$F(x) = \begin{cases} xp(\log_d |x|) & \text{for} \quad x \in \mathbb{R}\backslash\{0\}, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

3. Solutions of Eq. (2)

Remark 3.1. If a not identically zero function $f:\mathbb{R}\to\mathbb{R}$ satisfies Eq. (2) with some $A\subset\mathbb{R}$ such that $0\in A$ then, setting $y=0$ in (2), we get

$$f(x)=f(x)f(0)\quad\text{for}\quad x\in\mathbb{R}. \tag{15}$$

Hence

$$f(0)=1\quad\text{provided}\quad 0\in A. \tag{16}$$

Remark 3.2. If $A=\{0\}$, then (2) becomes (15). Hence, a function $f:\mathbb{R}\to\mathbb{R}$ satisfies Eq. (2) if and only if either $f=0$ or $f(0)=1$. So, from now on, dealing with the solutions of (2), we will assume that $A\backslash\{0\}\neq\emptyset$.

Remark 3.3. Assume that $f:\mathbb{R}\to\mathbb{R}$ is a non-constant continuous function satisfying Eq. (2) with some $A\subset\mathbb{R}$ such that $A\backslash\{0\}\neq\emptyset$. Then, taking (4) into account, for every $a\in A\backslash\{0\}$ and $x\in\mathbb{R}$, we obtain

$$\begin{aligned}(\phi_{(f,a)}\circ\phi_{(f,a)})(x)&=\phi_{(f,a)}(x)+(f\circ\phi_{(f,a)})(x)a=\phi_{(f,a)}(x)+f(x+f(x)a)a\\&=\phi_{(f,a)}(x)+f(x)f(a)a=(1+f(a))\phi_{(f,a)}(x)-f(a)x.\end{aligned}$$

Hence, for every $a\in A\backslash\{0\}$, the function $\phi_{(f,a)}$ is continuous and it satisfies

$$(\phi_{(f,a)}\circ\phi_{(f,a)})(x)=(1+f(a))\phi_{(f,a)}(x)-f(a)x\quad\text{for}\quad x\in\mathbb{R}. \tag{17}$$

Continuous solutions of Eq. (17) were studied in [16]. In the proof of the following lemma we will apply some results from [16].

Lemma 3.4. *Assume that $f:\mathbb{R}\to\mathbb{R}$ is a non-constant continuous function satisfying Eq.* (2) *with some $A\subset\mathbb{R}$ such that $A\backslash\{0\}\neq\emptyset$.*

(i) If $f(A\backslash\{0\})\cap(-\infty,0)$ contains an element different from -1 then there exist $c\in\mathbb{R}\backslash\{0\}$ and $d\in\mathbb{R}$ such that

$$f(x)=cx+d\quad\textit{for}\quad x\in\mathbb{R}. \tag{18}$$

(ii) If $f(A\backslash\{0\})\cap(-\infty,0)=\{-1\}$ then there exist a continuous decreasing involution $\phi:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}\backslash\{0\}$ such that $\phi(a)=0$ and

$$f(x)=\frac{\phi(x)-x}{a}\quad\textit{for}\quad x\in\mathbb{R}. \tag{19}$$

(iii) If $f(A\backslash\{0\})\cap(0,\infty)\neq\emptyset$ then either f is of the form (18) *with some $c\in\mathbb{R}\backslash\{0\}$ and $d\in\mathbb{R}$, or*

$$f(x)=\begin{cases}c(x-\alpha) & \textit{for}\quad x\in(-\infty,\alpha],\\ 0 & \textit{for}\quad x\in(\alpha,\beta),\\ c(x-\beta) & \textit{for}\quad x\in[\beta,\infty),\end{cases} \tag{20}$$

with some $c\in\mathbb{R}\backslash\{0\}$ and $-\infty\le\alpha<\beta\le\infty$.

Proof. According to Remark 3.3, for every $a \in A\backslash\{0\}$, the function $\phi_{(f,a)}$ satisfies (17). Suppose that $f(A\backslash\{0\}) \cap (-\infty, 0)$ contains an element different from -1 and fix an $a \in A\backslash\{0\}$ such that $f(a) \in (-\infty, 0)\backslash\{-1\}$. Then, as $\phi_{(f,a)}$ is continuous, applying [16, Theorem 9], we obtain that either $\phi_{(f,a)}(x) = x$ for $x \in \mathbb{R}$, or $\phi_{(f,a)}(x) = f(a)x + \delta$ for $x \in \mathbb{R}$ with some $\delta \in \mathbb{R}$. In view of (4), the first possibility implies that $f(x)a = 0$ for $x \in \mathbb{R}$, which yields a contradiction, as $a \neq 0$ and f is not identically 0. The second one gives (18) with $c := (f(a) - 1)/a \neq 0$ and $d := \delta/a$. Therefore, (*i*) holds.

Now, assume that $f(A\backslash\{0\}) \cap (-\infty, 0) = \{-1\}$. Fix $a \in A\backslash\{0\}$ with $f(a) = -1$ and put $\phi := \phi_{(f,a)}$. Then, according to (17), ϕ is a continuous involution. Furthermore, in view of (4), we get (19) and $\phi(a) = 0$. Note also that, as f is not identically 0 and the only increasing involution on $\mathbb{R}$ is the identity, from (19) it follows that ϕ is decreasing. Thus, (*ii*) is valid.

Finally, assume that $f(A\backslash\{0\}) \cap (0, \infty) \neq \emptyset$. Let $a \in A\backslash\{0\}$ be such that $f(a) > 0$. Suppose that $f(a) = 1$. Then, applying [16, Theorem 10], from (17) we derive that $\phi_{(f,a)}(x) = x + \delta$ for $x \in \mathbb{R}$ with some $\delta \in \mathbb{R}$, which yields a contradiction, as f is non-constant. Hence $f(a) \neq 1$ and so, according to [16, Theorem 8], either $\phi_{(f,a)}(x) = f(a)x + \delta$ for $x \in \mathbb{R}$ with some $\delta \in \mathbb{R}$, or

$$\phi_{(f,a)}(x) = \begin{cases} f(a)x + (1 - f(a))\alpha & \text{for} \quad x \in (-\infty, \alpha], \\ x & \text{for} \quad x \in (\alpha, \beta), \\ f(a)x + (1 - f(a))\beta & \text{for} \quad x \in [\beta, \infty) \end{cases} \tag{21}$$

with some $-\infty \leq \alpha < \beta \leq \infty$. Therefore, taking $c := (f(a) - 1)/a \neq 0$ and $d := \delta/a$, in the first case we get (18) and in the second one we obtain (20). Hence (*iii*) holds. □

Theorem 3.5. *Let $A \subset \mathbb{R}$ be such that $A\backslash\{0\} \neq \emptyset$. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.* (2) *if and only if either $f = 0$ or $f = 1$ or one of the following possibilities holds:*

(i) there exists $c \in \mathbb{R}\backslash\{0\}$ such that

$$f(x) = 1 + cx \quad \textit{for} \quad x \in \mathbb{R}; \tag{22}$$

(ii) there exists $c \in \mathbb{R}\backslash\{0\}$ such that

$$f(x) = \max\{1 + cx, 0\} \quad \textit{for} \quad x \in \mathbb{R}; \tag{23}$$

(iii) there exist $c \in (-\infty, 0)$ and $d \in (-1/c, \infty)$ such that $A \subset (-\infty, d]$, $(A\backslash\{0\}) \cap (-\infty, -1/c) \neq \emptyset$ and

$$f(x) = \begin{cases} 1 + cx & \textit{for} \quad x \in (-\infty, -1/c], \\ 0 & \textit{for} \quad x \in (-1/c, d), \\ c(x - d) & \textit{for} \quad x \in [d, \infty); \end{cases} \tag{24}$$

(iv) *there exist* $c \in (0, \infty)$ *and* $d \in (-\infty, -1/c)$ *such that* $A \subset [d, \infty)$, $(A \backslash \{0\}) \cap (-1/c, \infty) \neq \emptyset$ *and*

$$f(x) = \begin{cases} c(x-d) & \textit{for} \quad x \in (-\infty, d], \\ 0 & \textit{for} \quad x \in (d, -1/c), \\ 1 + cx & \textit{for} \quad x \in [-1/c, \infty); \end{cases} \tag{25}$$

(v) $A = \{a\}$ *or* $A = \{a, 0\}$ *with some* $a \in \mathbb{R}\backslash\{0\}$ *and* f *is of the form* (19)*, where* $\phi : \mathbb{R} \to \mathbb{R}$ *is a continuous decreasing involution such that* $\phi(a) = 0$ *and*

$$\phi(0) = a \quad \textit{whenever} \quad A = \{a, 0\}; \tag{26}$$

(vi) $A = \{a\}$ *or* $A = \{a, 0\}$ *with some* $a \in \mathbb{R}\backslash\{0\}$ *and* f *is of the form* (19)*, where* $\phi : \mathbb{R} \to \mathbb{R}$ *is a continuous projection such that* $\phi_{|A} = a$*;*

(vii) $A\backslash\{0\}$ *contains at least two elements,* (16) *is valid and there exist* $a \in (0, \infty)$ *and* $b \leq a$ *such that* $A\backslash\{0\} \subset [a, \infty)$, $f_{|[b,\infty)} = 0$ *and*

$$f(x) \geq \frac{b-x}{a} \quad \textit{for} \quad x \in (-\infty, b); \tag{27}$$

(viii) $A\backslash\{0\}$ *contains at least two elements,* (16) *is valid and there exist* $a \in (-\infty, 0)$ *and* $b \geq a$ *such that* $A\backslash\{0\} \subset (-\infty, a]$, $f_{|(-\infty,b]} = 0$ *and*

$$f(x) \geq \frac{b-x}{a} \quad \textit{for} \quad x \in (b, \infty);$$

(ix) $A\backslash\{0\}$ *contains at least two elements and there exist* $a, b \in (0, \infty)$*, with* $a < b$*, such that* $A \subset [a, b]$, $f_{|(-\infty,b]} = 0$ *and*

$$f(x) \leq \frac{b-x}{a} \quad \textit{for} \quad x \in (b, \infty);$$

(x) $A\backslash\{0\}$ *contains at least two elements and there exist* $a, b \in (-\infty, 0)$*, with* $b < a$*, such that* $A \subset [b, a]$, $f_{|[b,\infty)} = 0$ *and*

$$f(x) \leq \frac{b-x}{a} \quad \textit{for} \quad x \in (-\infty, b). \tag{28}$$

Proof. Assume that f satisfies Eq. (2). If f is constant then either $f = 0$ or $f = 1$. So, assume that f is not constant.

First consider the case where $f(A\backslash\{0\}) \cap (0, \infty) \neq \emptyset$. Then, by Lemma 3.4 (iii), f is either of the form (18) with some $c \in \mathbb{R}\backslash\{0\}$ and $d \in \mathbb{R}$, or it is of the form (20) with some $c \in \mathbb{R}\backslash\{0\}$ and $-\infty \leq \alpha < \beta \leq \infty$. In the first case, inserting into (2) f of the form (18), we get $(1-d)(cx+d) = 0$ for $x \in \mathbb{R}$. Thus $d = 1$ and so, in view of (18), f is of the form (22). Hence, (i) is valid. Assume that the second case holds. Then, making use of (20), we get

$$f(x) = 0 \quad \text{for} \quad x \in (\alpha, \beta). \tag{29}$$

Note that if $f(A\backslash\{0\}) \cap (-\infty, 0)$ contained an element different from -1 then, according to Lemma 3.4(i), f would be injective, which is excluded by (29). On the other hand, if $f(A\backslash\{0\}) \cap (-\infty, 0) = \{-1\}$, then applying Lemma 3.4(ii), we conclude that f is of the form (19) with a continuous decreasing involution $\phi : \mathbb{R} \to \mathbb{R}$ and some $a \in \mathbb{R}\backslash\{0\}$. Hence, in view of (29), we get $\phi(x) = x$ for $x \in (\alpha, \beta)$, which is not possible, as ϕ is decreasing. Therefore $\{0\} \neq f(A\backslash\{0\}) \subset [0, \infty)$. Let $a \in A\backslash\{0\}$ be such that $f(a) > 0$. If $c > 0$, then from (20) we derive that $\beta < a$ and $f(a) = c(a - \beta)$. Thus, applying (2) with $x = y = a$, we get $f(a + ac(a - \beta)) = c^2(a - \beta)^2 > 0$. Hence, in view of (20), we conclude that $a + ac(a - \beta) > \beta$ and so $f(a + ac(a - \beta)) = c(a + ac(a - \beta) - \beta)$. Therefore, we have $c(a + ac(a - \beta) - \beta) = c^2(a - \beta)^2$, which implies that $\beta = -1/c$. Thus, if $\alpha = -\infty$, then (20) becomes (23) and so (ii) holds. If $\alpha > -\infty$ then we get (25) with $d := \alpha < \beta = -1/c < 0$. Note also that as $\{0\} \neq f(A\backslash\{0\}) \subset [0, \infty)$, we have $A \subset [d, \infty)$ and $(A\backslash\{0\}) \cap (-1/c, \infty) \neq \emptyset$. Therefore, (iv) holds. For $c < 0$ similar arguments lead to (ii) whenever $\beta = \infty$ and to (iii) (with $d := \beta$), whenever $\beta < \infty$.

If $f(A\backslash\{0\}) \cap (-\infty, 0)$ contains an element different from -1 then, according to Lemma 3.4(i), f is of the form (18) with some $c \in \mathbb{R}\backslash\{0\}$ and $d \in \mathbb{R}$. Thus, repeating arguments from the beginning of the proof, we conclude that (i) holds.

Suppose that either $f(A\backslash\{0\}) = \{-1\}$ or $f(A\backslash\{0\}) = \{-1, 0\}$. Then, according to Lemma 3.4(ii), f is of the form (19) with a continuous decreasing involution $\phi : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$ such that $\phi(a) = 0$. Therefore, if $f(A \setminus \{0\}) = \{-1, 0\}$ then taking $z \in A \setminus \{0\}$ with $f(z) = 0$ and setting $y = z$ in (2), we obtain that $f(x + f(x)z) = 0$ for $x \in \mathbb{R}$. Thus $x + f(x)z \in Z_f$ for $x \in \mathbb{R}$. Moreover, in view of (19), every element of Z_f is a fixed point of ϕ. Since $z \in Z_f$ and ϕ, being decreasing, has at most one fixed point, this means that $x + f(x)z = z$ for $x \in \mathbb{R}$. Hence f is of the form (22) with $c := -1/z$ and so (i) holds. If $f(A \setminus \{0\}) = \{-1\}$ then, as $\phi(a) = 0$, taking $b \in A \setminus \{0\}$ and making use of (19), we obtain $\phi(b) - \phi(a) = \phi(b) = b + f(b)a = b - a$. Hence, if a and b were different, $\phi(b) - \phi(a)$ and $b - a$ would be of the same sign, which is not possible, as ϕ is decreasing. Thus $b = a$. In this way we have proved that either $A = \{a\}$ or $A = \{a, 0\}$. Furthermore, if $0 \in A$, we may derive from (16) and (19) that $\phi(0) = a$. Thus, (v) holds.

Finally, consider the case where $f(A \setminus \{0\}) = \{0\}$. If $A \setminus \{0\}$ is a singleton, say $A \setminus \{0\} = \{a\}$ with some $a \in \mathbb{R} \setminus \{0\}$, then in view of (4), we get (19) with $\phi = \phi_{(f,a)}$. Moreover, from (4) and (17) we derive that $\phi(a) = a$ and ϕ is a continuous projection, respectively. Furthermore, (4) and (16) imply that $\phi(0) = a$, provided $0 \in A$. Thus $\phi_{|A} = a$ and so (vi) holds. Assume that $A \setminus \{0\}$ contains at least two elements. From (2) it follows that $f \circ \phi_{(f,y)} = 0$ for $y \in A \setminus \{0\}$, that is $\phi_{(f,y)}(\mathbb{R}) \subset Z_f$ for $y \in A \setminus \{0\}$. On the other hand, if $x \in Z_f$ then according to (4), we get $x = x + f(x)y = \phi_{(f,y)}(x)$ for $y \in A \setminus \{0\}$, which means that $Z_f \subset \phi_{(f,y)}(\mathbb{R})$ for $y \in A \setminus \{0\}$. Therefore

$$\phi_{(f,y)}(\mathbb{R}) = Z_f \quad \text{for} \quad y \in A \setminus \{0\}. \tag{30}$$

Since f is continuous, (30) implies that Z_f is a closed interval. Suppose that it is bounded. Then, in virtue of (30), $\phi_{(f,y)}(\mathbb{R})$ is bounded for every $y \in A\setminus\{0\}$. Note that, taking $y_1, y_2 \in A \setminus \{0\}$ with $y_1 \neq y_2$, in view of (4), we get $f = \frac{\phi_{(f,y_1)}-\phi_{(f,y_2)}}{y_1-y_2}$. Hence f is bounded and so, applying (4) and (30), we obtain that $Z_f = \phi_{(f,y)}(\mathbb{R}) = \mathbb{R}$ for $y \in A \setminus \{0\}$, which yields a contradiction. Thus Z_f is an unbounded closed interval, that is there exists $b \in \mathbb{R}$ such that either $Z_f = [b, \infty)$ or $Z_f = (-\infty, b]$. Assume that the first possibility is valid. Then $f_{|[b,\infty)} = 0$, $A \setminus \{0\} \subset [b, \infty)$ and, in view of (4) and (30), we get

$$x + f(x)y \geq b \quad \text{for} \quad x \in \mathbb{R},\ y \in A \setminus \{0\}. \tag{31}$$

Suppose that there exist $a_1, a_2 \in A$ with $a_1 < 0 < a_2$. Then, applying (31), we obtain $(b-x)/a_2 \leq f(x) \leq (b-x)/a_1$ for $x \in \mathbb{R}$, which yields a contradiction. Therefore, by (16), either $b < 0$ and $A \subset [b, 0)$, or $A \subset [0, \infty)$. In the first case, putting $a := \sup\ A$, we get $b < a \leq 0$ and $A \subset [b, a]$. Furthermore, from (31) we derive that $x+f(x)a \geq b$ for $x \in (-\infty, b)$. Thus $a < 0$ and (28) holds, so (x) is valid. If $A \subset [0, \infty)$, then setting $a := \inf(A \setminus \{0\})$, we get $A \setminus \{0\} \subset [a, \infty)$. Moreover, arguing as previously, we conclude that $a > 0$, $b \leq a$ and (27) is valid. Since, according to Remark 3.1, (16) holds, we get (vii).

Using similar arguments, one can show that if $Z_f = (-\infty, b]$ then either $(viii)$ or (ix) holds.

The converse is easy to check. □

4. Solutions of Eq. (3)

Remark 4.1. Assume that $A \subset \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq. (3). Then, applying (3) with $y = 0$, we get $f(x) = f(x)f(0)$ for $x \in A$. Hence, either $f_{|A} = 0$ or $f(0) = 1$.

Proposition 4.2. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying equation (3) with some non-empty subset A of $\mathbb{R}$ and $f(A) \setminus \{-1, 0, 1\} \neq \emptyset$. Then there exists $c \in \mathbb{R} \setminus \{0\}$ such that*

$$f(x) = 1 + cx \quad \textit{for} \quad x \in A \setminus Z_f. \tag{32}$$

Proof. Since $f(A) \setminus \{-1, 0, 1\} \neq \emptyset$, according to Remark 4.1, we obtain that $f(0) = 1$. Therefore, taking $z \in A \setminus Z_f$ with $|f(z)| \neq 1$, we get $z \neq 0$. If $A \setminus Z_f = \{z\}$, then (32) trivially holds. So, assume that there is $x \in A \setminus Z_f$ with $x \neq z$. Then, applying (3), for every $y \in \mathbb{R}$, we obtain

$$f(x)(f(z)f(y)) = f(x)f(z + f(z)y) = f(x + f(x)z + f(x)f(z)y)$$

and

$$f(z)(f(x)f(y)) = f(z)f(x + f(x)y) = f(z + f(z)x + f(z)f(x)y).$$

Hence,

$$f(x + f(x)z + f(x)f(z)y) = f(z + f(z)x + f(z)f(x)y) \quad \text{for} \quad y \in \mathbb{R}.$$

Replacing y by $\frac{y-z-f(z)x}{f(z)f(x)}$, in this equality we get

$$f(y + (x + f(x)z - z - f(z)x)) = f(y) \quad \text{for} \quad y \in \mathbb{R}. \tag{33}$$

Suppose that $x + f(x)z - z - f(z)x \neq 0$. Then from (33) we derive that f is a periodic function. Thus f, being continuous, is bounded. Therefore, in view of (3), we obtain

$$\begin{aligned}\sup\{|f(y)| : y \in \mathbb{R}\} &= \sup\{|f(z + f(z)y)| : y \in \mathbb{R}\} \\ &= \sup\{|f(z)f(y)| : y \in \mathbb{R}\} = |f(z)| \sup\{|f(y)| : y \in \mathbb{R}\}.\end{aligned}$$

Since $|f(z)| \neq 1$, this means that $f = 0$, which yields a contradiction. In this way we have proved that

$$x + f(x)z - z - f(z)x = 0 \quad \text{for} \quad x \in A \setminus Z_f.$$

Hence, we obtain (32) with $c := (f(z) - 1)/z$. □

Theorem 4.3. *Assume that A is a non-empty subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then f satisfies Eq. (3) if and only if one of the following possibilities holds:*

(i) there exists $c \in \mathbb{R} \setminus \{0\}$ such that f is of the form (22);

(ii) there exist $c \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$A \subset (-\infty, -1/c] \text{ whenever } \alpha \neq 0, c < 0; \tag{34}$$

$$A \subset [-1/c, \infty) \text{ whenever } \alpha \neq 0, c > 0 \tag{35}$$

and

$$f(x) = \begin{cases} \alpha(1 + cx) & \text{whenever} \quad 1 + cx < 0, \\ 1 + cx & \text{otherwise;} \end{cases} \tag{36}$$

(iii) there exist (possibly empty) subsets A_{-1}, A_1 of A such that every non-zero element of A_1 is a period of f, for every $x \in A_{-1}$, the function $\psi_x : \mathbb{R} \to \mathbb{R}$ given by

$$\psi_x(y) = f(y + x/2) \quad \text{for} \quad y \in \mathbb{R}, \tag{37}$$

is odd and

$$f(x) = \begin{cases} i & \text{for} \quad x \in A_i, i \in \{-1, 1\}, \\ 0 & \text{for} \quad x \in A \setminus (A_{-1} \cup A_1); \end{cases} \tag{38}$$

(iv) there exist $B \subset A$, $c \in \mathbb{R} \setminus \{0\}$, $d \in (0, \infty) \setminus \{1\}$ and continuous 1-periodic functions $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ such that $p_1(0) = 1$,

$$\langle\{1 + cx : x \in B\}\rangle \cap (0, \infty) = \{d^n : n \in \mathbb{Z}\}, \tag{39}$$

$$p_1(\log_d(1 + cx)) = 0 \quad \text{for} \quad x \in A \setminus B \quad \text{with} \quad 1 + cx > 0, \tag{40}$$

$$p_2(\log_d(-(1 + cx))) = 0 \quad \text{for} \quad x \in A \setminus B \quad \text{with} \quad 1 + cx < 0, \tag{41}$$

$$p_2 = -p_1 \quad \text{whenever} \quad -(1 + d^n)/c \in B \quad \text{for some} \quad n \in \mathbb{Z}, \tag{42}$$

$$(9) \text{ holds whenever } \; -(1+d^{n/2})/c \in B \; \text{ for some odd } \; n \in \mathbb{Z} \tag{43}$$

and

$$f(x) = \begin{cases} -(1+cx)p_2(\log_d(-(1+cx))) & \text{whenever} \quad 1+cx<0, \\ 0 & \text{whenever} \quad x=-\frac{1}{c}, \\ (1+cx)p_1(\log_d(1+cx)), & \text{whenever} \quad 1+cx>0. \end{cases} \tag{44}$$

Proof. Assume that f satisfies (3). First consider the case where $f(A) \subset \{-1,0,1\}$. Let $A_i := \{x \in A : f(x) = i\}$ for $i \in \{-1,1\}$. Then (38) holds. Furthermore, if $x \in A_1 \setminus \{0\}$ then, in view of (3), we get $f(x+y) = f(y)$ for $y \in \mathbb{R}$. So, every non-zero element of A_1 is a period of f. Note also that if $x \in A_{-1}$ then, in view of (3) and (37), we get

$$\psi_x(-y) = f(-y+x/2) = f(x-(y+x/2)) = f(x+f(x)(y+x/2))$$
$$= f(x)f(y+x/2) = -f(y+x/2) = -\psi_x(y) \quad \text{for} \quad y \in \mathbb{R}.$$

Thus ψ_x is odd and so (iii) holds.

Now, assume that $f(A)\setminus\{-1,0,1\} \neq \emptyset$. Then, according to Proposition 4.2, (32) holds with some $c \in \mathbb{R}\setminus\{0\}$. Moreover, in view of (3) and (32), we obtain

$$f(x+(1+cx)y) = (1+cx)f(y) \quad \text{for} \quad x \in A \setminus Z_f, y \in \mathbb{R}.$$

Hence the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = f((x-1)/c) \quad \text{for} \quad x \in \mathbb{R}, \tag{45}$$

satisfies the equation

$$F((1+cx)(1+cy)) = (1+cx)F(1+cy) \quad \text{for} \quad x \in A \setminus Z_f, y \in \mathbb{R}.$$

Thus, replacing y by $(y-1)/c$, we obtain

$$F((1+cx)y) = (1+cx)F(y) \quad \text{for} \quad x \in A \setminus Z_f, y \in \mathbb{R},$$

that is F satisfies equation (6) with

$$C := \{1+cx : x \in A \setminus Z_f\}. \tag{46}$$

Obviously, F is continuous and, in view of (32), $C \subset \mathbb{R}\setminus\{0\}$. Note that, as $f_{|A}$ is not identically 0, applying Remark 4.1, we obtain $f(0) = 1$. Thus, making use of (45), we get $F(1) = f(0) = 1$. Since $f(A) \setminus \{-1,0,1\} \neq \emptyset$, we also have $C \setminus \{-1,1\} \neq \emptyset$. Moreover, in view of (45), we get

$$f(x) = F(1+cx) \quad \text{for} \quad x \in \mathbb{R}. \tag{47}$$

Suppose that $\langle C \rangle \cap (0,\infty)$ is dense in $(0,\infty)$. If $C \cap (-\infty,0) \neq \emptyset$, then applying Proposition 2.3 and making use of (47), we obtain (22). Hence (i) holds. If $C \subset (0,\infty)$ then from (46) we derive that

$$1+cx > 0 \quad \text{for} \quad x \in A \setminus Z_f. \tag{48}$$

Furthermore, taking (47) into account and applying Proposition 2.3, we obtain that f is of the form (36) with some $\alpha \in \mathbb{R}$. Note also that (36) and (48) imply (34) and (35) and so (ii) is valid.

If $\langle C\rangle \cap (0,\infty)$ is not dense in $(0,\infty)$ then from Remark 2.2 we deduce that $\langle C\rangle \cap (0,\infty) = \{d^n : n \in \mathbb{Z}\}$ for some $d \in (0,\infty) \setminus \{1\}$. Thus, in view of (8) and (47), we get (44). Moreover, taking $B := A \setminus Z_f$ and making use of (46), we obtain that (39) is valid,

$$p_2 = -p_1 \quad \text{whenever} \quad 1 + cx = -d^n \quad \text{for some} \quad x \in B,\ n \in \mathbb{Z}$$

and (9) holds whenever $1+cx = -d^{n/2}$ for some $x \in B$ and an odd $n \in \mathbb{Z}$. The last two assertions imply (42) and (43), respectively. Since $A \setminus B = A \cap Z_f$, from (44) we derive (40) and (41). Hence, (iv) holds.

The converse is easy to check. □

Remark 4.4. Applying Remark 4.1 we conclude that, if possibility (iii) in Theorem 4.3 is valid and $0 \in A$, then either $0 \in A_1$, or $A_{-1} = A_1 = \emptyset$ (and so $f_{|A} = 0$).

Example 4.5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \cos \frac{\pi}{2} x \quad \text{for} \quad x \in \mathbb{R}.$$

Then (38) holds with $A = \mathbb{Z}$, $A_1 = \{4k : k \in \mathbb{Z}\}$ and $A_{-1} = \{4k+2 : k \in \mathbb{Z}\}$. Moreover, every non-zero element of A_1 is a period of f and, for every $x \in A_{-1}$, we have

$$f\left(y + \frac{x}{2}\right) = \cos \frac{\pi}{2}\left(y + \frac{x}{2}\right) = (-1)^{(x+2)/4} \sin \frac{\pi}{2} y \quad \text{for} \quad y \in \mathbb{R}.$$

Thus, for every $x \in A_{-1}$, the function $\psi_x : \mathbb{R} \to \mathbb{R}$ given by (37) is odd. Therefore, according to Theorem 4.3, f satisfies the equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{for} \quad (x,y) \in \mathbb{Z} \times \mathbb{R}.$$

Example 4.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} |1+x| p(\log_8 |1+x|) & \text{for} \quad x \neq -1, \\ 0 & \text{for} \quad x = -1, \end{cases}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a continuous 1-periodic function such that $p(0) = 1$ and $p(x) = 0$ for $x \in [\log_8 6, \log_8 7]$. Then f is of the form (44) with $c = 1$, $d = 8$ and $p_1 = p_2 = p$. Moreover, it is easy to check that taking $A = (5,6) \cup \{7\}$ and $B = \{7\}$, we get (39)–(43). So, applying Theorem 4.3, we obtain that f satisfies (3).

It turns out that if A is a non-degenerate interval, then the description of solutions of Eq. (3) is significantly simpler. Namely, we have the following result.

Proposition 4.7. *Assume that $A \subset \mathbb{R}$ is a non-degenerate interval and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then f satisfies Eq. (3) if and only if either $f_{|A} = 0$ or $f = 1$ or one of the possibilities (i)–(ii) of Theorem 4.3 holds.*

Proof. Assume that f satisfies (3). Then, according to Theorem 4.3, one of the possibilities (i)–(iv) holds.

Assume that *(iii)* is valid. We are going to show that in this case either $f = 1$ or $f_{|A} = 0$. Since f is continuous, from (38) it follows that either $f_{|A} = 0$ or $A = A_i$ for some $i \in \{-1, 1\}$. If $A = A_1$ then every non-zero element of A is a period of f. As A is a non-degenerate interval, this means that f is constant and so $f = 1$. Suppose that $A = A_{-1}$. Then, for every $x \in A$, the function ψ_x given by (37) is odd. Thus, we have $f(x/2) = \psi_x(0) = 0$ for $x \in A$. So, taking $x_1 \in \operatorname{int} A$, for some $t \in (0, \infty)$, we get $\psi_{x_1}(y) = f(y + x_1/2) = 0$ for $y \in [-t, t]$. Hence, as f is continuous and not identically 0, the set $T := \{t \in [0, \infty) : \psi_{x_1}([-t, t]) = \{0\}\}$ is non-empty, closed and bounded with $M := \max T > 0$. Let $x_2 \in A$ be such that $0 < x_2 - x_1 < M$. Note that, according to (37), we get

$$\psi_{x_2}(y) = \psi_{x_1}(y + (x_2 - x_1)/2) \quad \text{for} \quad y \in \mathbb{R}. \tag{49}$$

Therefore, we have $\psi_{x_2}([-M - \frac{x_2 - x_1}{2}, M - \frac{x_2 - x_1}{2}]) = \psi_{x_1}([-M, M]) = \{0\}$. Since $0 < x_2 - x_1 < M$ and ψ_{x_2} is odd, this means that $\psi_{x_2}([-M - \frac{x_2 - x_1}{2}, M + \frac{x_2 - x_1}{2}]) = \{0\}$. Hence, in view of (49), we get $\psi_{x_1}([-M, M + (x_2 - x_1)]) = \{0\}$ and so, as $x_2 - x_1 > 0$ and ψ_{x_1} is odd, we obtain $\psi_{x_1}([-M - (x_2 - x_1), M + (x_2 - x_1)]) = \{0\}$, which contradicts the definition of M.

Now, assume that *(iv)* holds and suppose that $f_{|A}$ is not identically 0. Fix $x_0 \in A$ with $f(x_0) \neq 0$. Then, in view of (40), (41) and (44), we get $1 + cx_0 \neq 0$ and $x_0 \in B$. Suppose that $1 + cx_0 < 0$. Then, making use of (39) and applying Remark 2.2, we conclude that $1 + cx_0 = -d^{\frac{n}{2}}$ for some $n \in \mathbb{Z}$. Let (x_k) be a sequence of elements of $A \setminus \{x_0\}$ such that $\lim_{k\to\infty} x_k = x_0$ and $f(x_k) \neq 0$ for $k \in \mathbb{N}$. Then $x_k \in B$ for $k \in \mathbb{N}$ and $1 + cx_k < 0$ for sufficiently large $k \in \mathbb{N}$. So, arguing as previously, we obtain that for every sufficiently large $k \in \mathbb{N}$ there is an $n(k) \in \mathbb{Z}$ such that $1 + cx_k = -d^{n(k)/2}$. Then, we have

$$(1/c)(-d^{n/2} - 1) = x_0 = \lim_{k\to\infty} x_k = \lim_{k\to\infty} (1/c)(-d^{n(k)/2} - 1),$$

which implies that $\lim_{k\to\infty} n(k) = n$. Hence, for sufficiently large $k \in \mathbb{N}$, we get $n(k) = n$ and so $x_k = x_0$, which yields a contradiction.

If $1 + cx_0 > 0$ then, applying Remark 2.2, we get $1 + cx_0 = d^n$ for some $n \in \mathbb{Z}$. Thus, repeating the previous arguments, we again get a contradiction. In this way we have proved that $f_{|A} = 0$.

The converse is easy to check. □

Remark 4.8. In view of Example 4.6, the assumption that A is an interval is essential in Proposition 4.7.

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