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Strong convexity and separation theorems

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Dedicated to Professor Roman Ger on his 70th birthday

Abstract. Characterizations of pairs of functions that can be separated by a strongly convex, approximately concave or c-quadratic-affine function are presented. As consequences, stability results of the Hyers-Ulam type are obtained.

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1. Introduction

It is known, from the classical Hahn-Banach theorem, that if a function f is concave, g is convex and $f \le g$, then there exists an affine function h such that $f \le h \le g$. This separation (sandwich) theorem plays a crucial role especially in the field of convex analysis. Many other results about the separation of two given functions by a function from some special class (for instance, by a convex, affine, midconvex, Jensen, sublinear, linear, subadditive, additive, quasiconvex, monotonic, quadratic function) can be found in the literature (see, e.g. [1,3–5,7,10,12–16] and the references therein).

In this note we present a characterization of pairs of functions that can be separated by a strongly convex, approximately concave or c-quadratic-affine function. As consequences, we obtain stability results of the Hyers-Ulam type. Strongly convex functions have applications in optimization, mathematical economics and approximation theory. Many properties of them can be found, for instance, in [6,8,11,17–19].

Let $D \subset \mathbb{R}^n$ be a convex set and c be a positive number. A function $f : D \to \mathbb{R}$ is called:

– *strongly convex with modulus* c if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2, \qquad (1)$$

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for all $x, y \in D$ and $t \in [0, 1]$;

– *approximately concave with modulus* c if

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2, \tag{2}$$

for all $x, y \in D$ and $t \in [0, 1]$;

– *c-quadratic-affine* if it is strongly convex with modulus c and simultaneously approximately concave with modulus c, that is

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2, \tag{3}$$

for all $x, y \in D$ and $t \in [0, 1]$.

2. Connections with generalized convexity

In the case $n = 1$ the definitions of strong convexity and approximate concavity are strictly connected with the notion of generalized convexity introduced by Beckenbach [2]. Let us recall that a family $\mathcal{F}$ of continuous real functions defined on an interval $I \subset \mathbb{R}$ is called a two-parameter family if for any two points $(x_1, y_1),\ (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = y_i \quad \text{for} \quad i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ determined by the points $(x_1, y_1),\ (x_2, y_2)$ will be denoted by $\varphi_{(x_1,y_1),\ (x_2,y_2)}$. Following Beckenbach a function $f : I \to \mathbb{R}$ is said to be $\mathcal{F}$-convex if for any $x_1, x_2 \in I$, $x_1 < x_2$

$$f(x) \leq \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(x) \quad \text{for all} \quad x \in [x_1, x_2];$$

f is said to be $\mathcal{F}$-concave if for any $x_1, x_2 \in I$, $x_1 < x_2$

$$f(x) \geq \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(x) \quad \text{for all} \quad x \in [x_1, x_2].$$

Clearly, these definitions are motivated by the fact that if

$$\mathcal{F} = \{ax + b : a, b \in \mathbb{R}\},$$

then $\mathcal{F}$-convexity ($\mathcal{F}$-concavity) coincides with the classical convexity (concavity). In a similar way we can characterize strong convexity and approximate concavity. Let c be a positive number and

$$\mathcal{F}_c = \{cx^2 + ax + b : a, b \in \mathbb{R}\}.$$

Clearly, $\mathcal{F}_c$ is also a two parameter family. Moreover, the following theorem holds:

Theorem 1. *Let $f : I \to \mathbb{R}$. Then*

(1) *f is strongly convex with modulus c if and only if f is $\mathcal{F}_c$-convex;*
(2) *f is approximately concave with modulus c if and only if f is $\mathcal{F}_c$-concave;*
(3) *f is c-quadratic-affine if and only if $f \in \mathcal{F}_c$.*

Proof. Part (1) is proved in [8]. To prove (2) fix $x_1, x_2 \in I$ and take $\varphi = \varphi_{(x_1,f(x_1)),(x_2,f(x_2))} \in \mathcal{F}_c$. Then $\varphi(x) = cx^2 + ax + b$, where the coefficients a, b are uniquely determined by the conditions $\varphi(x_i) = f(x_i)$, $i = 1, 2$. Hence, for every $t \in [0, 1]$, we have

$$\begin{aligned}\varphi(tx_1 + (1-t)x_2) &= c(tx_1 + (1-t)x_2)^2 + a(tx_1 + (1-t)x_2) + b \\ &= c(t^2x_1^2 + 2t(1-t)x_1x_2 + (1-t)^2x_2^2) \\ &\quad + a(tx_1 + (1-t)x_2) + b \\ &= t(cx_1^2 + ax_1 + b) + (1-t)(cx_2^2 + ax_2 + b) \\ &\quad - ct(1-t)(x_1^2 - 2x_1x_2 + x_2^2) \\ &= tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2.\end{aligned}$$

Consequently, if f is approximately concave with modulus c, then

$$\begin{aligned}f(tx_1 + (1-t)x_2) &\geq tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2 \\ &= \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(tx_1 + (1-t)x_2),\end{aligned}$$

which means that f is $\mathcal{F}_c$-concave.

Conversely, if f is $\mathcal{F}_c$-convex, then

$$\begin{aligned}f(tx_1 + (1-t)x_2) &\geq \varphi_{(x_1,f(x_1)),(x_2,f(x_2))}(tx_1 + (1-t)x_2) \\ &= tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2,\end{aligned}$$

which shows that f is approximately concave with modulus c.
Part (3) follows from (1) and (2) and the fact that f is $\mathcal{F}_c$-convex and, simultaneously, $\mathcal{F}_c$-approximately concave if and only if $f \in \mathcal{F}_c$. □

3. Separation by strongly convex and approximately concave functions

In what follows we assume that D is a convex subset of $(\mathbb{R}^n, \|\cdot\|)$ and c is a positive number. We start with the following statement which is a useful tool in our investigations (see [6, Proposition 1.1.2.]; cf. also [10]).

Lemma 2. *Let $f : D \to \mathbb{R}$. Then*

(1) *f is strongly convex with modulus c if and only if $f - c\|x\|^2$ is convex;*
(2) *f is approximately concave with modulus c if and only if $f - c\|x\|^2$ is concave;*
(3) *f is c-quadratic-affine if and only if $f - c\|x\|^2$ is affine.*

Proof. It is enough to use the equality

$$\|tx + (1-t)y\|^2 + t(1-t)\|x - y\|^2 = t\|x\|^2 + (1-t)\|y\|^2.$$

□

The following result characterizes pairs of functions which can be separated by a strongly convex one. It is a counterpart of the sandwich theorem obtained in [1]. For $n=1$ an analogous result is given in [8].

Theorem 3. *Let $f,g:D\to\mathbb{R}$. There exists a function $h:D\to\mathbb{R}$ strongly convex with modulus c such that $f\le h\le g$ on D if and only if*

$$f\left(\sum_{i=1}^{n+1} t_i x_i\right) \le \sum_{i=1}^{n+1} t_i g(x_i) - c\sum_{i=1}^{n+1} t_i\|x_i-m\|^2 \tag{4}$$

for all $x_1,\dots,x_{n+1}\in D$, $t_1,\dots,t_{n+1}\ge 0$ with $t_1+\cdots+t_{n+1}=1$ and $m=t_1x_1+\cdots+t_{n+1}x_{n+1}$.

Proof. Assume first that $f\le h\le g$ where h is strongly convex with modulus c. Using the Jensen inequality for strongly convex functions (see [9, Theorem 2]), we get

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} t_i x_i\right) &\le h\left(\sum_{i=1}^{n+1} t_i x_i\right) \le \sum_{i=1}^{n+1} t_i h(x_i) - c\sum_{i=1}^{n+1} t_i\|x_i-m\|^2 \\ &\le \sum_{i=1}^{n+1} t_i g(x_i) - c\sum_{i=1}^{n+1} t_i\|x_i-m\|^2. \end{aligned}$$

To prove the converse implication, assume that f,g satisfy (4) and consider the functions $f_1,g_1:D\to\mathbb{R}$ defined by

$$f_1(x)=f(x)-c\|x\|^2,\quad g_1(x)=g(x)-c\|x\|^2,\quad x\in I.$$

Using (4) and the fact that

$$\sum_{i=1}^{n+1} t_i\|x_i-m\|^2 = \sum_{i=1}^{n+1} t_i\|x_i\|^2 - \|m\|^2,$$

we obtain

$$\begin{aligned} f_1\left(\sum_{i=1}^{n+1} t_i x_i\right) &= f\left(\sum_{i=1}^{n+1} t_i x_i\right) - c\Big\|\sum_{i=1}^{n+1} t_i x_i\Big\|^2 \le \sum_{i=1}^{n+1} t_i g(x_i) \\ &\quad -c\sum_{i=1}^{n+1} t_i\|x_i-m\|^2 - c\|m\|^2 \\ &= \sum_{i=1}^{n+1} t_i\left(g(x_i)-c\|x_i\|^2\right) = \sum_{i=1}^{n+1} t_i g_1(x_i). \end{aligned}$$

Hence, by the Baron-Matkowski-Nikodem sandwich theorem [1], there exists a convex function $h_1:D\to\mathbb{R}$ such that $f_1\le h_1\le g_1$ on D. Define $h(x)=h_1(x)+c\|x\|^2$ for $x\in D$. Then, by Lemma 2, h is strongly convex with modulus c and $f\le h\le g$ on D. $\square$

In a similar way we can characterize functions which can be separated by an approximately concave one.

Theorem 4. *Let $f, g : D \to \mathbb{R}$. There exists a function $h : D \to \mathbb{R}$ approximately concave with modulus c such that $f \le h \le g$ on D if and only if*

$$\sum_{i=1}^{n+1} t_i f(x_i) \ge g\left(\sum_{i=1}^{n+1} t_i x_i\right) + c\sum_{i=1}^{n+1} t_i \|x_i - m\|^2, \tag{5}$$

for all $x_1, \ldots, x_{n+1} \in D$, $t_1, \ldots, t_{n+1} \ge 0$ with $t_1 + \cdots + t_{n+1} = 1$ and $m = t_1 x_1 + \cdots + t_{n+1} x_{n+1}$.

As a consequence of the above theorems we obtain the following Hyers-Ulam-type stability results for strongly convex and approximately concave functions.

Corollary 5. *Let $\varepsilon > 0$. If $f : D \to \mathbb{R}$ satisfies the condition*

$$f\left(\sum_{i=1}^{n+1} t_i x_i\right) \le \sum_{i=1}^{n+1} t_i f(x_i) - c\sum_{i=1}^{n+1} t_i \|x_i - m\|^2 + \varepsilon \tag{6}$$

for all $x_1, \ldots, x_{n+1} \in D$, $t_1, \ldots, t_{n+1} \ge 0$ with $t_1 + \cdots + t_{n+1} = 1$ and $m = t_1 x_1 + \cdots + t_{n+1} x_{n+1}$, then there exists a function $h : D \to \mathbb{R}$ strongly convex with modulus c such that

$$|f(x) - h(x)| \le \frac{\varepsilon}{2}, \ x \in D. \tag{7}$$

Proof. Condition (6) means that f and $g = f + \varepsilon$ satisfy (4). Therefore, by Theorem 3, there exists a function h_1 strongly convex with modulus c such that $f \le h_1 \le f + \varepsilon$. Putting $h = h_1 - \varepsilon/2$, we get (7). $\square$

In an analogous way, using Theorem 4, we also get the next result.

Corollary 6. *Let $\varepsilon > 0$. If $f : D \to \mathbb{R}$ satisfies the condition*

$$f\left(\sum_{i=1}^{n+1} t_i x_i\right) \ge \sum_{i=1}^{n+1} t_i f(x_i) - c\sum_{i=1}^{n+1} t_i \|x_i - m\|^2 + \varepsilon$$

for all $x_1, \ldots, x_{n+1} \in D$, $t_1, \ldots, t_{n+1} \ge 0$ with $t_1 + \cdots + t_{n+1} = 1$ and $m = t_1 x_1 + \cdots + t_{n+1} x_{n+1}$, then there exists a function $h : D \to \mathbb{R}$ approximately concave with modulus c such that

$$|f(x) - h(x)| \le \frac{\varepsilon}{2}, \ x \in D.$$

4. Separation by c-quadratic-affine functions

In this section we consider the problem of separating two given functions by a c-quadratic-affine one. Obviously, if there exists a c-quadratic-affine function h such that $f \le h \le g$ on $D \subset \mathbb{R}^n$, then f and g satisfy conditions (4) and (5) (because h is strongly convex and approximately concave). For $n = 1$ the converse implication is also true (see Corollary 8 below). However, for $n > 1$ conditions (4) and (5) together are not sufficient for the separation of f and g by a c-quadratic-affine function (we can build a counterexample using the functions f and g described in [14, Remark 2] and Lemma 2). An appropriate necessary and sufficient condition is given in the following theorem. It is a counterpart of the result on separation by affine functions proved in [3].

Theorem 7. *Let $f, g : D \to \mathbb{R}$. There exists a c-quadratic-affine function $h : D \to \mathbb{R}$ such that $f \le h \le g$ on D if and only if*

$$\sum_{i=1}^{k} s_i\big(f(x_i) - c\|x_i - m\|^2\big) \le \sum_{j=k+1}^{n+2} t_j\big(g(x_j) - c\|x_j - m\|^2\big) \tag{8}$$

for all $x_1, \ldots, x_{n+2} \in D$, $k \in \{1, \ldots, n+1\}$, $s_1, \ldots, s_k, t_{k+1}, \ldots, t_{n+2} \ge 0$, such that $s_1 + \cdots + s_k = t_{k+1} + \cdots + t_{n+2} = 1$ and $m = s_1x_1 + \cdots + s_kx_k = t_{k+1}x_{k+1} + \cdots + t_{n+2}x_{n+2}$.

Proof. To prove the "only if" part assume that $f \le h \le g$ with a c-quadratic-affine function h and fix $x_1, \ldots, x_{n+2}$, k, $s_1, \ldots, s_k$ and $t_{k+1}, \ldots, t_{n+2}$ as above. Then

$$\begin{aligned}\sum_{i=1}^{k} s_i h(x_i) &= h\left(\sum_{i=1}^{k} s_i x_i\right) + c\sum_{i=1}^{k} s_i\|x_i - m\|^2 \\ &= h\left(\sum_{j=k+1}^{n+2} t_j x_i\right) + c\sum_{i=1}^{k} s_i\|x_i - m\|^2 \\ &= \sum_{j=k+1}^{n+2} t_j h(x_j) - c\sum_{j=k+1}^{n+2} t_j\|x_j - m\|^2 + c\sum_{i=1}^{k} s_i\|x_i - m\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=1}^{k} s_i\big(f(x_i) - c\|x_i - m\|^2\big) &\le \sum_{i=1}^{k} s_i h(x_i) - c\sum_{i=1}^{k} s_i\|x_i - m\|^2 \\ &= \sum_{j=k+1}^{n+2} t_j h(x_j) - c\sum_{j=k+1}^{n+2} t_j\|x_j - m\|^2 \\ &\le \sum_{j=k+1}^{n+2} t_j\big(g(x_j) - c\|x_j - m\|^2\big).\end{aligned}$$

To prove the "if" part consider the functions $f_1, g_1 : D \to \mathbb{R}$ defined by

$$f_1(x) = f(x) - c\|x\|^2, \quad g_1(x) = g(x) - c\|x\|^2, \quad x \in D.$$

Using (8) and the fact that

$$\begin{aligned}\sum_{i=1}^{k} s_i\|x_i - m\|^2 &= \sum_{i=1}^{k} s_i\|x_i\|^2 - \|m\|^2 \quad \text{and}\\ \sum_{j=k+1}^{n+2} t_i\|x_j - m\|^2 &= \sum_{j=k+1}^{n+2} t_j\|x_j\|^2 - \|m\|^2,\end{aligned}$$

we obtain

$$\begin{aligned}\sum_{i=1}^{k} s_i f_1(x_i) &= \sum_{i=1}^{k} s_i f(x_i) - c\sum_{i=1}^{k} s_i\|x_i\|^2\\ &= \sum_{i=1}^{k} s_i f(x_i) - c\left(\sum_{i=1}^{k} s_i\|x_i - m\|^2 - \|m\|^2\right)\\ &\le \sum_{j=k+1}^{n+2} t_j g(x_j) - c\left(\sum_{j=k+1}^{n+2} t_j\|x_j - m\|^2 - \|m\|^2\right)\\ &= \sum_{j=k+1}^{n+2} t_j g(x_j) - c\sum_{j=k+1}^{n+2} t_j\|x_j\|^2 = \sum_{j=k+1}^{n+2} t_j g_1(x_j).\end{aligned}$$

This implies, on account of the Behrends–Nikodem separation theorem [3], that there exists an affine function $h_1 : \mathbb{R}^n \to \mathbb{R}$ such that $f_1 \le h_1 \le g_1$ on D. Define $h(x) = h_1(x) + c\|x\|^2$ for $x \in D$. Then, by Lemma 2, h is strongly convex with modulus c and $f \le h \le g$ on D. □

In the case $n = 1$ condition (8) reduces to the system of two inequalities obtained for $k = 1$ and $k = 2$. Therefore, as a consequence of Theorem 7 we get the following counterpart of the sandwich theorem obtained in [14]. This result follows also from Lemma 2 and the separation theorem proved in [10].

Corollary 8. *Let $I \subset \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$. The following conditions are equivalent*

1. *there exists a c-quadratic-affine function $h : I \to \mathbb{R}$ such that $f \le h \le g$ on I;*
2. *there exist functions $h_1 : I \to \mathbb{R}$ strongly convex with modulus c and $h_2 : I \to \mathbb{R}$ approximately concave with modulus c such that $f \le h_1 \le g$ and $f \le h_2 \le g$ on I;*

3. $$f(tx_1 + (1-t)x_2) \le tg(x_1) + (1-t)g(x_2) - ct(1-t)(x_1 - x_2)^2$$
$$g(tx_1 + (1-t)x_2) \ge tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2,$$
for all $x_1, x_2 \in I$ *and* $t \in [0,1]$.

Proof. The implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are obvious, whereas $3 \Rightarrow 1$ follows from Theorem 7. □

As another consequence of Theorem 7 we also obtain the following Hyers-Ulam stability result for c-quadratic-affine functions.

Corollary 9. *Let* $\varepsilon > 0$. *If* $f : D \to \mathbb{R}$ *satisfies the condition*

$$\sum_{i=1}^{k} s_i\big(f(x_i) - c\|x_i - m\|^2\big) \le \sum_{j=k+1}^{n+2} t_j\big(f(x_j) - c\|x_j - m\|^2\big) + \varepsilon \tag{9}$$

for all $x_1, \dots, x_{n+2} \in D$, $k \in \{1, \dots, n+1\}$, $s_1, \dots, s_k, t_{k+1}, \dots, t_{n+2} \ge 0$, *such that* $s_1 + \dots + s_k = t_{k+1} + \dots + t_{n+2} = 1$ *and* $m = s_1x_1 + \dots + s_kx_k = t_{k+1}x_{k+1} + \dots + t_{n+2}x_{n+2}$, *then there exists a c-quadratic-affine function* $h : D \to \mathbb{R}$ *such that*

$$|f(x) - h(x)| \le \frac{\varepsilon}{2}, \quad x \in D. \tag{10}$$

Proof. Condition (9) implies that f and $g = f + \varepsilon$ satisfy (8). Therefore, by Theorem 7, there exists a c-quadratic-affine function h_1 such that $f \le h_1 \le f + \varepsilon$. Putting $h = h_1 - \varepsilon/2$, we get (10). □

References

[1] Baron, K., Matkowski, J., Nikodem, K.: A sandwich with convexity. Math. Pannonica **5**(1), 139–144 (1994)
[2] Beckenbach, E.F.: Generalized convex functions. Bull. Am. Math. Soc. **43**, 363–371 (1937)
[3] Behrends, E., Nikodem, K.: A selection theorem of Helly type and its applications. Stud. Math. **116**, 43–48 (1995)
[4] Bessenyei, M., Szokol, P.: Separation by convex interpolation families. J. Conv. Anal. **20**(4), 937–946 (2013)
[5] Förg-Rob, W., Nikodem, K., Páles, Z.: Separation by monotonic functions. Math. Pannonica **7**, 191–196 (1996)
[6] Hiriart–Urruty, J.-B., Lemaréchal, C.: Fundamentals of Convex Analysis. Springer–Verlag, Berlin–Heidelberg (2001)
[7] König, H.: On the Abstract Hahn–Banach Theorem due to Rodé. Aequationes Math. **34**, 89–95 (1987)

[8] Merentes, N., Nikodem, K.: Remarks on strongly convex functions. Aequationes Math. **80**, 193–199 (2010)
[9] Nikodem, K.: Strongly Convex Functions and Related Classes of Functions. In: Rassias, T.M. (eds.) Handbook of Functional Equations. Functional Inequalities, vol. 95, pp. 365–405. Springer Optimizations and Its Applications, New York (2015)
[10] Nikodem, K., Páles, Z.: Generalized convexity and separation theorems. J. Conv. Anal. **14**(2), 239–247 (2007)
[11] Nikodem, K., Páles, Z.: Characterizations of inner product spaces by strongly convex functions. Banach J. Math. Anal. **5**(1), 83–87 (2011)
[12] Nikodem, K., Páles, Z., Wąsowicz, S.: Abstract separation theorems of Rode type and their applications. Ann. Pol. Math. **72**, 207–217 (1999)
[13] Nikodem, K., Sadowska, E., Wąsowicz, S.: A note on separation by subadditive and sublinear functions. Ann. Math. Silesianae **14**, 33–39 (2000)
[14] Nikodem, K., Wąsowicz, S.: A sandwich theorem and Hyers–Ulam stability of affine functions. Aequationes Math. **49**, 160–164 (1995)
[15] Páles, Z.: Separation by approximately convex functions. Grazer Math. Ber. **344**, 43–50 (2001)
[16] Páles, Z., Zeiden, V.: Separation via quadratic functions. Aequationes Math. **51**, 209–229 (1996)
[17] Polovinkin, E.S.: Strongly convex analysis. Sb. Math. **187**(2), 103–130 (1996)
[18] Polyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. Soviet Math. Dokl. **7**, 72–75 (1966)
[19] Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York–London (1973)

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