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Improving regularity of solutions of a difference equation

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Dedicated to Professor János Aczél on the occasion of his 90th birthday

Abstract. Using some results on convex and almost convex functions defined on a locally compact Abelian group, we prove a theorem showing a "measurability implies continuity" effect for non-negative solutions of the difference equation $\varphi(x) = \sum_{i=1}^{k} p_i\varphi(x + a_i)$, where $p_1, \dots, p_k \in (0, \infty)$ and non-zero elements $a_1, \dots, a_k$ of the group are given.

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Introduction

Given an Abelian group G, non-zero elements $a_1, \dots, a_k \in G$ and positive numbers $p_1, \dots, p_k$ we are interested in non-negative solutions $\varphi\colon G \to \mathbb{R}$ of the difference equation

$$\varphi(x) = \sum_{i=1}^{k} p_i\varphi(x + a_i). \tag{E}$$

In the case when $G = \mathbb{R}$ all non-negative Lebesgue measurable solutions of (E) were determined by Laczkovich [12]. Later, another proof was given by the present author (see [5, Th. 3.1]), and then by Grinč [2] when $G = \mathbb{R}^n$.

The main step in the reasoning presented there (cf. [5, Prop. 3.3]) is an improvement of regularity of non-negative solutions of (E) provided the subgroup generated by $a_1, \dots, a_k$ is dense in G. Such a "measurability implies continuity" effect is well-known in the theory of functional equations in several variables (cf. for instance, the book [4] by A. Járai; also [1] by J. Aczél and [11] by M. Kuczma) but for equations in a single variable it is rather unexpected.

In the present paper we show how to improve regularity of non-negative solutions of (E) in the case when G is a locally compact Abelian group. Some arguments presented here take the pattern of those used in [5] in the case $G = \mathbb{R}$.

In the whole paper *measurability* of a function defined on G means $\mathcal{M}_\lambda$-measurability, where $\mathcal{M}_\lambda$ stands for the completion of the σ-algebra $\mathcal{B}(G)$ of Borel subsets of G with respect to the Haar measure λ. Equivalently this is *measurability in the sense of Carathéodory*, i.e.

$$\mathcal{M}_\lambda = \left\{A \subset G\colon \lambda^*(Z) \geq \lambda^*(Z \cap A) + \lambda^*(Z \setminus A) \text{ for every } Z \subset G\right\},$$

where $\lambda^*\colon 2^G \to [0,\infty]$ is the outer measure generated by λ:

$$\lambda^*(A) = \inf\left\{\lambda(B)\colon A \subset B \in \mathcal{B}(G)\right\}$$

for every $A \subset G$. Measurability of a function defined on G^2 is meant with respect to the λ^2-completion $\mathcal{M}_{\lambda^2}$ of the σ-algebra $\mathcal{B}\left(G^2\right)$, where λ^2 is the product measure built with two copies of λ.

The main result of the paper reads as follows.

Theorem. *Let G be an Abelian 2-divisible group, σ-compact and locally compact, with Haar measure λ. Assume that the subgroup generated by $a_1, \ldots, a_k$ is dense in G.*

If $\varphi\colon G \to \mathbb{R}$ is a non-negative measurable solution of equation (E), then either $\varphi = 0$ λ-a.e., or there is a positive continuous geometrically convex solution $\psi\colon G \to \mathbb{R}$ of (E) such that $\varphi = \psi$ λ-a.e.

Geometric convexity of $\psi\colon G \to \mathbb{R}$ means here that

$$\psi(x)^2 \leq \psi(x+h)\,\psi(x-h)$$

for all $x, h \in G$.

The proof of the Theorem is split into some lemmas presented in Sect. 2. Moreover, the following remarks will be recalled in Sect. 1 while proving some auxiliary facts.

Remark 0.1. It is well-known (cf. [3, (15.8) and (11.34)]) that the Haar measure on any Abelian locally compact group G is *regular, i.e.*

$$\lambda(B) = \inf\left\{\lambda(U)\colon B \subset U \subset G \text{ and } U \text{ is open}\right\}$$

for every $B \in \mathcal{B}(G)$ and

$$\lambda(B) = \sup\left\{\lambda(C)\colon C \subset B \text{ and } C \text{ is compact}\right\}$$

for every set $B \subset G$ which is open or of finite measure. Moreover, λ takes finite values on compacts. So, if $K \subset G$ is compact, then for all $B \in \mathcal{B}(K)$ and $\varepsilon \in (0,\infty)$ there exist a set U, open in K, and a compact set C such that

$$C \subset B \subset U \quad \text{and} \quad \lambda(U \setminus C) < \varepsilon.$$

The next two remarks concern folk-theorems.

Remark 0.2. Repeating the proof of [14, Th. 8.2] step by step we come to the following version of the classical Lusin's theorem.

Let X and Y be topological spaces, the second one with a countable base, and let μ be a measure defined on a σ-algebra $\mathcal{M}$ of subsets of X containing all Borel sets. Assume that for all $B \in \mathcal{M}$ and $\varepsilon \in (0, \infty)$ there exist an open set $U \subset X$ and a closed set $F \subset X$ such that

$$F \subset B \subset U \quad \text{and} \quad \mu(U \setminus F) < \varepsilon.$$

If $f: X \to Y$ is an $\mathcal{M}$-measurable function, then for every $\varepsilon \in (0, \infty)$ there exists a closed set $F \subset X$ such that

$$\mu(X \setminus F) < \varepsilon \quad \text{and the function } f|_F \text{ is continuous.}$$

Remark 0.3. The standard argument, proving that in a metric setting any continuous function defined on a compact set is uniformly continuous, allows to obtain the following group version of this fact.

Any continuous function f, mapping a compact subset C of an Abelian topological group G into an Abelian topological group H, is uniformly continuous: for every neighbourhood $W \subset H$ of 0 there exists a neighbourhood $V \subset G$ of 0 such that

$$\bigwedge_{x_1, x_2 \in C} \big(x_1 - x_2 \in V \Rightarrow f(x_1) - f(x_2) \in W\big).$$

1. Auxiliary results

We start with two general facts, not immediately connected with the problem of solutions of equation (E). The first one is a simple purely topological observation.

Lemma 1.1. *Let G be an Abelian σ-compact and locally compact group. Then there exists a sequence $(K_n)_{n \in \mathbb{N}}$ of compacts and a neighbourhood U of 0 such that* $\mathrm{cl}U$ *is compact,*

$$K_n + U \subset K_{n+1}, \quad n \in \mathbb{N}, \tag{1.1}$$

and

$$G = \bigcup_{n=1}^{\infty} K_n. \tag{1.2}$$

Proof. The group G, being σ-compact, is the union of a sequence $(C_n)_{n \in \mathbb{N}}$ of compacts. Take any neighbourhood U of 0 such that $\mathrm{cl}U$ is compact. For every $n \in \mathbb{N}$ put

$$K_n = \bigcup_{i=1}^{n} C_i + [n]\mathrm{cl}U,$$

where $[n]A$ stands for the sum $A+\cdots+A$ of n copies of A. Clearly the sets $K_n, n\in\mathbb{N}$, are compact. Moreover, for every $n\in\mathbb{N}$ we have

$$C_n\subset K_n\subset K_n+U=\bigcup_{i=1}^{n}C_i+[n]\mathrm{cl}U+U\subset K_{n+1}$$

and the desired properties (1.1) and (1.2) follow. □

The next result is an extension of [12, Lemma 2] to a group setting (see also [4, Theorems 19.3 and 19.5]).

Lemma 1.2. *Let G be an Abelian σ-compact and locally compact group with Haar measure λ and let $\varphi\colon G\to\mathbb{R}$ be a measurable function. Then, for every $y_0\in G$ and for every sequence $(y_n)_{n\in\mathbb{N}}$ of elements of G converging to y_0, there exists a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers such that*

$$\lim_{n\to\infty}\varphi\left(x+y_{m_n}\right)=\varphi\left(x+y_0\right)\quad\text{for }\lambda\text{–}a.a.\ x\in G.$$

Proof. Since λ is translation invariant, we may additionally assume that $y_0=0$. Define functions $\varphi_n\colon G\to\mathbb{R}$, $n\in\mathbb{N}$, by

$$\varphi_n\left(x\right)=\varphi\left(x+y_n\right).$$

By virtue of Lemma 1.1 we find a sequence $(K_i)_{i\in\mathbb{N}}$ of compacts in G and a neighbourhood $U\subset G$ of 0 satisfying (1.1) and (1.2). We prove that for every $i\in\mathbb{N}$ the sequence $(\varphi_n|_{K_i})_{n\in\mathbb{N}}$ converges in measure to the function $\varphi|_{K_i}$.

Fix any $i\in\mathbb{N}$ and a positive number ε. Following Remarks 0.1 and 0.2 we find a closed subset F of K_{i+1} such that

$$\lambda\left(K_{i+1}\setminus F\right)<\frac{\varepsilon}{2}\text{ and the function }\varphi|_F\text{ is continuous.}$$

Since the set F is compact, $\varphi|_F$ is actually uniformly continuous (cf. Remark 0.3). Thus we can find a neighbourhood $V\subset U$ of 0 such that $|\varphi(x_1)-\varphi(x_2)|<\varepsilon$ for all $x_1,x_2\in F$ satisfying $x_1-x_2\in V$.

Define a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of G by

$$A_n=\left\{x\in K_i\colon\ |\varphi_n\left(x\right)-\varphi\left(x\right)|\geq\varepsilon\right\}.$$

Since $(y_n)_{n\in\mathbb{N}}$ converges to 0, there exists a positive integer n_0 such that $y_n\in V$ for every $n\geq n_0$. Take any integer $n\geq n_0$ and point $x\in A_n$. Suppose that $x\in F\cap(F-y_n)$. Then $x, x+y_n\in F$ and $(x+y_n)-x=y_n\in V$, and thus

$$|\varphi_n\left(x\right)-\varphi\left(x\right)|=|\varphi\left(x+y_n\right)-\varphi(x)|<\varepsilon,$$

which is impossible. This shows that

$$A_n\subset\left(K_i\setminus F\right)\cup\left[\left(\left(K_i+V\right)\setminus F\right)-y_n\right]\subset\left(K_i\setminus F\right)\cup\left[\left(K_{i+1}\setminus F\right)-y_n\right].$$

Consequently, we have

$$\begin{aligned}\lambda\left(A_n\right)&\leq\lambda\left(K_i\setminus F\right)+\lambda\left(\left(K_{i+1}\setminus F\right)-y_n\right)\\&=\lambda\left(K_i\setminus F\right)+\lambda\left(K_{i+1}\setminus F\right)\leq2\lambda\left(K_{i+1}\setminus F\right)<\varepsilon.\end{aligned}$$

This proves that the sequence $(\varphi_n|_{K_i})_{n\in\mathbb{N}}$ converges in measure to $\varphi|_{K_i}$.

Consequently, every subsequence of $(\varphi_n|_{K_i})_{n\in\mathbb{N}}$ has a subsequence converging to $\varphi|_{K_i}$. Using induction and a standard diagonal method we complete the proof. $\square$

Now we remark that the group addition and substraction are transformations preserving measurability.

Lemma 1.3. *Let G be an Abelian locally compact group with Haar measure λ. Then*

$$\phi_+^{-1}(A)\in\mathcal{M}_{\lambda^2}\quad \textit{and}\quad \phi_-^{-1}(A)\in\mathcal{M}_{\lambda^2},\quad A\in\mathcal{M}_\lambda,$$

where $\phi_+\colon G^2\to G$ and $\phi_-\colon G^2\to G$ are given by $\phi_+(x,y)=x+y$ and $\phi_-(x,y)=x-y$, respectively.

Proof. Take any $A\in\mathcal{M}_\lambda$. Then $A=B\cup M$, where $B\in\mathcal{B}(G)$ and $M\subset N\in\mathcal{B}(G)$ with $\lambda(N)=0$. Clearly,

$$\phi_+^{-1}(A)=\phi_+^{-1}(B\cup M)=\phi_+^{-1}(B)\cup\phi_+^{-1}(M)$$

and $\phi_+^{-1}(M)\subset\phi_+^{-1}(N)$. Since ϕ_+ is continuous, we have $\phi_+^{-1}(B),\phi_+^{-1}(N)\in\mathcal{B}\left(G^2\right)$. Moreover, for any $x\in G$ the x-section $\left(\phi_+^{-1}(N)\right)_x$ of $\phi_+^{-1}(N)$ is

$$\left(\phi_+^{-1}(N)\right)_x=\left\{y\in G\colon \phi_+(x,y)\in N\right\}=\{y\in G\colon x+y\in N\}=N-x,$$

and thus, by Fubini's Theorem,

$$\begin{aligned}\lambda_2\left(\phi_+^{-1}(N)\right)&=\int_G\lambda\left(\left(\phi_+^{-1}(N)\right)_x\right)d\lambda(x)\\&=\int_G\lambda\left(N-x\right)d\lambda(x)=\int_G\lambda\left(N\right)d\lambda(x)=0.\end{aligned}$$

Consequently, $\phi_+^{-1}(A)\in\mathcal{M}_{\lambda^2}$. Similarly one can prove that $\phi_-^{-1}(A)\in\mathcal{M}_{\lambda^2}$. $\square$

2. Proof of the Theorem

The first of the lemmas, dealing with solutions of (E), is purely algebraic: no topology in the group G is assumed. However, the non-negativity of a solution turns out to be crucial for the assertion.

Lemma 2.1. *Let G be an Abelian group. If $\varphi\colon G\to\mathbb{R}$ is a non-negative solution of equation* (E)*, then*

$$\varphi(x)^2\le\varphi\left(x+h\right)\varphi\left(x-h\right)\tag{2.1}$$

for every $x\in G$ and all h's running through the subgroup of G generated by $a_1,\ldots,a_k$.

Proof. Take any $x \in G$ and define $c\colon \mathbb{Z}^k \to \mathbb{R}$ by

$$c(n) = \varphi\left(x + n_1 a_1 + \cdots + n_k a_k\right)$$

[here $n = (n_1, \ldots, n_k)$]. One can check that, by (E), c is a non-negative solution of the recurrent equation

$$c(n) = \sum_{i=1}^{k} p_i c\left(n + e_i\right),$$

where $(e_1, \ldots, e_k)$ stands for the canonical zero-one basis of the space $\mathbb{R}^k$. It follows from [5, Th. 1.1] that c is geometrically convex, that is

$$c(m)^2 \le c\left(m+n\right) c\left(m-n\right), \quad m, n \in \mathbb{Z}^k.$$

Putting here $m = (0, \ldots, 0)$ we see that

$$\varphi(x)^2 \le \varphi\left(x + n_1 a_1 + \cdots + n_k a_k\right) \varphi\left(x - n_1 a_1 - \cdots - n_k a_k\right), \quad n \in \mathbb{Z}^k,$$

which was to be proved. □

The next result shows that, under suitable assumptions on the group G and the function φ, if inequality (2.1) holds on a set which is large in a certain topological sense, then it is satisfied on a set of full measure.

Lemma 2.2. *Let G be an Abelian σ-compact and locally compact group with Haar measure λ. Let $\varphi\colon G \to \mathbb{R}$ be a measurable function. If inequality (2.1) holds for every $x \in G$ and h's running through a dense subset of G, then (2.1) is satisfied for all λ^2-a.a. $(x,h) \in G^2$.*

Proof. According to Lemma 1.3 the set

$$T = \left\{(x,h) \in G^2\colon \varphi(x)^2 > \varphi(x+h)\varphi(x-h)\right\} \tag{2.2}$$

is measurable. Fix an $h \in G$ and a sequence $(h_n)_{n\in\mathbb{N}}$ of elements of G converging to h and satisfying the condition

$$\varphi(x)^2 \le \varphi\left(x + h_n\right) \varphi\left(x - h_n\right), \quad x \in G, n \in \mathbb{N}. \tag{2.3}$$

On account of Lemma 1.2 there exists a strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers such that

$$\lim_{n\to\infty} \varphi\left(x + h_{m_n}\right) = \varphi(x+h), \quad x \in G \setminus E(h),$$

and

$$\lim_{n\to\infty} \varphi\left(x - h_{m_n}\right) = \varphi(x-h), \quad x \in G \setminus E(h).$$

Thus, by (2.3), we have

$$\varphi(x)^2 \le \varphi(x+h)\varphi(x-h), \quad x \in G \setminus E(h).$$

This means that

$$\left\{x \in G\colon \varphi(x)^2 > \varphi(x+h)\varphi(x-h)\right\} \subset E(h), \quad h \in \mathbb{R},$$

and, consequently, all h-sections of the set T are null sets. By Fubini's Theorem we infer that also T is a null set. □

The final lemma below completes the proof of the Theorem.

Lemma 2.3. *Let G be an Abelian 2-divisible group, locally compact, with Haar measure λ. Let $\varphi\colon G \to \mathbb{R}$ be a non-negative measurable function satisfying inequality (2.1) for λ^2-a.a. $(x,h) \in G^2$. Then either $\varphi = 0\,\lambda$-a.e., or there is a positive continuous geometrically convex function $\psi\colon G \to \mathbb{R}$ such that $\varphi = \psi\,\lambda$-a.e.*

Proof. The set $Z = \{x \in G\colon \varphi(x) = 0\}$ is measurable. If $\lambda(G \setminus Z) = 0$, then $\varphi(x) = 0$ for λ-a.a. $x \in G$. Now assume that $\lambda(G \setminus Z) > 0$. Since the set T defined by (2.2) is of measure λ^2 zero, Fubini's Theorem allows to find a null set $N \subset G$ such that

$$\lambda\big(\{h \in G\colon (x,h) \in T\}\big) = 0, \quad x \in G \setminus N.$$

As $\lambda\big(G \setminus (Z \cup N)\big) > 0$ we can take an $x_0 \in G \setminus (Z \cup N)$. Then $\varphi(x_0) > 0$ and $\lambda\big(\{h \in G\colon (x_0,h) \in T\}\big) = 0$ which means that

$$0 < \varphi(x_0)^2 \le \varphi(x_0 + h)\,\varphi(x_0 - h) \quad \text{for } \lambda\text{–}a.a.\ h \in G.$$

Thus $\varphi(x_0 + h) > 0$ for λ–*a.a.* $h \in G$, whence φ is positive a.e., that is Z is a null set.

Define the function $f\colon G \to \mathbb{R}$ by

$$f(x) = \begin{cases} \log\varphi(x), & \text{if } x \in G \setminus Z,\\ 0, & \text{if } x \in Z. \end{cases}$$

Since $\lambda^2(T) = 0$, where $T_0 = T \cup (Z \times G)$, we have

$$0 < \varphi(x)^2 \le \varphi(x+h)\,\varphi(x-h)\,, \quad (x,h) \in G^2 \setminus T_0,$$

whence

$$2f(x) \le f(x+h) + f(x-h) \quad \textit{for } \lambda^2\text{–}a.a.\ (x,h) \in G^2.$$

In other words, the function f is *almost convex*, and thus, by [9, Th. 1] (see also [7]), there exists a *convex* function $g\colon G \to \mathbb{R}$:

$$2g(x) \le g(x+h) + g(x-h)\,, \quad (x,h) \in G^2,$$

such that $g = f\lambda$-a.e. In particular, g is measurable. Making use of the extended version of the Blumberg–Sierpiński theorem [8, Th. 4.1] we infer that g is continuous. Now it is enough to observe that the function $\psi = \exp\circ g$ is positive, continuous, geometrically convex, and $\psi = \varphi\,\lambda$-a.e. □

3. Concluding remarks and an open problem

The main tools used here have topological counterparts. The topological version of Lusin's Theorem can be easily proved in topological spaces (cf. [14, Th. 8.1]). An analog of Lemma 1.2 for Baire measurable functions defined on an arbitrary linear topological space was given by M. Grinč (see [2, Lemma 2]); however, the argument used by him works for topological groups, too. Theorem 1 from [9], stating that every almost convex function is λ-a.e. equal to a convex function and used in the proof of Lemma 2.3, is a generalization of the Kuczma theorem [11, Th. 17.8.2] (cf. also [10]). Its topological version for functions defined on groups was proved in [6] (see also [7]). Finally, also the Blumberg–Sierpiński theorem has a topological version in a group setting which can be found in [8] (see also [7]). Making use of these results one can obtain a suitable counterpart of the Theorem where solutions of Eq. (E) are assumed to be Baire measurable. In the case $G = \mathbb{R}^n$ such a result was proved by Grinč in [2].

Using some versions of the Kuczma theorem one can prove also results improving the regularity of non-negative solutions of the following extension of Eq. (E), called the integrated Cauchy functional equation:

$$\varphi(x) = \int_G \varphi(x+y)d\mu(y); \tag{I}$$

here μ is a regular Borel measure on the group G. Its locally λ-integrable solutions were determined in [13] by Ka-Sing Lau and Wei-Bin Zeng.

Improvement of regularity of non-negative solutions of (E) is a crucial step in determining them in the cases $G = \mathbb{R}$ (see [5, Prop. 3.3]) and $G = \mathbb{R}^n$ (see [2, Theorem]). It seems that the Theorem could play an analogous role while looking for the form of solutions defined on groups. Also determining all non-negative measurable solutions of (I) might run in a similar way.

Open problem. The assumption of 2-divisibility of the group G in the Theorem is caused by the same condition imposed on G in [9, Th.1]. The authors of [9] still do not know if this assumption is essential there. However, the following question is natural: *is the* 2*-divisibility of* G *essential for the validity of the Theorem?*

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