| Aequationes Mathematicae

CrossMark

Mellish theorem for generalized constant width curves

Witold Mozgawa

Abstract. In this paper we give a generalization of the theorem characterizing ovals of constant width proved by Mellish (Ann Math (2) 32:181–190, 1931).

1. Introduction

In 1931 a paper [17] extensively focused (even for today's standards) on geometry was published containing some fragments found among the papers of an early deceased Canadian mathematician of great promise, Arthur Preston Mellish (June 10, 1905–February 7, 1930). His ideas influenced many interesting papers (for example, see [1,3,7,26,27]) and are fairly far from being exhausted. The paper we are speaking about begins with some considerations about ovals, where the author gives the following fascinating theorem.

Theorem 1.1. ([17]). *The statements*

(i) *a curve is of constant width;*
(ii) *a curve is of constant diameter;*
(iii) *all the normals of a curve (an oval) are double;*
(iv) *the sum of radii of curvature at opposite points of a curve (an oval) is constant;*

are equivalent, in the sense that whenever one of statements (i–iv) *holds true, all other statements also hold.*

(v) *All curves of the same (constant) width a have the same length L given by*

$$L = \pi a.$$

In the third section we will introduce a wide generalization of this theorem produced by means of isoptics theory, for which we give a necessary
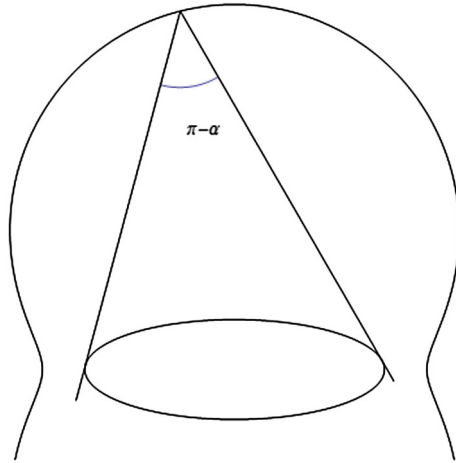
Figure 1. A construction of an isoptic

introduction as a fundamental tool in our constructions (see mainly [4,6]). Moreover, some interesting facts about isoptics are contained in [8–12, 16, 18, 21, 22, 25].

Definition 1.1. In the plane a simple, closed, positively oriented C^2-curve C of positive curvature is called an oval.

We take a coordinate system with origin O in the interior of C. Let $p(t)$, $t \in [0, 2\pi]$, be the distance from O to the tangent line of C perpendicular to the vector $e^{it} = \cos t + i \sin t$. It is well-known ([5, 23]) that the parametrization of C in terms of $p(t)$ is given by the formula

$$z(t) = p(t)e^{it} + p'(t)ie^{it}. \tag{1.1}$$

Note that the support function p can be extended to a smooth periodic function on $\mathbb{R}$ with the period 2π.

Definition 1.2. An α-isoptic C_α of C consists of points from where tangents to the oval meet at angle $\pi - \alpha$, where $\alpha \in (0, \pi)$.

In other words, an α-isoptic C_α of C is a set of points from which the curve is seen under the fixed angle $\pi - \alpha$ (Fig. 1). Following the considerations and results from [6] it is convenient to parametrize the α-isoptic C_α by the same angle t so that the equation of C_α has the form

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha) \right) ie^{it}. \tag{1.2}$$

Note that

$$z_\alpha(t) = z(t) + \lambda(t, \alpha)ie^{it} = z(t + \alpha) + \mu(t, \alpha)ie^{i(t+\alpha)} \qquad (1.3)$$
$$z'_\alpha(t) = -\lambda(t, \alpha)e^{it} + \varrho(t, \alpha)ie^{it} \qquad (1.4)$$

for suitable functions λ, μ and ϱ. Since $z_\alpha(t) = z(t+\alpha)+\mu(t, \alpha)ie^{i(t+\alpha)}$, $\mu(t, \alpha)$ is negative for all values t and α. Moreover, we have

$$\lambda(t, \alpha) = \frac{p(t + \alpha) - p'(t) \sin \alpha - p(t) \cos \alpha}{\sin \alpha}, \qquad (1.5)$$

$$\varrho(t, \alpha) = \frac{p(t) \sin \alpha + p'(t + \alpha) - p'(t) \cos \alpha}{\sin \alpha}, \qquad (1.6)$$

$$\mu(t, \alpha) = \lambda(t, \alpha) \cos \alpha - \varrho(t, \alpha) \sin \alpha. \qquad (1.7)$$

If we introduce the following notation

$$q(t, \alpha) = z(t) - z(t + \alpha) \qquad (1.8)$$

then the curvature of the α-isoptic is given by ([6])

$$k_\alpha(t) = \frac{\sin \alpha}{|q(t, \alpha)|^3}(2|q(t, \alpha)|^2 - [q(t, \alpha), q'(t, \alpha)]) \qquad (1.9)$$

where $[\,,\,]$ denotes the determinant of the arguments.

In the third section we will need the sine theorem proved in [6].

Theorem 1.2. *Under the notations in Fig. 2 the following identities hold*

$$\frac{|q|}{\sin \alpha} = \frac{\lambda}{\sin \alpha_1} = \frac{-\mu}{\sin \alpha_2}. \qquad (1.10)$$

2. Generalized constant width

As we have seen in the preceding section, an isoptic is formed as the locus of vertices of a fixed angle $\pi - \alpha$, formed by two tangents to the oval C. On the other hand, one knows that a constant width oval of width a can be freely rotated in a strip of width a. This means that parallel tangents to such an oval intersect "at infinity" giving the "π-isoptic". In the present paper we would like to bring together the concept of constant width and the concept of isoptics in the framework of plane Euclidean geometry, while avoiding the projective geometry concepts. Thus the isoptic "at infinity" giving the "π-isoptic" seen as the line at infinity is not applicable here, and in the further part of the paper we want to define and examine such isoptics. One of the possible ways to see this curve is given by the following definition.

Definition 2.1. For any oval C and any of its α-isoptic C_α, $0 < \alpha < \pi$, the function

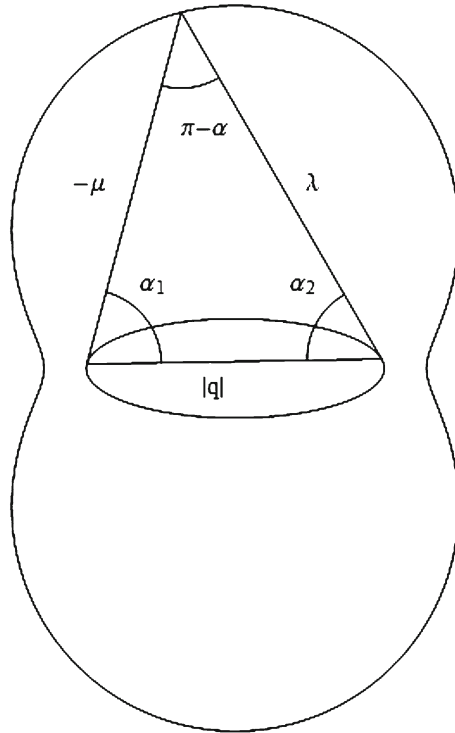$$\kappa_\alpha(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3} \qquad (2.1)$$

FIGURE 2. The sine theorem

is said to be a sine-curvature of C_α.

We could consider an isoptic of the oval C at infinity as a curve which is given by its curvature κ_π, but neither the π-isoptic nor its equation is known yet. Moreover, computer simulations show that such curves can be fairly complicated and possibly constructed in a non-geometric way.

For us the following definition will be useful.

Definition 2.2. An oval with $\kappa_\alpha \equiv const$ will be called the oval of constant α-width.

Note, that this notion makes sense for $\alpha = \pi$. Then, we can say that the oval is a curve of constant width at infinity.

Theorem 2.1. *An oval C is a curve of constant width at infinity iff it is an oval of constant width.*

Proof. If we take $\alpha = \pi$ then we get

$$q(t, \pi) = q(t) = z(t) - z(t+\pi) = (p(t) + p(t+\pi))e^{it} + (p'(t) + p'(t+\pi))ie^{it}. \quad (2.2)$$

And further if we put
$$a(t) = p(t) + p(t + \pi) \tag{2.3}$$
then
$$q(t) = a(t)e^{it} + a'(t)ie^{it} \tag{2.4}$$
and omitting the argument t we get
$$\kappa_\pi = \frac{a^2 + 2(a')^2 - aa''}{\sqrt{a^2 + (a')^2}^3}. \tag{2.5}$$
If the oval C is of constant (classical) width then $a = p(t)+p(t+\pi) \equiv const > 0$. Hence
$$\kappa_\pi = \frac{1}{a} \equiv const. \tag{2.6}$$
Thus we have proved that any oval of constant width is the one of constant width at infinity and its π-isoptic is a circle of radius a (for some analogy with this result see [20]).

Conversely, assume that
$$\frac{a^2 + 2(a')^2 - aa''}{\sqrt{a^2 + (a')^2}^3} \equiv const = c. \tag{2.7}$$
The expression (2.7) is in fact a curvature of a curve $(a(t), t)$ given in polar coordinates. From elementary differential geometry we know that the only possibilities are that the curve is a part of the straight line or a circle. Hence we have two cases:

Case $c = 0$. Then we get the differential equation
$$a^2 + 2(a')^2 - aa'' = 0 \tag{2.8}$$
and the solution is
$$a(t) = \frac{c_1}{\cos(t - c_2)} \tag{2.9}$$
which has to be rejected since the function $a(t) = p(t) + p(t + \pi)$ should be bounded as a continuous periodic function defined on the whole real line $\mathbb{R}$.

Case $c \neq 0$. The only possibility is when the curve $(a(t), t)$ is a part of a circle. Since in the polar coordinates (a, t) the general equation of a circle with center at a point (a_0, t_0) and radius A is
$$a^2 - 2aa_0 \cos(t_0 - t) + a_0^2 = A^2 \tag{2.10}$$
then its solutions
$$a(t) = a_0 \cos(t_0 - t) \pm \sqrt{-a_0^2 + A^2 + a_0^2 \cos^2(t_0 - t)} \tag{2.11}$$
are the only solutions of (2.7). Moreover, since the function $a(t)$ is periodic with the period of π, then from the formula (2.11) we get
$$a = \pm A \equiv const. \tag{2.12}$$
Hence $p(t) + p(t + \pi) = const$ and the starting curve is of constant width. $\quad\square$

From this theorem we immediately get

Corollary 2.1. *For $0 < \alpha \le \pi$ an oval is of constant α-width iff its α-isoptic is a circle.*

3. Generalized Mellish theorem

At the beginning we need a certain fact about the Steiner centroid of an isoptic. Recall ([24]) that for a closed simple plane parametric curve $z \colon [0, 2\pi] \to \mathbb{R}^2$ with the curvature $\kappa(t)$ its Steiner centroid S is given by

$$S = \frac{\int\limits_0^{2\pi} z(t)k(t)|z'(t)|dt}{\int\limits_0^{2\pi} k(t)|z'(t)|dt}. \tag{3.1}$$

Then for an oval C given by (1.1) we have $z'(t) = R(t)ie^{it}$, where $R(t) = \frac{1}{k(t)}$ is the curvature radius of C at t and hence

$$S = \frac{1}{2\pi} \int\limits_0^{2\pi} z(t)dt. \tag{3.2}$$

For derivation purposes we will assume from now on that $S = 0$ for the considered ovals, that is the coordinate origin will be chosen at the Steiner centroid of C.

Theorem 3.1. *Assume that for an oval C its α-isoptic for some $\alpha \in]0, \pi[$ is a circle. Then its center is a Steiner centroid of C.*

Before we proceed to the proof, note that it is an open problem whether the Steiner centroids of the oval C and any of its isoptics coincide. The above theorem suggests a hint.

Proof. Let this circle have center at $a + ib$ and curvature $k = \frac{1}{R}$. Then all osculating circles for all points of the isoptic coincide and their center is given by the formula

$$a + ib = z_\alpha(t) + \frac{1}{k} \cdot \frac{z'_\alpha(t)}{|z'_\alpha(t)|}. \tag{3.3}$$

Hence, integrating both sides we get

$$2\pi(a + ib) = \int\limits_0^{2\pi} z_\alpha(t)dt + R \int\limits_0^{2\pi} \frac{z'_\alpha(t)}{|z'_\alpha(t)|}dt. \tag{3.4}$$

But the last integral is equal to zero since for each value of the integrand its opposite value exists, which means that the areas determined by both

coordinates above and below x-axes are equal. Finally

$$a + ib = \frac{1}{2\pi} \int_0^{2\pi} z_\alpha(t) dt = \frac{1}{2\pi} \int_0^{2\pi} z(t) dt = S \tag{3.5}$$

since

$$\int_0^{2\pi} z_\alpha(t) dt = \int_0^{2\pi} p(t) e^{it} dt + \int_0^{2\pi} \left(-p(t) \cot \alpha + \frac{1}{\sin \alpha} p(t + \alpha) \right) i e^{it} dt$$

$$= 2 \int_0^{2\pi} p(t) e^{it} dt \tag{3.6}$$

and

$$\int_0^{2\pi} z(t) dt = \int_0^{2\pi} p(t) e^{it} dt + \int_0^{2\pi} p'(t) i e^{it} dt = 2 \int_0^{2\pi} p(t) e^{it} dt \tag{3.7}$$

by suitable integration by parts, substitution and properties of periodic functions. □

Before we formulate our main theorem we need two more ingredients—a generalization to our framework of the notion of width of an oval and a certain hedgehog associated with C.

In the classical projective definition the diameter is the polar line of the pole at infinity. Thus the generalization of the notion of width in this framework should rather be the section connecting the two touching points, that is $q(t, \alpha)$ defined in (1.8). But in the formula (2.2) we see that $q(t, \alpha)$ does not generalize to the diameter of an oval in the direction e^{it} when $\alpha = \pi$. Moreover, as we already said, we would like to work in the framework of plane Euclidean geometry, while avoiding the projective geometry concepts. Thus we introduced an adequate notion as it turns out in the course of the proof of Theorem 3.2 and which we will call α-spread of C at a point t and which also has some relevance to the diameter of the oval. For this purpose we introduce a vector $Q(t, \alpha)$ between the projections of the origin of the coordinate system onto support lines of C at points $z(t)$ and $z(t + \alpha)$, as it is shown in Fig. 3.

After some calculations one gets

$$Q(t, \alpha) = (p(t + \alpha) \cos \alpha - p(t)) e^{it} + p(t + \alpha) \sin \alpha \, i e^{it}. \tag{3.8}$$

Definition 3.1. A number $d_\alpha(t) = |Q(t, \alpha)|$
$= \sqrt{p(t)^2 + p^2(t + \alpha) - 2p(t)p(t + \alpha) \cos \alpha}$ is said to be an α-spread of C at a point t.

Note that if $\alpha = \pi$ then $d_\pi(t) = p(t) + p(t + \pi)$.

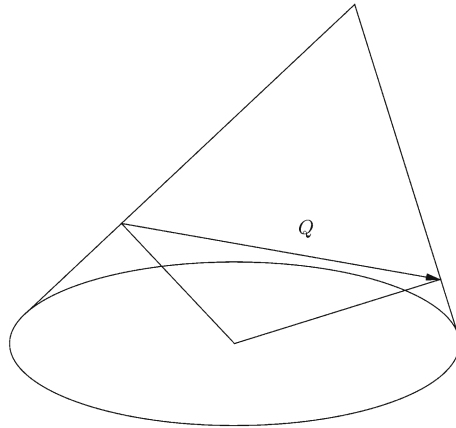FIGURE 3. Construction of vector Q

To define a necessary hedgehog we define a new support function $P(t) = \frac{|Q(t,\alpha)|}{2}$. Thus by definition *an α-hedgehog associated* to an oval C and the angle α is a curve $H_\alpha(t) = P(t)e^{it} + P'(t)ie^{it}$. Note that the hedgehogs are intensively investigated, see papers by Martinez-Maure ([15]) and Langevin ([13]) for references.

Theorem 3.2. *For fixed $\alpha \in]0, \pi]$ the statements*

  (i) *an oval is of constant α-width;*
 (ii) *an oval is of constant α-spread;*
(iii) *all the vectors $q(t, \alpha)$ are parallel to vectors $Q(t, \alpha)$;*
(iv) *with the notations from Fig. 2 the expression*

$$\frac{1}{|q(t,\alpha)|^2} \left(2|q(t,\alpha)| - \left(\frac{\sin \alpha_1}{k(t+\alpha)} + \frac{\sin \alpha_1}{k(t)} \right) \right) \tag{3.9}$$

  is constant;

are equivalent, in the sense that whenever one of statements (i–iv) *holds true, all other statements also hold.*

 (v) *For all curves of the same constant α-width a the associated hedgehogs H_α have the same length L given by*

$$L = \pi a.$$

Proof.
(i)$\Longleftrightarrow$(ii) In the case of $\alpha = \pi$ we get by Theorem 2.1 that $d_\pi = p(t)+p(t+\pi) \equiv const$. The constant α-width for $\alpha \in]0, \pi[$ means that the α-isoptic of the oval C is a circle and by theorem 3.1 its center coincides with $(0,0)$ thus we get

$$|z_\alpha(t)|^2 = R^2 \tag{3.10}$$

for some real number R. Thus substituting the Eq. (1.2) into (3.10) we get $d_\alpha(t) \equiv R\sin\alpha$.

Conversely, let

$$d_\alpha(t) = |Q(t,\alpha)| = \sqrt{p(t)^2 + p^2(t+\alpha) - 2p(t)p(t+\alpha)\cos\alpha} \equiv const = R.$$

Then in the case $\alpha = \pi$ we get

$$d_\pi = p(t) + p(t+\pi) \equiv const \tag{3.11}$$

and in the remaining cases one has

$$|z_\alpha(t)|^2 = p^2(t) + \frac{(p(t+\alpha) - p(t)\cos\alpha)^2}{\sin^2\alpha} = \frac{R^2}{\sin^2\alpha}, \tag{3.12}$$

so the α-isoptic of the oval C is a circle, and we are done.

(iii)$\Longleftrightarrow$(ii) After some calculations we get

$$[Q(t), q(t)] = \frac{1}{2}(d_\alpha^2(t))' \tag{3.13}$$

whence

$$[Q(t), q(t)] = 0 \Longleftrightarrow d_\alpha \equiv const, \tag{3.14}$$

where we omit α for simplicity of formulas. Note that when $\alpha = \pi$ then Q and q coincide and become double normals at $z(t)$ and $z(t+\pi)$, as stands in the Mellish theorem, and our generalization is well defined.

(iv)$\Longleftrightarrow$(i) First assume (i), that is

$$\kappa_\alpha(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3} \equiv const = c. \tag{3.15}$$

Let us perform some manipulations on this formula

$$\kappa_\alpha(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3} = \tag{3.16}$$

$$= \frac{1}{|q(t)|^3}\left(2|q(t)|^2 - \lambda(t,\alpha)R(t+\alpha)\sin\alpha + \mu(t,\alpha)R(t)\sin\alpha\right)$$

$$= \frac{1}{|q(t)|^2}\left(2|q(t)| - \left(\frac{1}{k(t+\alpha)}\frac{\lambda(t,\alpha)\sin\alpha}{|q(t)|} + \frac{1}{k(t)}\frac{-\mu(t,\alpha)\sin\alpha}{|q(t)|}\right)\right)$$

$$= \frac{1}{|q(t)|^2}\left(2|q(t)| - \left(\frac{\sin\alpha_1}{k(t+\alpha)} + \frac{\sin\alpha_2}{k(t)}\right)\right),$$

where the last equality follows form the sine formula (1.10). Thus we have obtained the claimed equivalence. However, it is worth the effort to see why it is a generalization of the Mellish formula. Let us substitute $\alpha = \pi$ and let C be an oval of constant width d then $\alpha_1 = \alpha_2 = \frac{\pi}{2}$ and $|q| = a$ thus we get

$$a = \frac{1}{k(t)} + \frac{1}{k(t+\pi)} \tag{3.17}$$

and conversely. To finish the proof we have to deal with the last property (v). However, if an oval C is of constant α-width, i.e. $d_\alpha \equiv const = a$ then the length L of the associated hedgehogs H_α is constant and given by the formula

$$L = \int\limits_0^{2\pi} P(t)dt = \int\limits_0^{2\pi} \frac{a}{2}dt = \pi a. \tag{3.18}$$

In the case of $\alpha = \pi$ we get $a = p(t) + p(t + \pi)$ which gives the reason for the Barbier theorem, and L is the perimeter of C in this case. $\qquad\square$

At the end of the paper we mention three still interesting questions:

(1) Is it possible to give a characterization of convex bodies of constant α-width analogous to that unexpected given by Makai and Martini in [14]?
(2) Is it possible to give a counterpart of a theorem given by Miernowski ([2,19]) showing that for closed, convex curves having circles as $\frac{\pi}{2}$-isoptics, the maximal perimeter of a parallelogram inscribed in this curve can be realized by a parallelogram with one vertex at any prescribed point of the curve?
(3) Is it possible to extend the above results to ovaloids, obtaining a suitable generalization of the second part of Mellish' paper [17]?

References

[1] Ait-Haddou, R., Herzog, W., Biard, L.: Pythagorean-hodograph ovals of constant width. Comput. Aided Geom. Des. **25**(4–5), 258–273 (2008)
[2] Anghel, N.: A maximal parallelogram characterization of ovals having circles as orthoptic curves. Forum Geom. **10**, 21–25 (2010) (electronic only)
[3] Araújo, P.V.: Representation of curves of constant width in the hyperbolic plane. Port. Math. **55**(3), 355–372 (1998)
[4] Benko, K., Cieślak, W., Góźdź, S., Mozgawa, W.: On isoptic curves. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Ser. Nouă. Mat. **36**(1), 47–54 (1990)
[5] Bonnesen, T., Fenchel, W.: Theory of convex bodies. Translated from the German and edited by L. Boron, C. Christenson and B. Smith, with the collaboration of W. Fenchel, vol. IX. BCS Associates, Moscow, Idaho (1987)
[6] Cieślak, W., Miernowski, A., Mozgawa, W.: Isoptics of a closed strictly convex curve. Global differential geometry and global analysis. Proc. Conf. Berlin/Germany 1990 Lect. Notes Math. **1481**, 28–35 (1991)
[7] Cieślak, W., Mozgawa, W.: On rosettes and almost rosettes. Geom. Dedicata **24**, 221–228 (1987)
[8] Csima, G., Szirmai, J.: Isoptic curves in the hyperbolic plane. Stud. Univ. Žilina Math. Ser. **24**(1), 15–22 (2010)
[9] Green, J.W.: Sets subtending a constant angle on a circle. Duke Math. J. **17**, 263–267 (1950)

[10] Hilton, H., Colomb, R.E.: On orthoptic and isoptic loci. Am. J. **39**, 86–94 (1917)
[11] Klamkin, M.S.: Conjectured isotropic characterization of a circle. Am. Math. Mon. **95**(9), 845 (1988)
[12] Kunkli, R., Papp, I., Hoffmann, M.: Isoptics of Bézier curves. Comput. Aided Geom. Des. **30**(1), 78–84 (2013)
[13] Langevin, R., Levitt, G., Rosenberg, H.: Hérissons et multihérissons (enveloppes parametrées par leur application de Gauss). Singularities. Banach Cent. Publ. **20**, 245–253 (1988)
[14] Makai, E. jun., Martini, H.: A new characterization of convex plates of constant width. Geom. Dedicata **34**(2), 199–209 (1990)
[15] Martinez-Maure, Y.: Hedgehogs of constant width and equichordal points. Ann. Pol. Math. **67**(3), 285–288 (1997)
[16] Matsuura, S.: On non-convex curves of constant angle. In: Komatsu, H. (ed.) Functional analysis and related topics, 1991. Proceedings of the International Conference in Memory of Professor Ksaku Yosida Held at RIMS, Kyoto University, Japan, 29 July–2 August 1991. Springer-Verlag, Berlin. Lecture Notes Mathematics, vol. 1540, pp. 251–268 (1993)
[17] Mellish, A.P.: Notes on differential geometry. Ann. Math. (2) **32**, 181–190 (1931)
[18] Michalska, M.: A sufficient condition for the convexity of the area of an isoptic curve of an oval. Rend. Semin. Mat. Univ. Padova **110**, 161–169 (2003)
[19] Miernowski, A.: Parallelograms inscribed in a curve having a circle as $\frac{\pi}{2}$-isoptic. Ann. Univ. Mariae Curie-Skodowska Sect. A **62**, 105–111 (2008)
[20] Mozgawa, W., Miernowski, A.: On the curves of constant relative width. Rend. Semin. Mat. Univ. Padova **107**, 57–65 (2002)
[21] Neville, E.H.: The isoptic point of a quadrangle. J. Lond. Math. Soc. **16**, 173–174 (1941)
[22] Nitsche, J.C.C.: Isoptic characterization of a circle (proof of a conjecture of M. S. Klamkin). Am. Math. Mon. **97**(1), 45–47 (1990)
[23] Santaló, L.A.: Integral geometry and geometric probability. Encyclopedia of Mathematics and Its Applications, vol. 1. Section: probability. reading, mass, etc. Addison-Wesley Publishing Company, Advanced Book Program XVII (1976)
[24] Su, B.: Lectures on differential geometry. Translated from Chinese by K. C. Chang, vol. 5. World Scientific Publishing Company, Singapore, pp. 149 (1980)
[25] Taylor, C.: Note of a theory of orthoptic and isoptic loci. Lond. R. S. Proc. **XXXVII**, 138–141 (1884)
[26] Tennison, R.L.: Smooth curves of constant width. Math. Gaz. **60**, 270–272 (1976)
[27] Wegner, B.: Analytic approximation of continuous ovals of constant width. J. Math. Soc. Jpn. **29**, 537–540 (1977)

Witold Mozgawa
Instytut Matematyki
Uniwersytet Marii Curie-Skłodowskiej
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: mozgawa@hektor.umcs.lublin.pl