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Aequationes Mathematicae

Conditionally δ-midconvex functions

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Abstract. Let X be a real linear space, V be a nonempty subset of X and δ be a nonnegative real number. A function $f : V \to \mathbb{R}$ is said to be conditionally δ-midconvex provided $f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2} + \delta$ for every $x, y \in V$ such that $\frac{x+y}{2} \in V$. We show that if V satisfies some reasonable assumptions, then for every bounded from above conditionally δ-midconvex function $f : V \to \mathbb{R}$ the following estimation holds: $\sup f(V) \le \sup f(ext\, V) + k(V)\delta$, where $ext\, V$ denotes the set of all extremal points of V and $k(V)$ is a respective constant depending on V.

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1. Introduction

The notion of convexity has been generalized in various directions. Some of them are motivated by the problems stemming from economics, optimization and mathematical programming. The main streams of generalizations consist in relaxing the inequality defining the convexity and in modifying the assumption concerning the domain of the function. In particular, the notion of δ-convexity and conditional convexity play a significant role in the applications. Approximately convex functions were studied for the first time by Hyers and Ulam [1]. Given a nonnegative real number δ, a function f defined on a convex subset V of a real linear space X is said to be δ-*convex* provided

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \delta \text{ for } x, y \in V, t \in [0, 1].$$

If the above inequality is assumed only for $t = \frac{1}{2}$, that is if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \delta \text{ for } x, y \in V,$$

then f is said to be δ-*midconvex*. The relations between local boundedness, δ-midconvexity and δ-convexity were studied by Ng and Nikodem [3].

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Another generalization of convexity, motivated by some applications in utility theory, has been considered by Peters and Wakker [5]. They studied convex functions on nonconvex sets. Inspired by these two approaches to generalized convexity, in [2] a notion of a conditionally δ-midconvex function has been introduced. A function $f : V \to \mathbb{R}$ is called *conditionally δ-midconvex* provided, for every $x, y \in V$ such that $\frac{x+y}{2} \in V$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \delta.$$

In this paper we study properties of such functions defined on an arbitrary nonempty subset of $\mathbb{R}^n$. It is known (cf. [6]) that every bounded above convex function defined on a convex compact domain in $\mathbb{R}^n$ achieve its supremum on the set of extremal points of the domain. We are going to prove the analogue of this fact for conditionally δ-midconvex functions. In particular, we show that if $V \subset \mathbb{R}^n$ is convex and compact, then for every bounded from above conditionally δ-midconvex function $f : V \to \mathbb{R}$ the following estimation holds:

$$\sup f(V) \leq \sup f(ext\, V) + k(V)\delta,$$

where $ext\, V$ denotes the set of all extremal points of V and $k(V)$ is a respective constant depending on V.

2. Jensen index

By $\mathbb{Z}$ and $\mathbb{N}$ we denote the sets of all integers and positive integers, respectively. In the whole paper we assume that X is a real linear space, V is a nonempty subset of X, W is a nonempty subset of V and δ is a given nonnegative real number.

Definition 2.1. A sequence $x = (x_0, \ldots, x_n)$ of elements of V is said to be a *Jensen chain* (or shortly, *J-chain*) *of length n* in V if either $n = 0$; or $n \geq 1$ and

$$2x_{k+1} - x_k \in V \text{ for } k \in \{0, \ldots, n-1\}.$$

Note that, for every $k \in \{0, \ldots, n-1\}$, the point $2x_{k+1} - x_k$ is symmetric to x_k with respect to x_{k+1}. So the condition in Definition 2.1 means that

$$x_{k+1} \in \frac{V + x_k}{2} \text{ for } k \in \{0, \ldots, n-1\},$$

that is x_{k+1} is the midpoint of x_k and a certain point from V.

Remark 2.1. If $(x_0, \ldots, x_n)$ is a J-chain in V then, for every $m \in \mathbb{N}$, a sequence $(x_0, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$, where $x_{n+j} = x_n$ for $j \in \{1, \ldots, m\}$, is also a J-chain in V.

Definition 2.2. Given $v \in V$, by $i_W(V; v)$ we denote the length of the shortest J-chain $x = (x_0, \ldots, x_N)$ in V such that $x_0 \in W$ and $x_N = v$. If there is no such a J-chain, then we put $i_W(V; v) = \infty$. Furthermore, we set $i_W(V) := \sup_{v \in V} i_W(V; v)$. We call $i_W(V)$ *the Jensen index of* V *with respect to* W.

Directly from Definitions 2.1 and 2.2, we obtain the following statements.

Remark 2.2. (i) $i_W(V; v) = 0$ if and only if $v \in W$,
(ii) $i_W(V) = 0$ if and only if $W = V$,
(iii) $i_W(X) \leq 1$.

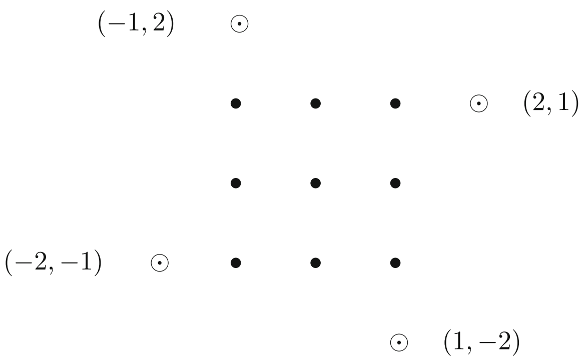
We illustrate the above defined notions by some simple examples.

Example 2.1. Let $X = \mathbb{R}$, $V = [0, 1]$ and $W = \{0, 1\}$. For every $x \in [0, 1]$, we put

$$\bar{x} = \begin{cases} 0 & \text{whenever} \quad x \in [0, \frac{1}{2}] \\ 1 & \text{otherwise.} \end{cases}$$

Then $\bar{x} \in W$ and $2x - \bar{x} \in [0, 1]$ for $x \in [0, 1]$. Hence, for every $x \in [0, 1]$, $(\bar{x}, x)$ is a J-chain in V. It proves that $i_W(V) \leq 1$. Moreover it is obvious that $i_W(V) > 0$. Hence $i_W(V) = 1$.

Example 2.2. Let $X = \mathbb{R}^2$, $V = \{(-2, -1), (-1, -1), (-1, 0), (-1, 1), (-1, 2), (0, -1), (0, 0), (0, 1), (1, -2), (1, -1), (1, 0), (1, 1), (2, 1)\}$ and $W = \{(-2, -1), (-1, 2), (1, -2), (2, 1)\}$.



Then one can easily notice that $i_W(V) = 2$.

The next result explains in a way the above introduced notions.

Proposition 2.1. *If ext* $V \setminus W \neq \emptyset$ *then* $i_W(V) = \infty$.

Proof. Assume that *ext* $V \setminus W \neq \emptyset$ and fix $v_0 \in$ *ext* $V \setminus W$. Suppose that there exists a $w_0 \in W$ and a J-chain $(w_0, v_1, \ldots, v_n)$ in V such that $v_n = v_0$.

Then $v_0 = v_n = \frac{v_{n-1}+v}{2}$ for some $v \in V$. Since $v_0 \in ext\ V$ this implies that $v_0 = v_n = v_{n-1}$. Repeating this procedure finally we get that $w_0 = v_0$, which gives a contradiction. □

Example 2.3. Let $X = \mathbb{R}$, $V = [0,1)$ and $W = \{0\}$. According to Remark 2.2(i), $i_W(V;0) = 0$. Furthermore, for every $v \in (0, \frac{1}{2})$, a sequence $(0, v)$ is a J-chain in V. So, taking Remark 2.2(i) into account, we obtain that $i_W(V;v) = 1$ for $v \in \left(0, \frac{1}{2}\right)$. Moreover, a standard calculation shows that, for every $k \in \mathbb{N}$ and $v \in [\sum_{i=1}^{k} \frac{1}{2^i}, \sum_{i=1}^{k+1} \frac{1}{2^i})$, a sequence $(0, x_1, \ldots, x_k, v)$, where

$$x_r := v - \sum_{i=r+1}^{k+1} \frac{1}{2^i} \text{ for } r \in \{1, \ldots, k\},$$

is a J-chain in V. Therefore, we get

$$i_W(V;v) \leq k+1 \text{ for } v \in \left[\sum_{i=1}^{k} \frac{1}{2^i}, \sum_{i=1}^{k+1} \frac{1}{2^i}\right). \tag{1}$$

Next, we show that if $(0, x_1, \ldots, x_N)$, where $N \in \mathbb{N}$, is a J-chain in V, then

$$x_N < \sum_{i=1}^{N} \frac{1}{2^i}. \tag{2}$$

We proceed by induction. If $(0, x_1)$ is a J-chain in V, then $2x_1 \in V$, that is $x_1 < \frac{1}{2}$. Assume that $(0, x_1, \ldots, x_N, x_{N+1})$ is a J-chain in V for some $N \in \mathbb{N}$. Then $(0, x_1, \ldots, x_N)$ is also a J-chain in V, so by the inductive argument, we get (2). Moreover, we have $2x_{N+1} - x_N \in V$, that is $2x_{N+1} - x_N < 1$. Thus

$$x_{N+1} < \frac{1}{2} + \frac{1}{2}x_N < \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{N} \frac{1}{2^i} = \sum_{i=1}^{N+1} \frac{1}{2^i},$$

which completes the inductive proof of (2). Now, taking (1) into account, we conclude that

$$i_W(V;v) = k+1 \text{ for } v \in \left[\sum_{i=1}^{k} \frac{1}{2^i}, \sum_{i=1}^{k+1} \frac{1}{2^i}\right).$$

Hence $i_W(V;v)$ is finite for every $v \in V$, but $i_W(V) = \infty$.

Theorem 2.1. *For every $n \in \mathbb{N}$ the following conditions are equivalent*

(i) $i_W(V) \leq n$,
(ii) $V \subset \frac{V}{2} + \frac{V}{4} + \cdots + \frac{V}{2^n} + \frac{W}{2^n}$.

Proof. Let $n \in \mathbb{N}$. Assume that (i) holds and fix a $v \in V$. Then there exists a J-chain $(v_0, v_1, \ldots, v_{\tilde{n}})$ in V such that $\tilde{n} \leq n$, $v_0 \in W$ and $v_{\tilde{n}} = v$. Then, according to Remark 2.1, the sequence $(v_0, v_1, \dots, v_n)$, where $v_i = v_{\tilde{n}}$ for $i \in \{\tilde{n}, \dots, n\}$ is a J-chain in V. Thus, we get

$$\begin{aligned} v_1 &\in \frac{V + v_0}{2} \subset \frac{V}{2} + \frac{W}{2}, \\ v_2 &\in \frac{V + v_1}{2} \subset \frac{V}{2} + \frac{V}{4} + \frac{W}{4}, \\ &\cdots \\ v = v_n &\in \frac{V + v_{n-1}}{2} \subset \frac{V + \frac{V}{2} + \frac{V}{4} + \cdots + \frac{V}{2^{n-1}} + \frac{W}{2^{n-1}}}{2} \\ &= \frac{V}{2} + \frac{V}{4} + \cdots + \frac{V}{2^n} + \frac{W}{2^n}. \end{aligned}$$

So, (ii) is valid.

Now, assume that (ii) holds and fix a $v \in V$. Then there exist $z_0 \in W$ and $z_i \in V$ for $i \in \{1, 2, \dots, n\}$ such that

$$v = \frac{z_0}{2^n} + \frac{z_1}{2^n} + \cdots + \frac{z_n}{2}.$$

Let

$$\begin{aligned} v_0 &:= z_0, \\ v_1 &:= \frac{v_0 + z_1}{2}, \\ v_2 &:= \frac{v_1 + z_2}{2} = \frac{z_0}{4} + \frac{z_1}{4} + \frac{z_2}{2}, \\ &\cdots \\ v_n &:= \frac{v_{n-1} + z_n}{2} = \frac{z_0}{2^n} + \frac{z_1}{2^n} + \cdots + \frac{z_n}{2} = v. \end{aligned}$$

Then clearly $(v_0, v_1, \dots, v_n)$ is a J-chain in V, $v_0 \in W$ and $v_n = v$, which proves (i). □

In the case where V is convex, from Theorem 2.1 we derive the following result.

Corollary 2.1. *If V is convex then for every $n \in \mathbb{N}$ the following conditions are equivalent*

(i) $i_W(V) \le n$,
(ii) $V \subset \left(1 - \frac{1}{2^n}\right) V + \frac{1}{2^n} W$.

Corollary 2.2. *Assume that X is a normed space, V is a closed unit ball in X and W is a unit sphere in X. Then $i_W(V) = 1$.*

Proof. First we prove that

$$V \subset \frac{1}{2} W + \frac{1}{2} V. \tag{3}$$

Fix an $x \in V$. If $x = 0$ then taking an arbitrary $v \in W$, we have

$$0 = \frac{1}{2}v - \frac{1}{2}v \in \frac{1}{2}W + \frac{1}{2}V.$$

If $x \neq 0$ then

$$x = \frac{1}{2\|x\|}x + \left(1 - \frac{1}{2\|x\|}\right)x.$$

Obviously $\frac{1}{2\|x\|}x \in \frac{1}{2}W$. Moreover

$$\left\|\left(1 - \frac{1}{2\|x\|}\right)x\right\| = \left|1 - \frac{1}{2\|x\|}\right|\|x\| = \left|\|x\| - \frac{1}{2}\right| \leq \frac{1}{2},$$

which means that $\left(1 - \frac{1}{2\|x\|}\right)x \in \frac{1}{2}V$. So, (3) is proved. Now, applying Theorem 2.1, we obtain that $i_W(V) \leq 1$. Thus, taking Remark 2.2(ii) into account, we get the assertion. □

Now, we are going to prove a result concerning the index of the Cartesian product of a finite family of sets.

Theorem 2.2. *Assume that $X_1, X_2, \ldots, X_k$ ($k \geq 2$) are real vector spaces and $\emptyset \neq W_j \subset V_j \subset X_j$ for $j \in \{1, \ldots, k\}$. Then*

$$i_{W_1 \times \cdots \times W_k}(V_1 \times \cdots \times V_k) = \max\{i_{W_j}(V_j) : j \in \{1, \ldots, k\}\}.$$

Proof. First we prove that

$$i_{W_1 \times \cdots \times W_k}(V_1 \times \cdots \times V_k) \leq \max\{i_{W_j}(V_j) : j \in \{1, \ldots, k\}\}. \tag{4}$$

If $\max\{i_{W_j}(V_j) : j \in \{1, \ldots, k\}\} = \infty$ then (4) trivially holds. So assume that $i_{W_j}(V_j) =: n_j < \infty$ for $j \in \{1, \ldots, k\}$. Let $n_0 := \max\{n_j : j \in \{1, \ldots, k\}\}$. Fix $(w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k$ and $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$. Then, for every $j \in \{1, \ldots, k\}$, there exists a J-chain $(w_j, v_j^1, \ldots, v_j^{n_j})$ in V_j such that $v_j^{n_j} = v_j$. Furthermore, according to Remark 2.1, for every $j \in \{1, \ldots, k\}$, a sequence $(w_j, v_j^1, \ldots, v_j^{n_j}, v_j^{n_j+1}, \ldots, v_j^{n_0})$, where $v_j^l = v_j^{n_j}$ for $l \in \{n_j + 1, \ldots, n_0\}$ is also a J-chain in V_j. Therefore the sequence

$$\big((w_1, \ldots, w_k), (v_1^1, \ldots, v_k^1), \ldots, (v_1^{n_0}, \ldots, v_k^{n_0})\big)$$

is a J-chain in $V_1 \times \cdots \times V_k$ and so (4) holds.

Now, we show that

$$i_{W_1 \times \cdots \times W_k}(V_1 \times \cdots \times V_k) \geq \max\{i_{W_j}(V_j) : j \in \{1, \ldots, k\}\}. \tag{5}$$

If $i_{W_1 \times \cdots \times W_k}(V_1 \times \cdots \times V_k) = \infty$ then the above estimation is obvious. So, assume that $i_{W_1 \times \cdots \times W_k}(V_1 \times \cdots \times V_k) =: n < \infty$. Let $(w_1, \ldots, w_k) \in W_1 \times \cdots \times W_k$ and $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$. Then there exists a J-chain

$$\big((w_1, \ldots, w_k), (v_1^1, \ldots, v_k^1), \ldots, (v_1^n, \ldots, v_k^n)\big)$$

in $V_1 \times \cdots \times V_k$ such that $v_j^n = v_j$ for $j \in \{1, \dots, k\}$. Hence, for every $j \in \{1, \dots, k\}$, $(w_j, v_j^1, \dots, v_j^n)$ is a J-chain in V_j with $v_j^n = v_j \in V_j$, which means that $i_{W_j}(V_j) \leq n$. This proves (5).

Finally, (4) and (5) imply the assertion. □

Corollary 2.3. *Let $X = \mathbb{R}^n$, $V = [0,1]^n$ and $W = \{0,1\}^n$. Then $i_W(V) = 1$.*

Proof. In Example 2.1 we have proved that $i_{\{0,1\}}([0,1]) = 1$. Therefore, applying Theorem 2.2, we get the assertion. □

In what follows $i(V)$ stands for $i_{extV}(V)$. The next result provides an upper bound for $i(V)$ in the case where V is a nonempty compact convex subset of $\mathbb{R}^N$ ($N \in \mathbb{N}$). We adopt the notation $\lceil x \rceil := \min\{k \in \mathbb{Z} | k \geq x\}$ for $x \in \mathbb{R}$.

Theorem 2.3. *Assume that V is a nonempty compact convex subset of $\mathbb{R}^N$. Then*

$$i(V) \leq \lceil \log_2(N+1) \rceil. \tag{6}$$

Proof. Let $W := \text{ext } V$ and $m := \lceil \log_2(N+1) \rceil$. Fix a $v \in V$. According to the Caratheodory Theorem there exist $w_0, w_1, \dots, w_N \in W$ and $\alpha_0, \dots, \alpha_N \in [0,1]$ such that $\sum_{i=0}^N \alpha_i = 1$ and $v = \sum_{i=0}^N \alpha_i w_i$. Without loss of generality we may assume that $\alpha_0 \geq \frac{1}{N+1}$. Note that $\frac{1}{2^m} \leq \frac{1}{N+1}$, so $\alpha_0 - \frac{1}{2^m} \geq 0$. Furthermore

$$\begin{aligned} v = \sum_{i=0}^N \alpha_i w_i &= \frac{1}{2^m} w_0 + \left(\alpha_0 - \frac{1}{2^m} \right) w_0 + \sum_{i=1}^N \alpha_i w_i \\ &= \frac{1}{2^m} w_0 + \left(1 - \frac{1}{2^m} \right) \left(\frac{\alpha_0 - \frac{1}{2^m}}{1 - \frac{1}{2^m}} w_0 + \sum_{i=1}^N \frac{\alpha_i}{1 - \frac{1}{2^m}} w_i \right). \end{aligned}$$

Therefore, as V is convex, we have

$$\frac{\alpha_0 - \frac{1}{2^m}}{1 - \frac{1}{2^m}} w_0 + \sum_{i=1}^N \frac{\alpha_i}{1 - \frac{1}{2^m}} w_i \in V$$

and so

$$v \in \frac{1}{2^m} W + \left(1 - \frac{1}{2^m} \right) V.$$

In this way we have proved that

$$V \subset \left(1 - \frac{1}{2^m} \right) V + \frac{1}{2^m} W.$$

Hence, applying Corollary 2.1, we obtain that $i_W(V) \leq m$. □

Remark 2.3. Example 2.3 shows that the compactness of V is an essential assumption in Theorem 2.3.

It happens that in the case where V is a simplex, inequality (6) can be replaced by equality.

Theorem 2.4. *Assume that $V \subset X$ is a simplex with vertices $w_0, \ldots, w_N$. Then*

$$i(V) = \lceil \log_2(N+1) \rceil. \tag{7}$$

Proof. Let $W := \{w_0, \ldots, w_N\}$ and $v := \frac{1}{N+1}\sum_{i=0}^N w_i$. Assume that $(v_0, \ldots, v_K)$ is a J-chain in V such that $v_0 \in W$ and $v_K = v$. We are going to find the lower estimate for K. We may assume that $v_0 = w_0$. Clearly, for every $x \in V$ there exist uniquely determined nonnegative real numbers $\alpha_0(x), \ldots, \alpha_N(x)$ such that $\sum_{i=0}^N \alpha_i(x) = 1$ and $x = \sum_{i=0}^N \alpha_i(x) w_i$. It is obvious that $\alpha_0(v_0) = \alpha_0(w_0) = 1$. Next, as $(v_0, \ldots, v_K)$ is a J-chain in V, we have $2v_{k+1} - v_k \in V$ for $k \in \{0, \ldots, K-1\}$. Since

$$2v_{k+1} - v_k = \sum_{i=0}^{N} (2\alpha_i(v_{k+1}) - \alpha_i(v_k)) w_i \quad \text{for} \quad k \in \{0, \ldots, K-1\},$$

this means, in particular, that $2\alpha_0(v_{k+1}) - \alpha_0(v_k) \geq 0$ for $k \in \{0, \ldots, K-1\}$. Hence

$$\alpha_0(v_K) \geq \frac{1}{2}\alpha_0(v_{K-1}) \geq \frac{1}{4}\alpha_0(v_{K-2}) \geq \ldots \geq \frac{1}{2^K}\alpha_0(v_0) = \frac{1}{2^K}.$$

On the other hand, we have $\alpha_0(v_K) = \alpha_0(v) = \frac{1}{N+1}$. Thus $\frac{1}{N+1} \geq \frac{1}{2^K}$ and so $K \geq \lceil \log_2(N+1) \rceil$. In this way we have proved that $i_W(V) \geq \lceil \log_2(N+1) \rceil$. So, taking Theorem 2.3 into account, we get (7). □

Remark 2.4. Theorem 2.4 proves that, in general, the estimation given in Theorem 2.3 is the best possible.

3. Bounded above conditionally δ-midconvex functions

In this section we study properties of bounded above conditionally δ-midconvex functions.

Proposition 3.1. *Assume that $f : V \to \mathbb{R}$ is a conditionally δ-midconvex function bounded from above conditionally by a constant M. Then, for every J-chain $(x_0, \ldots, x_N)$ in V, we have*

$$f(x_N) \leq \frac{1}{2^N} f(x_0) + \left(1 - \frac{1}{2^N}\right)(M + 2\delta).$$

Proof. The proof goes by induction over N. If $N = 0$, then the assertion is trivial. Suppose that the assertion holds for a given N. Let $(x_0, \ldots, x_{N+1})$ be a J-chain in V. Then

$$\begin{aligned} f(x_{N+1}) &\le \frac{f(x_N)+f(2x_{N+1}-x_N)}{2}+\delta \\ &\le \frac{\frac{1}{2^N}f(x_0)+(1-\frac{1}{2^N})(M+2\delta)+M}{2}+\delta \\ &= \frac{1}{2^{N+1}}f(x_0)+\left(1-\frac{1}{2^{N+1}}\right)(M+2\delta). \end{aligned}$$

□

Corollary 3.1. *Every bounded above midconvex function $f : X \to \mathbb{R}$ is constant.*

Proof. Let $f : X \to \mathbb{R}$ be a midconvex function bounded above. Put $M := \sup\{f(x)|x \in X\}$. Furthermore, let $x \in X$ be arbitrarily fixed. Then, for every $y \in X$, (x, y) is a J-chain in X. So, applying Proposition 3.1 with $V = X$ and $\delta = 0$, we get

$$f(y) \le \frac{1}{2}f(x)+\frac{1}{2}M \ \text{ for } \ y \in X.$$

Hence

$$M = \sup\{f(y)|y \in X\} \le \frac{1}{2}f(x)+\frac{1}{2}M \le M,$$

which yields that $f(x) = M$. Thus, as $x \in X$ was arbitrary, we conclude that f is constant. □

Theorem 3.1. *Assume that $f : V \to \mathbb{R}$ is a bounded from above conditionally δ-midconvex function and $i_W(V) < \infty$. Then*

$$\sup f(V) \le \sup f(W) + 2(2^{i_W(V)}-1)\delta. \tag{8}$$

Proof. Fix an arbitrary $\varepsilon > 0$. Let $M := \sup f(V)$. Then $f(v) > M - \varepsilon$ for some $v \in V$. Furthermore, by the assumptions, there exists a J-chain $(x_0, \dots, x_N)$ in V such that $x_0 \in W$, $x_N = v$ and $N \le i_W(V)$. Therefore, applying Proposition 3.1, we obtain

$$M-\varepsilon < f(v) = f(x_N) \le \frac{1}{2^N}f(x_0)+\left(1-\frac{1}{2^N}\right)(M+2\delta).$$

Hence

$$\frac{1}{2^N}M \le \frac{1}{2^N}f(x_0)+\varepsilon+2\left(1-\frac{1}{2^N}\right)\delta$$

and so

$$M \le f(x_0)+2^N\varepsilon+2(2^N-1)\delta \le f(x_0)+2^{i_W(V)}\varepsilon+2(2^{i_W(V)}-1)\delta.$$

Since $\varepsilon > 0$ was chosen arbitrarily and $x_0 \in W$, this implies (8). □

From Theorems 2.3 and 3.1 we obtain the following result.

Corollary 3.2. *If V is a nonempty compact convex subset of $\mathbb{R}^N$ and $f : V \to \mathbb{R}$ is a bounded from above δ-midconvex function, then*

$$\sup f(V) \leq \sup f(extV) + 2\left(2^{\lceil \log_2(N+1) \rceil} - 1\right)\delta. \tag{9}$$

Remark 3.1. Since $\lceil \log_2(N+1) \rceil \leq \log_2(N+1) + 1$ for $N \in \mathbb{N}$, (9) implies a weaker, but simpler estimation, which is sometimes more useful in applications

$$\sup f(V) \leq \sup f(extV) + (4N+2)\delta.$$

Corollary 3.3. *If V is a nonempty compact convex subset of $\mathbb{R}^N$ and $f : V \to \mathbb{R}$ is a bounded from above midconvex function, then* $\sup f(V) = \sup f(\text{ext}V)$.

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