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Aequationes Mathematicae

A support theorem for delta (s, t)-convex mappings

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Abstract. In the present paper a notion of delta (s, t)-convexity in the sense of Veselý and Zajiček is studied as a natural generalization of the classical (s, t)-convexity. The main result of this paper is a support theorem for delta (s, t)-convex mappings.

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1. Introduction and terminology

Throughout this paper $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ stand for two real Banach spaces.

Definition 1. Let $s, t \in (0, 1)$ be fixed real numbers, and let $D \subset X$ be a convex set. A function $f : D \to [-\infty, \infty)$ is said to be
(s, t)-convex, if

$$\bigwedge_{x,y\in D} f(sx + (1-s)y) \le tf(x) + (1-t)f(y), \tag{1}$$

(s, t)-affine, if

$$\bigwedge_{x,y\in D} f(sx + (1-s)y) = tf(x) + (1-t)f(y). \tag{2}$$

If the inequality (1) is satisfied for $t = s$ then we say that f is t-convex, if $t = s = \frac{1}{2}$ then f is said to be Jensen-convex. If the Eq. (2) is satisfied for $s = t$, and all $t \in [0, 1]$, then we say that f is affine.

In [7] Kuhn proved that every t-convex function is Jensen-convex (cf. Daróczy and Páles [1] for a simple argument). Some properties of (s, t)-convex functions are contained in [8]. In particular in [8] Kuhn remarks that f must be constant if t is rational. He also mentions that he does not know any example of a non constant (s, t)-convex functions for $s \neq t$. In a natural way a problem of a characterization of (s,t)-convex functions, for $s \neq t$ appears in this context (independently asked by Rolewicz [5]). The complete solution of this problem was given by Matkowski and Pycia in [10] (see also [5] for partial solution). To present the main result contained in [10] we need the following

Definition 2. The elements $s, t \in \mathbb{R}$ are said to be conjugate if either they are both transcendental, or they are algebraically conjugate, i.e. they are both algebraic and have the same minimal polynomial with rational coefficients.

The main result contained in [10] reads as follows

Theorem 1. ([10]) *Let $D \subset X$ be an open and convex set. Suppose that $s, t \in (0,1)$ are fixed.*
If s, t are conjugate then there exists a non-constant additive function $\phi : X \to \mathbb{R}$ such that

$$\phi(tx + (1-t)y) = s\phi(x) + (1-s)\phi(y), \qquad x, y \in X.$$

If s, t are not conjugate then every (s,t)-convex function $f : D \to [-\infty, \infty)$ i.e. such that

$$f(tx + (1-t)y) \leq sf(x) + (1-s)f(y), \qquad x, y \in D,$$

is a constant function.

In the sequel we will use the following remark contained in [10].(Actually in [10] the authors assume that D is an open interval but such a restriction is inessential).

Remark 1 ([10]). Let $D \subset X$ be a convex and open set, and let $f : D \to [-\infty, \infty)$ be an (s,t)-convex function. If there exists a point $x_0 \in D$ such that $f(x_0) = -\infty$ then $f \equiv -\infty$.

A survey of results concerning (s,t)-convex functions may be found in the papers [5,8,10], in particular, the following version of the theorem of Rodé (cf. [12]), has been proved by Kuhn in [8] and by Kominek in [5].

Theorem 2. *Let D be an open and convex subset of a real linear space endowed with a semi-linear topology and let $f : D \to \mathbb{R}$ be an (s,t)-convex function. Then for every $z \in D$ there exists a function $G_z : D \to \mathbb{R}$ such that*

(1) $G_z(tx + (1-t)y) = sG_z(x) + (1-s)G_z(y), \quad x, y \in D;$
(2) $G_z(z) = f(z);$
(3) $G_z(x) \leq f(x), \quad x \in D.$

In 1989 Veselý and Zajíček introduced an interesting generalization of functions which are representable as a difference of two convex functions. In the paper [13] the authors introduced the following definition

Definition 3. Let $D \subset X$ be a convex and open set. A map $F : D \to Y$ is called delta-convex if there exists a continuous and convex functional $f : D \to \mathbb{R}$ such that $f + y^* \circ F$ is continuous and convex for any member y^* of the space Y^* dual to Y with $\|y^*\| = 1$. If this is the case then we say that F is a delta-convex mapping with a control function f.

It turns out that a continuous function $F : D \to Y$ is a delta-convex mapping controlled by a continuous function $f : D \to \mathbb{R}$ if and only if the functional inequality

$$\left\|F\Big(\frac{x+y}{2}\Big) - \frac{F(x)+F(y)}{2}\right\| \le \frac{f(x)+f(y)}{2} - f\Big(\frac{x+y}{2}\Big), \tag{3}$$

is satisfied for all $x, y \in D$ (Corollary 1.18 in [13]).

In the paper [2] Ger generalized this result. He has shown that if the inequality (3) holds for all $x, y \in D$ and the function

$$D \ni x \Rightarrow f(x) + \|F(x)\|, \tag{4}$$

is upper bounded on a set $T \subset D$ whose $\mathbb{Q}$-convex hull $conv_{\mathbb{Q}}(T)$ forms a second cathegory Baire subset of X then F is locally Lipschtzian, in particular, F is a delta-convex mapping controlled by f. Moreover, if Y is a separable space and the function given by the formula (4) is Christensen measurable then it provides the same effect.

The inequality (3) may obviously be investigated without any regularity assumption upon F and f. Motivated by these two concepts we introduce the following definition.

Definition 4. Let $D \subset X$ be a convex set. A map $F : D \to Y$ is called delta (s,t)-convex with a control function $f : D \to \mathbb{R}$ if the inequality

$$\begin{aligned}\|tF(x) + (1-t)F(y) - F(sx + (1-s)y)\| &\le tf(x) + (1-t)f(y)\\ &\quad - f(sx + (1-s)y)\end{aligned} \tag{5}$$

holds for all $x, y \in D$. If the above inequality is satisfied for $t = s$, then we say that F is a delta t-convex mapping, if $t = s = \frac{1}{2}$ then F is said to be delta Jensen-convex.

As an immediate consequence of Theorem 1 we get

Corollary 1. *Every delta (s,t)-convex mapping with not conjugate $s, t \in (0,1)$ is constant.*

2. Results

In the proof of our first result we use the following corollary, which is a consequence of the Hahn–Banach theorem.

Corollary 2. *Let* $(X, \|\cdot\|)$ *be a real normed space. Then for each* $x \in X$

$$\|x\| = \sup\{x^*(x) : \ x^* \in X^*, \ \|x^*\| = 1\}.$$

The following result establishes necessary and sufficient conditions for a given map to be delta (s,t)-convex.

Theorem 3. *Let* $D \subset X$ *be a convex set and let* $F : D \to Y$, $f : D \to \mathbb{R}$. *The following conditions are pairwise equivalent:*

(i) F *is a delta* $(s,t)-$ *convex mapping controlled by* f,
(ii) *for every* $y^* \in Y^*$, $\|y^*\| = 1$, *the function* $y^* \circ F + f$ *is* (s,t)*-convex,*
(iii) *for every* $y^* \in Y^*$, $\|y^*\| = 1$, *the function* $y^* \circ F - f$ *is* (s,t)*-concave.*

Proof. (i) implies (iii). Assume that

$$\|tF(x) + (1-t)F(y) - F(sx + (1-s)y)\| \leq tf(x) + (1-t)f(y) \\ -f(sx + (1-s)y),$$

for all $x, y \in D$. Let $y^* \in Y^*$, $\|y^*\| = 1$ be arbitrary. From the above inequality and Corollary 2 it follows that

$$y^*(tF(x) + (1-t)F(y) - F(sx + (1-s)y)) \\ \leq tf(x) + (1-t)f(y) - f(sx + (1-s)y),$$

or, equivalently,

$$y^*(tF(x) + (1-t)F(y)) - tf(x) - (1-t)f(y) \leq y^*(F(sx + (1-s)y)) \\ -f(sx + (1-s)y).$$

(iii) implies (ii). Replace y^* by $-y^*$ in (iii).
(ii) implies (i). For every $y^* \in Y^*$, $\|y^*\| = 1$ we have

$$y^*(F(sx + (1-s)y)) + f(sx + (1-s)y) \leq \\ ty^*(F(x)) + (1-t)y^*(F(y)) - tf(x) - (1-t)f(y)$$

and, consequently,

$$\begin{aligned} \|tF(x) + (1-t)F(y) - F(sx + (1-s)y)\| &= \sup\{y^*(tF(x) + (1-t)F(y) \\ &\quad -F(sx + (1-s)y)) : \ \|y^*\| = 1\} \\ &\leq tf(x) + (1-t)f(y) \\ &\quad -f(sx + (1-s)y), \end{aligned}$$

which completes the proof. □

Let us observe, that delta (s,t)-convex mappings provide a generalization of functions which are representable as a difference of two (s,t)-convex functions.

Proposition 1. *Let* $D \subset X$ *be a convex set. In the case where* $Y = \mathbb{R}$, $\|\cdot\| = |\cdot|$ *a map* $F : D \to \mathbb{R}$ *is delta-*(s,t)*-convex, if and only if,* F *is a difference of two* (s,t)*-convex functions.*

Proof. Assume $f : D \to \mathbb{R}$ is a control function for F. For all $x, y \in D$ we have

$$|tF(x) + (1-t)F(y) - F(sx + (1-s)y)| \le tf(x) + (1-t)f(y) - f(sx + (1-s)y).$$

Put

$$\phi_1 := \frac{1}{2}(F + f) \qquad \text{and} \qquad \phi_2 := \frac{1}{2}(f - F).$$

It is easy to see that both ϕ_1 and ϕ_2 are (s,t)-convex functions, moreover, $F = \phi_1 - \phi_2$. Conversely let $F = \phi_1 - \phi_2$, where ϕ_1, ϕ_2 are (s,t)-convex. Setting $f := \phi_1 + \phi_2$ we infer that both $f - F$ and $f + F$ are (s,t)-convex, whence, for every $x, y \in D$ we obtain

$$|tF(x) + (1-t)F(y) - F(sx + (1-s)y)| \le tf(x) + (1-t)f(y) - f(sx + (1-s)y),$$

which finishes the proof. □

Using a well-known Daróczy and Páles representation of the mean $\frac{x+y}{2}$ we get the following

Lemma 1. *Let $D \subset X$ be a convex set. If a mapping $F : D \to Y$ is delta (s,t)-convex, then it is delta Jensen-convex.*

Proof. Take an arbitrary $y^* \in Y^*$ such that $\|y^*\| = 1$. From the identity (cf. Daróczy and Páles [1])

$$\frac{x+y}{2} = s\left[s\frac{x+y}{2} + (1-s)y\right] + (1-s)\left[sx + (1-s)\frac{x+y}{2}\right]$$

and (s,t)-convexity of the function $h := y^* \circ F + f$ we have for all $x, y \in D$

$$\begin{aligned} h\left(\frac{x+y}{2}\right) &= h\left(s\left[s\frac{x+y}{2} + (1-s)y\right] + (1-s)\left[sx + (1-s)\frac{x+y}{2}\right]\right) \\ &\le th\left(s\frac{x+y}{2} + (1-s)y\right) + (1-t)h\left(sx + (1-s)\frac{x+y}{2}\right) \\ &\le t^2 h\left(\frac{x+y}{2}\right) + t(1-t)h(x) + (1-t)th(y) \\ &\quad + (1-t)^2 h\left(\frac{x+y}{2}\right), \end{aligned}$$

which means that

$$t(1-t)h\left(\frac{x+y}{2}\right) \le t(1-t)\frac{h(x)+h(y)}{2}, \quad x, y \in D,$$

and consequently,

$$y^*\left(\frac{F(x)+F(y)}{2} - F\left(\frac{x+y}{2}\right)\right) \le \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right), \quad x, y \in D,$$

whence, in view of the arbitrariness of y^* and on account of Corollary 2 we get

$$\left\| \frac{F(x)+F(y)}{2} - F\left(\frac{x+y}{2}\right) \right\| \le \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right).$$

The proof of our lemma is finished. □

Observe that, on account of the above lemma the results obtained by Ger in [2] concerning delta Jensen-convex mappings are also true for delta (s,t)-convex mappings.

Now, we are able to prove our main result. The following theorem corresponds to a classical support theorem for (s,t)-convex functions.

Theorem 4. *Let $D \subset X$ be an open and convex set, and let $F : D \to Y$ be a delta (s,t)-convex map with a control function $f : D \to \mathbb{R}$. Then for an arbitrary point $y \in D$ there exist (s,t)-affine maps $A_y : D \to Y$ and $a_y : D \to \mathbb{R}$ such that*

$$\bigwedge_{x \in D} \|F(x) - A_y(x)\| \le f(x) - a_y(x),$$

moreover,

$$A_y(y) = F(y), \quad a_y(y) = f(y).$$

Proof. Fix an arbitrary point $y \in D$. Consider the following family of pairs of maps

$$\begin{aligned}\mathcal{H} := \{(H,h) :\ & H \text{ is delta } (s,t)\text{-convex with control function } h,\\ & h(y) = f(y),\ \ \|F(x) - H(x)\| \le f(x) - h(x),\ x \in D\}.\end{aligned}$$

Observe that $\mathcal{H} \neq \emptyset$, because $(F,f) \in \mathcal{H}$. Define an order relation $\preceq$ on $\mathcal{H}$ as follows

$$(H_1,h_1) \preceq (H_2,h_2) \Leftrightarrow \|H_1(x) - H_2(x)\| \le h_2(x) - h_1(x), \quad x \in D.$$

We will show that every chain has a lower bound in $\mathcal{H}$. Let $\mathcal{L} \subset \mathcal{H}$ be an arbitrary chain. Define the function $h_0 : D \to [-\infty, \infty)$ by the formula

$$h_0(x) := \inf\{h(x) :\ (H,h) \in \mathcal{L}\}, \quad x \in D.$$

Observe that h_0 is (s,t)-convex in D. To see it take arbitrary $x, z \in D$ and arbitrary $c_1, c_2 \in \mathbb{R}$ such that

$$h_0(x) < c_1, \qquad h_0(z) < c_2.$$

By the definition of h_0 there exist $(H_1,h_1), (H_2,h_2) \in \mathcal{L}$ such that

$$h_1(x) < c_1 \quad \text{and} \quad h_2(z) < c_2.$$

Therefore (say $(H_1,h_1) \preceq (H_2,h_2)$) we obtain by the (s,t)-convexity of h_1 and h_2

$$\begin{aligned}tc_1 + (1-t)c_2 > th_1(x) + (1-t)h_2(z) &\ge th_1(x) + (1-t)h_1(z)\\ &\ge h_1(sx + (1-s)z) \ge h_0(sx + (1-s)z).\end{aligned}$$

Tending in the above inequalities with $c_1 \to h_0(x)$, $c_2 \to h_0(z)$ we get the (s,t)-convexity of h_0. Since $h_0(y) = f(y) > -\infty$, then by Remark 1 h_0 has finite values.

There exists a sequence $\{(H_n, h_n)\}_{n\in\mathbb{N}} \subset \mathcal{L}$ such that

$$h_0(x) = \lim_{n\to\infty} h_n(x), \quad x \in D.$$

Since the sequence $\{h_n\}_{n\in\mathbb{N}}$ is convergent, in particular it is a Cauchy sequence, so

$$\bigwedge_{\varepsilon>0} \bigvee_{n_0\in\mathbb{N}} \bigwedge_{n,m\geq n_0} \|H_n(x) - H_m(x)\| \leq max\{h_n(x) - h_m(x), h_m(x) - h_n(x)\} < \varepsilon.$$

Therefore $\{H_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence and consequently it is convergent. Let H_0 be its limit. First we must check whether the definition of H_0 is correct. If a sequence $\{(K_n, k_n)\}_{n\in\mathbb{N}} \subset \mathcal{L}$ satisfies the condition

$$h_0(x) = \lim_{n\to\infty} k_n(x), \quad x \in D,$$

then using the same argumentation it is easy to check that $\{K_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and consequently converges. Let

$$K_0(x) = \lim_{n\to\infty} K_n(x), \quad x \in D,$$

whence

$$\begin{aligned}\|K_0(x) - H_0(x)\| &\leq \|K_0(x) - K_n(x)\| + \|K_n(x) - H_n(x)\| + \|H_n(x) - H_0(x)\| \\ &\leq \|K_0(x) - K_n(x)\| + max\{k_n(x) - h_n(x), h_n(x) - k_n(x)\} \\ &\quad + \|H_n(x) - H_0(x)\| \to_{n\to\infty} 0,\end{aligned}$$

and this implies that $K_0 = H_0$.

Let us observe that (H_0, h_0) is a lower bound of the chain $\mathcal{L}$. Fix an arbitrary $(H, h) \in \mathcal{L}$. Let a sequence $\{(H_n, h_n)\}_{n\in\mathbb{N}} \subset \mathcal{L}$ be such that

$$\lim_{n\to\infty} H_n = H_0 \quad \text{and} \quad \lim_{n\to\infty} h_n = h_0.$$

Since

$$\bigwedge_{x\in D} h_0(x) \leq h(x),$$

then, because $\mathcal{L}$ is a chain we have for all $n \in \mathbb{N}$

$$\|H_n(x) - H(x)\| \leq max\{h_n(x) - h(x), h(x) - h_n(x)\},$$

and letting $n \to \infty$ we obtain

$$\|H_0(x) - H(x)\| \leq h(x) - h_0(x), \quad x \in D,$$

and then $(H_0, h_0) \preceq (H, h)$. Due to the arbitrariness of $(H, h) \in \mathcal{L}$ we infer that (H_0, h_0) is a lower bound of the chain $\mathcal{L}$.

Now, we will show that $(H_0, h_0) \in \mathcal{H}$. Indeed, since $h_0(y) = f(y)$ also $H_0(y) = F(y)$. Note that, because

$$\bigwedge_{n\in\mathbb{N}} \|F(x) - H_n(x)\| \leq f(x) - h_n(x),$$

we have

$$\|F(x) - H_0(x)\| = \lim_{n\to\infty} \|F(x) - H_n(x)\| \leq f(x) - h_0(x), \quad x \in D.$$

To see that H_0 is delta (s,t)-convex with a control function h_0 fix arbitrary points $x, z \in D$. Let $x_1 := x,\ x_2 := z,\ x_3 := sx + (1-s)z$. There exist sequences $\{(H_n^j, h_n^j)\}_{n\in\mathbb{N}} \subset \mathcal{L},\ j = 1,2,3$ such that

$$\lim_{n\to\infty} H_n^j(x_j) = H_0(x_j) \quad \text{and} \quad \lim_{n\to\infty} h_n^j(x_j) = h_0(x_j), \qquad j = 1,2,3.$$

Let

$$(P_n, p_n) := min_{\preceq}\{(H_n^j, h_n^j):\ j = 1,2,3\}.$$

Observe that, because

$$\bigwedge_{n\in\mathbb{N}} \|P_n(x_j) - H_0(x_j)\| \leq p_n(x_j) - h_0(x_j) \leq h_n^j(x_j) - h_0(x_j) \to_{n\to\infty} 0,$$

and since for all $n \in \mathbb{N}$ we have

$$\begin{aligned}&\|tP_n(x) + (1-t)P_n(z) - P_n(sx + (1-s)z)\|\\ &\quad \leq tp_n(x) + (1-t)p_n(z) - p_n(sx + (1-s)z),\end{aligned}$$

then, letting $n \to \infty$ we obtain

$$\begin{aligned}&\|tH_0(x) + (1-t)H_0(z) - H_0(sx + (1-s)z)\|\\ &\quad \leq th_0(x) + (1-t)h_0(z) - h_0(sx + (1-s)z).\end{aligned}$$

By the Kuratowski–Zorn lemma there exists a minimal element $(\overline{H}, \overline{h}) \in \mathcal{H}$. For an arbitrary fixed point $z \in D$ define the following maps

$$\begin{aligned}\overline{H}_z(x) &:= \frac{1}{t}\Big[\overline{H}(sx + (1-s)z) - (1-t)\overline{H}(z)\Big],\\ \overline{h}_z(x) &:= \frac{1}{t}\Big[\overline{h}(sx + (1-s)z) - (1-t)\overline{h}(z)\Big].\end{aligned}$$

First we will show that $(\overline{H}_z, \overline{h}_z) \in \mathcal{H}$. For every $x \in D$ we have

$$\begin{aligned}\|F(x) - \overline{H}_z(x)\| &\leq \|F(x) - \overline{H}(x)\| + \|\overline{H}(x) - \overline{H}_z(x)\|\\ &\leq f(x) - \overline{h}(x) + \|\overline{H}(x) - \frac{1}{t}[\overline{H}(sx + (1-s)z) - (1-t)\overline{H}(z)]\|\\ &\leq f(x) - \overline{h}(x) + \overline{h}(x) - \frac{1}{t}[\overline{h}(sx + (1-s)z) - (1-t)\overline{h}(z)]\\ &= f(x) - \overline{h}_z(x).\end{aligned}$$

Let us observe that $\overline{H}_z$ is delta (s,t)-convex with a control function h_z. To see it fix $x, u \in D$ arbitrarily.. Since $\overline{H}$ is delta (s,t)-convex with a control function $\overline{h}$ we obtain

$$\begin{aligned}
&\Big\|t\overline{H}_z(x) + (1-t)\overline{H}_z(u) - \overline{H}_z(sx+(1-s)u)\Big\| \\
&\quad = \Big\|\overline{H}(sx+(1-s)z) - (1-t)\overline{H}(z) \\
&\qquad + \frac{1-t}{t}\overline{H}(su+(1-s)z) - \frac{(1-t)^2}{t}\overline{H}(z) \\
&\qquad - \frac{1}{t}\overline{H}(s[sx+(1-s)u]+(1-s)z) + \frac{1-t}{t}\overline{H}(z)\Big\| \\
&\quad = \Big\|\overline{H}(sx+(1-s)z) - (1-t)\overline{H}(z) + \frac{1-t}{t}\overline{H}(su+(1-s)z) \\
&\qquad - \frac{(1-t)^2}{t}\overline{H}(z) \\
&\qquad - \frac{1}{t}\overline{H}(s[sx+(1-s)z]+(1-s)[su+(1-s)z]) + \frac{1-t}{t}\overline{H}(z)\Big\| \\
&\quad = \frac{1}{t}\Big\|t\overline{H}(sx+(1-s)z) + (1-t)\overline{H}(su+(1-s)z) \\
&\qquad - \overline{H}(s[sx+(1-s)z]+(1-s)[su+(1-s)z])\Big\| \\
&\quad \le \frac{1}{t}\Big[t\overline{h}(sx+(1-s)z) + (1-t)\overline{h}(su+(1-s)z) \\
&\qquad - \overline{h}(s[sx+(1-s)z]+(1-s)[su+(1-s)z])\Big] \\
&\quad = t\Big[\frac{1}{t}\Big(\overline{h}(sx+(1-s)z) - \overline{h}(z)\Big)\Big] + (1-t)\Big[\frac{1}{t}\Big(\overline{h}(su+(1-s)z) - \overline{h}(z)\Big)\Big] \\
&\qquad - \frac{1}{t}\Big[\overline{h}(s[sx+(1-s)u]+(1-s)z) - \overline{h}(z)\Big] \\
&\quad = t\overline{h}_z(x) + (1-t)\overline{h}_z(u) - \overline{h}_z(sx+(1-s)u).
\end{aligned}$$

Since, in particular, $\overline{h}_y(y) = f(y)$, $\overline{H}_y(y) = F(y)$ and

$$\|\overline{H}(x) - \overline{H}_y(x)\| \le \overline{h}(x) - \overline{h}_y(x), \qquad x \in D,$$

then $(\overline{H}_y, \overline{h}_y) \in \mathcal{H}$ and because $(\overline{H}_y, \overline{h}_y) \preceq (\overline{H}, \overline{h})$ then by the minimality of $(\overline{H}, \overline{h})$ we get

$$\overline{h}_y(x) = \overline{h}(x) \quad \text{and} \quad \overline{H}_y(x) = \overline{H}(x), \quad x \in D.$$

Hence for every $z \in D$ we have

$$\overline{h}_z(y) = \frac{1}{t}[\overline{h}(sy+(1-s)z) - (1-t)\overline{h}(z)] = \overline{h}_y(y) = f(y),$$

then evidently $\overline{H}_z(y) = F(y)$, and we infer that $(\overline{H}_z, \overline{h}_z) \in \mathcal{H}$, and using again the minimality of $(\overline{H}, \overline{h})$ we get that for all $x, z \in D$

$$\overline{h}(sx + (1-s)z) = t\overline{h}(x) + (1-t)\overline{h}(z), \quad \text{and,}$$
$$\overline{H}(sx + (1-s)z) = t\overline{H}(x) + (1-t)\overline{H}(z).$$

This completes the proof. □

The following Theorem states that the existence of a support mapping at an arbitrary point in fact characterizes delta (s,t)-convexity.

Theorem 5. *Let $D \subset X$ be an open and convex set. A map $F : D \to Y$ is delta (s,t)-convex with a control function $f : D \to \mathbb{R}$ if and only if, for every point $y \in D$ there exist (s,t)-affine maps $A_y : D \to Y$ and $a_y : D \to \mathbb{R}$ such that*

$$\|F(x) - A_y(x)\| \le f(x) - a_y(x), \quad x \in D,$$

moreover,

$$A_y(y) = F(y), \quad a_y(y) = f(y).$$

Proof. The sufficiency results from Theorem 4. To prove the necessity fix arbitrary $x, z \in D$. Put $y := sx + (1-s)z$. By our assumption we get

$$\begin{aligned} \|tF(x) + (1-t)F(z) - F(y)\| &= \big\|t(F(x) - A_y(x)) + (1-t)(F(z) - A_y(z)) \\ &\quad -(F(y) - A_y(y))\big\| \\ &\quad \le t\|F(x) - A_y(x)\| + (1-t)\|F(z) - A_y(z)\| \\ &\le t(f(x) - a_y(x)) + (1-t)(f(z) - a_y(z)) \\ &\quad -(f(y) - a_y(y)) \\ &= tf(x) + (1-t)f(z) - f(y). \end{aligned}$$

The proof is complete. □

It follows from the proof of Theorems 4 and 5 that for delta convex mappings the following theorem holds true

Theorem 6. *Let $D \subset X$ be an open and convex set. A map $F : D \to Y$ is delta convex with a control function $f : D \to \mathbb{R}$ i.e. it satisfies for all $x, y \in D$ and $\alpha \in [0,1]$ the inequality*

$$\|\alpha F(x) + (1-\alpha)F(y) - F(\alpha x + (1-\alpha)y)\| \le \alpha f(x) + (1-\alpha)f(y) \\ -f(\alpha x + (1-\alpha)y)$$

if and only if, for every point $y \in D$ there exist affine maps $A_y : D \to Y$ and $a_y : D \to \mathbb{R}$ such that

$$\bigwedge_{x \in D} \|F(x) - A_y(x)\| \le f(x) - a_y(x),$$

moreover,

$$A_y(y) = F(y), \quad a_y(y) = f(y).$$

Remarks

1. Substituting $F := 0$ in our theorems we obtain the well-known results concerning classical (s,t)-convexity.
2. Let us observe that there is a close relationship between the theory of delta convex mappings and the problems of Hyers–Ulam stability, developed by studying the following functional inequality

$$\|tF(x) + (1-t)F(y) - F(sx + (1-s)y)\| \le \Phi(x,y), \quad x,y \in D,$$

where

$$\Phi(x,y) := tf(x) + (1-t)f(y) - f(sx + (1-s)y).$$

In Theorem 4 we have shown that there exists the solution $A : D \to Y$ of the equation

$$F(sx + (1-s)y) = tF(x) + (1-t)F(y), \quad x,y \in D,$$

such that

$$\|F(x) - A(x)\| \le \phi(x), \quad x \in D,$$

where $\phi(x) := f(x) - a(x)$, for some (s,t)-affine function $a : D \to \mathbb{R}$. In the case $s = t = \frac{1}{2}$ we get the stability result for Jensen equation.

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References

[1] Daróczy, Z., Páles, Zs.: Convexity with given infinite weight sequences. Stochastica **11**, 5–12 (1987)
[2] Ger, R.: Stability aspects of delta-convexity. In: Rassias, Th.M., Tabor, J. (eds.) Stability of Hyers–Ulam type, pp. 99–109. Hardonic Press, Palm Harbor (1994)
[3] Ger, R.: Stability of polynomial mappings controlled by n-convex functionals. In: Agorwal, R.P. (ed.) Inequalities and applications, a volume dedicated to W.Walter, World Scientific Series in Applied Analysis, pp. 255–268. World Scientific Publishing Company, Singapore (1994)
[4] Kominek, Z.: Convex Functions in Linear Spaces. Prace Naukowe Uniwersytetu Śląskiego w Katowicach nr 1087, Katowice (1989)
[5] Kominek, Z.: On (a,b)-convex functions. Arch. Math. **58**, 64–69 (1992)
[6] König, H.: On the abstract Hahn–Banach theorem due to Rodé. Aequ. Math. **34**, 89–95 (1987)
[7] Kuhn, N.: A note on t-convex functions. General Inequal. **4**, 269–276 (1984)

[8] Kuhn, N.: On the structure of (s,t)-convex functions. General Inequalities 5. Internat. Ser. Numer. Math. **80**, 161–174 (1987)
[9] Kuczma, M.: An Introduction to the Theory of Functional Equations and Inequalities. Birkhäuser, Basel, Boston, Berlin (2009)
[10] Matkowski, J., Pycia, M.: On (α,a)-convex functions. Arch. Math. **64**, 132–138 (1995)
[11] Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York, London (1973)
[12] Rodé, G.: Eine abstrakte Version des Satzes von Hahn–Banach. Arch. Math. **31** (1978)
[13] Veselý, L., Zajiček, L.: Delta-convex mappings between Banach spaces and applications. Dissertationes Math., Polish Scientific Publishers (289), Warszawa (1989)

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