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Ohlin's lemma and some inequalities of the Hermite–Hadamard type

Tomasz Szostok

Abstract. Using the Ohlin lemma on convex stochastic ordering we prove inequalities of the Hermite–Hadamard type. Namely, we determine all numbers $a, \alpha, \beta \in [0,1]$ such that for all convex functions f the inequality

$$af(\alpha x + (1-\alpha)y) + (1-a)f(\beta x + (1-\beta)y) \le \frac{1}{y-x}\int_x^y f(t)dt$$

is satisfied and all $a, b, c, \alpha \in (0,1)$ with $a+b+c=1$ for which we have

$$af(x) + bf(\alpha x + (1-\alpha)y) + cf(y) \ge \frac{1}{y-x}\int_x^y f(t)dt.$$

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1. Introduction

Inspired by the paper [8] we use the Ohlin lemma to prove some new inequalities of the Hermite–Hadamard type. Namely we find all numbers $a, \alpha, \beta \in [0,1]$ such that the inequality

$$af(\alpha x + (1-\alpha)y) + (1-a)f(\beta x + (1-\beta)y) \le \frac{1}{y-x}\int_x^y f(t)dt \tag{1.1}$$

is satisfied for all convex functions $f : [x, y] \to \mathbb{R}$. In the second part we deal with the second inequality of the Hermite–Hadamard type

$$af(x) + bf(\alpha x + (1-\alpha)y) + cf(y) \ge \frac{1}{y-x}\int_x^y f(t)dt.$$

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We also show that some inequalities which may be found in literature (with quite long proofs) are particular cases of our results.

For the sake of completeness we cite the main tool which we will use throughout the whole paper.

Lemma 1.1. (Ohlin [7]) *Let X_1, X_2 be two random variables such that $\mathbb{E}X_1 = \mathbb{E}X_2$ and let F_1, F_2 be their distribution functions. If F_1, F_2 satisfy for some x_0 the following inequalities*

$$F_1(x) \le F_2(x) \text{ if } x < x_0 \ \text{ and } \ F_1(x) \ge F_2(x) \text{ if } x > x_0$$

then

$$\mathbb{E}f(X_1) \le \mathbb{E}f(X_2) \tag{1.2}$$

for all continuous and convex functions $f : \mathbb{R} \to \mathbb{R}$.

Remark 1.2. A careful inspection of the proof of this lemma shows that if measures μ_1, μ_2 corresponding to F_1, F_2 are concentrated on the interval $[x, y]$ then, in fact, inequality (1.2) is satisfied for all convex functions $f : [x, y] \to \mathbb{R}$.

Remark 1.3. The classical Hermite–Hadamard inequality

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2} \tag{1.3}$$

may easily be obtained with the use of the Ohlin lemma. Indeed, let F_1 be the distribution function of $\delta_{\frac{x+y}{2}}$, F_2 the distribution function of the measure which is equally distributed in $[x, y]$ and let F_3 be the distribution function of $\frac{\delta_x + \delta_y}{2}$. Then it is easy to see that pairs (F_1, F_2) and (F_2, F_3) satisfy the assumptions of the Ohlin lemma and, using (1.2), we get (1.3).

As we can see, the Ohlin lemma is a strong tool, however, it is worth noticing that in the case of inequalities of the type which we consider the distribution functions involved cross more than once. Therefore a simple application of the Ohlin lemma is impossible and some additional idea is needed.

2. Results

First we prove the following simple lemma.

Lemma 2.1. *If for two random variables X_1, X_2 we have $\mathbb{E}f(X_1) \le \mathbb{E}f(X_2)$ for all continuous and convex functions $f : \mathbb{R} \to \mathbb{R}$ then $\mathbb{E}X_1 = \mathbb{E}X_2$.*

Proof. If $\mathbb{E}f(X_1) \le \mathbb{E}f(X_2)$ is satisfied for all continuous and convex functions then it is satisfied for $f(x) = x$ which yields $\mathbb{E}X_1 \le \mathbb{E}X_2$ and for $g(x) = -x$ gives us $\mathbb{E}X_1 \ge \mathbb{E}X_2$. □

Now, observe that if we want inequality (1.1) to be satisfied then we cannot use the endpoints of the interval at the left-hand side of (1.1).

Remark 2.2. If inequality (1.1) with some $a \in (0,1), \alpha, \beta \in [0,1]$, $\alpha > \beta$ is satisfied for all $x, y \in \mathbb{R}$ and all continuous and convex functions $f : [x,y] \to \mathbb{R}$ then $\alpha, \beta \in (0,1)$. Indeed, suppose that $\alpha = 1$, take $x = 0, y = 1$ and

$$f(t) := \begin{cases} 1 - \frac{t}{a} & \text{if} \quad x < a \\ 0 & \text{if} \quad x \geq a. \end{cases}$$

Then f is convex and

$$af(\alpha x + (1-\alpha)y) + (1-a)f(\beta x + (1-\beta y) \geq af(0) = a > \frac{a}{2} = \int_0^1 f(t)dt.$$

Remark 2.3. Note that if inequality (1.1) is satisfied for all continuous and convex functions defined on the interval $[0,1]$ then it is satisfied for all such functions defined on a given interval $[x,y]$.

Indeed, taking a convex function f defined on a given interval $[x,y]$, we may define a function

$$f_1(t) := f(t(y-x)+x), t \in [0,1].$$

This function is convex and using (1.1) for f_1, we obtain this inequality for f.

Theorem 2.4. *Inequality (1.1) with some $a, \alpha, \beta \in [0,1]$, $\alpha > \beta$ is satisfied for all $x, y \in \mathbb{R}$ and all continuous and convex functions $f : [x,y] \to \mathbb{R}$ if and only if*

$$a\alpha + (1-a)\beta = \frac{1}{2} \tag{2.1}$$

and one of the following conditions holds true:

(i) $a + \alpha \leq 1$
(ii) $a + \beta \geq 1$
(iii) $a + \alpha > 1$, $a + \beta < 1$ *and* $a + 2\alpha \leq 2$.

Proof. In view of Remark 2.3, we may assume that $x = 0, y = 1$. Let X_1 be the random variable such that

$$\mu_{X_1} = a\delta_{1-\alpha} + (1-a)\delta_{1-\beta}$$

and let F_1 be its distribution function. Further let X_2 be such that

$$\mu_{X_2}(A) = l_1(A \cap [0,1]).$$

Assume that (1.1) is satisfied for all continuous and convex functions $f : [0,1] \to \mathbb{R}$. Then from Lemma 2.1 we know that

$$a(1-\alpha)+(1-a)(1-\beta)=\int_0^1 t dF_1=\int_0^1 t dF_2=\frac{1}{2},$$

which means that we have proved (2.1).

We consider three cases. If $a+\alpha \le 1$ then

$$F_1(t) \le F_2(t), \;\; t \in (-\infty, 1-\beta) \;\text{ and }\; F_1(t) \ge F_2(t) \;\; t \in (1-\beta, \infty),$$

together with (2.1) this means that our assertion follows from the Ohlin lemma.

Similarly, if $a+\beta \ge 1$ then

$$F_1(t) \le F_2(t), \;\; t \in (-\infty, 1-\alpha) \;\text{ and }\; F_1(t) \ge F_2(t) \;\; t \in (1-\alpha, \infty)$$

and, also in this case, we get the inequality (1.1) using the Ohlin lemma.

Now we pass to the most interesting case. If $a+\alpha>1$ and $a+\beta<1$ then functions F_1, F_2 cross three times and it is no longer possible to use the Ohlin lemma.

Assume first that $a+2\alpha=2$. Then we define functions G_1 and H_1 by the following formulas

$$G_1(t):=\begin{cases} 0 & \text{if} \quad t<1-\alpha \\ a & \text{if} \quad t\in[1-\alpha, a) \\ t & \text{if} \quad t \in [a,1) \\ 1 & \text{if} \quad t \ge 1 \end{cases}$$

and

$$H_1(t):=\begin{cases} 0 & \text{if} \quad t<0 \\ t & \text{if} \quad t\in[0, a) \\ a & \text{if} \quad t \in [a,1-\beta) \\ 1 & \text{if} \quad t \ge 1-\beta. \end{cases}$$

Then pairs G_1, F_2 and H_1, F_2 satisfy the assumptions of the Ohlin lemma thus we obtain for every convex function f

$$af(1-\alpha)+\int_a^1 f(t)dt=\int_{-\infty}^{\infty} f dG_1 \le \int_{-\infty}^{\infty} f dF_1=\int_0^1 f(t)dt$$

and

$$(1-a)f(1-\beta)+\int_0^a f(t)dt=\int_{-\infty}^{\infty} f dH_1 \le \int_{-\infty}^{\infty} f dF_1=\int_0^1 f(t)dt.$$

Using the above inequalities we arrive at

$$af(1-\alpha)+(1-a)f(1-\beta) \le \int_0^1 f(t)dt$$

which means that the proof in this case is finished.

Now we consider the case $a + 2\alpha < 2$, which means that

$$2a\alpha < 2a - a^2 \tag{2.2}$$

on the other hand from (2.1) we get $2a\alpha = 1 - 2(1-a)\beta$ thus, in view of (2.2), we get

$$1 - 2(1-a)\beta < 2a - a^2$$

which gives us

$$a + 2\beta > 1. \tag{2.3}$$

Now we shall show that there exists $t_1 \in (1-\beta, 1)$ such that

$$t_1(1-t_1) + \int_a^{t_1} tdt = (1-\beta)(1-a). \tag{2.4}$$

To this end, define a function $g : [1-\beta, 1] \to \mathbb{R}$ by $g(s) := s(1-s) + \int_a^s tdt$. Then g is continuous and, using (2.3), we obtain

$$g(1) = \int_a^1 tdt = \frac{1-a^2}{2} = (1-a)\frac{1+a}{2} > (1-a)(1-\beta).$$

On the other hand, since $a < 1 - \beta$

$$\begin{aligned} g(1-\beta) &= \beta(1-\beta) + \int_a^{1-\beta} tdt = \beta(1-\beta) + \frac{(1-\beta)^2 - a^2}{2} \\ &= \beta(1-\beta) + (1-\beta-a)\frac{1-\beta+a}{2} < \beta(1-\beta) + (1-\beta-a)(1-\beta) \\ &= (1-a)(1-\beta). \end{aligned}$$

We have proved that $g(1-\beta) < (1-\beta)(1-a) < g(1)$, which means that there exists $t_1 \in (1-\beta, 1)$ such that (2.4) is satisfied. Now, using this t_1, we shall define functions G_1, H_1 and H_2 by the following formulas:

$$G_1(t) := \begin{cases} 0 & \text{if} \quad t < 1-\alpha \\ a & \text{if} \quad t \in [1-\alpha, a) \\ t & \text{if} \quad t \in [a, t_1) \\ 1 & \text{if} \quad t \geq t_1, \end{cases}$$

$$H_1(t) := \begin{cases} 0 & \text{if} \quad t < 0 \\ t & \text{if} \quad t \in [0, a) \\ a & \text{if} \quad t \in [a, 1-\beta) \\ 1 & \text{if} \quad t \geq 1-\beta, \end{cases}$$

and

$$H_2(t) := \begin{cases} 0 & \text{if} \quad t < 0 \\ t & \text{if} \quad t \in [0, t_1) \\ 1 & \text{if} \quad t \geq t_1. \end{cases}$$

It is easy to see that pairs (G_1, F_2) and (H_1, H_2) satisfy the assumptions of the Ohlin lemma. Indeed, using (2.4), we may write

$$\int_{-\infty}^{\infty} t dG_1(t) = (1-\alpha)a + \int_a^{t_1} t dt + t_1(1-t_1)$$
$$= (1-\alpha)a + (1-\beta)(1-a) = \int_{-\infty}^{\infty} t dF_1(t) = \int_{-\infty}^{\infty} t dF_2(t)$$

and

$$\int_{-\infty}^{\infty} t dH_2(t) = \int_0^{t_1} t dt + t_1(1-t_1)$$
$$= \int_0^a t dt + \int_a^{t_1} t dt + t_1(1-t_1)$$
$$= \int_0^a t dt + (1-\beta)(1-a) = \int_{-\infty}^{\infty} t dH_1(t).$$

This means that for both pairs we may use the Ohlin lemma, getting for every convex f the following inequalities

$$\int_{-\infty}^{\infty} f dG_1 \leq \int_{-\infty}^{\infty} f dF_2 \tag{2.5}$$

and

$$\int_{-\infty}^{\infty} f dH_1 \leq \int_{-\infty}^{\infty} f dH_2. \tag{2.6}$$

Now from (2.5) and (2.6) we get

$$af(1-\alpha) + \int_a^{t_1} f(t) dt + f(t_1)(1-t_1) \leq \int_0^1 f(t) dt$$

and

$$\int_0^a f(t)dt + (1-a)f(1-\beta) \leq \int_0^{t_1} f(t)dt + f(t_1)(1-t_1).$$

From the above inequalities we get

$$af(1-\alpha) + (1-a)f(1-\beta) \leq \int_0^1 f(t)dt.$$

To finish the proof it remains to show that, in the case $a+2\alpha > 2$, inequality (1.1) is not satisfied for some convex function h. To this end it suffices to take

$$h(t) := \begin{cases} a-t & \text{if} \quad x < a \\ 0 & \text{if} \quad x \geq a. \end{cases}$$

We get

$$ah(1-\alpha) + (1-a)h(1-\beta) = ah(1-\alpha) = a(a-(1-\alpha)) > \frac{a^2}{2} = \int_0^1 h(t)dt$$

□

Remark 2.5. If we take $a = \frac{1}{2}$ in Theorem 2.4 then we get the inequality proved in [6] which may be found also in [5] [inequality (2.73)].

On the other hand, considering the particular case $\alpha = \frac{a}{2}$, we obtain an inequality proved in [4].

Now we turn our attention to the second of the Hermite–Hadamard inequalities. It is easy to see that expressions of the form

$$af(\alpha x + (1-\alpha)y) + (1-a)f(\beta x + (1-\beta y)$$

do not majorize $\frac{1}{y-x}\int_x^y f(t)dt$ for all convex f (if $\alpha, \beta \in (0,1)$). Therefore we shall consider expressions of the form

$$af(x) + bf(\alpha x + (1-\alpha)y) + cf(y),$$

where $a, b, c, \alpha \in (0,1), a+b+c = 1$. One of the examples is the inequality

$$\frac{1}{y-x}\int_x^y f(t)dt \leq \frac{1}{4}f(x) + \frac{1}{2}f\left(\frac{x+y}{2}\right) + \frac{1}{4}f(y) \tag{2.7}$$

(satisfied for all continuous and convex f) which was proved in [2], see also [3].

Therefore now we formulate the following result.

Theorem 2.6. *Numbers* $a, b, c, \alpha \in (0,1)$, *such that* $a+b+c=1$ *satisfy the inequality*

$$af(x) + bf(\alpha x + (1-\alpha)y) + cf(y) \geq \frac{1}{y-x}\int_x^y f(t)dt \tag{2.8}$$

for all $x, y \in \mathbb{R}$ *and all continuous and convex functions* $f : [x,y] \to \mathbb{R}$ *if and only if*

$$b(1-\alpha) + c = \frac{1}{2} \tag{2.9}$$

and one of the following conditions holds true:

(i) $a + \alpha \geq 1$,
(ii) $a + b + \alpha \leq 1$,
(iii) $a + \alpha < 1$, $a + b + \alpha > 1$ *and* $2a + \alpha \geq 1$.

Proof. So as in the proof of Theorem 2.4 we assume that $x = 0, y = 1$. Let X_1 be the random variable such that

$$\mu_{X_1} = a\delta_0 + b\delta_{1-\alpha} + c\delta_1$$

and let F_1 be its distribution function. Further let X_2 be such that

$$\mu_{X_2}(A) = l_1(A \cap [0,1])$$

and let F_2 be its distribution function. Equality (2.9) is an easy consequence of Lemma 2.1. The proof in cases (i) and (ii) is very similar to the respective part of the proof of Theorem 2.4. Thus we assume that (iii) is satisfied. Moreover we assume that the third inequality occurring in (iii) is strict (the equality case is similar to that of Theorem 1). We shall show that there exists $t_1 \in (a+b, 1)$ such that

$$t_1(t_1 - a - b) + \int_{t_1}^1 tdt = (1-\alpha)(1-\alpha-a) + c - \int_0^{1-\alpha} tdt. \tag{2.10}$$

To this end we define

$$g(s) := s(s-a-b) + \int_s^1 tdt,\ s \in [a+b, 1].$$

Then, since $2a + \alpha > 1$,

$$g(1) = 1 - a - b = c > c + (1-\alpha)(1-\alpha-a) - \int_0^{1-\alpha} tdt.$$

On the other hand, we have

$$a^2 + (1-\alpha+a)(1-\alpha-a) = (2-2\alpha-a)a + (1-\alpha-a)^2 \tag{2.11}$$

further from (2.9) we obtain

$$a^2 + (a+b-1+\alpha)^2 = (1-\alpha-a)^2 + c^2. \tag{2.12}$$

From (2.11) and (2.12) we get

$$(2-2\alpha-a)a - c^2 = (1-\alpha-a)(1-\alpha+a) - (a+b-1+\alpha)^2,$$

which means that, in particular,

$$(2-2\alpha-a)a - c^2 < (1-\alpha-a)(1-\alpha+a)$$

and, consequently,

$$(2-2\alpha-a)a < (1-\alpha-a)(1-\alpha+a) + c^2. \tag{2.13}$$

On the other hand we have

$$\int_0^{1-\alpha} t dt = \frac{1}{2}\big((2-2\alpha-a)a + (1-\alpha-a)^2\big)$$

and

$$\int_{a+b}^{1} t dt = c(a+b) + \frac{c^2}{2}.$$

The above two equalities together with (2.13) give us

$$\begin{aligned}\int_0^{1-\alpha} t dt + \int_{a+b}^{1} t dt &< \frac{1}{2}\big((1-\alpha-a)(1-\alpha+a) + (1-\alpha-a)^2 \\ &\quad + c^2\big) + c(a+b) + \frac{c^2}{2} = (1-\alpha)(1-\alpha-a) + c^2 + c(a+b) \\ &= (1-\alpha)(1-\alpha-a) + c \end{aligned} \tag{2.14}$$

i.e.

$$g(a+b) = \int_{a+b}^{1} t dt < (1-\alpha)(1-\alpha-a) + c - \int_0^{1-\alpha} t dt.$$

Thus the existence of t_1 such that (2.10) is satisfied has been proved.

Using this t_1, we define functions:

$$G_1(t) := \begin{cases} 0 & \text{if} \quad t < 0 \\ a & \text{if} \quad t \in [0, 1-\alpha) \\ t & \text{if} \quad t \in [1-\alpha, a+b) \\ a+b & \text{if} \quad t \in [a+b, 1) \\ 1 & \text{if} \quad t \geq 1, \end{cases}$$

$$G_2(t) := \begin{cases} 0 & \text{if} \quad t < 0 \\ t & \text{if} \quad t \in [0, a+b) \\ a+b & \text{if} \quad t \in [a+b, t_1) \\ t & \text{if} \quad t \in [t_1, 1) \\ 1 & \text{if} \quad t \geq 1, \end{cases}$$

and

$$H_1(t) := \begin{cases} 0 & \text{if} \quad t < 0 \\ t & \text{if} \quad t \in [0, 1-\alpha) \\ a+b & \text{if} \quad t \in [1-\alpha, t_1) \\ t & \text{if} \quad t \in [t_1, 1) \\ 1 & \text{if} \quad t \geq 1. \end{cases}$$

Then pairs (G_2, G_1) and (F_2, H_1) satisfy the assumptions of the Ohlin lemma. Indeed, we have

$$\int_{-\infty}^{\infty} t dG_1(t) = (1-\alpha)(1-\alpha-a) + \int_{1-\alpha}^{a+b} t dt + c$$

and, in view of (2.10)

$$\int_{-\infty}^{\infty} t dG_2(t) = \int_{0}^{a+b} t dt + t_1(t_1 - a - b) + \int_{t_1}^{1} t dt = \int_{-\infty}^{\infty} t dG_1(t).$$

Using (2.10) once more, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} t dH_1(t) &= \int_{0}^{1-\alpha} t dt + (1-\alpha)(a+b-(1-\alpha)) + t_1(t_1-a-b) + \int_{t_1}^{1} t dt \\ &= \int_{0}^{1-\alpha} t dt + (1-\alpha)(a+b-(1-\alpha)) + (1-\alpha)(1-\alpha-a) + c \\ &\quad - \int_{0}^{1-\alpha} t dt = (1-\alpha)b + c = \frac{1}{2} = \int_{-\infty}^{\infty} t dF_2(t). \end{aligned}$$

This means that we may use the Ohlin lemma, getting for every convex function $f : [0,1] \to \mathbb{R}$ the following inequalities:

$$\int_0^{a+b} f(t)dt + f(t_1)(t_1 - a - b) + \int_{t_1}^{1} f(t)dt = \int_{-\infty}^{\infty} f dG_2$$
$$\leq \int_{-\infty}^{\infty} f dG_1 = af(0) + f(1-\alpha)(1-\alpha-a) + \int_{1-\alpha}^{a+b} f(t)dt + cf(1)$$

and

$$\int_0^1 f(t)dt = \int_{-\infty}^{\infty} f dF_2 \leq \int_{-\infty}^{\infty} f dH_1$$
$$= \int_0^{1-\alpha} f(t)dt + f(1-\alpha)(a+b-(1-\alpha)) + f(t_1)(t_1-a-b) + \int_{t_1}^{1} f(t)dt.$$

The above inequalities give us the inequality

$$af(0) + bf(1-\alpha) + cf(1) \geq \int_0^1 f(t)dt,$$

as claimed.

It is easy to show that in the case $2a + \alpha < 1$ there exists a convex function f such that inequality (2.8) is not satisfied. This observation finishes the proof. □

Note that the original Hermite–Hadamard inequality consists of two parts. We treated these cases separately. However it is possible to formulate a result containing both inequalities.

Corollary 2.7. *If $a, \alpha, \beta \in (0,1)$ satisfy (2.1) and one of the conditions $(i), (ii), (iii)$ of Theorem 2.4 then the inequality*

$$af(\alpha x + (1-\alpha y) + (1-a)f(\beta x + (1-\beta)y) \leq \frac{1}{y-x}\int_x^y f(t)dt \leq$$
$$(1-\alpha)f(x) + (\alpha-\beta)f(ax + (1-a)y) + \beta f(y)$$

is satisfied for all continuous and convex functions $f : \mathbb{R} \to \mathbb{R}$.

To prove this corollary it suffices to check that the constants occurring at the right-hand side satisfy the assumptions of Theorem 2.6.

Remark 2.8. As it is known from the paper [1], if a continuous function satisfies inequalities of the type which we have considered then such a function must be convex.

Therefore inequalities obtained in this paper characterize convex functions (in the class of continuous functions).

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Tomasz Szostok
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: tszostok@math.us.edu.pl

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