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Aequationes Mathematicae

Width of spherical convex bodies

Marek Lassak

Abstract. For every hemisphere K supporting a convex body C on the sphere S^d we define the width of C determined by K. We show that it is a continuous function of the position of K. We prove that the diameter of every convex body $C \subset S^d$ equals the maximum of the widths of C provided the diameter of C is at most $\frac{\pi}{2}$. In a natural way, we define spherical bodies of constant width. We also consider the thickness $\Delta(C)$ of C, i.e., the minimum width of C. A convex body $R \subset S^d$ is said to be reduced if $\Delta(Z) < \Delta(R)$ for every convex body Z properly contained in R. For instance, bodies of constant width on S^d and regular spherical odd-gons of thickness at most $\frac{\pi}{2}$ on S^2 are reduced. We prove that every reduced smooth spherical convex body is of constant width.

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1. Introduction

Let S^d be the unit sphere in the $(d+1)$-dimensional Euclidean space E^{d+1}, where $d \geq 2$. By a *great circle* of S^d we mean the intersection of S^d with any two-dimensional subspace of E^{d+1}. The common part of the sphere S^d with any hyper-subspace of E^{d+1} is called a $(d-1)$*-dimensional great sphere* of S^d. In particular, for S^2 the $(d-1)$-dimensional great spheres are great circles. By a pair of *antipodes* of S^d we mean any pair of points of intersection of S^d with a straight line through the origin of E^{d+1}. Observe that if two different points are not antipodes, there is exactly one great circle containing them.

If two different points $a, b \in S^d$ are not antipodes, by the *arc ab* connecting them we mean the shorter part of the great circle containing a and b. By the *spherical distance* $|ab|$, or shortly *distance*, of these points we understand the length of the arc connecting them. Moreover, we put π, if the points are antipodes and 0 if the points coincide.

By a *spherical ball of radius* $\rho \in (0, \frac{\pi}{2}]$, or shorter *a ball*, we mean the set of points of S^d having distance at most ρ from a fixed point, called the *center* of this ball. An *open ball* is the set of points of S^d having distance smaller than ρ from a point. Balls on S^2 are called *disks*. Spherical balls of radius $\frac{\pi}{2}$ are called *hemispheres*. In other words, by a *hemisphere* of S^d we mean the common part of S^d with any closed half-space of E^{d+1}. We denote by $H(m)$ the hemisphere whose center is m. Two hemispheres whose centers are antipodes are called *opposite hemispheres*. By an *open hemisphere* we mean the set of points having distance less than $\frac{\pi}{2}$ from a fixed point.

By a *spherical* $(d-1)$*-dimensional ball of radius* $\rho \in (0, \frac{\pi}{2}]$ we mean the set of points of a $(d-1)$-dimensional great sphere of S^d which are at distance at most ρ, from a fixed point, called the *center* of this ball. The $(d-1)$-dimensional balls of radius $\frac{\pi}{2}$ are called $(d-1)$*-dimensional hemispheres*. If $d = 2$, we call them *semicircles*.

We say that a set $C \subset S^d$ is *convex* if it does not contain any pair of antipodes and if together with every two points it contains the whole arc connecting them. By a *convex body* on S^d we mean a closed convex set with non-empty interior. Observe that a set $C \subset S^d$ is a convex body if and only if it is contained in an open hemisphere and is an intersection of hemispheres. For a short survey of definitions of convexity on S^d we refer to Sect. 9.1 of [1]. The literature concerning this subject is very large. For instance see [2,3] and [4].

Clearly, the intersection of every family of convex sets is also convex. Thus for every set $Q \subset S^d$ contained in an open hemisphere of S^d there exists the unique smallest convex set containing Q. It is called *the convex hull of* Q and it is denoted by $\mathrm{conv}(Q)$.

Lemma 1. *If* $Q \subset S^d$ *is a closed subset of an open hemisphere, then* $\mathrm{conv}(Q)$ *is also closed.*

This lemma follows by applying an analogous theorem for compact sets in E^{d+1}.

If a $(d-1)$-dimensional great sphere G of S^d has a common point t with a convex body $C \subset S^d$ and if its intersection with the interior of C is empty, we say that G is a *supporting* $(d-1)$*-dimensional great sphere of* C *passing through* t. We also say that G *supports* C at t. If H is the hemisphere bounded by G and containing C, we say that H *supports* C at t. If at every boundary point of a convex body $C \subset S^d$ exactly one hemisphere supports C, we say that the body is *smooth*.

By the well known fact that a set $C \subset S^d$ is convex if and only if the cone generated by it in E^{d+1} is convex and from the classic separation theorem in Euclidean space we obtain the following analogous fact for S^d.

Lemma 2. *Every two convex bodies on the sphere S^d with empty intersection of their interiors are subsets of some two opposite hemispheres.*

Let $P \subset S^d$ be a convex body. Let $Q \subset S^d$ be a convex body or a hemisphere. We say that P *touches* Q *from outside* if $P \cap Q \neq \emptyset$ and $\text{int}(P) \cap \text{int}(Q) = \emptyset$. We say that P *touches* Q *from inside* if $P \subset Q$ and $\text{bd}(P) \cap \text{bd}(Q) \neq \emptyset$. In both cases, points of $\text{bd}(P) \cap \text{bd}(Q)$ are called *points of touching*.

The convex hull V of $k \geq 3$ points on S^2 such that none of them belongs to the convex hull of the remaining points is called a *spherical convex k-gon*. The mentioned points are called the *vertices* of V. We write $V = v_1 v_2 \dots v_k$ provided $v_1, v_2, \dots, v_k$ are successive vertices of V when we go around V on the boundary of V. In particular, when we take $k \geq 3$ successive points in a spherical circle of radius less than $\frac{\pi}{2}$ on S^2 with equal distances of every two successive points, we obtain a *regular spherical k-gon*.

2. Lunes

If hemispheres G and H of S^d are different and not opposite, then $L = G \cap H$ is called *a lune* of S^d. This notion is considered in many books and papers, for lunes on S^2 see e.g. [5], p. 18. The $(d-1)$-dimensional hemispheres bounding the lune L and contained in G and H, respectively, are denoted by G/H and H/G.

Claim 1. *Every pair of different points a, b which are not antipodes determines exactly one lune L such that a, b are the centers of the $(d-1)$-dimensional hemispheres bounding L.*

Proof. If $|ab| \leq \frac{\pi}{2}$, then on the great circle containing the arc ab we find points p and q such that $a \in pb$, $b \in aq$, $|pb| = |qa| = \frac{\pi}{2}$. If $|ab| > \frac{\pi}{2}$, then on the great circle containing the arc ab we find points p and q such that $q \in pb$, $p \in aq$, $|pb| = |qa| = \frac{\pi}{2}$. The lune $L = H(p) \cap H(q)$ is the one that we are looking for. □

Since every lune L determines exactly one pair of centers of the $(d-1)$-dimensional hemispheres bounding L, from Claim 1 we see that there is a one-to-one correspondence between lunes and pairs of points (different and not antipodes) of S^d.

Clearly, $(G/H) \cup (H/G)$ is the boundary of the lune $G \cap H$. In particular, every lune of S^2 is bounded by two different semicircles. Denote by $c_{G/H}, c_{H/G}$ the centers of G/H and H/G, respectively. Points of $(G/H) \cap (H/G)$ are called *corners* of the lune $G \cap H$. Of course, $r \in (G/H) \cup (H/G)$ is a corner of $G \cap H$ if and only if r is equidistant from $c_{G/H}$ and $c_{H/G}$. In particular, every lune on S^2 has exactly two corners. They are antipodes.

By the *thickness* $\Delta(L)$ *of a lune* $L = G \cap H \subset S^d$ we mean the spherical distance of the centers of the $(d-1)$-dimensional hemispheres G/H and H/G bounding L. Observe that it equals each of the non-oriented angles $\angle c_{G/H} r c_{H/G}$, where r is any corner of L.

We omit the simple proof of the following lemma.

Lemma 3. *Let H and G be different and not opposite hemispheres. Consider the lune $L = H \cap G$. Let $x \neq c_{G/H}$ belong to G/H. If $\Delta(L) < \frac{\pi}{2}$, we have $|xc_{H/G}| > |c_{G/H}c_{H/G}|$. If $\Delta(L) = \frac{\pi}{2}$, we have $|xc_{H/G}| = |c_{G/H}c_{H/G}|$. If $\Delta(L) > \frac{\pi}{2}$, we have $|xc_{H/G}| < |c_{G/H}c_{H/G}|$.*

For a convex body $C \subset S^d$ they matter lunes containing it, and in particular such lunes for which both $(d-1)$-dimensional hemispheres bounding the lune have non-empty intersection with C. We say that *a lune passes through a boundary point p of a convex body $C \subset S^d$* if the lune contains C and if the boundary of the lune contains p. If the centers of both $(d-1)$-dimensional hemispheres bounding a lune belong to C, then we call such a lune an *orthogonally supporting lune of C*.

By applying the classical Blaschke selection theorem (e.g., see [6], p. 64) in E^{d+1} we easily obtain its spherical analogue and also the following lemma.

Lemma 4. *From every sequence of lunes on S^d we may select a subsequence of lunes convergent to a lune.*

3. Width and thickness of a spherical convex body

For every hemisphere K supporting a convex body $C \subset S^d$ we are looking for hemispheres K^* supporting C such that the lunes $K \cap K^*$ are of the minimum thickness, i.e., which are the "narrowest" lunes of the form $K \cap K'$ over all hemispheres K' supporting C. By compactness arguments we immediately see that at least one such hemisphere K^* exists, and thus at least one corresponding lune $K \cap K^*$ exists. Denote by $\mathrm{width}_K(C)$ its thickness and we call it *the width of C determined by K*. This notion of width of $C \subset S^d$ is an analogue of the notion of width of a convex body of E^d. How to find the width of C determined by a given hemisphere K? Theorem 1 presented below and its proof present a procedure for establishing $\mathrm{width}_K(C)$.

First let us prove a lemma needed for the proof of Theorem 1.

Lemma 5. *Let G and H be different and not opposite hemispheres, and let g denote the center of G. If $g \notin \mathrm{bd}(H)$, then by B denote the ball with center g which touches H (from inside or outside) and by t the point of touching. If $g \in \mathrm{bd}(H)$, we put $t = g$. We claim that t is always at the center of the $(d-1)$-dimensional hemisphere H/G.*

Proof. If $g \notin \mathrm{bd}(H)$ consider any two corners r_1 and r_2 of the lune $G \cap H$. Look at the triangles gtr_1 and gtr_2. Below we explain that they have three equal elements. They have the length of the common side gt. Since g is the center of G, we have $|gr_1| = \frac{\pi}{2} = |gr_2|$. By the orthogonality of gt and H it follows that $\angle gtr_1 = \frac{\pi}{2} = \angle gtr_2$. Since these three elements are equal, we have $|tr_1| = |tr_2|$ (by the way, they are both equal to $\frac{\pi}{2}$ since $|r_1r_2| = \pi$). When $g \in \mathrm{bd}(H)$, we have $|gr_1| = |gr_2|$; still r_1, r_2 belong to the boundary of H. Thus t is always the center of the hemisphere H/G. □

Theorem 1. *Let K be a hemisphere which supports a convex body $C \subset S^d$. Denote by k the center of K.*

I. *If $k \notin C$, then there exists a unique hemisphere K^* supporting C such that the lune $L = K \cap K^*$ contains C and has thickness* $\mathrm{width}_K(C)$. *This hemisphere supports C at the point t at which the largest ball B with center k touches C from outside. We have $\Delta(K \cap K^*) = \frac{\pi}{2} - \rho_B$, where ρ_B denotes the radius of B.*

II. *If $k \in \mathrm{bd}(C)$, then there exists at least one hemisphere K^* supporting C such that $L = K \cap K^*$ is a lune containing C which has thickness* $\mathrm{width}_K(C)$. *This hemisphere supports C at $t = k$. We have $\Delta(K \cap K^*) = \frac{\pi}{2}$.*

III. *If $k \in \mathrm{int}(C)$, then there exists at least one hemisphere K^* supporting C such that $L = K \cap K^*$ is a lune containing C which has thickness* $\mathrm{width}_K(C)$. *Every such K^* supports C at exactly one point $t \in \mathrm{bd}(C) \cap B$, where B denotes the largest ball with center k contained in C, and for every such t this hemisphere K^*, denoted K^*_t, is unique. For every t we have $\Delta(K \cap K^*_t) = \frac{\pi}{2} + \rho_B$, where ρ_B denotes the radius of B.*

Proof. Figures 1 and 2 illustrate this theorem and its proof. They show the orthogonal look to the hemisphere K from outside.

Part I.

Since C is a convex body and B is a ball, we see that B touches C from outside and the point of touching is unique. Denote it by t (see Fig. 1). By Lemma 2, the bodies C and B are in some two opposite hemispheres. What is more, since B is a ball touching C from outside, this pair of hemispheres is unique. Denote by K^*_t the one which contains C. We intend to show that K^*_t is nothing else but the promised K^*.

Denote by k^* the center of K^*_t. Since k is also the center of B and since B and K^*_t touch from outside at t, we have $t \in kk^*$. From Lemma 5 we see that t is the center of the $(d-1)$-dimensional hemisphere K^*_t/K. Analogously, from this lemma we conclude that the common point u of kk^* and the boundary of K is the center of K/K^*_t. Since t and u are centers of the $(d-1)$-dimensional hemispheres bounding the lune $K \cap K^*_t$, we have $|tu| = \Delta(K \cap K^*_t)$. This and $|kt| + |tu| = |ku| = \frac{\pi}{2}$ imply $\Delta(K \cap K^*_t) = \frac{\pi}{2} - \rho_B$.

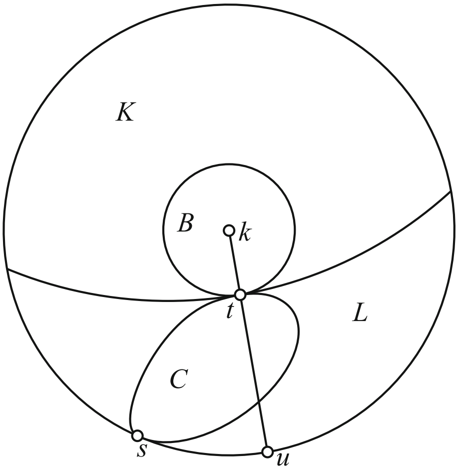

Figure 1. Illustration to Part I of Theorem 1 and to Theorem 3

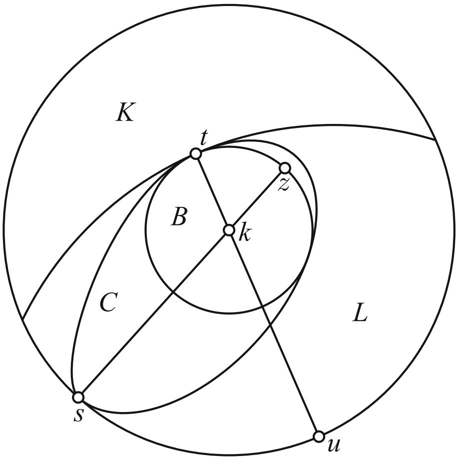

Figure 2. Illustration to Part III of Theorem 1 and to Proposition 1

If we assume that there exists a hemisphere $M \supset C$ with $\Delta(K \cap M) < \frac{\pi}{2} - \rho_B$, then the lune $K \cap M$ must be disjoint with B, and hence it does not contain C. A contradiction. Thus $K \cap K_t^*$ is a narrowest lune of the form $K \cap N$ containing C. It is the unique lune of this form by the uniqueness of t and K_t^* explained at the beginning of the proof of Part I.

Part II.

Clearly, there is at least one hemisphere K^* supporting C at k. Of course, $\Delta(K \cap K^*) = \frac{\pi}{2}$. By Lemma 5 we see that k is the center of K^*/K.

Part III.

Take the largest ball $B \subset C$ with center k. Clearly, there is at least one boundary point t of C which is also a boundary point of B (see Fig. 2). We find a hemisphere K_t^* which supports C at t. Of course, it also supports B and thus, for given t, it is unique.

For every t there is a unique point $u \in K/K_t^*$ such that $k \in tu$. This, $|ku| = \frac{\pi}{2}$ and $|kt| = \rho_B$ imply $|tu| = \frac{\pi}{2} + \rho_B$. Hence the facts, resulting from Lemma 5, that t is the center of K_t^*/K and that u is the center of K/K_t^* give $\Delta(K \cap K_t^*) = \frac{\pi}{2} + \rho_B$.

If we assume that there exists a hemisphere $M \supset C$ such that the lune $K \cap M$ is narrower than $\frac{\pi}{2} + \rho_B$, then this lune does not contain B, and hence it does not contain C either. A contradiction. Thus the narrowest lunes of the form $K \cap N$ containing C are of the form $K \cap K_t^*$. □

Let us point out that in Part I, so if the center k of K does not belong to C, the lune $K \cap K^*$ is unique. In Part II this narrowest lune $K \cap K^*$ containing C is sometimes unique and sometimes not. This depends on the point $k = t$ of C which belongs to the boundary of B. In Part III for any given point t of touching C by B from inside (we may have one, or finitely many, or infinitely many such points t), the lune $K \cap K_t^*$ is unique.

For instance, if $C \subset S^2$ is a regular spherical triangle of sides $\frac{\pi}{2}$ and the circle bounding a hemisphere K contains a side of this triangle, then $K \cap K^*$ is not unique. Namely, as K^* we may take any hemisphere containing C, whose boundary contains this vertex of C which does not belong to K. The thickness of every such lune $K \cap K^*$ equals $\frac{\pi}{2}$. If C is a regular spherical triangle of sides over $\frac{\pi}{2}$ and the boundary of K contains a side of this triangle, then $K \cap K^*$ is not unique either. This time the boundary of K^* contains a side of C different from the side which is in K. So we have exactly two positions of K^*.

Here are two corollaries from Theorem 1 (for the second we also apply Lemma 5).

Corollary 1. *If $k \notin C$, then* $\text{width}_K(C) = \frac{\pi}{2} - \rho_B$. *If $k \in \text{bd}(C)$, we have* $\text{width}_K(C) = \frac{\pi}{2}$. *If $k \in \text{int}(C)$, then* $\text{width}_K(C) = \frac{\pi}{2} + \rho_B$.

Corollary 2. *The point t of support in Theorem 1 is the center of the $(d-1)$-dimensional hemisphere K^*/K.*

We define the *thickness* $\Delta(C)$ *of a convex body* $C \subset S^d$ as follows:

$$\Delta(C) = \inf\{\text{width}_K(C); K \text{ is a supporting hemisphere of } C\}.$$

Compactness arguments show that the infimum is realized. As a consequence, $\Delta(C) = \min\{\text{width}_K(C); K \text{ is a supporting hemisphere of } C\}$. By the definitions of width and thickness we conclude that *the thickness of every convex body $C \subset S^d$ is equal to the minimum thickness of a lune containing C.*

At this moment observe that our definition of $\mathrm{width}_K(C)$ has an advantage, when applied to find the thickness of a convex body $C \subset S^d$. Namely, it is sufficient to find the minimum of the values of $\mathrm{width}_K(C)$ over all hemispheres K supporting C. Theorem 1 helps to establish every $\mathrm{width}_K(C)$.

Example 1. *Applying Theorem 1 we easily find the thickness of any regular triangle T_α of angles α. Formulas of spherical trigonometry imply that $\Delta(T_\alpha) = \arccos\frac{\cos\alpha}{\sin\alpha/2}$ for $\alpha < \frac{\pi}{2}$. If $\alpha \geq \frac{\pi}{2}$ (but, of course, $\alpha < \frac{2}{3}\pi$), then $\Delta(T_\alpha) = \alpha$. In both cases $\Delta(T_\alpha)$ is realized for $\mathrm{width}_K(T_\alpha)$, where K is a hemisphere whose bounding semicircle contains a side of T_α. In the first case T_α is symmetric with respect to the arc A connecting the centers of K/K^* and K^*/K, while in the second T_α is symmetric with respect to the arc passing through the middle of A and having endpoints at the corners of the lune $K \cap K^*$. For $\alpha = \frac{\pi}{2}$ there are infinitely many positions in K^*.*

Theorem 2. *As the position of the $(d-1)$-dimensional supporting hemisphere of a convex body $C \subset S^d$ changes, the width of C determined by this hemisphere changes continuously.*

Proof. We keep the notation of Theorem 1. Of course, the positions of k and thus of B depend continuously on K. Hence $\frac{\pi}{2} - \rho_B$ and $\frac{\pi}{2} + \rho_B$ change continously. This and Corollary 1 imply the thesis of our theorem. It does not matter here that for a fixed K sometimes the lunes $K \cap K^*$ are not unique; still they are all of equal. □

Claim 2. *Consider a convex body $C \subset S^d$ and any lune L of thickness $\Delta(C)$ containing C. Both centers of the $(d-1)$-dimensional hemispheres bounding L belong to C.*

This claim results immediately from Corollary 2 applied twice: for each of the two $(d-1)$-dimensional hemispheres bounding L.

If for every hemisphere supporting a convex body $C \subset S^d$ the width of C determined by K is the same, we say that C is a *body of constant width.* In particular, spherical balls of radius smaller than $\frac{\pi}{2}$ are bodies of constant width.

Also every spherical Reuleaux odd-gon is a convex body of constant width. Recall this notion. Take a regular spherical k-gon $v_1 v_2 \dots v_k \subset S^2$, where $k \geq 3$ is odd. Clearly, all the distances $|v_i v_{i+\frac{k-1}{2}}|$ and $|v_i v_{i+\frac{k+1}{2}}|$ for $i = 1, \dots, k$ are equal (the indices are taken modulo k). Denote them by δ. Assume that $\delta \leq \frac{\pi}{2}$. Let B_i, where $i = 1, \dots, k$, be the disk with center v_i and radius δ. The set $B_1 \cap \dots \cap B_k$ is just a *spherical Reuleaux k-gon.*

By the definition of width and by Claim 2, if $C \subset S^d$ is a body of constant width, then every supporting hemisphere G of C determines a supporting hemisphere H of C for which $G \cap H$ is a lune such that the centers of G/H and H/G belong to the boundary of C. Is the opposite true? More precisely, is a convex body $C \subset S^d$ of constant width provided every supporting hemisphere G of C determines at least one hemisphere H supporting C such that $G \cap H$ is a lune with the centers of G/H and H/G in the boundary of C?

4. Diameter

By the *diameter* $\mathrm{diam}(C)$ of a set $C \subset S^d$ we mean the supremum of the spherical distances between pairs of points of C. Clearly, if C is closed, the diameter of C is realized for at least one pair of points of C.

Claim 3. *Let* $\mathrm{diam}(C) \leq \frac{\pi}{2}$ *for a convex body* $C \subset S^d$ *and assume that* $\mathrm{diam}(C) = |ab|$ *for some points* $a, b \in C$. *Denote by* L *the lune such that* a *and* b *are the centers of the* $(d-1)$*-dimensional hemispheres bounding* L. *We have* $C \subset L$.

Proof. We apply Claim 1. Let us keep the notation of its proof. Since $|ab| = \mathrm{diam}(C)$, for every $x \in C$ we have $|ax| \leq |ab|$. Moreover $\mathrm{diam}(C) \leq \frac{\pi}{2}$ implies $a \in pb$. Hence $|px| \leq |pa| + |ax| \leq |pa| + |ab| = |pb| = \frac{\pi}{2}$. Thus $C \subset H(p)$. Similarly, $C \subset H(q)$. Consequently, $C \subset H(p) \cap H(q) = L$. □

Remark 1. In general, Claim 3 does not hold true without the assumption that $\mathrm{diam}(C) \leq \frac{\pi}{2}$. A simple counterexample is the triangle $T = abc$ with $|ab| = \frac{2}{3}\pi \approx 2.0944$, $|bc| = \frac{\pi}{6} \approx 0.5236$ and $\angle abc = 95°$. From the Al Battani formulas, also called law of cosines for sides, (see, e.g., [5], p. 45), we get $|ac| \approx 2.0609$. Consequently, $|ab| = \frac{2}{3}\pi$ is the diameter of T. Since $\angle abc = 95°$, the lune with centers a and b of the semicircles bounding it does not contain c. Still its thickness is $\frac{2}{3}\pi$. Thus this lune does not contain T.

Compactness arguments lead to the conclusion that for every convex body $C \subset S^d$ the supremum of $\mathrm{width}_H(C)$ over all hemispheres H supporting C is realized for a supporting hemisphere of C, that is, we may take here the maximum instead of supremum.

The following theorem is an analog of the classic theorem for Euclidean space.

Theorem 3. *Let* $\mathrm{diam}(C) \leq \frac{\pi}{2}$ *for a convex body* $C \subset S^d$. *We have*

$$\max\{\mathrm{width}_K(C); K \text{ is a supporting hemisphere of } C\} = \mathrm{diam}(C).$$

Proof. Let K be an arbitrary hemisphere supporting C and let $s \in C$ be a point of support by K (see Fig. 1). Take k, t , u and K^* like in Parts I and II of Theorem 1. By Lemma 3 we have $|st| \geq |ut|$. Hence $\mathrm{diam}(C) \geq |st| \geq |ut| = \mathrm{width}_K(C)$. This and the assumption that K is an arbitrary hemisphere supporting C imply that $\operatorname{diam}(C)$ is at least the maximum of $\operatorname{width}_K(C)$ over all supporting hemispheres K of C.

Let $a, b \in C$ be such that $|ab| = \operatorname{diam}(C)$. Take the lune L from Claim 3, i.e., $H(p) \cap H(q)$ like in its proof. Thus $\operatorname{diam}(C)$ equals the thickness of L, i.e., $\operatorname{width}_{H(p)}(C)$. Hence $\operatorname{diam}(C)$ is at most the maximum of $\operatorname{width}_K(C)$ over all supporting hemispheres K of C. □

The following example shows that Theorem 3 requires the assumption $\operatorname{diam}(C) \leq \frac{\pi}{2}$.

Example 2. Let T be an isosceles triangle with base of length λ close to 0 and the height perpendicular to it of length $\mu \in (\frac{\pi}{2}, \pi)$. Denote by w the center of the base and by v the opposite vertex of T. Lemma 3 implies that wv is the diametrical segment of T. Take the hemisphere K supporting T at w. Denote by k the center of K. Clearly, $k \in wv$, so k is in the interior of T. Let ρ be the radius of the largest disk B with center k contained in T. The radius ρ of B is arbitrarily close to 0, as λ is sufficiently small. Applying Part III of Theorem 1 we conclude that the width of T determined by K is $\frac{\pi}{2} + \rho$. Hence it may be arbitrarily close to $\frac{\pi}{2}$, as λ is sufficiently small. On the other hand, the diameter $|wv|$ of T may be arbitrarily close to π, as μ is sufficiently close to π.

Proposition 1. *Let* $\operatorname{diam}(C) > \frac{\pi}{2}$ *for a convex body* $C \subset S^d$*. We have*

$$\max\{\operatorname{width}_K(C); K \text{ is a supporting hemisphere of } C\} \leq \operatorname{diam}(C).$$

Proof. Let K be an arbitrary hemisphere supporting C and let $s \in C$ be a point of support by K (see Fig. 2). Take k, t and K^* like in Parts I–III of Theorem 1.

If $k \notin \operatorname{int}(C)$, so if we apply Parts I and II of Theorem 1, we repeat the consideration of the first paragraph of the proof of Theorem 3 which gives $\operatorname{width}_K(C) \leq \operatorname{diam}(C)$.

Assume that $k \in \operatorname{int}(C)$, so that we apply Part III of Theorem 1. Clearly, $|sk| = \frac{\pi}{2}$. Take the largest ball B with center k contained in C. Denote by ρ its radius. By Part III we have $\operatorname{width}_K(C) = \frac{\pi}{2} + \rho$. Provide the great circle through s and k. It intersects the boundary of B at two points. Denote by z this from these two points for which $k \in sz$. We have $|sz| = |sk| + |kz| = \frac{\pi}{2} + \rho$, which, by Part III, equals $\operatorname{width}_K(C)$. This and $|sz| \leq \operatorname{diam}(C)$ lead to the conclusion that $\operatorname{width}_K(C) \leq \operatorname{diam}(C)$.

Since K is an arbitrary hemisphere supporting C, we get the thesis. □

5. Reduced bodies

In analogy to the definition of reduced bodies in Euclidean space E^d introduced in [7] (see also [8–10] and [11]), we define reduced convex bodies on S^d. We say that a convex body $R \subset S^d$ is *reduced* if $\Delta(Z) < \Delta(R)$ for every convex body $Z \subset R$ different from R.

By our definition of bodies of constant width on S^d we see that they are reduced bodies. In particular, every Reuleaux polygon on S^2 is a reduced body.

It is easy to show that all regular odd-gons on S^2 of thickness at most $\frac{\pi}{2}$ are reduced bodies. The assumption that the thickness is at most $\frac{\pi}{2}$ matters here. For instance take the regular triangle T_α of angles $\alpha > \frac{\pi}{2}$ (see Example 1). Take the hemisphere K whose boundary contains a side of T_α and apply Part III of Theorem 1. The corresponding ball $B \subset T_\alpha$ touches T_α from inside at exactly two points t_1, t_2. Cutting off a part of T_α by the shorter arc of the boundary of B between t_1 and t_2 we obtain a convex body $Z \subset T_\alpha$. We have $\Delta(Z) = \Delta(T_\alpha)$, which implies that T_α is not reduced.

Dissect a disk on S^2 by two orthogonal great circles through its center. The four obtained parts are called *quarters of disks*. In particular, the triangle of sides and angles $\frac{\pi}{2}$ is a quarter of a disk. It is easy to see that every quarter of a disk is a reduced body and that the thickness of it is equal to the radius of the original disk. More generally, each of the 2^d parts of a spherical ball on S^d dissected by d pairwise orthogonal great $(d-1)$-dimensional spheres through the center of this ball is a reduced body of S^d. We call it $\frac{1}{2^d}$*-th part of a ball.* Clearly, its thickness is equal to the radius of the above ball.

We say that e is an *extreme* point of a convex body $C \subset S^d$ provided the set $C \setminus \{e\}$ is convex. From the analogue of the Krein–Milman theorem for convex cones (e.g., see [12]) its analogue for spherical convex bodies follows: every convex body $C \subset S^d$ is the convex hull of its extreme points. This and the fact that the common part of any closed convex body $C \subset S^d$ with any of its supporting $(d-1)$-dimensional great sphere is a closed convex set imply the following lemma.

Lemma 6. *The boundary of every supporting hemisphere of a convex body $C \subset S^d$ passes through an extreme point of C.*

Theorem 4. *Through every extreme point e of a reduced body $R \subset S^d$ a lune $L \supset R$ of thickness $\Delta(R)$ passes with e as the center of one of the two $(d-1)$-dimensional hemispheres bounding L.*

Proof. Let B_i be the open ball of radius $\Delta(R)/i$ centered at e and let $R_i = \operatorname{conv}(R \setminus B_i)$ for $i = 2, 3, \ldots$. By Lemma 1 every R_i is a convex body. Moreover, since e is an extreme point of R, R_i is a proper subset of R. So, since R is reduced, $\Delta(R_i) < \Delta(R)$. By the definition of thickness of a convex body, R_i is contained in a lune L_i of thickness $\Delta(R_i)$.

From Lemma 4 we conclude that there exists a subsequence of the sequence $L_2, L_3, \ldots$ converging to a lune L. Since $R_i \subset L_i$ for $i = 2, 3, \ldots$, we obtain that $R \subset L$. Since $\Delta(L_i) = \Delta(R_i) < \Delta(R)$ for every i, we get $\Delta(L) \leq \Delta(R)$. This and $R \subset L$ imply $\Delta(L) = \Delta(R)$.

Let m_i, m_i' be the centers of the $(d-1)$-dimensional hemispheres H_i, H_i' bounding L_i. We claim that at least one of these two centers, say m_i, belongs to the closure of $R \setminus R_i$. The reason is that in the opposite case, there would be a neighborhood N_i of m_i such that $N_i \cap R_i = N_i \cap R$, which would imply that H_i supports R at m_i. Moreover, H_i' supports R at m_i'. Hence $\Delta(R) = \Delta(L_i) = \Delta(R_i)$, in contradiction with $\Delta(R_i) < \Delta(R)$.

Since $m_i \in R \setminus R_i$ for $i = 2, 3, \ldots$, we see that the sequence of points $m_2, m_3, \ldots$ tends to e. Consequently, e is the center of a $(d-1)$-dimensional hemisphere bounding L. □

Remark 2. Besides the lune from Theorem 4, sometimes we have additional lunes $L' \supset R$ of thickness $\Delta(R)$ through e for which e is not in the middle of a $(d-1)$-dimensional hemisphere bounding L'. This happens, for instance, when R is a spherical regular triangle T_α with $\alpha \leq \frac{\pi}{2}$.

Theorem 4 leads to the following questions. Is it true that through every boundary point p of a reduced body $R \subset S^2$ a lune $L \supset R$ of thickness $\Delta(R)$ passes? A consequence would be that every reduced body $R \subset S^2$ is an intersection of lunes of thickness $\Delta(R)$. Is a stronger version of the preceding question true, namely, that there is always such a lune L with p at the center of one of the two $(d-1)$-dimensional hemispheres bounding L?

There are more questions on spherical reduced bodies. For instance, which properties of reduced bodies in E^d, and especially in E^2 (see [9] and [10]), may be reformulated and proved for reduced bodies on S^d? Are there reduced spherical polytopes on S^d, where $d \geq 3$, different from $\frac{1}{2^d}$-th part of a ball? Or at least a spherical simplex different from $\frac{1}{2^d}$ of S^d (comp. [11]).

By Theorem 4 we obtain the following spherical analog of a theorem from [8], see also Corollary 1 in [9] and [10].

Theorem 5. *Every smooth reduced body on S^d is of constant width.*

Proof. Let $R \subset S^d$ be a smooth reduced body. Take any supporting hemisphere K of R. By Lemma 6 the boundary of K contains an extreme point e of R. Since R is smooth, K is the unique supporting hemisphere of R at e. Moreover, from Theorem 4 we see that through e a lune $L \supset R$ of thickness $\Delta(R)$ passes. Thus $L = K \cap K^*$ and hence $\mathrm{width}_K(R) = \Delta(R)$. This and the arbitrariness of K imply the thesis of our theorem. □

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Marek Lassak
Institute of Mathematics and Physics
University of Technology and Life Sciences
al. prof. Kaliskiego 7
85-789 Bydgoszcz
Poland
e-mail: lassak@utp.edu.pl

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