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Solvability of Age-Structured Epidemiological Models with Intracohort Transmission

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Abstract. The standard version of the epidemiological model with continuous age structure (Iannelli, Mathematical Theory of Age-Structured Population Dynamics, 1995) consists of the linear McKendrick model for the evolution of a disease-free population, coupled with one of the classical (SIS, SIR, etc.) models for the spread of the disease. A natural functional space in which the linear McKendrick model is well posed is the space of integrable functions. However, in the so-called intracohort models, the disease term contains pointwise products of the unknown functions; that is, of the age-specific densities of susceptibles, infectives and other classes (if applicable) which render the standard semilinear perturbation technique of proving the well-posedness of the full model not applicable in that space. This is due to the fact that the product of two integrable functions need not be integrable. Therefore, most works on the well-posedness of such problems have been done under additional assumption that the disease-free population is stable and the equilibrium has been reached (Busenberg et al. SIAM J Math Anal 22(4):1065–1080, 1991, Prüß, J Math Biol 11:65–84, 1981). This allowed for showing, after some algebraic manipulations, that the order interval $[0, 1]$ in the space of integrable functions was invariant under the action of the linear McKendrick semigroup and thus the usual iteration technique could be applied in this interval, yielding a bounded solution. An additional advantage of adopting the stability assumption was that it eliminated from the model the death rate which, in all realistic cases, is unbounded and creates serious technical difficulties. The aim of this note is to show that a careful modification of the standard Picard iteration procedure allows for proving the same result without the stability assumption. Since in such a case we are not able to suppress the unbounded death rate, we will briefly present the necessary linear results in a way which simplifies and unifies some classical results existing in the literature (Inaba, Math Popul Stud 1:49–77, 1988, Prüß, J Math Biol 11:65–84, 1981, Webb, Theory of Nonlinear Age Dependent Population Dynamics, 1985).

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1. Introduction

In this note, we consider the age-structured SIS model with intracohort infection, as introduced in [9],

$$(\partial_t + \partial_a)\, s(a,t) = -\mu(a)s(a,t) - K_0(a)i(a,t)s(a,t) + \delta(a)i(a,t), \tag{1.1a}$$

$$(\partial_t + \partial_a)\, i(a,t) = -\mu(a)i(a,t) + K_0(a)i(a,t))s(a,t) - \delta(a)i(a,t), \tag{1.1b}$$

$$s(0,t) = \int_0^\omega \beta(a)\left\{s(a,t) + (1-q)i(a,t)\right\} \mathrm{d}a, \tag{1.1c}$$

$$i(0,t) = q\int_0^\omega \beta(a)i(a,t)\,\mathrm{d}a, \tag{1.1d}$$

$$s(a,0) = s_0(a), \quad i(a,0) = i_0(a), \tag{1.1e}$$

for $0 \le t \le T \le \infty$, $0 \le a \le \omega < +\infty$. Here, i and s are, respectively, the age-specific densities of infective and susceptible individuals, μ is the death rate, K_0 is the force of infection and δ is the recovery rate. In the McKendrick boundary conditions (1.1c) and (1.1d), β is the birth rate, whilst $q \in [0,1]$ is the coefficient of the vertical transmission of the disease. Further, $\omega < \infty$ is the maximum age in the population. Denoting $\mathrm{I} = (0,\omega)$, the natural space for the problem is $\mathbf{X}_1 = \mathrm{L}^1(\mathrm{I}) \times \mathrm{L}^1(\mathrm{I})$ with the norm $\|(p_1,p_2)\|_{\mathbf{X}_1} = \|p_1\|_1 + \|p_2\|_1$, where the norm $\|\cdot\|_1$ refers to the norm in $\mathrm{L}^1(\mathrm{I})$; in $\mathbb{R}^2$ we use the norm $\|(x,y)\| = |x| + |y|$ for $x, y \in \mathbb{R}$.

Solvability results for problems of this type seem to belong to mathematical folklore. However, a closer look at classical papers reveals that they often adopt some assumptions which are not universally valid. One of the typical problems is that if one considers a realistic finite life span of individuals ω, then necessarily $\mu(a)$ must be unbounded as $a \to \omega^-$, see [3,9], which introduces another unbounded operator into the problem. This often has been circumvented by assuming that $\omega = \infty$, whereupon μ could be bounded, see [17]. On the other hand, in [10] the author assumes that the reproductive period is shorter than the life span of the individual and uses the reducibility of the McKendrick semigroup to consider the problem on this shorter age interval. Furthermore, as mentioned in the abstract, the structure of the intracohort infection term creates difficulties for analysis of the full equation in $\mathbf{X}_1$. Therefore, (1.1) usually is considered with the so-called intercohort model of infections [9,17], where in contrast to (1.1), the infections can occur amongst different age groups and thus are modelled by an integral operator. A notable exception is [7] where, however, the authors used a simplifying assumption that the total population is in equilibrium.

In this note, we briefly indicate the necessary linear results with a novel elementary proof that the linear McKendrick operator is densely defined and then we give a proof of the existence of global integrable classical solutions of (1.1) in which we do not require that there exists a closed convex order bounded subset of $\mathbf{X}_1$, which is invariant under the action of the McKendrick semigroup, making the result more general than that in [7].

2. Definitions, Notation and Assumptions

Let us recall that our state space is $\mathbf{X}_1 = \mathrm{L}^1(\mathrm{I}) \times \mathrm{L}^1(\mathrm{I})$. For any Banach lattice we denote by $\mathbf{X}_+$ the positive cone of $\mathbf{X}$; in our case $\mathbf{f} = (f_1, f_2) \geq 0$ if $f_i(a) \geq 0$ for almost any $a \in [0, \omega]$ and $i = 1, 2$. By $\mathbb{B}(\mathbf{X}, \mathbf{Y})$ we denote the space of bounded linear operators between Banach spaces $\mathbf{X}, \mathbf{Y}$; we use notation $\mathbb{B}(\mathbf{X})$ when $\mathbf{X} = \mathbf{Y}$. Then, we make the following assumptions on the coefficients of (1.1).

(H1) $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}([0, \omega[)$ and satisfies $\int_0^\omega \mu(r)\, dr = +\infty$, with $\underline{\mu} > 0$;
(H2) $0 \leq \beta \in \mathrm{L}^\infty(\mathrm{I})$;
(H3) $0 \leq \delta \in \mathrm{W}^{1,\infty}(\mathrm{I})$;
(H4) $K_0 \in \mathrm{L}^\infty(\mathrm{I})$.

For any measurable function α on $[0, \omega]$, we introduce the notation

$$\overline{\alpha} = \operatorname*{esssup}_{a \in \mathrm{I}} \alpha(a), \quad \underline{\alpha} = \operatorname*{essinf}_{a \in \mathrm{I}} \alpha(a).$$

We denote $\mathbf{Y}_1 := \mathrm{W}^{1,1}(\mathrm{I}) \times \mathrm{W}^{1,1}(\mathrm{I})$ be the Sobolev space of vector-valued functions with integrable first derivatives. Further, define $\mathbf{S} = \mathrm{diag}\,\{-\partial_a, -\partial_a\}$ on $D(\mathbf{S}) = \mathbf{Y}_1$, $\mathbf{M}_\mu = \mathrm{diag}\,\{-\mu, -\mu\}$ on $D(\mathbf{M}_\mu) = \{\boldsymbol{\varphi} \in \mathbf{X}_1 : \mu\boldsymbol{\varphi} \in \mathbf{X}_1\}$,

$$\mathbf{M}_\delta = \begin{pmatrix} 0 & \delta \\ 0 & -\delta \end{pmatrix} \tag{2.1}$$

in $\mathbb{B}(\mathbf{X}_1)$ and

$$\mathbf{B} = \begin{pmatrix} \beta & (1-q)\beta \\ 0 & q\beta \end{pmatrix} \tag{2.2}$$

with

$$\boldsymbol{\mathcal{B}}\boldsymbol{\varphi} = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\, \mathrm{d}a;$$

$\boldsymbol{\mathcal{B}} \in \mathbb{B}(\mathbf{X}_1, \mathbb{R}^2)$ with $\|\boldsymbol{\mathcal{B}}\|_{\mathbb{B}(\mathbf{X}_1, \mathbb{R}^2)} \leq \overline{\beta}$. Then, we introduce $\mathbf{A}$ on the domain

$$D\,(\mathbf{A}) = \{\boldsymbol{\varphi} \in D(\mathbf{S}) \cap D(\mathbf{M}_\mu);\ \boldsymbol{\varphi}(0) = \boldsymbol{\mathcal{B}}\boldsymbol{\varphi}\} \tag{2.3}$$

by $\mathbf{A} = \mathbf{S} + \mathbf{M}_\mu$ and $\boldsymbol{\mathcal{Q}}$ by $\boldsymbol{\mathcal{Q}} = \mathbf{A} + \mathbf{M}_\delta$ on $D(\boldsymbol{\mathcal{Q}}) = D(\mathbf{A})$. Finally, for $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbf{X}_1$ we formally define

$$\mathfrak{F}(\boldsymbol{\varphi}) = \begin{pmatrix} 0 & -K_0\varphi_1 \\ 0 & K_0\varphi_1 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{2.4}$$

In what follows we denote $\mathbf{u} = (s, i)$. Using the above notation, we re-write (1.1a)–(1.1e) in the following compact form

$$\partial_t \mathbf{u} = \mathfrak{Q}\mathbf{u} + \mathfrak{F}(\mathbf{u}), \tag{2.5a}$$

$$\mathbf{u}(0,t) = \int_0^\omega \mathbf{B}(a)\mathbf{u}(a,t)\,da, \tag{2.5b}$$

$$\mathbf{u}(a,0) = \mathbf{u}_0(a) = (s_0(a), i_0(a)). \tag{2.5c}$$

3. The Linear Part

The main result of this section is:

Theorem 3.1. *The linear operator* $\mathfrak{Q}$ *generates a strongly continuous positive quasi-contractive semigroup* $\left(e^{t\mathfrak{Q}}\right)_{t\geq 0}$ *in* $\mathbf{X}_1$ *that satisfies the estimate*

$$\left\|e^{t\mathfrak{Q}}\right\|_{\mathbb{B}(\mathbf{X}_1)} \leq e^{t(\overline{\beta}-\underline{\mu})}. \tag{3.1}$$

Since $\mathbf{M}_\delta$ is bounded, it is sufficient to prove the same result for the operator $\mathbf{A}$.

Theorem 3.2. *The linear operator* $\mathbf{A}$ *generates a strongly continuous positive semigroup* $\left(e^{t\mathbf{A}}\right)_{t\geq 0}$ *in* $\mathbf{X}_1$ *such that*

$$\|e^{t\mathbf{A}}\|_{\mathbb{B}(\mathbf{X}_1)} \leq e^{(\overline{\beta}-\underline{\mu})t}. \tag{3.2}$$

As we noted, parts of this theorem belong to folklore of the field and thus we only briefly sketch results which are different than in the classical literature [10,17]. The details can be found in [14]. Theorem 3.2 is proved in a series of lemmas in which we construct and estimate the resolvent of $\mathbf{A}$, showing that it satisfies the Hille–Yosida estimates. Then, we show the density of the domain of $D(\mathbf{A})$, hence completing the proof. First we introduce the survival rate matrix $\mathbf{L}(a)$, which corresponds to the survival probability in a single population. $\mathbf{L}(a)$ is a solution of the matrix differential equation:

$$\partial_a \mathbf{L}(a) = \mathbf{M}_\mu(a)\mathbf{L}(a), \quad \mathbf{L}(0) = \mathbf{I}, \tag{3.3}$$

where $\mathbf{I}$ is the identity matrix. The solution of (3.3) is a diagonal matrix given by

$$\mathbf{L}(a) = \begin{pmatrix} e^{-\int_0^a \mu(r)\,dr} & 0 \\ 0 & e^{-\int_0^a \mu(r)\,dr} \end{pmatrix} = e^{-\int_0^a \mu(r)\,dr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e^{-\int_0^a \mu(r)\,dr}\mathbf{I}. \tag{3.4}$$

We see that $\mathbf{L}(a)$ invertible and hence we can define $\mathbf{L}(a,b)$ by

$$\mathbf{L}(a,b) = \mathbf{L}(a)\mathbf{L}^{-1}(b). \tag{3.5}$$

The next result is an extension of [10] to the case of finite maximum life span ω. Its proof follows the usual lines but requires more care because of the unbounded coefficient $\mu(a)$. This can be handled thanks to the diagonal structure of the survival rate function $\mathbf{L}(a)$, see [14].

Lemma 3.3. *If $\lambda > \overline{\beta} - \underline{\mu}$, then $(\lambda\mathbf{I} - \mathbf{A})^{-1}$ is given by*

$$\begin{aligned}\boldsymbol{\varphi} &= (\lambda\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\psi}\\ &= e^{-\lambda a}\mathbf{L}(a)\left(\mathbf{I} - \int_0^\omega e^{-\lambda\sigma}\mathbf{B}(\sigma)\mathbf{L}(\sigma)\,d\sigma\right)^{-1}\int_0^\omega e^{-\lambda\tau}\mathbf{B}(\tau)\mathbf{L}(\tau)\\ &\quad\times\int_0^\tau e^{\lambda\sigma}\mathbf{L}^{-1}(\sigma)\boldsymbol{\psi}(\sigma)\,d\sigma\,d\tau + e^{-\lambda a}\mathbf{L}(a)\int_0^a e^{\lambda\sigma}\mathbf{L}^{-1}(\sigma)\boldsymbol{\psi}(\sigma)\,\mathrm{d}\sigma,\end{aligned} \tag{3.6}$$

where $\boldsymbol{\psi} \in \mathbf{X}_1$. Furthermore,

$$\left\|(\lambda\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\psi}\right\|_{\mathbf{X}_1} \le \frac{1}{\lambda - (\overline{\beta} - \underline{\mu})}\|\boldsymbol{\psi}\|_{\mathbf{X}_1} \tag{3.7}$$

and $\mathbf{A}$ is closed.

The following lemma shows that the operator $\mathbf{A}$ is densely defined on $\mathbf{X}_1$. A proof of this result (with some gaps) is provided in [10, p. 60]. A correct but more technically involved proof can be found in [17]. We present a much simpler proof which, actually, shows that $D(\mathbf{A})_+$ is dense in $\mathbf{X}_{1,+}$.

Lemma 3.4. $\overline{D(\mathbf{A})_+} = \mathbf{X}_{1,+}$.

Proof. Fix $\mathbf{f} \in \mathbf{X}_{1,+}$. For any given $\epsilon > 0$ there exists a mollifier $\boldsymbol{\varphi} \in \mathcal{C}_0^\infty(\mathrm{I}) \times \mathcal{C}_0^\infty(\mathrm{I})$, which is positive, such that $\|\mathbf{f} - \boldsymbol{\varphi}\|_{\mathbf{X}_1} \le \epsilon$. We see that $\boldsymbol{\varphi} \in D(\mathbf{M}_\mu)$, but $0 = \boldsymbol{\varphi}(0) \ne \mathcal{B}\varphi$ unless supports of $\mathbf{B}$ and $\boldsymbol{\varphi}$ are disjoint. Take a nonnegative function $\eta \in \mathcal{C}_0^\infty([0,\omega[)$ with $\eta(0) = 1$ and let $\eta_\epsilon(a) = \eta(a/\epsilon)$, then $\eta(a/\epsilon) = 0$ for $a > \epsilon\omega$ and hence $\operatorname{supp}\eta_\epsilon \subseteq [0, \epsilon\omega]$. Further, let $\boldsymbol{\alpha}$ be an arbitrary vector and consider

$$\boldsymbol{\psi} = \boldsymbol{\varphi} + \eta_\epsilon\boldsymbol{\alpha}.$$

We see that $\boldsymbol{\psi} \in \mathbf{Y}_1 \cap D(\mathbf{M}_\mu)$. Now, we need to find $\boldsymbol{\alpha}$ such that $\boldsymbol{\psi}$ satisfies the compatibility condition $\boldsymbol{\psi}(0) = \mathcal{B}\psi$. Since $\operatorname{supp}\eta_\epsilon \subseteq [0, \epsilon\omega]$, $\boldsymbol{\alpha}$ is determined from

$$\boldsymbol{\alpha} = \int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\,da + \left(\int_0^{\epsilon\omega}\eta_\epsilon(a)\mathbf{B}(a)\,da\right)\boldsymbol{\alpha}, \tag{3.8}$$

where

$$\eta_\epsilon(a)\mathbf{B}(a) = \begin{pmatrix}\beta(a)\eta_\epsilon(a) & (1-q)\beta(a)\eta_\epsilon(a)\\ 0 & q\beta(a)\eta_\epsilon(a)\end{pmatrix}.$$

Now, the matrix l_1-norm of $\int_0^{\epsilon\omega}\eta_\epsilon(a)\mathbf{B}(a)\,da$ satisfies

$$\left\|\int_0^\omega \begin{pmatrix}\epsilon\beta(\epsilon s)\eta(s) & \epsilon(1-q)\beta(\epsilon s)\eta(s)\\ 0 & \epsilon q\beta(\epsilon s)\eta(s)\end{pmatrix}ds\right\|_{\mathbb{B}(\mathbb{R}^2)} \le \epsilon\overline{\beta}\|\eta\|_1,$$

hence (3.8) is solvable for sufficiently small ϵ giving $\boldsymbol{\alpha} \ge 0$ if $\boldsymbol{\varphi} \ge 0$, with

$$\|\boldsymbol{\alpha}\| \le \left\|\int_0^\omega \mathbf{B}(a)\boldsymbol{\varphi}(a)\,da\right\|(1 - \epsilon\overline{\beta}\|\eta\|_1)^{-1} \le C$$

for some constant C, which is independent of ϵ for sufficiently small ϵ. Hence

$$\|\mathbf{f}-\boldsymbol{\psi}\|_{\mathbf{X}_1}=\|(\mathbf{f}-\boldsymbol{\varphi})+(\boldsymbol{\varphi}-\boldsymbol{\psi})\|_{\mathbf{X}_1}\leq\|\mathbf{f}-\boldsymbol{\varphi}\|_{\mathbf{X}_1}+\epsilon\|\boldsymbol{\alpha}\|\|\eta\|_1\leq(1+C\|\eta\|_1)\epsilon.$$

□

Now we can complete proofs of Theorems 3.2 and 3.1.

Proof of Theorem 3.2. Using the above lemmas with the estimate (3.7), we can see that $\mathbf{A}$ satisfies the assumptions of the Hille–Yosida theorem. Hence, it generates a strongly continuous semigroup $\left(e^{t\mathbf{A}}\right)_{t\geq 0}$ satisfying (3.2). Since the resolvent is positive, the semigroup is positive as well. □

Proof of Theorem 3.1. Since $\mathbf{M}_\delta(a)$ may be considered as an operator in $\mathbb{B}\left(\mathbf{X}_1\right)$ with $\left\|\mathbf{M}_\delta\right\|_{\mathbb{B}(\mathbf{X}_1)}\leq 2\overline{\delta}$, the Bounded Perturbation Theorem [15] yields that $(\boldsymbol{\mathcal{Q}}, D(\mathbf{A}))$ generates a C_0-semigroup, denoted by $\left(e^{t\boldsymbol{\mathcal{Q}}}\right)_{t\geq 0}$ that satisfies

$$\|e^{t\boldsymbol{\mathcal{Q}}}\|_{\mathbb{B}(\mathbf{X}_1)}\leq e^{t(\overline{\beta}-\underline{\mu}+2\overline{\delta})}.$$

Thanks to the structure of $\mathbf{M}_\delta$, we can improve the above estimate and also show that the semigroup $\left(e^{t\boldsymbol{\mathcal{Q}}}\right)_{t\geq 0}$ generated by $\boldsymbol{\mathcal{Q}}$ is positive. Indeed, the semigroup generated by $\mathbf{M}_\delta$ is given by

$$e^{t\mathbf{M}_\delta}=\begin{pmatrix}1 & 1-e^{-t\delta(a)}\\ 0 & e^{-t\delta(a)}\end{pmatrix}$$

so that it is positive and

$$\left\|e^{t\mathbf{M}_\delta}\right\|_{\mathbb{B}(\mathbf{X}_1)}=1.$$

Hence, by the Trotter product formula [15], we obtain positivity of $\left(e^{t\boldsymbol{\mathcal{Q}}}\right)_{t\geq 0}$ and

$$\|e^{t\boldsymbol{\mathcal{Q}}}\|_{\mathbb{B}(\mathbf{X}_1)}\leq e^{t(\overline{\beta}-\underline{\mu})}. \tag{3.9}$$

□

Remark 3.5. The estimates (3.2) and (3.9) are not optimal. In fact, see [9], for the scalar linear McKendrick problem

$$\partial_t u(a,t)=-\partial_a u(a,t)-\mu(a)u(a,t),\quad t>0,\ a\in(0,\omega), \tag{3.10a}$$

$$u(0,t)=\int_0^\omega \beta(a)u(a,t)\,da, \tag{3.10b}$$

$$u(a,0)=u_0(a), \tag{3.10c}$$

there exists a unique real eigenvalue λ^* of (3.10) such that

$$\|u(t)\|_1\leq Ne^{t\lambda^*}\|u_0\|_1,$$

for some constant N.

Consider now an initial condition $(s_0,i_0)\in D(\mathbf{A})_+$ for (1.1). Since the semigroup $\left(e^{t\boldsymbol{\mathcal{Q}}}\right)_{t\geq 0}$ is positive, the strict solution (s,i) of the linear part of (1.1) is nonnegative and the total population $0\leq s(a,t)+i(a,t)=:u(a,t)$

satisfies (3.10). Using nonnegativity, we find $s(a,t) \leq u(a,t)$ and $i(a,t) \leq u(a,t)$ and consequently

$$\|e^{t\mathfrak{Q}}(s_0,i_0)\|_{\mathbf{X}_1} \leq Ne^{t\lambda^*}\|(s_0,i_0)\|_{\mathbf{X}_1}$$

(provided $(s_0,i_0) \in D(\mathbf{A})_+$). However, by Lemma 3.4, the above estimate can be extended to $\mathbf{X}_{1,+}$, and by [2, Proposition 2. 67], to

$$\|e^{t\mathfrak{Q}}\|_{\mathbb{B}(\mathbf{X}_1)} \leq Ne^{t\lambda^*}. \tag{3.11}$$

Note that the crucial role in the above argument is played by the fact that (s,i) satisfies the differential equation (1.1)—if it was only a mild solution, it would be difficult to directly prove that the sum $s+i$ is the mild solution to (3.10).

4. The Nonlinear Problem

We recall that in the considered here intracohort infection mechanism, the infection term $\mathfrak{F}$ in (2.5) is defined by

$$\mathfrak{F}\big(\mathbf{u}(a,t)\big) = (-K_0(a)s(a,t)i(a,t), K_0(a)s(a,t)i(a,t)), \quad \mathbf{u} = (s,i)$$

with K_0 satisfying assumption (H4). We adopt the notation $\mathbf{X}_\infty = \mathrm{L}^\infty(\mathrm{I}) \times \mathrm{L}^\infty(\mathrm{I})$ equipped with the norm $\|(u_1,u_2)\|_{\mathbf{X}_\infty} = \|u_1\|_\infty + \|u_2\|_\infty$, where $\|\cdot\|_\infty = \|\cdot\|_{\mathrm{L}^\infty([0,\omega])}$, and for $\mathbf{u}_0 \in \mathbf{X}_\infty, \rho > 0, i = 1, \infty$ we denote

$$\mathbf{B}_i(\mathbf{u}_0, \varrho) = \{\mathbf{u} \in \mathbf{X}_i : \|\mathbf{u}-\mathbf{u}_0\|_{\mathbf{X}_i} \leq \varrho\}.$$

Finally, we denote $\mathbf{Z}_{i,T} = \mathcal{C}\big([0,T], \mathbf{X}_i\big)$ for a given $0 < T < \infty$ and $i = 1, \infty$.

The main problem with the intracohort transmission is that, in general, $\mathfrak{F}(\mathbf{u}) \notin \mathbf{X}_1$ for $\mathbf{u} \in \mathbf{X}_1$. Multiplication is well defined in $\mathrm{L}^\infty(\mathrm{I})$ but then the latter space is not suitable for the semigroup techniques—any strongly continuous semigroup on $\mathrm{L}^\infty(\mathrm{I})$ is uniformly continuous [1, Theorem 3.6]. To handle this nonlinearity, we use the fact that for $\omega < \infty$, $\mathbf{X}_\infty$ is densely and continuously embedded in $\mathbf{X}_1$ but, on the other hand, $\mathbf{B}_\infty(\mathbf{u}_0,\rho)$ is closed in $\mathbf{X}_1$ (this follows as a sequence converging in $\mathbf{X}_1$ has a subsequence converging almost everywhere). We shall show that we can carry out the analysis on balls $\mathbf{B}_\infty$, though no ball is invariant under the action of $\big(e^{t\mathfrak{Q}}\big)_{t\geq 0}$, the latter being a standard requirement in nonlinear problems, e.g. [7,13].

We begin with the following result.

Proposition 4.1. *For any $t \geq 0, \rho > 0$,*

$$e^{t\mathfrak{Q}}\left(\mathbf{B}_\infty(0,\rho)\right) \subset \mathbf{B}_\infty(0, \max\{1, \overline{\beta}\omega e^{\overline{\beta}t}\}e^{-\underline{\mu}t}\rho).$$

Proof. We consider the explicit representation of the semigroup $\big(e^{t\mathfrak{Q}}\big)_{t\geq 0}$, see [16, p. 69], [10, p. 62]):

$$e^{t\mathfrak{Q}}\mathbf{u}_0(a) = \begin{cases} \mathbf{L}_\delta(a,a-t)\mathbf{u}_0(a-t), & a \in (t,\omega), \\ \mathbf{L}_\delta(a,0)\mathbf{b}(t-a;\mathbf{u}_0), & a \in (0,t), \end{cases} \tag{4.1}$$

where $\mathbf{L}_\delta(a,b) = (l_{ij}(a,b))_{1\leq i,j\leq 2}$ is defined similarly to (3.5), but using the solution to

$$\partial_a \mathbf{L}_\delta = (\mathbf{M}_\mu + \mathbf{M}_\delta)\mathbf{L}_\delta, \quad \mathbf{L}_\delta(0) = \mathbf{I}. \tag{4.2}$$

and $\mathbf{b}(t;\mathbf{u}_0)$ is the solution of the integral equation

$$\mathbf{b}(t;\mathbf{u}_0) = \mathbf{J}(t) + \int_0^t \mathbf{B}(s)\mathbf{L}_\delta(s,0)\mathbf{b}(t-s;\mathbf{u}_0)\,ds, \tag{4.3}$$

with

$$\mathbf{J}(t) = \int_t^\omega \mathbf{B}(s)\mathbf{L}_\delta(s,s-t)\mathbf{u}_0(s-t)\,ds \tag{4.4}$$

for $0 \le t \le \omega$ and $\mathbf{J}(t) = 0$ for $t > \omega$. Since the coefficient matrix of (4.2) is non-negative off-diagonal, $l_{ij} \ge 0$, and by adding appropriate rows,

$$\sum_{i=1}^{2} l_{ij}(a,b) = e^{-\int_b^a \mu(s)\mathrm{d}a} \le e^{-\underline{\mu}(a-b)}, \qquad j = 1,2, \quad b < a.$$

Using this, and the fact that we use the l_1 norm in $\mathbb{R}^2$, we obtain

$$\|\mathbf{J}(t)\| \le \overline{\beta}\omega e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}$$

(clearly also valid for $t \ge \omega$) which, upon substitution into (4.3), gives

$$\|\mathbf{b}(t;\mathbf{u}_0)\| \le \overline{\beta}\omega e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} + \overline{\beta}e^{-\underline{\mu}t}\int_0^t e^{\underline{\mu}s}\|\mathbf{b}(s;\mathbf{u}_0)\|\,\mathrm{d}s.$$

Gronwall's lemma yields

$$\|\mathbf{b}(t;\mathbf{u}_0)\| \le \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}$$

hence, by (4.1), we obtain

$$\|\mathbf{L}_\delta(a,0)\mathbf{b}(t-a;\mathbf{u}_0)\| \le \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}e^{-\overline{\beta}a}\|\mathbf{u}_0\|_{\mathbf{X}_\infty} \le \overline{\beta}\omega e^{(\overline{\beta}-\underline{\mu})t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty},$$

for $a < t$ and

$$\|\mathbf{L}_\delta(a,a-t)\mathbf{u}_0(a-t)\| \le e^{-\underline{\mu}t}\sup_{s\in[0,a]}\|\mathbf{u}_0(s)\| \le e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}, \quad \text{for } a > t.$$

Hence,

$$\left\|e^{t\mathfrak{Q}}\mathbf{u}_0\right\|_{\mathbf{X}_\infty} \le \max\{1, \overline{\beta}\omega e^{\overline{\beta}t}\}e^{-\underline{\mu}t}\|\mathbf{u}_0\|_{\mathbf{X}_\infty}. \tag{4.5}$$

□

In what follows we list the relevant properties of the nonlinear term $\mathfrak{F}$ which are straightforward to prove.

Lemma 4.2. 1. *$\mathfrak{F}(\mathbf{X}_\infty) \subset \mathbf{X}_\infty$ with $\|\mathfrak{F}(\mathbf{u})\|_{\mathbf{X}_\infty} \le \|K_0\|_\infty\|\mathbf{u}\|^2_{\mathbf{X}_\infty}$.*

2. *$\mathfrak{F}$ is locally Lipschitz continuous on $\mathbf{X}_\infty$ with*

$$\left\|\mathfrak{F}(\mathbf{u}_1) - \mathfrak{F}(\mathbf{u}_2)\right\|_{\mathbf{X}_\infty} \le 2\|K_0\|_\infty R\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{X}_\infty}, \quad \mathbf{u}_1,\mathbf{u}_2 \in \mathbf{B}_\infty(0,R). \tag{4.6}$$

3. *$\mathfrak{F}$, restricted to $\mathbf{B}_\infty(0,R) \subset \mathbf{X}_1$, $R > 0$ is Lipschitz continuous on $\mathbf{X}_1$ with*

$$\left\|\mathfrak{F}(\mathbf{u}_1) - \mathfrak{F}(\mathbf{u}_2)\right\|_{\mathbf{X}_1} \le 2\|K_0\|_\infty R\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{X}_1}, \quad \mathbf{u}_1,\mathbf{u}_2 \in \mathbf{B}_\infty(0,R). \tag{4.7}$$

4. $\mathfrak{F}$ *is continuously Fréchet differentiable on* $\boldsymbol{\varphi} \in \mathbf{X}_\infty$ *and for any* $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$, $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \mathbf{X}_\infty$, *the Fréchet derivative at* $\boldsymbol{\varphi}$ *is given by*

$$\left(\mathfrak{F}_{\boldsymbol{\varphi}}\boldsymbol{\psi}\right)(a) := \begin{pmatrix} -K_0(a)\psi_1(a)\varphi_2(a) - K_0(a)\varphi_1(a)\psi_2(a) \\ K_0(a)\psi_1(a)\varphi_2(a) + K_0(a)\varphi_1(a)\psi_2(a) \end{pmatrix}. \tag{4.8}$$

Furthermore, we have the expansion

$$\mathfrak{F}(\boldsymbol{\varphi} + \boldsymbol{\psi})(a) = \mathfrak{F}(\boldsymbol{\varphi})(a) + \mathfrak{F}_{\boldsymbol{\varphi}}(\boldsymbol{\psi})(a) + \mathbf{G}(\boldsymbol{\psi}, \boldsymbol{\psi})(a),$$

where

$$\mathbf{G}(\boldsymbol{\psi}, \boldsymbol{\psi})(a) = \begin{pmatrix} -K_0(a)\psi_1(a)\psi_2(a) \\ K_0(a)\psi_1(a)\psi_2(a) \end{pmatrix}.$$

Note that $\mathfrak{F}$, even restricted to $\mathbf{B}_\infty(0, R)$, is not differentiable in $\mathbf{X}_1$ since convergence to 0 in $\mathbf{X}_1$ does not imply the same in $\mathbf{X}_\infty$.

The above properties of $\mathfrak{F}$ allow to consider (2.5) in $\mathbf{X}_1$ provided the solution with bounded initial data stays bounded. We consider the integral formulation of (2.5),

$$\mathbf{u}(t) = e^{t\mathfrak{Q}}\mathbf{u}_0 + \int_0^t e^{(t-s)\mathfrak{Q}}\mathfrak{F}\big(\mathbf{u}(s)\big)\, ds, \quad 0 < t < T, \tag{4.9}$$

denote by $\mathsf{B}_T(0, R)$ the closed ball in $\mathbf{Z}_{\infty,T}$ and introduce the integral operator $\mathcal{J}$ defined on $\mathsf{B}_T(0, R)$ by

$$\left(\mathcal{J}\mathbf{u}\right)(t) = e^{t\mathfrak{Q}}\mathbf{u}_0 + \int_0^t e^{(t-s)\mathfrak{Q}}\mathfrak{F}\big(\mathbf{u}(s)\big)\, ds, \quad 0 < t < T. \tag{4.10}$$

We aim to show that $\mathcal{J}$ has a fixed point in $\mathsf{B}_T(0, R)$ for sufficiently large R and small T. For this, let us fix an initial datum $\mathbf{u}_0 \in \mathbf{X}_\infty$ we take R such that

$$\frac{R}{2\max\{L, 1\}} > \|\mathbf{u}_0\|_{\mathbf{X}_\infty}, \tag{4.11}$$

where $L = \overline{\beta}\omega$. Further, denote $\Psi(t) = \max\{1, \overline{\beta}\omega e^{\overline{\beta}t}\}e^{-\underline{\mu}t}$.

Lemma 4.3. *Let* R *satisfies (4.11). Then, there exists* $T > 0$ *such that the operator* $\mathcal{J}$ *maps* $\mathsf{B}_T(0, R)$ *into itself. Moreover, for* $\mathbf{u}$, $\mathbf{v} \in \mathsf{B}_T(0, R)$, *we have*

$$\|\mathcal{J}\mathbf{u} - \mathcal{J}\mathbf{v}\|_{\mathbf{Z}_{\infty,T}} \le C_R T\|\mathbf{u} - \mathbf{v}\|_{\mathbf{Z}_{\infty,T}}, \tag{4.12}$$

where $C_R = 2\|K_0\|_\infty R \sup_{0\le t\le T}\Psi(t)$.

Proof. Let $\mathbf{u}$, $\mathbf{v} \in \mathsf{B}_\tau(0, R)$ for some τ. We immediately have, for $0 \le t \le \tau$,

$$\begin{aligned} \Big\|\big(\mathcal{J}\mathbf{u}\big)(t) - \big(\mathcal{J}\mathbf{v}\big)(t)\Big\|_{\mathbf{X}_\infty} &\le \sup_{0\le s\le t}\Psi(s)\int_0^t \Big\|\mathfrak{F}\big(\mathbf{u}(s)\big) - \mathfrak{F}\big(\mathbf{v}(s)\big)\Big\|_{\mathbf{X}_\infty} \mathrm{d}s \\ &\le 2\|K_0\|_\infty R \sup_{0\le s\le t}\Psi(s)\int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_{\mathbf{X}_\infty} \mathrm{d}s. \end{aligned} \tag{4.13}$$

Using (4.10), we see that

$$\begin{aligned}\left\|(\mathcal{J}\mathbf{u})(t)\right\|_{\mathbf{X}_\infty} &\le \Psi(t)\|\mathbf{u}_0\|_{\mathbf{X}_\infty} + \sup_{0\le s\le t}\Psi(s)\int_0^t \left\|\mathfrak{F}(\mathbf{u}(s))\right\|_{\mathbf{X}_\infty}\mathrm{d}s\\ &\le R\sup_{0\le s\le t}\Psi(s)\left(\frac{1}{2\max\{1,L\}} + 2\|K_0\|_\infty Rt\right).\end{aligned}$$

It is easy to see that $\lim_{t\to 0}\sup_{0\le s\le t}\Psi(s) = \max\{1, L\}$; hence, there is $T > 0$ such that

$$\sup_{0\le s\le T}\Psi(s)\left(\frac{1}{2\max\{1,L\}} + 2\|K_0\|_\infty RT\right) \le 1 \tag{4.14}$$

so that $\mathcal{J}\mathbf{u} \in \mathsf{B}_T(0,R)$ for any $\mathbf{u} \in \mathsf{B}_T(0,R)$. With this T we also get (4.12) upon taking supremum of both sides (4.13) with respect to t over $[0,T]$. □

Consequently, we have the following result:

Theorem 4.4. *For R and T satisfying, respectively, (4.11) and (4.14), the integral operator $\mathcal{J} : \mathsf{B}_T(0,R) \to \mathsf{B}_T(0,R)$ given by (4.10) has a unique fixed point.*

Proof. Let us denote by $\mathcal{J}^N$ N compositions of $\mathcal{J}$, with N a positive integer. Taking standard Picard iterations, for $\mathbf{u}$, $\mathbf{v} \in \mathsf{B}_T(0,R)$ we get the estimate

$$\left\|\mathcal{J}^N\mathbf{v} - \mathcal{J}^N\mathbf{u}\right\|_{\mathbf{Z}_{\infty,T}} \le \frac{T^N}{N!}C_R^N\|\mathbf{v}-\mathbf{u}\|_{\mathbf{Z}_{\infty,T}}, \tag{4.15}$$

where $\frac{T^N}{N!}C_R^N < 1$, provided N is sufficiently large. Hence, $\mathcal{J}^N$ is a contraction on $\mathsf{B}_T(0,R)$, and so it possesses a fixed point $\mathbf{u} \in \mathsf{B}_T(0,R)$. By a standard argument, [11, Lemma 5.4-3], this is a unique fixed point of $\mathcal{J}$. □

Theorem 4.4 can be re-phrased by saying that for any $\mathbf{u}_0 \in \mathbf{X}_\infty$ there is a unique mild solution to (2.5) in $\mathbf{X}_1$ on some time interval $[0,T]$ which is also in $\mathbf{X}_\infty$. We observe that, by the construction, T depends on the parameters of the problem and the selected L_∞ bound R. Clearly, the solution can be extended in $\mathbf{X}_\infty$ to a maximal interval $[0, T_{max}[$. Then, a standard argument gives

Corollary 4.5. *If $T_{\max} < \infty$, then $\|\mathbf{u}(t)\|_{\mathbf{X}_\infty}$ is unbounded as $t \to T_{\max}^-$.*

To prove that the solutions are global, we first have to address the question of regularity of the solution. We observe that this is not an obvious question as, on the one hand, $\mathfrak{F}$ is not differentiable in $\mathbf{X}_1$, even when restricted to $\mathbf{B}_\infty(0,R)$, and on the other hand, we do not have a C_0-semigroup in $\mathbf{X}_\infty$. However, taking advantage of the properties of the problem in both $\mathbf{X}_1$ and $\mathbf{X}_\infty$ allows to use essentially the same steps as in the standard proof, see e.g [5,15], to prove the regularity of the solution.

Let $t \to \mathbf{u}(t)$ be a mild solution emanating from $\mathbf{u}_0 \in D(\mathbf{A}) \cap X_\infty$ such that $\mathbf{A}\mathbf{u}_0 \in \mathbf{X}_\infty$ (so that also $\mathcal{Q}\mathbf{u}_0 \in \mathbf{X}_\infty$) and satisfying $u(t) \in \mathbf{B}_\infty(0,R)$ on $[0,T]$ for some R satisfying (4.11), with T determined by (4.14). First, we show

Lemma 4.6. $\mathbf{u}$ *is Lipschitz continuous on $[0,T]$ in both $\mathbf{X}_1$ and $\mathbf{X}_\infty$ norms.*

Proof. Let $h > 0$ and $t \in [0, T - h]$. Then

$$\mathbf{u}(t+h) - \mathbf{u}(t) = e^{t\boldsymbol{\mathfrak{Q}}}(e^{h\boldsymbol{\mathfrak{Q}}}\mathbf{u}_0 - \mathbf{u}_0) + \int_0^h e^{(t+h-s)\boldsymbol{\mathfrak{Q}}}\boldsymbol{\mathfrak{F}}\big(\mathbf{u}(s)\big)\,\mathrm{d}s + \int_0^t e^{(t-s)\boldsymbol{\mathfrak{Q}}}(\boldsymbol{\mathfrak{F}}\big(\mathbf{u}(s+h) - \boldsymbol{\mathfrak{F}}\big(\mathbf{u}(s)\big)\,\mathrm{d}s. \tag{4.16}$$

Thus, using

$$e^{h\boldsymbol{\mathfrak{Q}}}\mathbf{u}_0 - \mathbf{u}_0 = \int_0^h e^{s\boldsymbol{\mathfrak{Q}}}\boldsymbol{\mathfrak{Q}}\mathbf{u}_0\,\mathrm{d}s$$

and (4.7), and denoting $M_i(t) = \sup_{0\le s\le t}\|e^{s\boldsymbol{\mathfrak{Q}}}\|_{\mathbb{B}(\mathbf{X}_i)}$, $i = 1, \infty$,

$$\|\mathbf{u}(t+h) - \mathbf{u}(t)\|_{\mathbf{X}_i} \le M_i(t)M_i(h)\|\boldsymbol{\mathfrak{Q}}\mathbf{u}_0\|_{\mathbf{X}_i}h + M_i(t+h)\|K_0\|_\infty c_i R^2 h + M_i(t)2\|K_0\|_\infty R\int_0^t \|\mathbf{u}(s+h) - \mathbf{u}(s)\|_{\mathbf{X}_i}\mathrm{d}s,$$

where $c_1 = \omega$ and $c_\infty = 1$. Hence, by Gronwall's lemma,

$$\|\mathbf{u}(t+h) - \mathbf{u}(t)\|_{\mathbf{X}_i} \le C_i h, \tag{4.17}$$

where C_i are constants depending on T, $\mathbf{u}_0$ and the coefficients of the problem. □

Unfortunately, L_1 spaces do not have the Radon–Nikodym property and thus Lipschitz continuity of $\mathbf{u}$ does not imply that it is differentiable almost everywhere. Therefore, we need to prove it separately.

First, we observe that at a point $\boldsymbol{\varphi} \in \mathbf{B}_\infty(0, R)$, we have $\boldsymbol{\psi} \to \boldsymbol{\mathfrak{F}}_{\boldsymbol{\varphi}}\boldsymbol{\psi} \in \mathbb{B}(\mathbf{X}_1)$ with $\|\boldsymbol{\mathfrak{F}}_{\boldsymbol{\varphi}}\|_{\mathbb{B}(\mathbf{X}_1)} \le 2\|K_0\|_\infty R$. Then, it is easily seen that the equation for the 'formal' derivative of $\mathbf{u}$, denoted by $\mathbf{v}$

$$\mathbf{v}(t) = e^{t\boldsymbol{\mathfrak{Q}}}[\boldsymbol{\mathfrak{Q}}\mathbf{u}_0 + \boldsymbol{\mathfrak{F}}(\mathbf{u}_0)] + \int_0^t e^{(t-s)\boldsymbol{\mathfrak{Q}}}\boldsymbol{\mathfrak{F}}_{\mathbf{u}(s)}\big(\mathbf{v}(s)\big)\,\mathrm{d}s,$$

has a unique continuous solution, both in $\mathbf{X}_1$ and $\mathbf{X}_\infty$. Then we have:

Theorem 4.7. *Let $\mathbf{u}_0 \in D(\mathbf{A}) \cap X_\infty$ such that $\mathbf{A}\mathbf{u}_0 \in \mathbf{X}_\infty$ and let $t \to \mathbf{u}(t)$ be a mild solution emanating from $\mathbf{u}_0$ which satisfies $u(t) \in \mathbf{B}_\infty(0, R)$ on $[0, T]$ for some R. Then, $\mathbf{u}$ is a classical solution of (2.5) on $[0, T[$.*

Proof. We refine (4.16) to include the 'formal' derivative $\mathbf{v}$,

$$\begin{aligned}\frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} - \mathbf{v}(t) &= e^{t\boldsymbol{\mathfrak{Q}}}\left(\frac{e^{h\boldsymbol{\mathfrak{Q}}}\mathbf{u}_0 - \mathbf{u}_0}{h} - \boldsymbol{\mathfrak{Q}}\mathbf{u}_0 + \frac{1}{h}\int_0^h e^{(h-s)\boldsymbol{\mathfrak{Q}}}\boldsymbol{\mathfrak{F}}\big(\mathbf{u}(s)\big)\,ds - \boldsymbol{\mathfrak{F}}\big(\mathbf{u}_0\big)\right) \\ &\quad + \int_0^t e^{(t-s)\boldsymbol{\mathfrak{Q}}}\left(\frac{\boldsymbol{\mathfrak{F}}\big(\mathbf{u}(s+h)\big) - \boldsymbol{\mathfrak{F}}\big(\mathbf{u}(s)\big)}{h} - \boldsymbol{\mathfrak{F}}_{\mathbf{u}(s)}\big(\mathbf{v}(s)\big)\right)\mathrm{d}s.\end{aligned}$$

The $\mathbf{X}_1$ norm of the first two terms can be estimated as in the standard case: since $\mathbf{u}_0 \in D(\mathbf{A})$, for any $\epsilon > 0$ we find h_0 such that for all $0 < h < h_0$

$$\left\| \frac{e^{h\mathfrak{Q}}\mathbf{u}_0 - \mathbf{u}_0}{h} - \mathfrak{Q}\mathbf{u}_0 \right\|_{\mathbf{X}_1} \leq C_1\epsilon$$

and

$$\begin{aligned} &\left\| \frac{1}{h}\int_0^h e^{(h-s)\mathfrak{Q}}\mathfrak{F}\big(\mathbf{u}(s)\big)\,ds - \mathfrak{F}\big(\mathbf{u}_0\big) \right\|_{\mathbf{X}_1} \\ &\quad \leq \left\| \int_0^h e^{\sigma\mathfrak{Q}}\frac{\mathfrak{F}\big(\mathbf{u}(h-\sigma)\big) - \mathfrak{F}\big(\mathbf{u}(\sigma)\big)}{h}\,d\sigma \right\|_{\mathbf{X}_1} \\ &\qquad + \left\| \frac{1}{h}\int_0^h e^{\sigma\mathfrak{Q}}\mathfrak{F}\big(\mathbf{u}(\sigma)\big)\,d\sigma - \mathfrak{F}\big(\mathbf{u}_0\big) \right\|_{\mathbf{X}_1} \\ &\quad \leq \frac{C_2}{h}\int_0^h |h-2\sigma|d\sigma + C_3\epsilon \leq C_4\epsilon, \end{aligned}$$

where the estimate of the first term is due to (4.7) and Lemma 4.6, whilst the second follows since, by the continuity of the integrand, the integral is a differentiable function of its upper limit. Finally, by Lemma 4.2 and (4.17)

$$\begin{aligned} &\left\| \frac{\mathfrak{F}\big(\mathbf{u}(s+h)\big) - \mathfrak{F}\big(\mathbf{u}(s)\big)}{h} - \mathfrak{F}_{\mathbf{u}(s)}\big(\mathbf{v}(s)\big) \right\|_{\mathbf{X}_1} \\ &\quad \leq \left\| \frac{\mathfrak{F}_{\mathbf{u}(s)}\big(\mathbf{u}(s+h) - \mathbf{u}(s)\big)}{h} - \mathfrak{F}_{\mathbf{u}(s)}\big(\mathbf{v}(s)\big) \right\|_{\mathbf{X}_1} \\ &\qquad + \left\| \frac{\mathbf{G}\left(\mathbf{u}(s+h) - \mathbf{u}(s), \mathbf{u}(s+h) - \mathbf{u}(s)\right)}{h} \right\|_{\mathbf{X}_1} \\ &\quad \leq 2\|K_0\|_\infty R \left\| \frac{\mathbf{u}(s+h) - \mathbf{u}(s)}{h} - \mathbf{v}(s) \right\|_{\mathbf{X}_1} \\ &\qquad + \frac{\|K_0\|_\infty}{h}\int_0^\omega ||\mathbf{u}(a,s+h) - \mathbf{u}(a,s)||^2\,\mathrm{d}a \\ &\quad \leq 2\|K_0\|_\infty R \left\| \frac{\mathbf{u}(s+h) - \mathbf{u}(s)}{h} - \mathbf{v}(s) \right\|_{\mathbf{X}_1} \\ &\qquad + \|K_0\|_\infty C_1 C_\infty h. \end{aligned}$$

Summarizing, for any $\epsilon > 0$, there are $h_0 > 0$ and constants L_1, L_2 such that for any $0 < h < h_0$

$$\left\| \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} - \mathbf{v}(t) \right\|_{\mathbf{X}_1} \leq L_1\epsilon + L_2\int_0^t \left\| \frac{\mathbf{u}(s+h) - \mathbf{u}(s)}{h} - \mathbf{v}(s) \right\|_{\mathbf{X}_1} \mathrm{d}s,$$

which, by Gronwall's inequality, yields

$$\left\| \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} - \mathbf{v}(t) \right\|_{\mathbf{X}_1} \leq L_3\epsilon$$

for some constant L_3. Analogous estimates hold for $t > 0, h < 0$ with $t+h > 0$, showing that $\mathbf{u}$ is strongly differentiable in $\mathbf{X}_1$ and $\partial_t \mathbf{u} = \mathbf{v}$.

To complete the proof, we observe that $t \to \mathfrak{F}\big(\mathbf{u}(t)\big)$ is continuously differentiable in X_1 norm on $[0, T[$. Indeed, for $\mathbf{u} = (s, i)$, we have

$$\Bigg\| \frac{(-K_0 s(t+h)i(t+h), K_0 s(t+h)i(t+h)) - (-K_0 s(t)i(t), K_0 s(t)i(t))}{h}$$

$$\frac{1}{h} - (-K_0(\partial_t s(t)i(t) + \partial_t i(t)s(t)), K_0(\partial_t s(t)i(t) + \partial_t i(t)s(t))) \Bigg\|_{\mathbf{X}_1}$$

$$\leq 2\|K_0\|_\infty \left(\left\| s(t+h)\frac{i(t+h)-i(t)}{h} - s(t)\partial_t i \right\|_1 \right.$$

$$\left. + \left\| i(t)\left(\frac{s(t+h)-s(t)}{h} - \partial_t s \right) \right\|_1 \right)$$

and the statement follows from differentiability of (s, i) in $\mathbf{X}_1$ and continuity in $\mathbf{X}_\infty$. Thus, $\mathbf{u}$ is a mild solution of a nonhomogeneous Cauchy problem in $\mathbf{X}_1$ with differentiable inhomogeneity, and therefore, $\mathbf{u}(t) \in D(\mathbf{A})$ for any $t \in [0, T[$, see the argument in the proof [15, Theorem 6.1.5]. Hence, $\mathbf{u}$ is a classical solution to (2.5). □

In the last part, we address the question whether the constructed solution is global in time. We begin with the positivity of solutions. First, we observe that the iterates of (4.9) are not necessarily nonnegative, even if we start from $\mathbf{u}_0 \geq 0$ since $\mathfrak{F}$ is not nonnegative. However, as in [4], we can consider an equivalent formulation of (2.5),

$$\begin{cases} \partial_t \mathbf{u} \;= (\mathcal{Q} - \kappa\mathbf{I})\,\mathbf{u} + (\kappa\mathbf{I} + \mathfrak{F})\,(\mathbf{u}), \quad t > 0, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{4.18}$$

Using the fact that the iterates stay in, say $\mathsf{B}_T(0, R)$; that is, $\|\mathbf{u}(t)\|_{\mathbf{X}_\infty} \leq R$ for $t \in [0, T]$, the fact that $\mathcal{Q} - \kappa\mathbf{I}$ generates a positive semigroup for any $\kappa \in \mathbb{R}$ and that $\kappa\mathbf{I} + \mathfrak{F}$ is nonnegative on $\mathbf{B}_\infty(0, R)$ for $\kappa > \|K_0\|_\infty R$, we see that the Picard iterates of the integral formulation of (4.18) stay nonnegative provided $\mathbf{u}_0 \geq 0$. Hence, we have

Corollary 4.8. *Assume that $\mathbf{u}_0 \in \mathbf{X}_{\infty,+}$ and let $\mathbf{u} : [0, T_{\max}[\to \mathbf{X}_\infty$ be the unique mild solution of (2.5). Then, this solution is nonnegative on the maximal interval of its existence.*

Then we have

Theorem 4.9. *Let $0 \leq \mathbf{u}_0 \in D(\mathbf{A}) \cap X_\infty$ such that $\mathbf{A}\mathbf{u}_0 \in \mathbf{X}_\infty$. Then, the classical solution to (2.5) originating in $\mathbf{u}_0$ is global in time.*

Proof. Consider a classical solution $\mathbf{u}(t) = \big(s(t), i(t)\big)$ to (2.5), and hence of (1.1), in $\mathbf{X}_{\infty,+}$ defined on $[0, T_{\max}]$. Then we have $\big\|\mathbf{u}(a,t)\big\| = s(a,t) + i(a,t) = u(a,t)$, where $u(a,t)$ is the solution to the McKendrick equation (3.10) with the initial condition $u_0(a) = s_0(a) + i_0(a) = \|\mathbf{u}_0(a)\|$. Thus, using a scalar version of Proposition 4.1, we obtain

$$\big\|\mathbf{u}(t)\|_{\mathbf{X}_\infty} \leq \Psi(t)\|\mathbf{u}_0\|_{X_\infty}.$$

where, recall, $\Psi(t) = \max\{1, \overline{\beta}\omega e^{\overline{\beta} t}\} e^{-\underline{\mu} t}$. Since Ψ is bounded on finite time intervals, Corollary 4.5 yields $T_{\max} = \infty$. □

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