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A Note on Jacobsthal Quaternions

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Abstract. In this paper we introduce the Jacobsthal quaternions and the Jacobsthal–Lucas quaternions and give some of their properties. We derive the relations between Jacobsthal quaternions and Jacobsthal–Lucas quaternions and we give the matrix representation of these quaternions.
Keywords. Jacobsthal numbers, Jacobsthal–Lucas numbers, quaternions, recurrence relations.

1. Introduction

Let $\mathbb{H}$ be the set of quaternions q of the form

$$q = a + bi + cj + dk, \tag{1}$$

where $a, b, c, d \in \mathbb{R}$.

If $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ are any two quaternions then equality, addition, substraction and multiplication by scalar are defined.

Equality: $q_1 = q_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$,
substraction: $q_1 - q_2 = (a_1 - a_2) + (b_1 - b_2)i + (c_1 - c_2)j + (d_1 - d_2)k$,
multiplication by scalar $s \in \mathbb{R}$: $sq_1 = sa_1 + sb_1 i + sc_1 j + sd_1 k$.
The quaternion multiplication is defined using the rule

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

Note that (2) implies

$$ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j. \tag{3}$$

The conjugate of a quaternion is defined by

$$q^* = (a + bi + cj + dk)^* = a - bi - cj - dk. \tag{4}$$

The norm of a quaternion is defined by

$$N(q) = a^2 + b^2 + c^2 + d^2. \tag{5}$$

For the basics on quaternions theory, see [10].

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Numbers of the Fibonacci type are defined by the second-order linear recurrence relation of the form $a_n = b_1 a_{n-1} + b_2 a_{n-2}$, where $b_i \in \mathbb{N}, i = 1, 2$. For special b_i, $i = 1, 2$ we obtain the recurrence equation which defines the Fibonacci numbers and the like. Among numbers of the Fibonacci type we list the well-known as follows

- $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$—the Lucas numbers,
- $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$—the Pell numbers,
- $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ with $Q_0 = 2$, $Q_1 = 2$—the Pell–Lucas numbers,
- $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0$, $J_1 = 1$—the Jacobsthal numbers,
- $j_n = j_{n-1} + 2j_{n-2}$, for $n \geq 2$ with $j_0 = 2$, $j_1 = 1$—the Jacobsthal–Lucas numbers.

These numbers have many applications in distinct areas of mathematics also in quaternions theory.

In 1963 Horadam [4] introduced nth Fibonacci and Lucas quaternions. Many interesting properties of Fibonacci and Lucas quaternions can be found in [3,7]. Nurkan and Gv̈en in [8] defined dual Fibonacci quaternions and dual Lucas quaternions. In [2] Halici investigated complex Fibonacci quaternions. In [6] Horadam mentioned the possibility of introducing Pell quaternions and generalized Pell quaternions. Interesting results of Pell quaternions, Pell–Lucas quaternions obtained recently can be find in [1,9].

In this paper we introduce and study the Jacobsthal quaternions and Jacobsthal–Lucas quaternions. We describe their properties also using a matrix representations.

For convenience initial Jacobsthal numbers and Jacobsthal–Lucas numbers are presented in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
J_n	0	1	1	3	5	11	21	43	85	171	341
j_n	2	1	5	7	17	31	65	127	257	511	1025

For Jacobsthal numbers and Jacobsthal–Lucas numbers many identities are given, see [5]. In this paper we need some of them.

$$j_{n+1} + j_n = 3\left(J_{n+1} + J_n\right) = 3 \cdot 2^n, \tag{6}$$

$$j_{n+1} - j_n = 3\left(J_{n+1} - J_n\right) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \tag{7}$$

$$j_{n+r} + j_{n-r} = 3\left(J_{n+r} + J_{n-r}\right) + 4(-1)^{n-r} = 2^{n-r}\left(2^{2r} + 1\right) + 2(-1)^{n-r}, \tag{8}$$

$$j_{n+r} - j_{n-r} = 3\left(J_{n+r} - J_{n-r}\right) = 2^{n-r}\left(2^{2r} - 1\right), \tag{9}$$

$$J_n + j_n = 2J_{n+1}, \tag{10}$$

$$3J_n + j_n = 2^{n+1}, \tag{11}$$

$$j_n J_n = J_{2n}, \tag{12}$$

$$J_m j_n + J_n j_m = 2J_{m+n}, \tag{13}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \tag{14}$$

$$J_n = \frac{1}{3}\left(2^n - (-1)^n\right), \tag{15}$$

$$j_n = 2^n - (-1)^n. \tag{16}$$

The formulas (15) and (16) are named as the Binet formula for Jacobsthal numbers and Jacobsthal–Lucas numbers, respectively.

2. The Jacobsthal and Jacobsthal–Lucas Quaternions

The nth Jacobsthal quaternion JQ_n and the nth Jacobsthal–Lucas quaternion JLQ_n are defined as

$$JQ_n = J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3} \tag{17}$$

$$JLQ_n = j_n + ij_{n+1} + jj_{n+2} + kj_{n+3} \tag{18}$$

respectively.

Theorem 1. *Let $n \geq 1$, $r \geq 1$ be integer. Then*

$$JQ_{n+1} + JQ_n = 2^n\left(1 + 2i + 4j + 8k\right), \tag{19}$$

$$JQ_{n+1} - JQ_n = \frac{1}{3}\left[2^n\left(1 + 2i + 4j + 8k\right) + 2(-1)^n\left(1 - i + j - k\right)\right], \tag{20}$$

$$\begin{aligned} JQ_{n+r} + JQ_{n-r} = \frac{1}{3}\Big[&2^{n-r}\left(2^{2r} + 1\right)\left(1 + 2i + 4j + 8k\right) \\ &-2(-1)^{n-r}(1 - i + j - k)\Big], \end{aligned} \tag{21}$$

$$JQ_{n+r} - JQ_{n-r} = \frac{1}{3} \cdot 2^{n-r}\left(2^{2r} - 1\right)\left(1 + 2i + 4j + 8k\right), \tag{22}$$

$$N(JQ_n) = \frac{1}{9}\left[85 \cdot 2^{2n} + 10 \cdot 2^n(-1)^n + 4\right]. \tag{23}$$

Proof. (19) Let $JQ_n = J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}$ and $JQ_{n+1} = J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}$. Then $JQ_{n+1} + JQ_n = (J_{n+1} + J_n) + i(J_{n+2} + J_{n+1}) + j(J_{n+3} + J_{n+2}) + k(J_{n+4} + J_{n+3})$. Using (6) we obtain that $JQ_{n+1} + JQ_n = 2^n + i \cdot 2^{n+1} + j \cdot 2^{n+2} + k \cdot 2^{n+3} = 2^n\left(1 + 2i + 4j + 8k\right)$. Analogously we can prove formulas (20), (21), (22). To prove (23) we use the definition of the norm of a quaternion and we obtain that

$$N(JQ_n) = \sum_{s=0}^{3} J_{n+s}^2.$$

Moreover by the Binet formula (15) we have

$$\begin{aligned} N(JQ_n) &= \frac{1}{9}\Big[\left(2^n-(-1)^n\right)^2+\left(2^{n+1}-(-1)^{n+1}\right)^2 \\ &\quad +\left(2^{n+2}-(-1)^{n+2}\right)^2+\left(2^{n+3}-(-1)^{n+3}\right)^2\Big] \\ &= \frac{1}{9}\left[2^{2n}-2\cdot 2^n(-1)^n+(-1)^{2n}+4\cdot 2^{2n}+4\cdot 2^n(-1)^n\right. \\ &\quad +(-1)^{2n}+16\cdot 2^{2n}-8\cdot 2^n(-1)^n+(-1)^{2n} \\ &\quad \left.+64\cdot 2^{2n}+16\cdot 2^n(-1)^n+(-1)^{2n}\right] \\ &= \frac{1}{9}\left[85\cdot 2^{2n}+10\cdot 2^n(-1)^n+4\right], \end{aligned}$$

which ends the proof. □

In the same way, using (6)–(11) and (16) one can easily prove Theorems 2 and 3.

Theorem 2. *Let $n \geq 1$, $r \geq 1$ be integer. Then*

$$JLQ_{n+1}+JLQ_n = 3\cdot 2^n\left(1+2i+4j+8k\right), \tag{24}$$

$$JLQ_{n+1}-JLQ_n = 2^n\left(1+2i+4j+8k\right)+2(-1)^n\left(1-i+j-k\right), \tag{25}$$

$$JLQ_{n+r}+JLQ_{n-r} = 2^{n-r}\left(2^{2r}+1\right)(1+2i+4j+8k)-2(-1)^{n-r}(1-i+j-k), \tag{26}$$

$$JLQ_{n+r}-JLQ_{n-r} = 2^{n-r}\left(2^{2r}-1\right)(1+2i+4j+8k), \tag{27}$$

$$N(JLQ_n) = 85\cdot 2^{2n}-10\cdot 2^n(-1)^n+4. \tag{28}$$

Theorem 3. *Let $n \geq 1$ be integer. Then*

$$JQ_n+JLQ_n = 2\cdot JQ_{n+1}, \tag{29}$$

$$3\cdot JQ_n+JLQ_n = 2^{n+1}\left(1+2i+4j+8k\right). \tag{30}$$

Theorem 4. *Let $n \geq 1$ be integer. Then*

$$\begin{aligned} JLQ_n\cdot JQ_n &= \frac{1}{3}\left(-83\cdot 2^{2n}+2\right) \\ &\quad +i\left(\frac{2}{3}\left(2^{2n+1}+1\right)+(-1)^{n+2}\cdot 2^{n+3}\right) \\ &\quad +j\left(\frac{2}{3}\left(2^{2n+2}-1\right)-(-1)^{n+1}\cdot 2^{n+2}\right) \\ &\quad +k\left(\frac{2}{3}\left(2^{2n+3}+1\right)+(-1)^{n+1}\cdot 2^{n+2}\right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} JQ_n \cdot JLQ_n = & \frac{1}{3}\left(-83 \cdot 2^{2n} + 2\right) \\ & +i\left(\frac{2}{3}\left(2^{2n+1}+1\right) - (-1)^{n+2} \cdot 2^{n+3}\right) \\ & +j\left(\frac{2}{3}\left(2^{2n+2}-1\right) + (-1)^{n+1} \cdot 2^{n+2}\right) \\ & +k\left(\frac{2}{3}\left(2^{2n+3}+1\right) - (-1)^{n+1} \cdot 2^{n+2}\right). \end{aligned} \tag{32}$$

Proof. Multiplying JLQ_n by JQ_n we have

$$\begin{aligned} JLQ_n \cdot JQ_n = & j_n J_n - j_{n+1}J_{n+1} - j_{n+2}J_{n+2} - j_{n+3}J_{n+3} \\ & +i\left(j_n J_{n+1} + j_{n+1}J_n + j_{n+2}J_{n+3} - j_{n+3}J_{n+2}\right) \\ & +j\left(j_n J_{n+2} + j_{n+2}J_n - j_{n+1}J_{n+3} + j_{n+3}J_{n+1}\right) \\ & +k\left(j_n J_{n+3} + j_{n+3}J_n + j_{n+1}J_{n+2} - j_{n+2}J_{n+1}\right). \end{aligned}$$

Using identities (12), (13) and (14) we obtain

$$\begin{aligned} JLQ_n \cdot JQ_n = & \frac{1}{3}\left(2^{2n}-1\right) - \frac{1}{3}\left(2^{2(n+1)}-1\right) \\ & -\frac{1}{3}\left(2^{2(n+2)}-1\right) - \frac{1}{3}\left(2^{2(n+3)}-1\right) \\ & +i\left(2J_{2n+1} + (-1)^{n+2} \cdot 2^{n+3}J_1\right) \\ & +j\left(2J_{2n+2} - (-1)^{n+1} \cdot 2^{n+2}J_2\right) \\ & +k\left(2J_{2n+3} + (-1)^{n+1} \cdot 2^{n+2}J_1\right) \end{aligned}$$

and finally

$$\begin{aligned} JLQ_n \cdot JQ_n = & \frac{1}{3} \cdot 2^{2n}(1-4-16-64) + \frac{2}{3} \\ & +i\left(2 \cdot \frac{1}{3}\left(2^{2n+1} - (-1)^{2n+1}\right) + (-1)^{n+1} \cdot 2^{n+3}\right) \\ & +j\left(2 \cdot \frac{1}{3}\left(2^{2n+2} - (-1)^{2n+2}\right) - (-1)^{n+1} \cdot 2^{n+2}\right) \\ & +k\left(2 \cdot \frac{1}{3}\left(2^{2n+3} - (-1)^{2n+3}\right) + (-1)^{n+1} \cdot 2^{n+2}\right). \end{aligned}$$

In the same way we can prove (32). □

Now we give the matrix representation of Jacobsthal and Jacobsthal–Lucas quaternions.

Theorem 5. *Let $n \geq 1$ be integer. Then*

$$\begin{bmatrix} JQ_n & JQ_{n-1} \\ JQ_{n+1} & JQ_n \end{bmatrix} = \begin{bmatrix} JQ_1 & JQ_0 \\ JQ_2 & JQ_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{n-1}.$$

Proof. (by induction on n) If $n = 1$ then the result is obvious. Assume that

$$\begin{bmatrix} JQ_n & JQ_{n-1} \\ JQ_{n+1} & JQ_n \end{bmatrix} = \begin{bmatrix} JQ_1 & JQ_0 \\ JQ_2 & JQ_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{n-1}.$$

We shall show that

$$\begin{bmatrix} JQ_{n+1} & JQ_n \\ JQ_{n+2} & JQ_{n+1} \end{bmatrix} = \begin{bmatrix} JQ_1 & JQ_0 \\ JQ_2 & JQ_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{n}.$$

By simple calculation using induction's hypothesis we have

$$\begin{bmatrix} JQ_1 & JQ_0 \\ JQ_2 & JQ_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} JQ_n & JQ_{n-1} \\ JQ_{n+1} & JQ_n \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} JQ_n + 2 \cdot JQ_{n-1} & JQ_n \\ JQ_{n+1} + 2 \cdot JQ_n & JQ_{n+1} \end{bmatrix} = \begin{bmatrix} JQ_{n+1} & JQ_n \\ JQ_{n+2} & JQ_{n+1} \end{bmatrix},$$

which ends the proof. □

Theorem 6. *Let $n \geq 1$ be integer. Then*

$$\begin{aligned} JQ_n^2 - JQ_{n+1} \cdot JQ_{n-1} &= (7 - 3i + 3j + 9k) \cdot (-2)^{n-1} \\ &= \left(JQ_1^2 - JQ_2 \cdot JQ_0\right) \cdot (-2)^{n-1}. \end{aligned}$$

Proof. Let $JQ_n = J_n + iJ_{n+1} + jJ_{n+2} + kJ_{n+3}$, $JQ_{n+1} = J_{n+1} + iJ_{n+2} + jJ_{n+3} + kJ_{n+4}$ and $JQ_{n-1} = J_{n-1} + iJ_n + jJ_{n+1} + kJ_{n+2}$. Then

$$\begin{aligned} JQ_n^2 - JQ_{n+1} \cdot JQ_{n-1} = {} & J_n^2 - J_{n+1}^2 - J_{n+2}^2 - J_{n+3}^2 - J_{n+1}J_{n-1} \\ & + J_{n+2}J_n + J_{n+3}J_{n+1} + J_{n+4}J_{n+2} \\ & + i\left(J_{n+1}J_n - J_{n+2}J_{n-1} - J_{n+3}J_{n+2} + J_{n+4}J_{n+1}\right) \\ & + j\left(2J_{n+2}J_n - J_{n+1}^2 + J_{n+2}^2 - J_{n+3}J_{n-1} - J_{n+4}J_n\right) \\ & + k\left(3J_nJ_{n+3} - 2J_{n+1}J_{n+2} - J_{n+4}J_{n-1}\right). \end{aligned}$$

Using Binet formula (15) we obtain

$$JQ_n^2 - JQ_{n+1} \cdot JQ_{n-1} = (7 - 3i + 3j + 9k) \cdot (-2)^{n-1}.$$

Moreover,

$$JQ_1^2 - JQ_2 \cdot JQ_0 = 7 - 3i + 3j + 9k,$$

which ends the proof. □

Compliance with Ethical Standards

Conflict of Interest
The authors declare that they have no conflict of interest.

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