Eliana Manuel Pinho

CMUP – Centro de Matemática da Universidade do Porto
Faculdade de Arquitectura da Universidade do Porto
Rua do Gólgota, 215
4150-755 Porto, PORTUGAL
empinho@gmail.com

João Pedro Xavier*

*Corresponding author
CEAU – Centro de Estudos em Arquitectura e Urbanismo da Faculdade de Arquitectura da Universidade do Porto
Rua do Gólgota, 215
4150-755 Porto, PORTUGAL
jpx@arq.up.pt

Research

Grid-Based Design in Roman Villas: A Method of Analysis

Presented at Nexus 2012: Relationships Between Architecture and Mathematics, Milan, 11-14 June 2012

Abstract. The perceivable regularity of some Roman villas can be understood in the context of a grid-based design. In this paper we try to clarify the requirements under which we may consider that a villa has an outline based on a grid and we quantify the accuracy of the correspondence between a villa's plan and a given grid. We follow this approach with some Roman villas in Portugal and use the grids as tools for the analysis and the reconstruction of their plans.

Keywords: Roman architecture, grids, modules, design analysis, Pythagorean triangle, villas

1 Introduction

When studying Roman villas, we often find that their overall layout exhibits a perceivable degree of regularity, which becomes even more evident when we superimpose a grid on the plan of a villa. We also observe that the measures of individual rooms usually do not yield integer numbers of Roman feet, or of any others ancient units of length, because the thickness of the walls occupies part of the standard areas provided by the regular division of the space.

Generally, given a grid, we accept that a wall is in accordance with the grid if its direction corresponds to one direction of the grid, and there is a line that either goes through the inside of the wall, or along one of its faces. This wall-by-wall study requires a careful translation into a global analysis of the villa. Therefore, our first goal is to suggest a method to measure the accuracy with which a particular villa fits into a grid-based structure.

The process of looking for a grid that describes the regularity of a villa is a good method of analysis, revealing different sections of the plans – for example, only one section fits a given grid while the others are irregular or fit another grid, or there are portions of the plan whose correspondence to a grid is much more accurate than the rest – and enhancing particular strategies in the definition of the original plan, such as stretching, shrinking or distorting shapes. After having accepted a grid-based design, and established a grid with a given tolerance, that grid is a basic tool for the reconstruction of the villa's plan.

1.1 A non-ideal grid

Given a villa, we ask if there was the previous definition of a grid, on the terrain, when it was laid out. This could be due either to the existence of a grid-based project or to a practical procedure, as from a layout whose measures were not totally determined.

When we look for proportions, or other mathematical ideas, in a given building, we must be aware that rigorous mathematical content is garbled by the several steps between

DOI 10.1007/s00004-012-0138-0; *published online* 22 December 2012

the initial idea to its material achievement. In particular, the plotting of a grid on the ground is subject to errors and deviations of measurements, features that we must take into account when analysing the remains in the search for a grid pattern that structures the building. One way of dealing with the accuracy of the implementation of a conceptual tool, such as a grid, is to quantify the tolerance we expect in the actual measurements, given the ideal ones.

Identifying a conceptual intention from material vestiges always involves a degree of tolerance, even if this is not taken explicitly. Faced with the regular remains of a Roman villa, we can accept that they match a grid-based design within a certain tolerance, and reject the same hypothesis in the context of a narrower admission of variation. This is particularly relevant when several cases are to be compared in order to draw conclusions.

Our analysis rests on the value of a tolerance, which we denote by λ, for the measurements in the layout of the grid on the terrain. A tolerance of λ, or 100λ per cent, means that the bias in measuring the ideal length d, and marking it on the ground, can reach up to $d\lambda$. For example, $\lambda=1/100=0.01$ if we work with 1% of tolerance. This overall tolerance is naturally reflected in the linear dimensions of the building, but also in its angles, i.e., parallelism and perpendicularity between lines. With regard to a grid, the equally spaced parallel lines turn out to form acute angles, perpendicularity is slightly distorted, and the characteristic periodicity of measures also suffer some deviations. We describe the variations of these observable dimensions that are due to inaccuracies, and express them as a function of the tolerance λ: to describe how inaccuracy in the definition of the grid affects a length, a set of ideally equal lengths, and the angles formed by two directions that were intended to be either the same or perpendicular.

With this tool, the acceptance of a deviation from the ideal grid is framed within a certain tolerance, allowing the comparison between cases, and the determination of the required tolerance in order to accept the grid-based design for a particular building. Moreover, the control of the maximum effects of inaccuracy in the measurements, provides a method for distinguishing between inaccuracy and other factors that can cause deviations. For example: to decide whether two nonparallel walls forming a small angle were or were not, intended to be parallel (see Pisões villa in section 3.1); to justify, in the context of a grid-based design villa, the deviation of a small set of walls, or rooms (see the villa in Quinta das Longas, in section 3.2).

The method described above quantifies the maximum errors in the layout of a construction as a percentage of the measures. This is a robust way of dealing with the deviations from an ideal plotting, particularly if the actual procedures of the constructors for the definition of the lengths are partially unknown. Although other statistical measurements may change, the maximum deviations from the intended length, in the context of a given tolerance, are the same either for an additive process or for the definition of total lengths followed by a subdivision. Analogously, the use of different gauges during the process do not influence the maximum deviations as a percentage of the lengths (see the end of section 2.1 for some considerations on this subject).

For the analysis, a given ideal grid is translated into a “blur” containing the ideal grid and whose thickness depends on the considered tolerance λ. Is this “smudged” grid that we superimpose on the remains to look for a correspondence. If we accept this correspondence we are accepting the existence of a grid-based design under the hypothesis that the original constructors had established the grid with an accuracy quantified by λ.

1.2 Description of the present article

In section 2 we make the calculations for the dependence on the tolerance of the marked lengths (section 2.1), of the determination of perpendicular directions (section 2.2) and of the overall relations between the elements that characterise a grid: equal lengths, parallelism and perpendicularity (section 2.3). We conclude this section with a summary, shown in tables 4 and 5, both for the general case, as functions of λ, and for the particular values of 0.5% and 0.6%. The detailed calculations are performed in Appendices 1, 2 and 3, to lighten the pathway to the main conclusions of section 2. The appendices present step-by-step calculations, and will be useful for those who attempt to adapt our approach to analysis of different features.

In section 3 we follow the several phases of analysis when searching for the existence of a grid-based design in a villa. We separate the analysis in two main steps, according to the inner nature of a grid – its orientation (section 3.1), and the size of its basic cell, i.e., the standard area that is defined by the repetitive drawing of equally spaced lines along two perpendicular directions (section 3.2). We present examples of Roman villas in Portugal that fit within a grid-based design, for a square grid measuring 5 Roman feet.

2 Accuracy and grids

What was the accuracy in the plotting of a building in Roman times? Mark Wilson Jones warns that in Roman architecture the "standards of accuracy differ enormously from building to building, and even within the same one" [2000: 71]. Within this wide range, Jones claims that "normally, however, tolerances may be expected of up to 0.5 per cent for medium-to-large distances, and perhaps a bit more where the materials are brick and concrete as opposed to good quality stone" [2000: 72].

It is necessary to take various factors into consideration, especially when far away from Rome: the different regions of the Roman world, different periods, types of buildings or sections of buildings (public/private, cult/domestic, representational/utilitarian), since all these characteristics play a hierarchical role that will be reflected in the relative accuracies in different sections, and on the actual shape of the remains.

For a grid-based design, we expect to find a bias in the plotting of the grid that rules the plan.

We consider either square or rectangular grids, and focus on two main features that depend on the accuracy of measures:

- the definition of the lengths, in particular the plotting of equally spaced marks along a line, and
- the determination, given a line, of its perpendicular direction.

2.1 Marking the rhythm of the grid

Suppose we are plotting a grid-based design building. The definition, on the ground, of the basic grid starts with a line having equally spaced points (fig. 1).

Fig. 1. The ideal length *d* defines equally spaced points along the horizontal direction

After drawing the first line with a point we call the origin (point with abscissa zero in fig. 2), we intend to mark n points along a given direction (the horizontal direction in figs. 1 to 3), equally spaced the distance d. However, the plotting of these initial markings will be affected by inaccuracies. The actual marks p_0, p_1, p_2, ... , p_n are spaced, respectively, the approximated values d_1, d_2, ... , d_n.

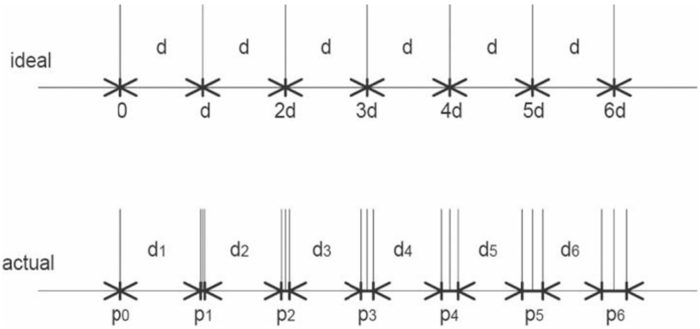


Fig. 2. The ideal marks 0, d, $2d$, ... , $6d$ are well defined points at a distance d, while the actual ones 0, p_1, p_2, ... , p_6 depend on the measures d_1, d_2, ... , d_6 affected by a random error, and are, thus, somewhere at the represented intervals. In this example the inaccuracy is up to 2.5%

For a given tolerance λ, the markings on the ground have the following properties (we refer to Appendix 1 for a step-by-step explanation):

A. The distance between two consecutive marks is larger than $d(1-\lambda)$ and smaller than $d(1+\lambda)$, i.e., varies in a range of $2\lambda d$.

B. The distance between a given mark p_i and the origin is larger than $id(1-\lambda)$ and smaller than $id(1+\lambda)$, i.e., varies in a range of $2i\lambda$.

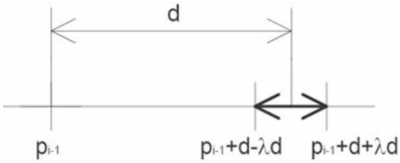


Fig. 3. Given the mark p_{i-1}, the next mark is p_i in the interval $[p_{i-1}+d-\lambda d, p_{i-1}+d+\lambda d]$, represented in bold

Suppose the constructors change the gauge during this process. For example, after measuring two times the length d, they begin to use a new gauge with length $2d$. Shall we define tolerance in a different way in order to capture this other possible procedure? In fact, as far as the maximum deviations are concerned, given the tolerance λ, everything remains. Since the maximum inaccuracy for the new gauge is $2d\lambda$, the defined lengths would correspond to the even markings in fig. 2. After using the $2d$ gauge, the workers could define the intermediate odd markings, a procedure where property A holds. Similarly, if the first gauge is the total length, nd, its maximum error is $nd\lambda$ and the subsequent divisions follow the properties described above, with regard to maximum deviations. That is one major advantage of quantifying the maximum errors by way of a percentage of the measures. Thus, the percentages will establish the same maximum deviations whether the lengths are marked by additive processes or if a large length is divided into smaller parts.

2.2 Determining perpendicular directions

In this section, we are going to consider a given accuracy and ask how it affects the perpendicularity of the lines defining a grid. The translation of this accuracy to the angles amplitudes, must rest on the procedures that the Romans followed in the definition of right angles.

Some known constructions of the right angle are based on particular right triangles:

A. Pythagorean triangles, i.e., right triangles with integer side lengths, in particular the 3-4-5 triangle;

B. the isosceles right triangle, whose hypotenuse corresponds to the diagonal of a square defined by the catheti [Schneider 2002].

As an example of procedure A, consider the description by Vitruvius:

> If we take three rules, one three feet, the second four feet, and the third five feet in length, and join these rules together with their tips touching each other so as to make a triangular figure, they will form a right angle [1960: IX, Introduction, p. 253].

The right angle is of course that formed by the 3-foot and 4-foot rods. Vitruvius describes the important quality of this device:

> … a right angle can be formed without the contrivances of the artisan. Thus, the result which carpenters reach very laboriously, but scarcely to exactness, with their squares, can be demonstrated to perfection [1960, IX, Introduction, p. 252-253].

Instead of three rods, or ropes, a procedure that prevailed until now, we can use a circular rope with 12 equally spaced knots, to which is given a triangular shape with sides 3, 4 and 5. Other right triangles with integer side lengths were known since Babylonian times [Kline 1972: 10]. However, we have found no reference concerning their use in Roman building procedures.

Given a line segment with, say, 10 feet, we can draw a perpendicular line if we know how to measure a $10\sqrt{2}$ feet line segment, the diagonal of the square with sides 10 feet long. Thus, procedure B depends on the approximations of the ratio $1:\sqrt{2}$ or, analogously, the ratio of side and diagonal of a square.

Faventius, a third century commentator on Vitruvius, describes a way of establishing a right angle based on the approximate ratio 12:17 of side and diagonal of a square:

> Since the principle of the square was a clever discovery and useful for all purposes – since, indeed, nothing can be done very practically without it, this is how you will prepare one. Take three scales, two of them each 2 foot long, the third 2 foot 10 inches. They are all to be of one uniform width, and are to be joined at the ends to give the shape of a triangle. Your square will thus be made to professional standards [Plommer 1973: 81].

Since one inch is 1/12 of the foot, Faventius suggests the ratio

$$\frac{2}{2+\frac{10}{12}}=\frac{24}{24+10}=\frac{24}{34}=\frac{12}{17}$$

for side and diagonal of the square.

Approximations for the ratio $1:\sqrt{2}$ existed in Mesopotamia and Egypt, long before Rome [Kline 1972], and the successive approximations of 5:7=1:1.4, 12:17=1:1.41(6), 29:41≈1:1.4138, … , were known in Roman times [Fowler 1999, in particular the passage from Theon of Smyrna, pp. 56-57]. A direct reference to their use by Roman constructors is found in the description of Faventius.

The right angles so obtained are affected by errors that depend on the accuracy of the measurements. We now describe how the errors on the sides of a right triangle affect the accuracy of the definition of its right angle.

Consider a general right triangle with sides *a*, *b* and *c*, where *c* is the hypothenuse (fig. 4). The drawing of such a triangle will be affected by the accuracy we can achieve in the lengths of the sides. The drawn triangle $a_1b_1c_1$ differs from the ideal one; in particular, to the ideal right angle corresponds an angle measuring α+90°. For the purposes of our study, it is important to relate the bias α with the tolerance in the measuring of the sides *a*, *b* and *c*.

Let λ be the overall tolerance in the determination of the lengths. To the intended length *a* corresponds a real length, say a_1, that is bigger than $a-\lambda a$ and smaller than $a+\lambda a$ or, formally,

$$a_1 \in [a(1-\lambda), a(1+\lambda)].$$

An analogous interval of possible values exists for each length, also defined up to an error that depends on λ.

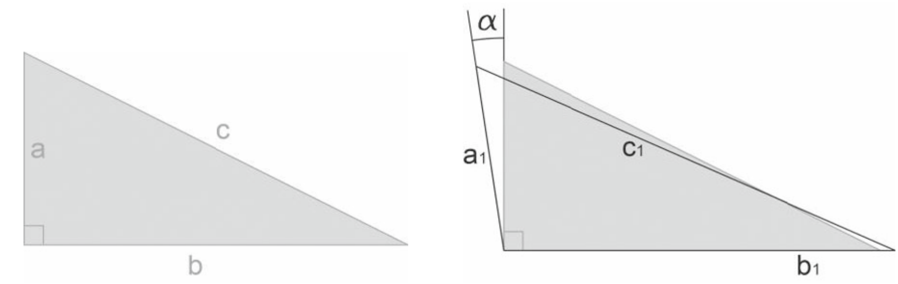


Fig. 4. The ideal right triangle *abc* and another triangle $a_1b_1c_1$, whose corresponding sides differ slightly, having the angle α+90° opposed to side c_1. In this example we have $\alpha>0$

The inaccuracy in the definition of the lengths *a*, *b* and *c* causes a bias α from the right angle. We show in Appendix 2 that method A, based on the 3-4-5 Pythagorean triangle, leads to a variation in the angle that is expected to be 90° in the interval $[-\alpha_m, \alpha_m]$, where

$$\frac{2(a^2+b^2)}{ab} \cdot \frac{180}{\pi} \cdot \lambda$$

is a good approximation for α_m, for small values of λ; see table 1.

tolerance measuring sides	deviation
0.5%	[-1.2°–1.2°]
1%	[-2.4°–2.4°]

Table 1. The range of deviation from the right angle, as a function of the tolerance in the measurements, when it is defined by way of the 3-4-5 triangle

For the second method, and using 5:7 as the approximation to the ratio1:√2, we have the range of deviations from the right angle described in table 2.

tolerance measuring sides	deviation
0%	−1.1°
0.5%	[−2.3°,0.0°]
1%	[−3.4°,1.1°]

Table 2. The range of deviation from the right angle, as a function of the tolerance in the measurements, when it is defined by way of the triangle whose side and diagonal have the ratio 5:7

If the ratio 1:√2 is approximated by 12:17, we obtain the values in table 3.

tolerance measuring sides	deviation
0%	0.2°
0.5%	[−0.9°,1.3°]
1%	[−2.1°,2.5°]

Table 3. The range of deviation from the right angle, as a function of the tolerance in the measurements, when it is defined by way of the triangle whose side and diagonal have the ratio 12:17

2.3 The overall grid construction

Now we present an estimate for some of the expected deviations in the plotting of a grid on the terrain. In fig. 5 we describe a possible way of marking a grid. Then we describe the several steps as well as the deviations that can occur.

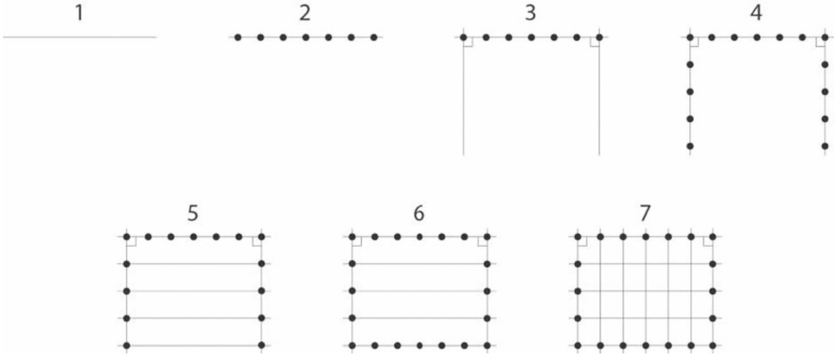


Fig. 5. Marking, step by step, a grid on the terrain

Step 1. Definition of a main direction. In this article we suppose that there are no errors in the drawing of this straight line. However, when we want to compare the orientation of a building with the cardinal points, we must consider a possible error of orientation in this first step.

Step 2. Marking the length of the first line, along with its subdivisions. In this step we consider the deviations described in section 2.1. Thus, in the context of a tolerance λ, any marking with intended length d will, at most, suffer a deviation of λd. Larger deviations should be justified. Recall that the maximum deviations that we are taking into account do not change for different procedures, whether an additive process, the change of the gauge during the process, or the marking of a total length followed by subdivisions.

Step 3. In this step we consider the determination of right angles and suppose that at least two lines must be plotted which are perpendicular to the first line. The maximum deviation from the right angle, α_m, treated in section 2.2, implies that the two lines drawn in this step can form an angle up to 2 α_m (fig. 6). Recall that

$$\alpha_m = \frac{25}{12} \cdot 2\lambda \cdot \frac{180}{\pi}$$

if the right angle is determined via the 3-4-5 Pythagorian triangle. If an approximate isosceles right triangle is used, with side and diagonal ratio 12:17, then

$$\alpha_m = \left[\frac{289}{288}(1 + 4\lambda) - 1 \right] \cdot \frac{180}{\pi} \cdot$$

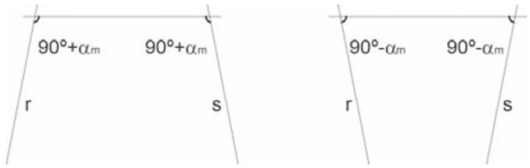


Fig. 6. Lines r and s form, with the initial line, an angle close to the right angle, with a maximum deviation of α_m. The maximum angle between r and s is, therefore, 2 α_m and happens in the two cases depicted in this figure

Step 4. In this step, as in step 2, the maximum deviation is a function of the tolerance and does not depend on the particular procedures of the constructors. The errors in the markings of the two lines can cause one of them to be larger than the other. If the total length is intended to be d then the maximum difference between the lengths of the two lines, occur when one of them measures $d(1-\lambda)$ and the other measures $d(1+\lambda)$. Therefore, the difference between the lengths of the two lines is smaller than $2d\lambda$. However, if we measure the distances between the first wall and the end of each one of the walls defined in this step, they can differ even more, see fig. 7. These distances, measured in a direction perpendicular to the first wall, can vary up to $d[(1+\lambda)-(1-\lambda)\cos(\alpha_m)]$, which, for small values of λ is approximately $2d\lambda$.

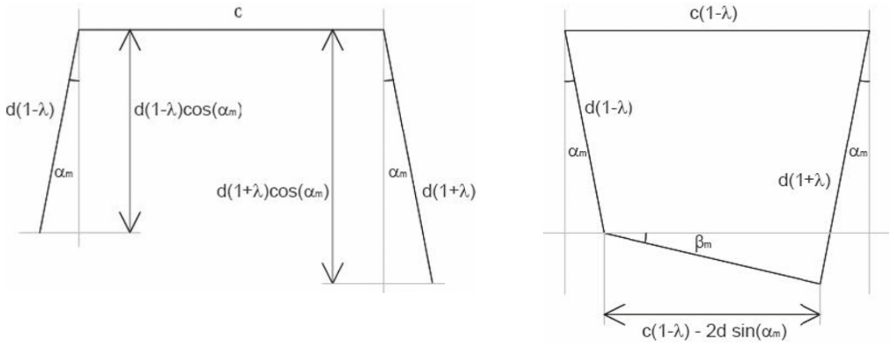


Fig. 7. In steps 4 and 5, deviations in the measurements of lengths and angles affect the distances and thus lead to the existence of an angle α_m where parallel lines were intended

Step 5. The new lines drawn in this step depend, both in length and direction, on the previous marks. If the line at the top measures *c*, then the maximum and minimum possible lengths of the line at the bottom are $c \pm 2d(1+\lambda)\sin(\alpha_m)$, and correspond to the cases depicted in fig. 6, with lines r and s measuring $d(1+\lambda)$. The maximum angle between the line at the top and that at the bottom is β_m, which, for small values of λ is

$$\beta_m \approx 2k\lambda \cdot \frac{180}{\pi},$$

where k is the ratio between the lengths d and c (fig. 7 and Appendix 3). Unless the length d is larger than $4c$, this angle will be smaller than the one calculated in step 3, the maximum angle between two lines that were intended to be parallel. The maximum angle between the lines at the bottom and at left, intended to be perpendicular lines, is $90°+\alpha_m+\beta_m$, which is larger than α_m, calculated in step 3.

Step 6. In this step some errors can possibly be cancelled but we do not deal with this hypothesis since our approach quantifies the maximum deviations from the intended ideal grid.

Step 7. In this step we consider that no errors are added to the construction.

After this analysis, we summarise the largest possible deviations for each of the observable dimensions in the next tables.

	length d	difference between two lengths d	right angle deviation	angle between parallel lines
λ	$d\lambda$	$2d\lambda$	$\alpha_m+\beta_m$	$2\alpha_m$
0.005	0.005λ	$0.01d$	1.8°	2.4°
0.006	0.006λ	$0.012d$	2.1°	2.9°

Table 4. Maximum deviations for each of the observable dimensions, calculating right angles with the 3-4-5 Pythagorean triangle

	length d	difference between two lengths d	right angle deviation	angle between parallel lines
λ	$d\lambda$	$2d\lambda$	$\alpha_m+\beta_m$	$2\alpha_m$
0.005	0.005λ	$0.01d$	1.9°	2.7°
0.006	0.006λ	$0.012d$	2.3°	3.2°

Table 5. Maximum deviations for each of the observable dimensions, calculating right angles with the approximate isosceles right triangle with catheti and hypothenuse ratio 12:17

3 Analysis with examples

Now we consider the process of analysing the remains of a Roman villa, looking for the possibility of a grid-based design.

3.1 Orientation

The first decision has to do with the orientation of the orthogonal grid. We look for some correspondence with a grid whenever a villa exhibits a certain regular pattern with the walls mainly along two orthogonal directions. Sometimes a villa is divided into sections that clearly correspond to different orthogonal grids. This is the case of the Roman villa of Pisões, located near Beja, Portugal – a richly decorated villa that was inhabited from the first and until the fourth centuries. In Fig. 8 two sections, with clearly distinct orientations, are indicated.

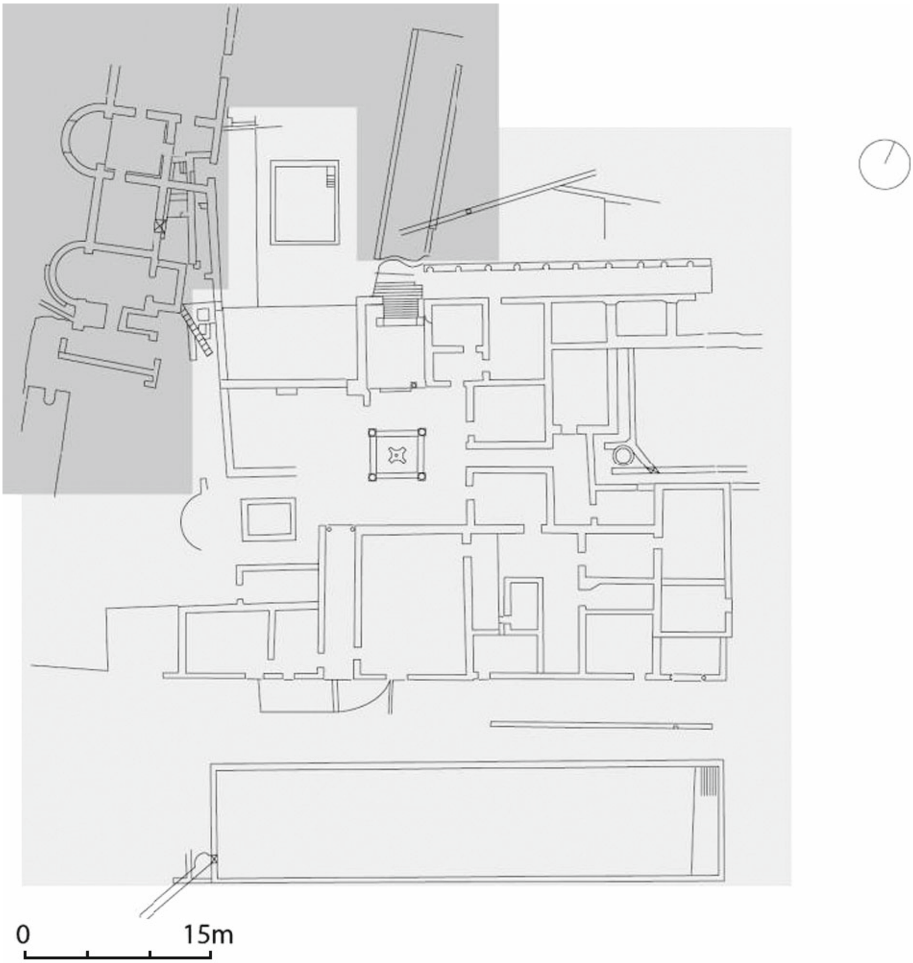


Fig. 8. Two sections with distinct orientations of the walls

However, if the difference between two directions is very small we must try to understand, based on available data on the construction (dating of the ruins, several phases of construction), if this follows from an accidental deviation or if it can be justified by two different directions.

What should the threshold be for distinguishing between a deviation from a given grid and a different section of the villa, in the absence of evidence proving more than one phase of construction? We use tables 4 and 5 to have the magnitude of the deviations due to the lack of accuracy in the measures. Thus, for a tolerance of 0.5%, we see that ideally parallel lines can form an angle up to 2.4° or 2.7°, depending on the procedures used to determine right angles. We posed this question about Pisões, since there is a small angle between two sets of elements, depicted in fig. 9. We see that the deviation from parallelism is 1.3°, which is framed within a tolerance smaller than 0.5%. In fact, we can calculate the minimum tolerance required to accept this deviation from parallelism, λ such that $2\alpha_m$=1.3°, approximately 0.2%.

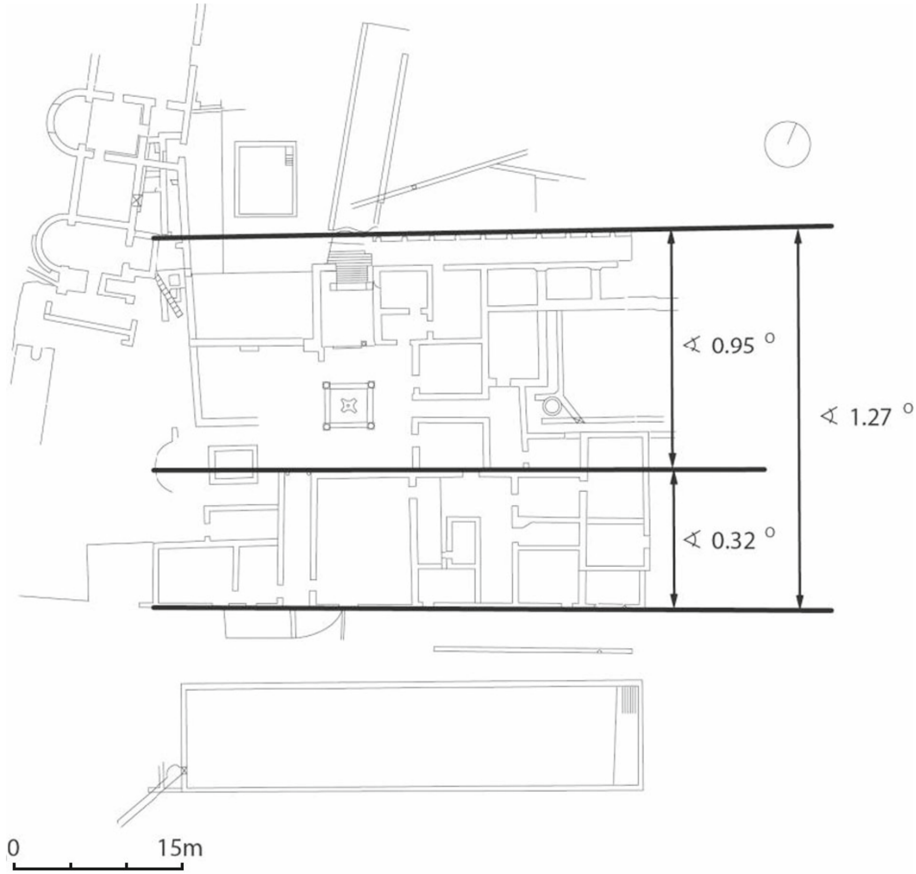


Fig. 9. Walls that are close but not exactly parallel. Can the angle of deviation be justified by the accuracy of measure in the plotting, or should we look for other reasons supporting an intended different direction?

After this analysis, we end up with sections of the plan that have well defined orientations and proceed to the next step.

3.2 Measure of the grid

Suppose we are analysing a section of a villa with walls along two orthogonal directions, i.e., a section where the orientation is well defined, up to the deviations that we accept in the context of a given tolerance. We must look for a rhythm that fits the regularity of the walls or, equivalently, we must try to determine the measure of a grid that fits the design of the remains.

Consider the Roman villa in Quinta das Longas, a farm near the small village São Vicente e Ventosa, Alentejo. In fig. 10 we present the plan of the remains of the second Roman villa that was built in this place, inhabited from the end of the third century and until the beginning of the fifth century. We easily find that a square grid measuring 10 Roman feet fits broadly some sections of the villa. Thus, we proceed to a more accurate analysis using grids related to this, with lengths of either 5 or 15 Roman feet.

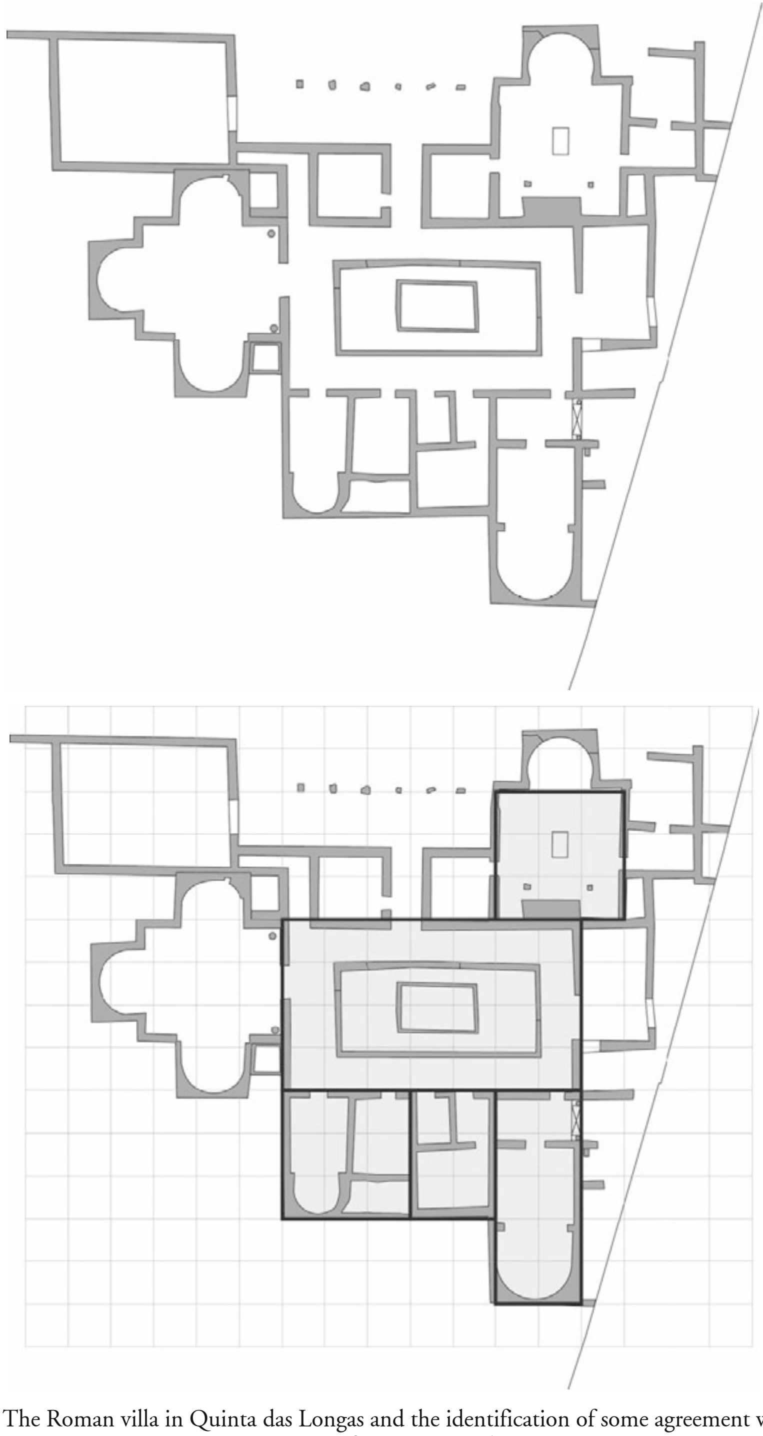


Fig. 10. The Roman villa in Quinta das Longas and the identification of some agreement with a 10 Roman foot square grid

There is a good agreement with a 5 Roman foot square grid (fig. 11). In this figure we used a dark grey colour for the walls that lie on the lines of that grid, either because their boundaries are provided by those lines or because there is a line passing in the interior of the wall along its length. These would be accepted in the context of a 0% tolerance in the plotting of the grid. However, for a non-zero actual tolerance, we consider a smudge grid, whose ideal lines are replaced by narrow stripes whose widths are determined by the intervals described in section 2.1. In practice, we perform the analysis with grids that are close to the 5 Roman foot square grid, up to a given tolerance, and the set of lines of all these grids, when seen altogether, define the stripes we referred. We concluded that a tolerance of 0.5%, which leads to grids with standard lengths in the interval [5−5*0.005, 5+5*0.005]=[4.975, 5.025], in Roman feet, is enough to include the majority of the walls.

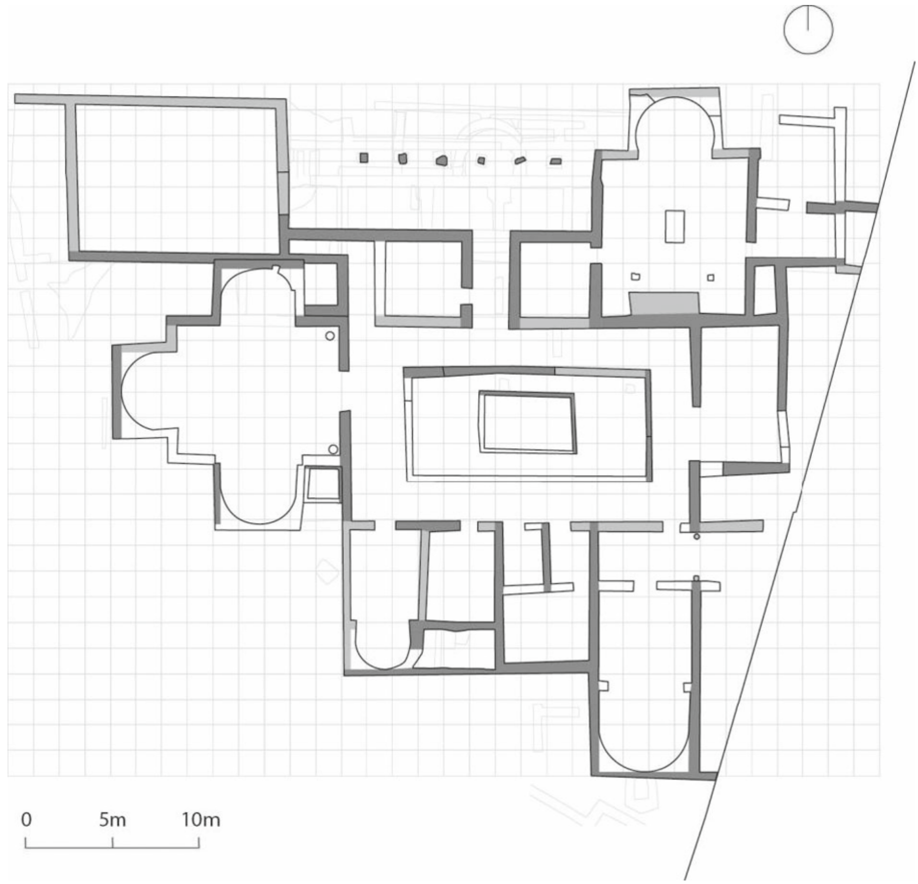


Fig. 11. The Roman villa in Quinta das Longas with a 5 Roman foot square grid. The walls in dark grey coincide with lines of the grid, while the ones in light grey are within a tolerance of 0.5%

There are some walls that do not fit the 5 Roman foot square grid with tolerance 0.5%, the ones left in blank in the figure. However, they can be framed within metrics that are in agreement with that grid, as we now describe. The semi-circular walls in the triple-apsidal triclinium and in the semi-circular recess of the north inner courtyard, have diameters of 15 Roman feet. Moreover, the semi-circular form that lies on the axis of symmetry for each of these two divisions has a centre on that axis. Further, the axis is itself a line of the accepted grid.

The walls facing east are irregular. This may be due to the constructions of the diagonal wall, which partially destroyed the remains. The garden bed and lake in the central rectangular open yard are symmetrically disposed in relation to the east-west axis that crosses this space. The antechamber, with two columns, of the winter triclinium at the southeast area, has width of 10 Roman feet, while the room next to the colonnade is broadly a square measuring 15 Roman feet. As we remarked in the introduction, following a grid leads to irregular sizes of the rooms. So, if regular sizes are intended, to ensure symmetry, for example, we expect that some walls are taken out the grid scheme, as in these last examples. However the measurements are in agreement with the overall approach of a regular division of space having a basic length of 5 Roman feet.

The distance between the axis of two consecutive columns, in the north colonnade, is 7.5 Roman feet. This is in agreement with a 15 Roman foot grid, which reinforces our conclusion that a 5 Roman foot module was intended.

3.3 grids as tools

The superimposition of grids on the plan of a villa's remains constitutes a method of analysis in itself. Recalling the example in fig. 8, the two sections with different orientations correspond to two distinct phases of construction. Although in Pisões the two sections are easily observed, other villas can exhibit subtler vestiges of their different phases. In São Cucufate, Vidigueira, three Roman villas were constructed, respectively in the first, second and fourth centuries. The remains of Villa 2 are shown in fig. 12, and their correspondence with a 5 Roman foot square grid are colour coded. Dark grey walls, fitting the grid, are mainly in the right side of the plan, and highlight a section where the structure of the earlier villa is used. In fact, the pars urbana of Villa 1 has a grid-based design, for a 5 Roman foot square grid, a feature that is not followed in Villa 2, for the sections not using the previous existing walls. The grid is, thus, one more tool for the analysis and classification of the remains.

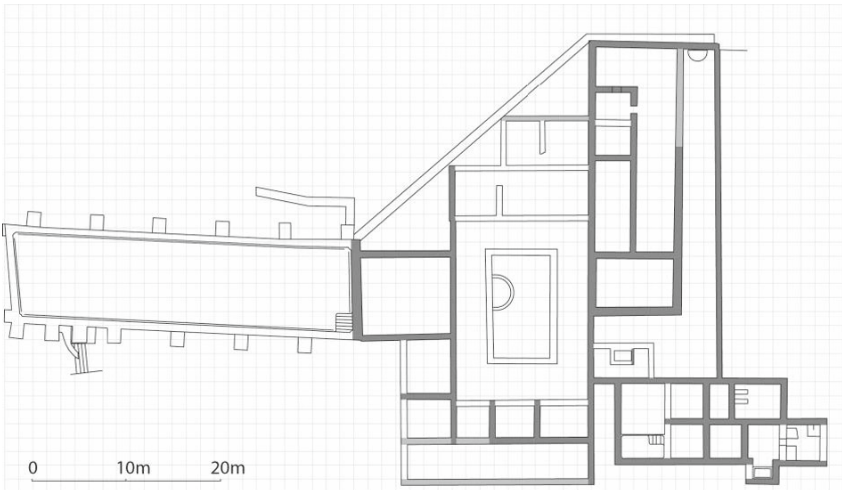


Fig. 12. *Pars urbana* of Roman Villa 2 in São Cucufate, with a 5 Roman foot square grid. The walls in dark grey coincide with lines of the grid, the ones in light grey are within a tolerance of 1%, and the walls left in blank are within a larger tolerance. The correspondence with the grid is very accurate on the right side – a trace of the earlier villa, Villa 1, from the first century

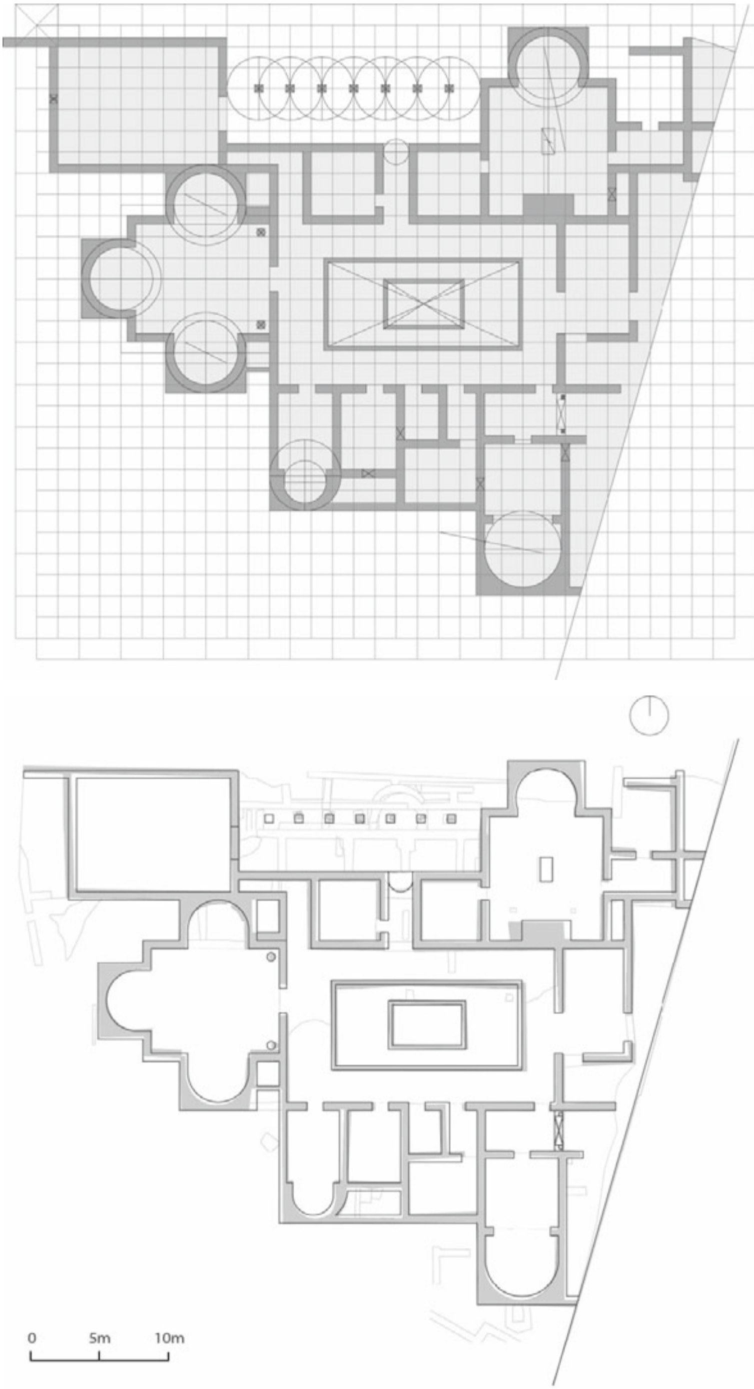


Fig. 13. The reconstruction of the Roman villa in Quinta das Longas, based on the acceptance of a grid-based design with a 5 Roman foot square grid. The key for the reconstruction is at the top and, at the bottom, the comparison between the reconstruction (black lines) and the remains

3.4 Reconstruction of grid-based design plans

The reconstruction of the plan of a given villa rests on the results of the analysis previously performed. If a grid-based design is accepted, the underlying grid is a basic outline for the drawing of the ideal plan. In fig. 13 we make a proposal for the plan of the villa in Quinta das Longas according to a plan based on a grid, and compare it, at the bottom, with the actual remains. At the top we present the key to the drawing. The main walls face lines of the grid, except for the winter triclinium, at south-east, whose walls have a grid line along the middle of their width. The axes of the divisions are then determined, and the features that are intended to have central positions – lake, apse – are added. We consider integer measures to define some missing shapes, like the width of doors, and the antechamber with two columns. The apses and the north colonnade are based on 15 Roman foot circumferences.

Appendix 1

In this appendix we refer to figs. 1 to 3 and properties A and B in section 2.1.

The abscissae of the intended marks would form the sequence

$$0$$
$$d$$
$$2d$$
$$\dots$$
$$nd$$

as depicted in the upper part of fig. 2.

However, instead of measuring n times the exact distance d, we will measure d_1, d_2, … , d_n, where

$$d_1 = d + \varepsilon_1$$
$$d_2 = d + \varepsilon_2$$
$$\dots$$
$$d_n = d + \varepsilon_n$$

and the random small errors ε_i vary up to a value that is a percentage of the distance d. Let λ be the accepted percentage of error in a measure, or tolerance. Thus the error in the measure of d is up to λd, and $\varepsilon_i \in [-\lambda d, \lambda d]$, for all i = 1, 2, … , n, which are intervals with width $2\lambda d$.

Instead of equally spaced marks we plot the set of points

$$p_0 = 0$$
$$p_1 = d_1$$
$$p_2 = d_1 + d_2$$
$$\dots$$
$$p_n = d_1 + d_2 + \dots + d_n$$

whose properties A and B we now justify.

Property A is a direct consequence of measuring the distance d once. The abscissae of two consecutive marks, p_{i-1} and p_i, are related by $p_{i-1}+d_i=p_i$, and thus are distant

$p_i - p_{i-1} = d_i = d + \varepsilon_i$, a value that belongs to the interval $[d-\lambda d, d+\lambda d] = [d(1-\lambda), d(1+\lambda)]$ with width $2\lambda d$ (see fig. 3).

Property B is a consequence of measuring i times the distance d. The marks ideally placed at the points 0, d, $2d$, … , nd will be affected by an error and their actual position will be

$$\begin{aligned} 0 &= p_0 & & \\ d_1 &= p_1 & \in \quad & [d(1-\lambda), d(1+\lambda)] \\ d_1 + d_2 &= p_2 & \in \quad & [2d(1-\lambda), 2d(1+\lambda)] \\ & & \ldots & \\ d_1 + d_2 + \ldots + d_n &= p_n & \in \quad & [nd(1-\lambda), nd(1+\lambda)] \end{aligned}$$

where the intervals have the sequence of widths

$$\begin{gathered} 0 \\ 2\lambda d \\ 4\lambda d \\ \ldots \\ 2n\lambda d \end{gathered}$$

as illustrated in fig. 2 for $n = 6$ and $\lambda = 0.0125 = 1.25\%$.

Appendix 2

Pythagorian triangle

Consider the ideal right triangle abc and the real triangle $a_1 b_1 c_1$, that we construct when trying to achieve the first triangle, under the accuracy of λ in the determination of lengths. Let $\alpha + 90°$ be the angle, opposite to side c_1, that corresponds to the ideal right angle, according to fig. 4. Thus, the lengths of the sides are in the intervals

$$\begin{aligned} a_1 \quad & \in \quad [a(1-\lambda), a(1+\lambda)] \\ b_1 \quad & \in \quad [b(1-\lambda), b(1+\lambda)] \\ c_1 \quad & \in \quad [c(1-\lambda), c(1+\lambda)] \end{aligned}$$

and the law of cosines states

$$\cos(\alpha + 90°) = \frac{a_1^2 + b_1^2 - c_1^2}{2 a_1 b_1},$$

for each triangle $a_1 b_1 c_1$. The minimum angle opposite to side c_1, denoted by $-\alpha_m + 90°$, corresponds to the triangle having the minimum length of the opposite side c_1, i.e., $c(1-\lambda)$ and the maximum lengths of the sides that correspond to the catheti, $a(\lambda+1)$ and $b(\lambda+1)$. According to the equation above, the value of α_m is given by:

$$\cos(-\alpha_m + 90°) = \frac{a^2(1+\lambda)^2 + b^2(1+\lambda)^2 - c^2(1-\lambda)^2}{2a(1+\lambda)b(1+\lambda)}$$

$$= \frac{(a^2+b^2)(1+\lambda)^2 - c^2(1-\lambda)^2}{2ab(1+\lambda)^2}$$

$$= \frac{(a^2+b^2)(1+\lambda)^2 - (a^2+b^2)(1-\lambda)^2}{2ab(1+\lambda)^2}, \quad \text{by the Pythagorean theorem}$$

$$= \frac{(a^2+b^2)}{ab} \cdot \frac{(1+\lambda)^2 - (1-\lambda)^2}{2(1+\lambda)^2}$$

$$= \frac{(a^2+b^2)}{ab} \cdot \frac{2\lambda}{(1+\lambda)^2}.$$

Since $\cos(-\alpha_m+90°) = \sin(\alpha_m)$, we have

$$\alpha_m = \arcsin\left(\frac{a^2+b^2}{ab} \cdot \frac{2\lambda}{(1+\lambda)^2}\right) \cdot \frac{180}{\pi}.$$

This expression can be simplified since we are considering small values of λ. Under this condition we have the following good approximation (the Taylor polynomial of degree one):

$$\arcsin\left(\frac{a^2+b^2}{ab} \cdot \frac{2\lambda}{(1+\lambda)^2}\right) \approx 2\lambda \cdot \frac{a^2+b^2}{ab}$$

and thus instead of the expression for α_m above, we can use the following:

$$\alpha_m = 2\lambda \cdot \frac{a^2+b^2}{ab} \cdot \frac{180}{\pi}.$$

Notice that the approximation is better for small values of $(a^2+b^2)/ab$, i.e., for triangles where the lengths a and b are similar. The best approximation therefore occurs for isosceles triangles and becomes worse for triangles with pointed tips. However, this is not relevant when we consider angle deviations in degrees up to hundredths.

Analogously, the maximum angle opposite to side c_1, denoted by $\alpha_m+90°$, corresponds to the triangle with the greatest c_1, $c(1+\lambda)$, and the smallest lengths a_1 and b_1, respectively $a(1-\lambda)$ and $b(1-\lambda)$. Thus

$$\cos(\alpha_m + 90°) = -\frac{a^2+b^2}{ab} \cdot \frac{2\lambda}{(1-\lambda)^2}$$

or, equivalently,

$$\alpha_m = \arcsin\left(\frac{a^2+b^2}{ab} \cdot \frac{2\lambda}{(1-\lambda)^2}\right) \cdot \frac{180}{\pi}$$

which can be approximated by

$$\alpha_m = 2\lambda \cdot \frac{a^2+b^2}{ab} \cdot \frac{180}{\pi}.$$

Approximate isosceles right triangle

Consider the isosceles triangle with two sides measuring a and the third measuring c, such that a/c approximates $1/\sqrt{2}$. We follow the previous calculations to determine how the deviations in lengths a and c affect the approximate right angle, opposite to side c. The extreme values of this angle are α+90° such that, for small values of λ, we have

$$\alpha \approx \left(\frac{c^2}{2a^2} - 1 \pm \frac{4c}{\sqrt{4a^2-c^2}} \cdot \lambda \right) \cdot \frac{180}{\pi}.$$

This results from the extreme deviations of lengths, as described above, either

$$\cos(-\alpha + 90°) = \frac{a^2(1+\lambda)^2 + a^2(1+\lambda)^2 - c^2(1-\lambda)^2}{2a(1+\lambda)a(1+\lambda)}$$

$$\Leftrightarrow \alpha = \arcsin\left(\frac{c^2}{2a^2} \cdot \frac{(1-\lambda)^2}{(1+\lambda)^2} - 1 \right) \cdot \frac{180}{\pi}$$

or

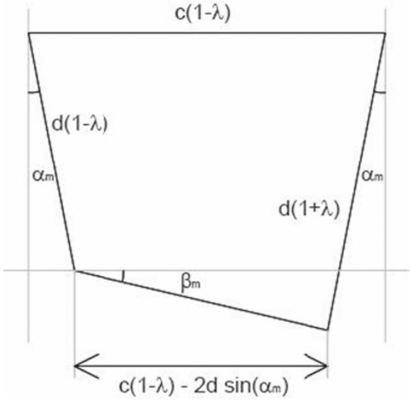
$$\cos(\alpha + 90°) = \frac{a^2(1-\lambda)^2 + a^2(1-\lambda)^2 - c^2(1+\lambda)^2}{2a(1-\lambda)a(1-\lambda)}$$

$$\Leftrightarrow \alpha = \arcsin\left(\frac{c^2}{2a^2} \cdot \frac{(1+\lambda)^2}{(1-\lambda)^2} - 1 \right) \cdot \frac{180}{\pi}.$$

To conclude, we calculate the Taylor polynomial of degree one and since $\frac{c^2}{2a^2} - 1 \approx 0$, we use the approximation $\arcsin\left(\frac{c}{2a^2} - 1 \right) \approx \frac{c^2}{2a^2} - 1$.

Appendix 3

We now calculate the angle β_m, described in the figure below:



Let r be the length of the line at the bottom. Since

$$r\cos(\beta_m)=c(1-\lambda)-2d\sin(\alpha_m)$$

and

$$r\sin(\beta_m) = d(1+\lambda)\cos(\alpha_m) - d(1-\lambda)\cos(\alpha_m) = 2d\lambda\cos(\alpha_m),$$

we have

$$\tan(\beta_m) = \frac{2d\lambda\cos(\alpha_m)}{c(1-\lambda)-2d\sin(\alpha_m)}$$

which leads to

$$\beta_m = \arctan\left(\frac{2k\lambda\cos(\alpha_m)}{(1-\lambda)-2k\sin(\alpha_m)}\right)\cdot\frac{180}{\pi},$$

where $k = d/c$.

For small values of λ we can use the approximation $\beta_m \approx 2k\lambda\cdot\frac{180}{\pi}$.

References

ALMEIDA, Maria José de and António CARVALHO. 1998. Ânforas da uilla romana da Quinta das Longas (S. Vicente e Ventosa, Elvas): resultados de 1990-1998. *Revista Portuguesa de Arqueologia* 1, 2: 137-163.

———. 2005. Villa romana da Quinta das Longas (Elvas, Portugal): a lixeira baixo-imperial. *Revista Portuguesa de Arqueologia* 8, 1: 299-368.

FOWLER, David. 1999. *The Mathematics of Plato's Academy, A New Reconstruction*, 2nd ed. Oxford: Oxford University Press

KLINE, Morris. 1972. *Mathematical Thought from Ancient to Modern Times*, vol. 1. Oxford: Oxford University Press.

NUNES RIBEIRO, Fernando. 1972. *A Villa Romana de Pisões*. Beja.

PLOMMER, Hugh. 1973. *Vitruvius and Later Roman Building Manuals*. Cambridge: Cambridge University Press.

SCHNEIDER, Peter. 2002. Old Shoes, New Feet, and the Puzzle of the First Square in Ancient Egyptian Architecture. Pp. 207-221 in *Nexus IV: Architecture and Mathematics*, José Francisco Rodrigues and Kim Williams, eds. Fucecchio (Florence): Kim Williams Books.

VITRUVIUS. 1960. *The Ten Books of Architecture*. Morris Hicky Morgan, trans. New York: Dover Publications.

WILSON JONES, Mark. 2000. *Principles of Roman Architecture*. New Haven: Yale University Press.

About the authors

Eliana Manuel Pinho has a degree (Licenciatura) in Physics/Applied Mathematics (astronomy) and a Ph.D. in Mathematics from the Faculty of Sciences, University of Porto. After some years doing research in mathematics (patterns, symmetry and coupled cell networks and teaching mathematics in the Universities of Beira Interior and Aveiro, Portugal, she is currently studying the geometrical content in the domestic Roman architecture in Portugal, in particular in the plans and mosaics of Roman villas. This work is developed with a post-doctoral grant (SFRH/BDP/61266/2009) from FCT in the Faculty of Architecture, University of Porto, under the supervision of João Pedro Xavier. Eliana is interested in the subjects shared by art and mathematics and has an incomplete degree in sculpture from the Faculty of Fine Arts, University of Porto.

João Pedro Xavier, an architect and geometry teacher, is Associate Professor at the Faculty of Architecture, University of Porto (FAUP), where he received his degree in 1985 and his Ph.D. in Architecture in 2005. He worked in Álvaro Siza's office from 1986 to 1999. At the same time, he

established his own practice as an architect. Xavier has always been interested in the relationship between architecture and mathematics, especially perspective. He is the author of *Perspectiva, perspectiva acelerada e contraperspectiva* (FAUP Publicações, 1997) and *Sobre as origens da perspectiva em Portugal* (FAUP Publicações, 2006). He has published several works and papers on the subject, presented conferences and lectures and taught courses to high school teachers. He is a member of the editorial board of the *Nexus Network Journal* and member of the executive board of *Resdomus*.