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Research

### *Polygons, Diagonals, and the Bronze Mean*

**Abstract.** This article furthers the study of the Metallic Means and investigates the question of whether or not there exists a polygon corresponding to the Bronze Mean as the pentagon and the octagon correspond respectively to the Golden and Silver Means.

The Metallic Means Family (MMF) was introduced in 1998 by Vera W. de Spinadel [1998; 1999]. Its members are the positive solutions of the quadratic equations  $x^2 - px - q = 0$ , where the parameters  $p$  and  $q$  are positive integer numbers. The more relevant of them are the Golden Mean and the Silver Mean, the first members of the subfamily which is obtained by considering  $q=1$ . The members  $\sigma_p$ ,  $p=1,2,3\dots$  of this subfamily share properties which are the generalization of the Golden Mean properties. For instance, they all may be obtained by the limit of consecutive terms of certain “*generalized secondary Fibonacci sequences*” (GSFS) and they are the only numbers which yield geometric sequences:

$$\dots, \frac{1}{(\sigma_p)^3}, \frac{1}{(\sigma_p)^2}, \frac{1}{\sigma_p}, 1, \sigma_p, (\sigma_p)^2, (\sigma_p)^3, \dots$$

with additive properties

$$1 + p\sigma_p = (\sigma_p)^2, (\sigma_p)^k + p(\sigma_p)^{k+1} = (\sigma_p)^{k+2}$$

$$\frac{1}{(\sigma_p)^k} = \frac{p}{(\sigma_p)^{k+1}} + \frac{1}{(\sigma_p)^{k+2}}, k = 1, 2, 3, \dots$$

However, the generalization of geometrical aspects presents some differences. If we consider the rectangles of ratio  $\sigma_p$ , they all have the property that the corresponding gnomon is the union of  $p$  squares. Figs. 1, 2 and 3 show the three first metallic rectangles.

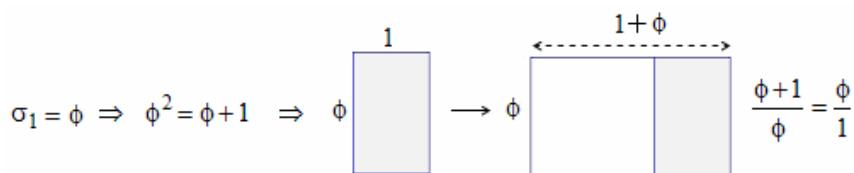


Fig. 1

$$(\sigma_{Ag})^2 = 2\sigma_{Ag} + 1 \Rightarrow \sigma_{Ag} \begin{array}{|c|}\hline 1 \\ \hline \end{array} \longrightarrow \sigma_{Ag} \begin{array}{|c|c|c|}\hline & & 2\sigma_{Ag}+1 \\ \hline \end{array} \frac{2\sigma_{Ag}+1}{\sigma_{Ag}} = \frac{\sigma_{Ag}}{1}$$

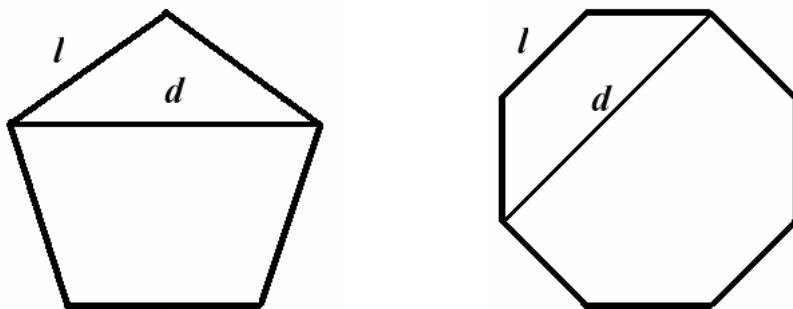
Fig. 2

$$(\sigma_3)^2 = 3\sigma_3 + 1 \Rightarrow \sigma_3 \begin{array}{|c|}\hline 1 \\ \hline \end{array} \longrightarrow \sigma_3 \begin{array}{|c|c|c|c|}\hline & & & 3\sigma_3+1 \\ \hline \end{array} \frac{3\sigma_3+1}{\sigma_3} = \frac{\sigma_3}{1}$$

Fig. 3

Also, the Silver Mean,  $\sigma_{Ag} = \sigma_2$ , and the Bronze Mean,  $\sigma_{Br} = \sigma_3$ , allow us to construct spirals which generalize to that of the Golden Mean [Redondo Buitrago 2006].

On the other hand, the Golden Mean is linked to pentagonal symmetry, and the Silver Mean to octagonal symmetry. Actually, in the regular pentagon, the ratio of the lengths of the first diagonal to that of the side is  $\sigma_1 = \phi$ , and the Silver Mean,  $\sigma_2 = \sigma_{Ag}$ , is the ratio of the lengths of the second diagonal to that of the side in the regular octagon.



$$\frac{d}{l} = \frac{1+\sqrt{5}}{2}$$

$$\frac{d}{l} = 1 + \sqrt{2}$$

Fig. 4

So, it is natural to expect that there exists some regular polygon linked with the Bronze Mean  $\sigma_3 = \sigma_{Br}$ . However, in the classical literature we have not been able to find any reference to this fact. Next, we are going to prove that it is not possible to construct some diagonal in some regular polygon, with a ratio equal to the Bronze Mean. We will only need very elementary geometrical arguments.

Let there be a regular polygon with  $n$  sides. When we draw and number its diagonals  $d_1, d_2, \dots, d_{n-1}$ , including the sides of the polygon as  $d_1$  and  $d_{n-1}$ , as in Figs. 5 and 6, we observe that the length of  $d_i$  is equal to  $d_{n-1}$ .

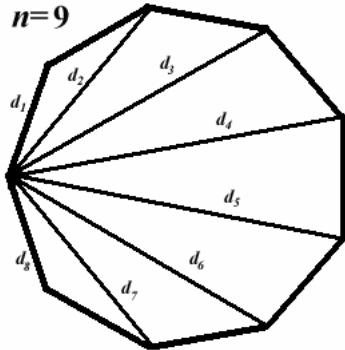


Fig. 5

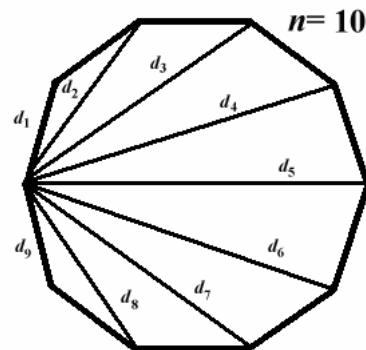


Fig. 6

So, we are going to consider only the lengths of diagonals  $d_1, d_2, \dots, d_{n_0}$ , where  $n_0$  stands for the integer part of  $(n-2):2$ . Obviously, if  $n$  is an even number, as in fig. 6, then the largest diagonal coincides with the diameter of the circumscribed circumference.

The ratios of the lengths of the diagonals are given by the law of cosines (fig. 7).

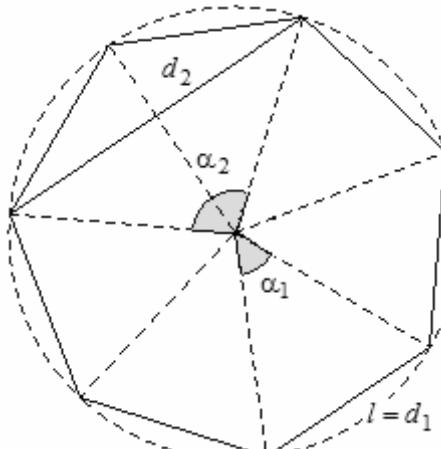


Fig. 7

$$d_k = \sqrt{2 - 2 \cos \alpha_k} = \sqrt{2 - 2 \cos \frac{2\pi k}{n}}$$

$$k = 1, 2, 3, \dots, \left[ \frac{n-2}{2} \right] + 1$$

In particular, taking into account that the side of the polygon is  $d_1$ , we have

$$\frac{d_k}{l} = \sqrt{\frac{1 - \cos \frac{2\pi k}{n}}{1 - \cos \frac{2\pi}{n}}} = \frac{\sin \frac{\pi k}{n}}{\sin \frac{\pi}{n}}.$$

So, in order to research the possibility of existence of diagonals in some regular polygon with  $n \geq 5$  sides, satisfying

$$\frac{d_k}{l} = \frac{3 + \sqrt{13}}{2} = 3.30277563\dots, \quad k = 2, 3, \dots, \left[ \frac{n-2}{2} \right] + 1$$

we will study the following periodic function family

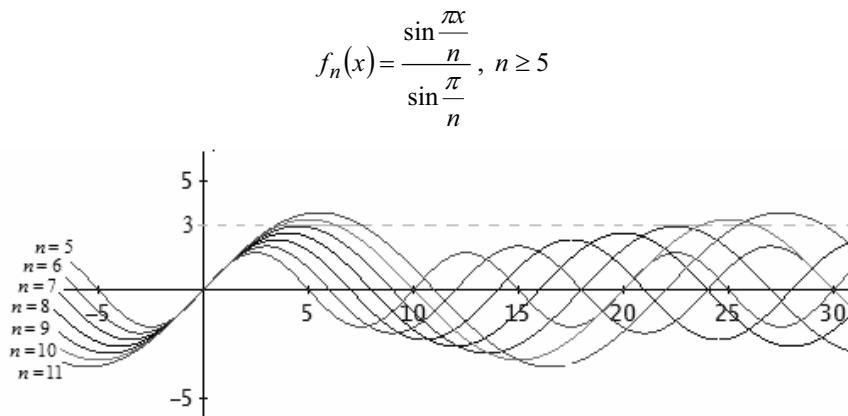


Fig. 8

First, we observe that all functions have period  $T = 2n$ . Moreover, in  $[0, 2n]$  the maximum value of the function is  $f(n/2) = (\sin(\pi/n))^{-1}$  (fig. 9).

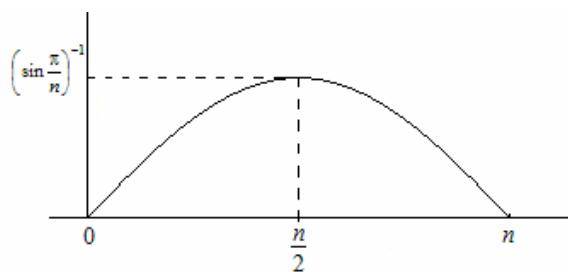


Fig. 9

Notice that when the number of sides,  $n$ , is fixed by computing the values of the corresponding function at the integer values of the interval  $[0, n]$ , we obtain the ratio of the successive lengths of the side and diagonals of the corresponding polygon. Nevertheless, only we need consider the first half of them. That is, the interval  $[0, n/2]$ .

If we take a look at the function graph above, we observe that the maximum for  $n < 10$  is less than 3. Therefore, we must search out polygons with  $n \geq 10$ . But, by checking  $n = 10, 11, 12, 13$  y 14, we deduce that it is not possible to get

$$f_n(k) = \frac{3 + \sqrt{13}}{2} = 3.30277563\ldots , \quad k = 2, 3, \dots, \left[ \frac{n-2}{2} \right] + 1$$

$n$	10	11	12	13	14
	3.236 ( $k=5$ )	3.229 ( $k=4$ )	2.732 ( $k=3$ )	2.771 ( $k=3$ )	2.802 ( $k=3$ )
$f_n(k)$		3.513 ( $k=5$ )	3.334 ( $k=4$ )	3.439 ( $k=4$ )	3.513 ( $k=4$ )
			3.732 ( $k=5$ )	3.907 ( $k=5$ )	4.049 ( $k=5$ )
			3.863 ( $k=6$ )	4.148 ( $k=6$ )	4.381 ( $k=6$ )
					4.494 ( $k=7$ )

On the other hand, when  $k$  is fixed, the function  $g_k(n) = f_n(k)$  is an increasing function, and fig. 10 shows that we need to study only the values 2, 3 and 4, that is the second, third and fourth diagonals.

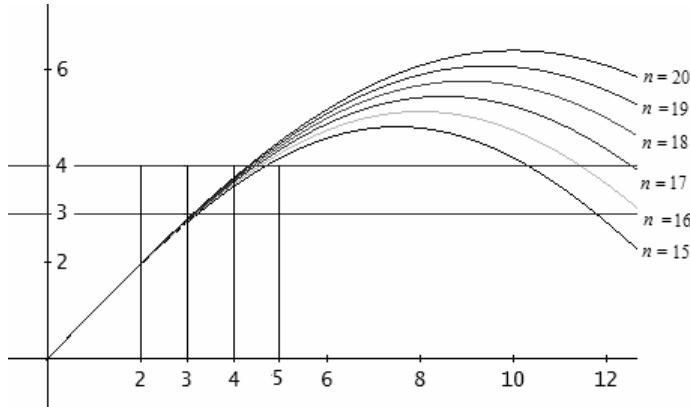


Fig. 10

Hence, we find that

$$1.95\dots = f_{15}(2) < f_{16}(2) < f_{17}(2) < \dots < \lim_{n \rightarrow +\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} = 2 < \frac{3 + \sqrt{13}}{2}$$

$$2.82\dots = f_{15}(3) < f_{16}(3) < f_{17}(3) < \dots < \lim_{n \rightarrow +\infty} \frac{\sin\left(\frac{3\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} = 3 < \frac{3 + \sqrt{13}}{2}$$

$$\frac{3 + \sqrt{13}}{2} < 3.57\dots = f_{15}(4) < f_{16}(4) < f_{17}(4) < \dots < \lim_{n \rightarrow +\infty} \frac{\sin\left(\frac{4\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} = 4$$

Consequently, we can conclude: There exists no regular polygon in which the ratio of the length of the diagonal to the side of the polygon is equal to the Bronze Mean.

### References

- REDONDO BUITRAGO, Antonia. 2006. Algunos resultados sobre Números Metálicos. *Journal of Mathematics & Design* 6, 1: 29-45.
- SPINADEL, Vera W. de. 1998. *From the Golden Mean to Chaos*. 2nd ed. Buenos Aires: Nueva Librería
- \_\_\_\_\_. 1999. The family of metallic means. *Visual Mathematics* 1, 3.  
<http://members.tripod.com/vismath1/spinadel/>

### About the author

Antonia Redondo Buitrago teaches Mathematics in a high school of Albacete (Spain). She is a doctor in Applied Mathematics by University of Valencia (Spain). Her research interests and her contributions in international journals and congresses include works about the fractional powers of operators, continued fractions and the Metallic Means. At the present, in the domain of mathematics and design, her collaborations with Vera W. Spinadel in the research of new properties of Metallic Number Family are the most relevant.