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Geometric Graphs with Few Disjoint Edges*

G. Tóth[1,2] and P. Valtr[2,3]

[1] Courant Institute, New York University,
251 Mercer Street, New York, NY 10012, USA
toth@cims.nyu.edu

[2] DIMACS Center, Rutgers University,
Piscataway, NJ 08855, USA

[3]Department of Applied Mathematics, Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
valtr@kam.mff.cuni.cz

Abstract. A *geometric graph* is a graph drawn in the plane so that the vertices are represented by points in general position, the edges are represented by straight line segments connecting the corresponding points.

Improving a result of Pach and Törőcsik, we show that a geometric graph on n vertices with no $k+1$ pairwise disjoint edges has at most $k^3(n+1)$ edges. On the other hand, we construct geometric graphs with n vertices and approximately $\frac{3}{2}(k-1)n$ edges, containing no $k+1$ pairwise disjoint edges.

We also improve both the lower and upper bounds of Goddard, Katchalski, and Kleitman on the maximum number of edges in a geometric graph with no four pairwise disjoint edges.

1. Introduction

A *geometric graph G* is a graph drawn in the plane by (possibly crossing) straight line segments, i.e., it is defined as a pair $G = (V, E)$, where V is a set of points in general position in the plane and E is a set of closed segments whose endpoints belong to V.

The following problem was raised by Avital and Hanani [AH], Kupitz [K], Erdős, and Perles. Determine the smallest number $e_k(n)$ such that any geometric graph with n vertices and $m > e_k(n)$ edges contains $k+1$ pairwise disjoint edges. By a result

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of Hopf and Pannwitz [HP] and Erdős, $e_1(n) = n$. Alon and Erdős [AE] showed that $e_2(n) \le 6n - 5$ which was improved by Goddard et al. [GKK] to $e_2(n) \le 3n$. The best known lower bound, $e_2(n) \ge 2.5n - 4$, is due to Perles (see [PA] and [MP]). It has also been known that $3.5n - 6 \le e_3(n) \le 10n$ and that $e_k(n) \le c_k n(\log n)^{k-3}$ $(k \ge 4)$, see [GKK]. For any fixed k, Pach and Törőcsik [PT] were first to prove that $e_k(n)$ is linear in n; their bound was $e_k(n) \le k^4 n$. It follows from a result of Kupitz [K] that $e_k(n) \ge kn$. In this paper we further improve both the upper and lower bounds for general k.

Theorem 1. *For $k \le n/2$,*

$$\tfrac{3}{2}(k-1)n - 2k^2 \le e_k(n) \le k^3(n+1).$$

We also improve the above mentioned bounds on $e_3(n)$.

Theorem 2. *For any $n \ge 6$,*

$$4n - 9 \le e_3(n) \le 8.5n.$$

Theorems 1 and 2 are proved in Sections 2 and 3, respectively. Throughout the paper, we do not make any notational distinction between an edge and the segment representing it.

Very recently, using similar methods, Tóth has further improved the upper bound in Theorem 1 to $e_k \le 2^9 k^2 n$ [T]. Two related problems were studied in [LMPT] and [V]. In [V] it was shown that the number of edges in a geometric graph with no k pairwise crossing edges is at most $c_k n \log n$. In [LMPT] it was proved that among any k^5 convex sets in the plane, one can find k of them which are either pairwise disjoint or pairwise intersecting (in particular, this also holds for line segments).

2. The General Case

2.1. *The Upper Bound*

Our proof, as the proof of Pach and Törőcsik [PT], is based on Dilworth's theorem [D].

Dilworth's Theorem. *Let P be a partially ordered set containing no chain* (*totally ordered subset*) *of size $k+1$. Then P can be covered by k antichains* (*subsets of pairwise incomparable elements*).

Let $G = (V, E)$ be a geometric graph on n vertices, containing no $k+1$ pairwise disjoint edges. For a vertex v, let $x(v)$ and $y(v)$ denote its x- and y-coordinate, respectively. We can assume without loss of generality that no two vertices have the same x-coordinate.

An edge e is said to *lie below* an edge e', if no vertical line crossing both e and e' crosses e strictly above e'. Finally, let $\pi(e)$ denote the orthogonal projection of e to the x-axis.

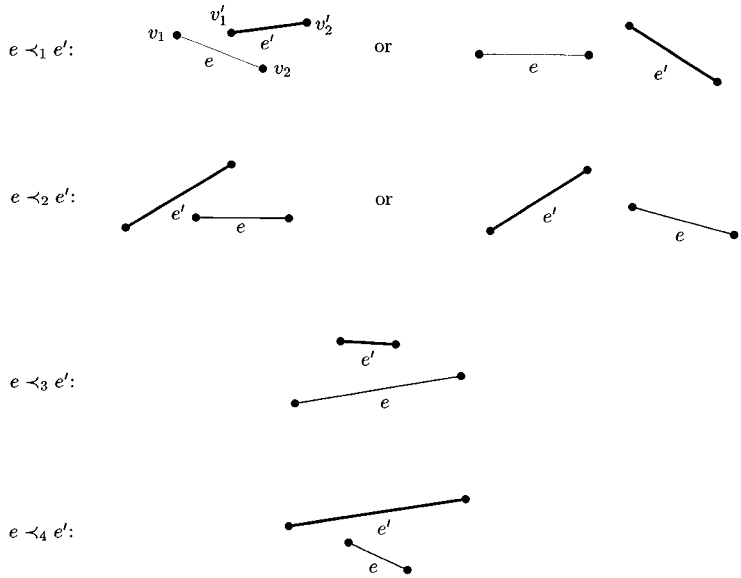

Fig. 1. The relations $\prec_i$.

Define four binary relations $\prec_i$ $(i = 1, \ldots, 4)$ on the edge set E as follows (see also [PT] and [PA]). Let $e = v_1v_2$ and $e' = v_1'v_2'$ be two disjoint edges of G, where $x(v_1) < x(v_2)$ and $x(v_1') < x(v_2')$. Then (see Fig. 1)

$$\begin{array}{llll} e \prec_1 e', & \text{if} \quad x(v_1) < x(v_1'), & x(v_2) < x(v_2'), & \text{and } e \text{ lies below } e', \\ e \prec_2 e', & \text{if} \quad x(v_1) > x(v_1'), & x(v_2) > x(v_2'), & \text{and } e \text{ lies below } e', \\ e \prec_3 e', & \text{if} \quad x(v_1) < x(v_1'), & x(v_2) > x(v_2'), & \text{and } e \text{ lies below } e', \\ e \prec_4 e', & \text{if} \quad x(v_1) > x(v_1'), & x(v_2) < x(v_2'), & \text{and } e \text{ lies below } e'. \end{array}$$

Obviously, each of the relations $\prec_i$ is a partial ordering, and any pair of disjoint edges in G is comparable by at least one of them.

Since G does not contain $k + 1$ disjoint edges, $(E, \prec_1)$ does not contain a chain of length $k + 1$. Therefore, by Dilworth's theorem, E can be covered by k antichains with respect to $\prec_1$. Let E_1 be the largest of these antichains, thus $|E_1| \geq |E|/k$. Applying Dilworth's theorem on $(E_1, \prec_2)$, we similarly get an antichain (with respect to $\prec_2$) $E_2 \subseteq E_1$ of size $|E_2| \geq |E_1|/k \geq |E|/k^2$. In the rest of the proof we estimate the size of E_2 from above.

Since E_2 is an antichain with respect to $\prec_1$ and $\prec_2$, $\pi(e) \cap \pi(e') \neq \emptyset$ for any $e, e' \in E_2$. Therefore, $\bigcap_{e \in E_2} \pi(e) \neq \emptyset$, so there is a vertical line ℓ which intersects all edges in E_2.

Let $\overrightarrow{G_2} = (V, \overrightarrow{E_2})$ be a directed geometric graph obtained from (V, E_2) by replacing each edge $e = v_1v_2$ in E_2 by the two oriented edges $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_2v_1}$. For two edges $e_1 = \overrightarrow{v_0v_1}$, $e_2 = \overrightarrow{v_1v_2}$ forming a path in $\overrightarrow{G_2}$, we say that e_2 is a *zag of* e_1, if the following

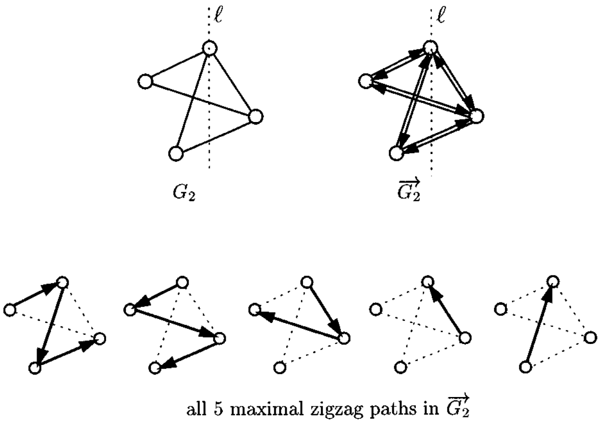

Fig. 2. Maximal zigzag paths.

two conditions hold:

(i) $\pi(e_1) \cap \pi(e_2)$ has positive length,

(ii) for any $z \in (\pi(e_1) \cap \pi(e_2)) \setminus \{\pi(v_1)\}$, the vertical line through z intersects e_2 below e_1, and it intersects no other edge going from v_1 between e_1 and e_2.

Observe that each edge in $\overrightarrow{G_2}$ has at most one zag. We call an oriented path $e_1 e_2 \cdots e_r$ in $\overrightarrow{G_2}$ a *zigzag path*, if e_{i+1} is the zag of e_i, for each $i = 1, \ldots, r-1$ (see Fig. 2).

Lemma 3. *Every zigzag path in* $\overrightarrow{G_2}$ *has at most* $2k$ *edges*.

Lemma 4. $\overrightarrow{G_2}$ *has at most* $n+1$ *maximal zigzag paths*.

Lemmas 3 and 4 immediately give Theorem 1: every edge in $\overrightarrow{G_2}$ is contained in a maximal zigzag path; therefore, by Lemmas 3 and 4,

$$|\overrightarrow{E_2}| \le 2k(n+1),$$

and, consequently,

$$|E| \le k^2 |E_2| = k^2 |\overrightarrow{E_2}|/2 \le k^3(n+1).$$

It remains to prove Lemmas 3 and 4.

Proof of Lemma 3. Let $P = e_1 e_2 \cdots e_r$ be a zigzag path in $\overrightarrow{G_2}$, and let $e_i = \overrightarrow{v_{i-1} v_i}$ for $i = 1, \ldots, r$. We need the following claim.

Claim. *One of the two sequences*

$$S_0 = x(v_0), x(v_2), \ldots, \qquad S_1 = x(v_1), x(v_3), \ldots$$

is decreasing, and the other one is increasing.

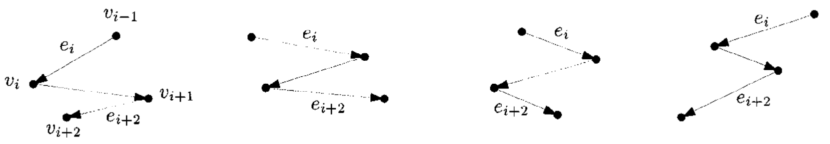

Fig. 3. $e_{i+2} \prec_1 e_i$ or $e_{i+2} \prec_2 e_i$.

Proof of the Claim. Suppose the claim is false. Then there is an index i such that either $x(v_{i-1}) < x(v_{i+1})$ and $x(v_i) < x(v_{i+2})$, or $x(v_{i-1}) > x(v_{i+1})$ and $x(v_i) > x(v_{i+2})$. Consequently, $e_{i+2} \prec_2 e_i$ or $e_{i+2} \prec_1 e_i$ (see Fig. 3 for all four possible cases), which is a contradiction with the definition of E_2. The claim is proved. □

The length of P is at most $2k$ since otherwise $e_1, e_3, \ldots, e_{2k+1}$ would be $k+1$ disjoint edges according to the Claim (see Fig. 4). □

Proof of Lemma 4. Let $v \in V$ be a vertex lying to the right of ℓ, and let P_1, P_2 be two different zigzag paths in $\overrightarrow{G_2}$ ending in v. If the slope of the last edge in P_1 is, say, smaller than the slope of the last edge in P_2, then the last edge of P_1 has a zag and, consequently, P_1 can be extended to a longer zigzag path. It follows that at most one maximal zigzag path ends in v, and this is similarly true for any vertex not lying on ℓ. Similarly, at most two zigzag paths end in any vertex v on ℓ (one coming to v from the left, the other one from the right). The lemma now follows from the assumption that no pair of vertices lies on a vertical line. □

2.2. *The Lower Bound*

For simplicity, suppose that k is even and n is odd. Set $z = (n-k+1)/2$. Let P be a set of z points $p_1, \ldots, p_z$ placed equidistantly in this order from left to right on a horizontal line p. Let $Q = \{q_1, \ldots, q_z\}$ be a translation of P such that q_i always corresponds to p_i and $p_1 p_z q_z q_1$ is a square. Let q be the line containing Q, and let r be the line parallel

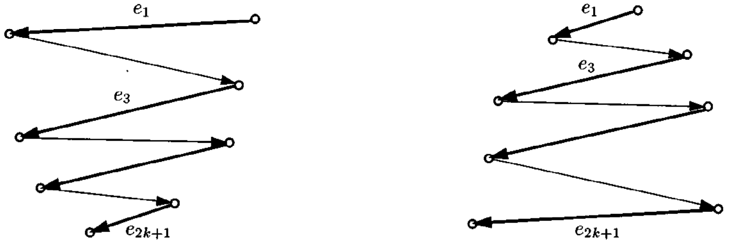

Fig. 4. $e_1, e_3, \ldots, e_{2k+1}$ are disjoint.

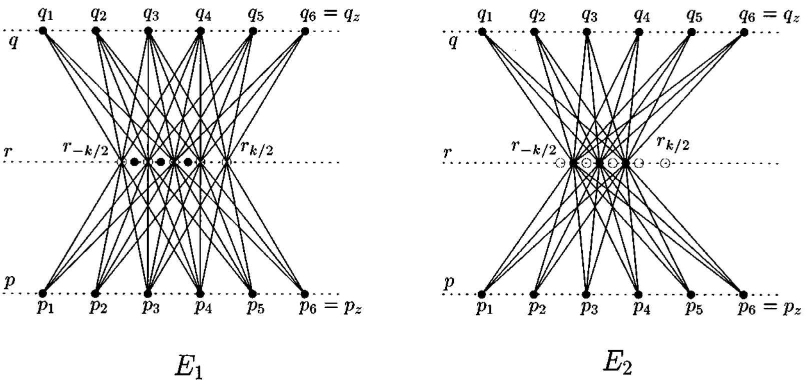

Fig. 5. The edges of E_1 and E_2 (for $k = 4,\ n = 15,\ z = 6$).

to p and q, halfway between them. Let E_1 be the set of all segments (edges) $p_i q_j$ with $-k/2 \le i + j - (z+1) \le k/2$ (see Fig. 5).

The segments of E_1 intersect the line r in $k+1$ points $r_{-k/2}, r_{-k/2+1}, \ldots, r_{k/2}$ such that r_t always lies on the segments $p_i q_j$ with $i + j - (z+1) = t$. Let R be the set of the centers of the segments $r_t r_{t+1}$ $(t = -k/2, -k/2+1, \ldots, k/2-2)$, and let E_2 be the set of all edges (segments) joining vertices of R with vertices of $P \cup Q$ (see Fig. 5).

We now show that the geometric graph $G = (P \cup Q \cup R, E_1 \cup E_2)$ gives the lower bound in Theorem 1.

First, observe that G has $z + z + (k-1) = n$ vertices and

$$|E_1| + |E_2| = \frac{(n-k+1)(k+1) - 4(1+2+\cdots+k/2)}{2} + (n-k+1)(k-1)$$
$$= (n-k+1)\left(\frac{3k}{2} - \frac{1}{2}\right) - \frac{k(k+2)}{4} > \tfrac{3}{2}(k-1)n - 2k^2$$

edges. It remains to show that G contains no $k+1$ pairwise disjoint edges. Suppose that D is a set of disjoint edges in G. If $|D \cap E_1| \le 1$, then $|D| \le |R| + 1 = k$. Otherwise, let the leftmost edge of $D \cap E_1$, e_1, intersect r in a point r_s, and the rightmost one, e_2, in a point r_t. Since there are $s + k/2$ vertices of R to the left of e_1, D contains at most $s + k/2$ edges to the left of e_1. Similarly, D contains at most $k/2 - t$ edges to the right of e_2. Since there are $t - s - 2$ vertices of $P \cup Q$ between e_1 and e_2, D contains at most $t - s - 2$ edges between e_1 and e_2. Altogether, D contains at most $(s + k/2) + (k/2 - t) + (t - s - 2) + 2 = k$ edges. □

3. The Case *k* = 3

3.1. *The Upper Bound*

We use the following result.

Lemma 5 [GKK]. *If a geometric graph G of n vertices does not contain four pairwise disjoint edges and there is a line which intersects every edge of G and contains no vertex of G, then G has at most $7n$ edges.*

Lemma 5 is not stated in [GKK] explicitly. However, its proof (relatively long case-analysis) is readily contained in the proof of Theorem 2 in [GKK].

Let $G = (V, E)$ be a geometric graph without four pairwise disjoint edges. Denote the vertices by $v_1, \ldots, v_n$ from left to right, and assume that no pair of them lies on a vertical line. For any $1 \leq i < n$ let $\overline{G}_i$ be the subgraph of G which contains only those edges $v_\alpha v_\beta$ of G where $\alpha \leq i < \beta$.

It follows from Lemma 5 and the assumption that, for any $1 \leq i < n$, $|E(\overline{G}_i)| \leq 7n$. For any $1 \leq i < n$, let G_i^- and G_i^+ be the subgraph of G induced by the vertices $v_1, \ldots, v_i$ and by $v_i, \ldots, v_n$, respectively. Let

$$I = \max\left\{i \mid G_i^- \text{ does not contain two disjoint edges}\right\}.$$

Since G_{I+1}^- contains two disjoint edges, G_{I+2}^+ does not. Suppose without loss of generality that $I < n/2$. Then

$$E(G) = E(G_I^-) \cup E(G_{I+2}^+) \cup E(\overline{G}_{I+1}) \cup \{v_i v_{I+1} \in E(G) \mid i \leq I\}.$$

Therefore,

$$|E(G)| \leq e_1(I) + e_1(n - I - 1) + 7n + I < 8.5n. \qquad \square$$

3.2. *The Lower Bound*

First, let n be odd. Take the vertices of a regular $(n-2)$-gon, and join each of them with the furthermost four vertices. Add two vertices near the center of the $(n-2)$-gon, and join each of them by an edge to all the other vertices. The resulting geometric graph on n vertices has $4(n-2)/2 + 2(n-1) - 1 = 4n - 7$ edges. It is easy to see that no four edges are pairwise disjoint, see Fig. 6.

For n even, we take the above geometric graph on $n+1$ vertices, and remove a vertex and the six edges incident to it. The resulting graph on n vertices has $4n - 9$ edges. $\square$

4. Remarks

By a little modification, it is possible to improve the upper bound of Theorem 1 slightly.

Theorem 6. *For any $k \leq n/2$,*

$$e_k(n) \leq \tfrac{4}{21}k^3 n + O(k^2 n).$$

Proof. (Sketch) We use the same partial orderings, $\prec_1, \prec_2, \prec_3, \prec_4$, as in the proof of Theorem 1 (see Fig. 1). For any edge e of a geometric graph $G = (V, E)$, let

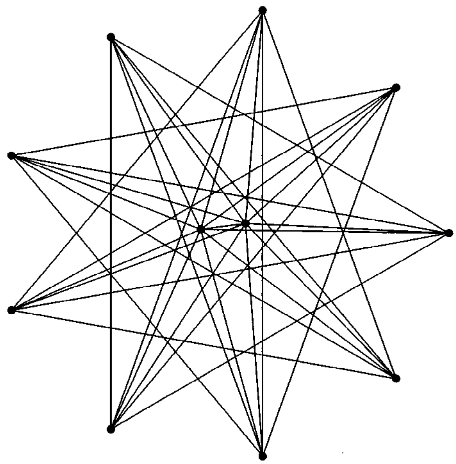

Fig. 6. A geometric graph with no four pairwise disjoint edges.

$\mathrm{Rank}_3^+(e)$ be the maximum number $a \geq 0$ such that there exist $e_1, e_2, \ldots, \prec e_a \in E$, $e \prec_3 e_1 \prec_3 e_2 \prec_3 e_3 \prec \cdots \prec_3 e_a$, and let $\mathrm{Rank}_4^-(e)$ be the maximum number $b \geq 0$ such that there exist $e_1, e_2, \ldots, e_b \in E$, $e_1 \prec_4 e_2 \prec_4 e_3 \prec \cdots \prec_4 e_b \prec_4 e$.

Lemma 7. *Let $G = (V, E)$ be a geometric graph with n vertices and with no $k+1$ pairwise disjoint edges. Suppose that there is a line which avoids all vertices and crosses all edges of G. Then*

$$|E| \leq \tfrac{1}{6}k^3 n + O(k^2 n).$$

Proof of Lemma 7. For any $0 \leq a, b \leq k-1$ let

$$E_{a,b} = \left\{ e \in E \mid \mathrm{Rank}_3^+(e) = a, \mathrm{Rank}_4^-(e) = b \right\}.$$

Clearly, $E_{a,b}$ is an antichain with respect to $\prec_3$ and $\prec_4$. We can define $\overrightarrow{E_{a,b}}$ and its maximal zigzag paths just as in the proof of Theorem 1. It is easy to see that there are at most n maximal zigzag paths and every edge of $\overrightarrow{E_{a,b}}$ is contained in at least one of them (in fact, in exactly one of them, but here we do not need that). Suppose that there is a zigzag path $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_x}$ of $x = 2(k-a-b)+1$ edges. Then $\overrightarrow{e_1}, \overrightarrow{e_3}, \overrightarrow{e_5}, \ldots, \overrightarrow{e_x}$ are $k-a-b+1$ pairwise disjoint edges. Moreover, we can add a edges above $\overrightarrow{e_1}$ and b edges below $\overrightarrow{e_x}$ to get $k+1$ pairwise disjoint edges, contradicting our assumption. (See Fig. 7.) Therefore, every maximal zigzag path of $\overrightarrow{E}_{a,b}$ has at most $2(k-a-b)$ edges, so

$$|E_{a,b}| \leq n(k-a-b),$$

$$|E| = \sum_{\substack{0 \leq a,b \\ a+b<k}} |E_{a,b}| \leq n \sum_{\substack{0 \leq a,b \\ a+b<k}} (k-a-b) = \binom{k+2}{3} n. \qquad \square$$

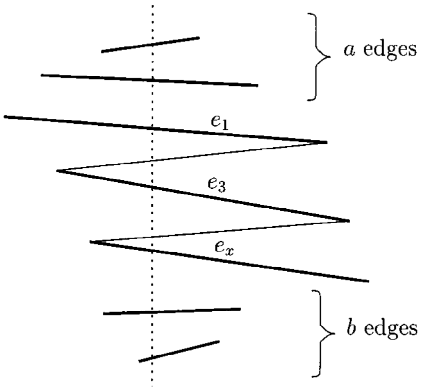

Fig. 7. $k + 1$ disjoint edges.

Return to the proof of Theorem 6. It is not hard to see that there is a line ℓ which avoids all vertices of G and on one side of ℓ there are $\lceil n/2 \rceil$ vertices and at most $\lceil k/2 \rceil$ pairwise disjoint edges, while on the other side of ℓ there are $\lfloor n/2 \rfloor$ vertices and at most $\lfloor k/2 \rfloor$ pairwise disjoint edges of G. We get the recursion

$$e_k(n) \leq \tfrac{1}{6}k^3 n + O(k^2)n + e_{\lceil k/2 \rceil}(\lceil n/2 \rceil) + e_{\lfloor k/2 \rfloor}(\lfloor n/2 \rfloor)$$

and Theorem 6 follows. $\square$

If the number of edges in a geometric graph is at least $\Omega(n^2)$, then Theorem 1 guarantees $\Omega(n^{1/3})$ pairwise disjoint edges. This is improved by the following result of Pach [P].

Theorem 8 [P]. *For any $c > 0$ there is a $c' > 0$ such that every geometric graph of n vertices and at least cn^2 edges has at least $c'\sqrt{n}$ pairwise disjoint edges.*

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