Discrete Comput Geom 20:131–138 (1998)

Discrete & Computational
Geometry

Mapping $\mathbf{Z}^r$ into $\mathbf{Z}^s$ with Maximal Contraction*

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Abstract. Given any bijection $f\colon \mathbf{Z}^r \to \mathbf{Z}^s$ with $s \geq r$, easy volume comparisons show that there must be a universal constant $K > 0$ (depending only on r and s) and infinitely many pairs of points $x, y \in \mathbf{Z}^r$ such that $\| f(x) - f(y)\| > K\|x - y\|^{r/s}$. This puts a bound on how much contraction can be achieved for any such bijection. We show that, conversely, for any $s \geq r$ there is a bijection $f\colon \mathbf{Z}^r \to \mathbf{Z}^s$ and a constant $C > 0$ such that for all $x, y \in \mathbf{Z}^r$ we have $\| f(x) - f(y)\| < C\|x - y\|^{r/s}$. Phrased differently there is a bijection $f\colon \mathbf{Z}^r \to \mathbf{Z}^s$ which shrinks the distance between the images of any two points as much as possible, up to a constant factor. This generalizes a construction in fractal image processing and answers in the affirmative a question of Michael Freedman.

1. Introduction

About five years ago, Michael Freedman proposed the following question. Does there exist a bijection $f\colon \mathbf{Z}^r \to \mathbf{Z}^s$ and a constant $C > 0$ such that for all $x, y \in \mathbf{Z}^r$ we have $\| f(x) - f(y)\| < C\|x - y\|^{r/s}$? Such a bijection would map the ball of radius R about x into the ball of radius $C \cdot R^{r/s}$ about $f(x)$. Thus, up to a constant, such an f would achieve the most contraction possible for a map of $\mathbf{Z}^r$ into $\mathbf{Z}^s$. Note that the condition $\| f(x) - f(y)\| < C\|x - y\|^{r/s}$ for $\|x - y\| = 1$ implies that the bijection f is Lipschitz. Thus any such f also has bounded stretching. In the case $r = 1$ (or more generally if r divides s), then such a bijection is easy to construct. However $\mathbf{Z}^2$ has more rigidity than $\mathbf{Z}$. It was hoped that for $r = 2$ and $s = 3$ no such bijection would exist and that this would be a consequence of a discrete version of curvature on $\mathbf{Z}^2$. The purpose of this paper is to show that in fact such a bijection always exists. That is we have the following theorem.

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Theorem. *For all positive integers r and s with $s \geq r$, there exists a bijection $f\colon \mathbf{Z}^r \to \mathbf{Z}^s$ and a constant $C > 0$ such that for all $x, y \in \mathbf{Z}^r$ we have $\|f(x) - f(y)\| < C\|x - y\|^{r/s}$.*

Such bijections have potential applications. The following application has been noted by Barnsley and Hurd [1] and independently by Freedman [2] and was in fact one of the motivations for proposing the problem. Suppose we have a square array of data with correlated neighbors such as a television signal. We can use such a bijection in the case $r = 1$ and $s = 2$ to record the data in a linear array. Since the bijection stretches distances as little as possible nearby elements of the array should be as highly correlated as possible. Thus it should be possible to compress the data effectively. More discussion of this technique is given in [3].

2. Proof of the Theorem

We now turn to the proof of the theorem. To prove the theorem it clearly suffices to build the bijections in the case $s = r + 1$. Compositions of such bijections give the general case. The proof is divided into two steps. The first is constructing a subset $X_r \subset \mathbf{Z}^2$ which should be thought of as $(r+1)/r$ dimensional and a Lipschitz bijection $F\colon (X_r)^r \to \mathbf{Z}^{r+1}$. The second step is constructing a bijection $g\colon \mathbf{Z} \to X_r$ which shrinks distances by a power of $r/(r+1)$. Then f is the composition $F \circ g^r$.

The subset $X_r \subset \mathbf{Z}^2$ can be defined in either of two equivalent ways and we will use both constructions. First, if n is an integer we define the generalized base b expansion $\cdots a_k a_{k-1} \cdots a_1 a_0$ of n as follows. If n is nonnegative, then it is the ordinary base b expansion padded by infinitely many leading zeros. If n is negative, then it is the limit as k goes to infinity of the base b expansion of $b^k + n$. Alternately, we take the base b expansion of $-n-1$ and replace every digit d by its complement $b-1-d$. Alternately we can phrase this as saying that we take the usual b-adic expansion restricted to the integers. Note that an infinite string of digits $0, \ldots, b-1$ is a generalized base b expansion of some integer if and only if it is eventually all 0 or all $b - 1$. Also notice that restricting the generalized base b expansion of n to the lowest r digits gives $n \bmod b^r$. Therefore one can formally add generalized base b expansions.

With this terminology we can define

$$\begin{aligned} X_r = \{(m, n) | \text{ If } m = \cdots b_2 b_1 b_0 \text{ and } n = \cdots a_2 a_1 a_0 \text{ base } 2^r, \\ \text{then for all } i, b_i = a_i \text{ or } 2^r - 1 - a_i\}. \end{aligned}$$

That is, X_r consists of all pairs (m, n) where each digit of m base 2^r either agrees with the corresponding digit of n or is the complement (in the binary sense) of the corresponding digit of n. Note that $X_1 = \mathbf{Z}^2$.

For an alternate definition of X_r consider $\mathbf{R}^2$ as being divided up into squares with vertices at the lattice points. Let S_0 be the union of the 2^{r+1} squares inside $[-2^{r-1}, 2^{r-1}] \times [-2^{r-1}, 2^{r-1}]$ and along one of the two diagonals (see Fig. 1). Define S_n, $n \geq 1$, inductively as follows. Start with S_{n-1} and dilate by a factor of 2^r (see Fig. 2). The result is a union of squares of side length 2^r. Replace each such square by a copy of S_0. Notice that $S_1 \cap [-2^{r-1}, 2^{r-1}] \times [-2^{r-1}, 2^{r-1}] = S_0$, therefore $S_k \cap [-2^{(j+1)r-1}, 2^{(j+1)r-1}] \times$

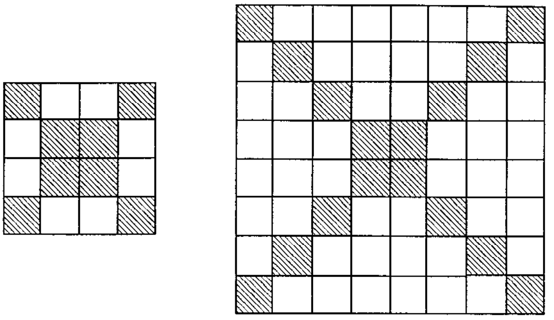

Fig. 1. The patterns of squares S_0 for $r = 2$ and $r = 3$.

$[-2^{(j+1)r-1}, 2^{(j+1)r-1}] = S_j$ for all $k > j$. Let S_∞ be the nested union of the S_i. Then S_∞ is a union of squares in the plane and X_r is the set of lower left-hand corners of these squares.

To see that this agrees with the previous definition note that the lower left-hand corners of the squares of S_0 are lattice points (m, n) where either $m \equiv n \pmod{2^r}$ or $m \equiv -n-1 \pmod{2^r}$. That is, the lowest digits in their base 2^r expansions satisfy the right property. When we dilate by 2^r we transfer this property to the more significant digits and when we replace each large square by S_0 we restore it in the lowest digit.

The first step of the proof is completed by the following proposition.

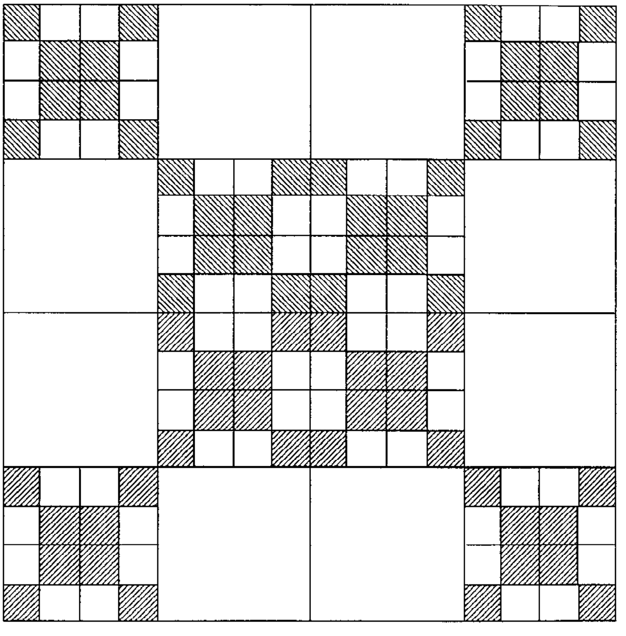

Fig. 2. The pattern of squares S_1 for $r = 2$.

Proposition 1. *The map* $F\colon (X_r)^r \to \mathbf{Z}^{r+1}$ *given by*

$$F((m_1,n_1),(m_2,n_2),\ldots,(m_r,n_r)) = (m_1+2m_2+4m_3+\cdots+2^{r-1}m_r, n_1, n_2, \ldots, n_r)$$

is a bijection and stretches distances by at most $\sqrt{(2^{2r}-1)/3}$.

Proof. F is the restriction of a linear function $\mathbf{Z}^{2r} \to \mathbf{Z}^{r+1}$ hence it is easy to see that F stretches distances by at most the constant claimed. To see that it is a bijection fix values for the n_i. Let P be any other integer and let $P = \cdots p_2 p_1 p_0$ be its generalized base 2 expansion. We wish to solve for m_i which will give $m_1+2m_2+4m_3+\cdots+2^{r-1}m_r = P$ and obey the constraints that $(m_i,n_i) \in X_r$. Let $m_i = \cdots a_2^i a_1^i a_0^i$ be the generalized base 2 expansion of m_i. Then we have the following system to solve, where everything is written in base 2,

$$\begin{array}{ccccccccc}
 & \cdots & a_{r+1}^1 & a_r^1 & a_{r-1}^1 & \cdots & a_2^1 & a_1^1 & a_0^1 \\
 & \cdots & a_r^2 & a_{r-1}^2 & a_{r-2}^2 & \cdots & a_1^2 & a_0^2 & 0 \\
 & & & & \vdots & & & & \\
+ & \cdots & a_2^r & a_1^r & a_0^r & \cdots & 0 & 0 & 0 \\
\hline
 & \cdots & p_{r+1} & p_r & p_{r-1} & \cdots & p_2 & p_1 & p_0
\end{array}$$

Notice that from this linear equation p_0 determines a_0^1. The r base 2 digits $a_{r-1}^1 \cdots a_2^1 a_1^1 a_0^1$ together form the first base 2^r digit of m_1. This base 2^r digit must agree with or be the complement of the first base 2^r digit of n_1. Thus from the constraint $(m_1,n_1) \in X_r$, a_0^1 determines all these digits. Thus p_1 determines a_0^2 and by the same argument a_0^2 determines $a_{r-1}^2, \ldots, a_1^2$. Continuing in this way we see that the p_i determine all the digits of the m_i.

This shows that F is injective. Each point $(P, n_1, n_2, \ldots, n_r)$ determines a unique formal collection of bits $\{a_k^i\}$, or equivalently a unique solution in the 2-adics. However it is not immediately obvious that these form the generalized base 2 expansions of r integers $m_1, \ldots, m_r$. To see this we must show that for all i the sequence $(a_k^i)_{k=0}^\infty$ is eventually all zeros or eventually all ones. Since the digits of n_i base 2^r are eventually 0 or 2^r-1 the a_k^i eventually come in blocks of length r which are all zeros or ones. We must show that in fact they are eventually the same such block.

Suppose we are far enough out in the calculation that all digits of P and the n_i have stabilized. Consider the calculation we do to use p_k to determine some of the a's. Write $k+1 = qr+l$ where $1 \le l \le r$. Then we are trying to use p_k to determine a_{qr}^l. At this point we have already determined $a_{(q+1)r-1}^1 \cdots a_0^1, a_{(q+1)r-1}^2 \cdots a_0^2 0, \ldots, a_{(q+1)r-1}^{l-1} \cdots a_0^{l-1} 0 \cdots 0$ and $a_{qr-1}^l \cdots a_0^l 0 \cdots 0, \ldots, a_{qr-1}^r \cdots a_0^r 0 \cdots 0$. Write the sum of these numbers already determined as

$$a_{(q+1)r-1}^1 \cdots a_0^1 + \cdots + a_{qr-1}^r \cdots a_0^r 0 \cdots 0 = N_k 2^k + p_{k-1} p_{k-2} \cdots p_0.$$

Note that the left-hand side is bounded above by $(2^k-1)+\cdots+(2^{k+r-1}-1) < (2^r-1)2^k$. Hence $0 \le N_k \le 2^r-2$. The equation we want to solve for a_{qr}^l is $N_k + a_{qr}^l \equiv p_k \pmod 2$. We need to see how N_k varies with k.

There are two cases to consider. Suppose the p_k stabilized to be all zeros. If N_k is even, then we will get $a^l_{qr} = 0$. Since the n_i have also stabilized the next $r - 1$ values $a^l_{qr+1}, \ldots, a^l_{(q+1)r-1}$ are also zeros. Thus N_{k+1} is simply $N_k/2$. If N_k is odd, then we get $a^l_{qr} = 1$ and hence $a^l_{qr+1}, \ldots, a^l_{(q+1)r-1} = 1$. Thus the new sum is

$$(2^r - 1)2^k + N_k 2^k + p_{k-1}p_{k-2}\cdots p_0 = ((2^r - 1 + N_k)/2)2^{k+1} + p_k p_{k-1}\cdots p_0.$$

Thus $N_{k+1} = (2^r - 1 + N_k)/2$. Note that in either case N_{k+1} is obtained from N_k by writing N_k as an r digit number base 2 and rotating its digits cyclically. Thus the sequence (N_k) is periodic with period r. This is what we wanted. The next time we wish to compute a value of the same sequence $(a^l_k)^\infty_{k=0}$ will be r steps later. Thus the sequence $(a^l_k)^\infty_{k=0}$ is eventually constant.

Now suppose the p_k stabilized to be all ones. If N_k is odd, then we will get $a^l_{qr} = 0$ and hence $a^l_{qr+1}, \ldots, a^l_{(q+1)r-1} = 0$. Thus $N_{k+1} = (N_k - 1)/2 = ((2^r - 2 + N_k) - (2^r - 1))/2$. If N_k is even, then we get $a^l_{qr} = 1$ and hence $a^l_{qr+1}, \ldots, a^l_{(q+1)r-1} = 1$. Thus the new sum is

$$(2^r - 1)2^k + N_{k+1}2^k + p_{k-1}p_{k-2}\cdots p_0 = ((2^r - 2 + N_k)/2)2^{k+1} + p_k p_{k-1}\cdots p_0.$$

Thus $N_{k+1} = (2^r - 2 + N_k)/2$. This recurrence function is a conjugate of the recurrence function for the case $p_k = 0$ (conjugated by the function $x \mapsto 2^r - 2 - x$). Thus in this case, (N_k) is also periodic with period r. Hence as above the sequences $(a^l_k)^\infty_{k=0}$ are eventually constant. $\square$

Now we turn to the second step in the proof, building the map $g\colon \mathbf{Z} \to X_r$. For this we will use the second construction of X_r given above. In particular, let S_k be the set of squares in $\mathbf{R}^2$ described above. Let G_k be the graph whose vertex set is the set of squares in S_k and two squares are adjacent if they share a common edge or vertex. The first step will be to construct inductively paths in the G_k which go through every vertex at most twice and use each edge at most twice.

For $r = 2$ the path in G_1 is shown in Fig. 3. For other r just extend (or if $r = 1$ contract) the crosses that make up each copy of G_0. To build the path in G_k from the path in G_{k-1}, replace the middle block of eight vertices (essentially G_0) by the path in G_1 shown in Fig. 3. Replace each other vertex by the appropriate block of eight vertices shown in Fig. 4. By convention, if an edge joins two squares of S_{k-1} which share a common edge, we will agree to join them through the corner closer to the origin. The nested union of these paths is a path P in G_∞ which goes through every vertex at most twice and over every edge at most twice.

Now we want to use P to build a bijection $g\colon \mathbf{Z} \to X_r$. We will construct g as an ordered list of the elements of X_r. Notice that dilating S_∞ by a factor of 2^r and replacing each $2^r \times 2^r$ square by a copy of Fig. 1 gives back S_∞. Thus we may regard X_r as being a union of copies of S_0 and the path P as describing the order in which we want to go through these copies. To list the elements of X_r, we walk following P in both directions. If P goes through a vertex only once, then write down all 2^{r+1} vertices in that copy of S_0 in any order and go to the next vertex of P. If P goes through a vertex twice, we write down 2^r of the 2^{r+1} vertices each time we pass through it.

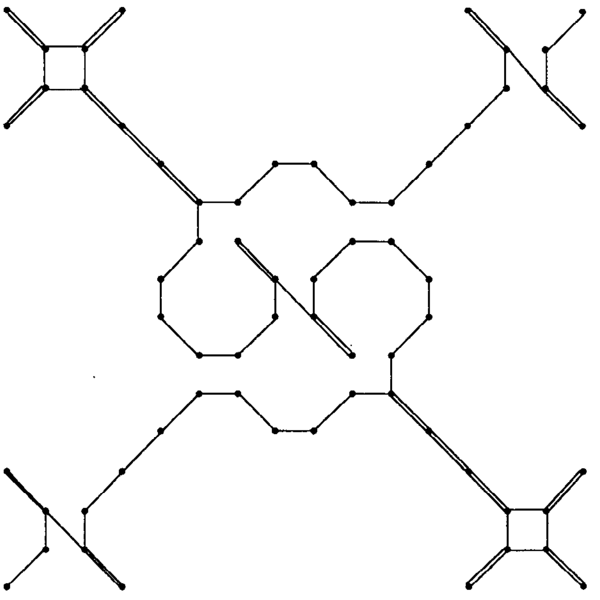

Fig. 3. The basic path G_1 for $r = 2$.

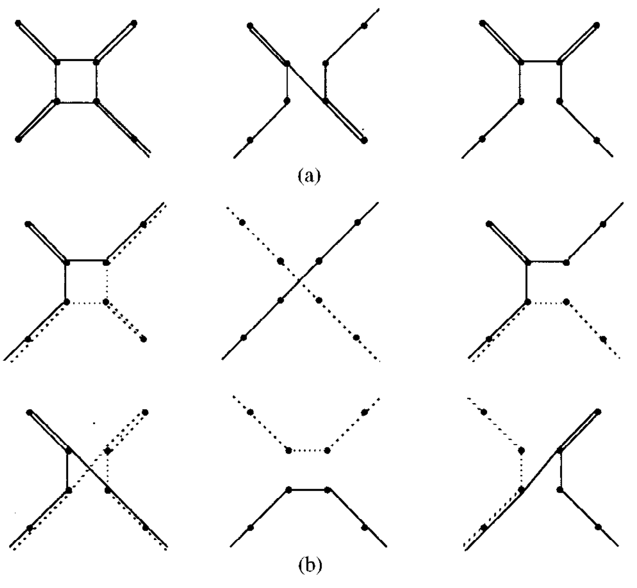

Fig. 4. The patterns used to extend the path inductively, up to rotational symmetry: (a) if the vertex is passed through only once; and (b) if the vertex is passed through twice.

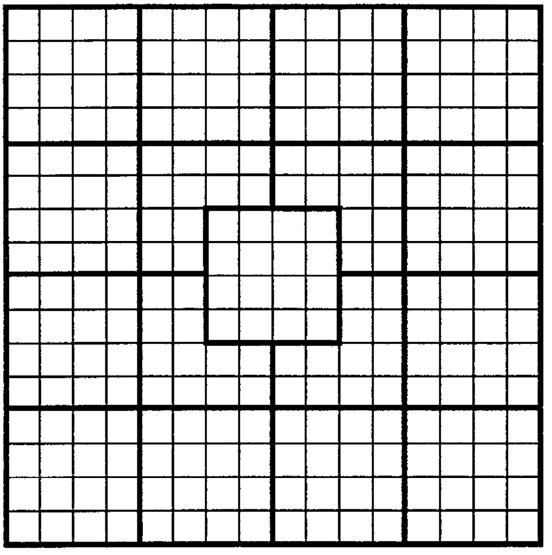

Fig. 5. The basic regions in S_∞.

To complete the construction, we need only show that g shrinks distances by the appropriate power. To see this we think of X_r as being built out of blocks of various scales. Call a single point a block of degree 0. Call any set of 2^{r+1} vertices coming from a single square of S_∞ a block of degree 1. (These are the blocks used above.) Let a degree 2 block be the vertices of X_r corresponding to any of the regions in S_∞ shown in Fig. 5. Except for the five center regions they are just the obvious $2^r \times 2^r$ squares. Let a degree k block be the vertices of X_r corresponding to any of the regions of S_∞ in the dilation of Fig. 5 by a factor of $2^{r(k-2)}$. Note that a (nonempty) degree k block contains either $2^{k(r+1)}$ or $3\cdot 2^{k(r+1)-2}$ vertices of X_r. Also the ordered list of vertices of X_r built above enters and leaves each block of any degree at most twice. Furthermore, the paths drawn in Figs. 3 and 4 have the following property. When the path enters any (degree 2) block there are at least 2^{r-1} vertices of the block which it goes through and which are gone through only once. Each of these corresponds to 2^{r+1} vertices of X_r and each dilation multiplies the number of vertices by 2^{r+1}. Thus whenever the ordered list enters a block of degree k it must go through at least $2^{k(r+1)-2}$ vertices in that block before leaving the block.

Now suppose we are given any two distinct integers m and n. Let k be the smallest integer such that $f(m)$ and $f(n)$ are contained in the same or adjacent degree k blocks. Thus the ordered list must pass through at least one degree $k-1$ block between m and n. Hence $|m-n| \geq 2^{(k-1)(r+1)-2}$. Also $f(m)$ and $f(n)$ are contained in adjacent degree k blocks. Any degree k block is contained in a $2^{kr} \times 2^{kr}$ square. In the worst case the two blocks containing $f(n)$ and $f(m)$ share only a common corner. Hence $\| f(m) - f(n)\| \leq 2\sqrt{2}\cdot 2^{kr}$. Combining these two bounds we get

$$\| f(m) - f(n)\| \leq 2^{r+3/2+2r/(r+1)}|m-n|^{r/(r+1)}.$$

This completes the proof. □

Acknowledgments

The author would like to thank Michael Freedman for his assistance.

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Received May 15, 1996.