Discrete Comput Geom 19:197–227 (1998)

Discrete & Computational
Geometry

Finite Packings of Spheres*

U. Betke[1] and M. Henk[2]

[1]Fachbereich 6, Universität Siegen,
Hölderlinstrasse 3, D-57068 Siegen, Germany
betke@mathematik.uni-siegen.de

[2]Konrad-Zuse-Zentrum (ZIB) Berlin,
Combinatorial Optimization,
Takustr. 7, D-14195 Berlin, Germany
henk@zib.de

Abstract. We show that the sausage conjecture of László Fejes Tóth on finite sphere packings is true in dimension 42 and above.

1. Introduction

Throughout this paper E^d denotes the d-dimensional Euclidean space equipped with the Euclidean norm $|\cdot|$ and the scalar product $\langle \cdot, \cdot \rangle$. B^d denotes the d-dimensional unit ball with boundary S^{d-1} and conv P (lin P) denotes the convex (linear) hull of a set $P \subset E^d$. The interior of P is denoted by int P and the volume of P with respect to the affine hull of P is denoted by $V(P)$. The spherical volume is denoted by $V_\star(\cdot)$. Further, let $\kappa_d = V(B^d)$.

$C \subset E^d$ is called a *packing arrangement* or simply a *packing* (of spheres), if for every pair $x, y \in C$, $x \neq y$, we have $\text{int}(x + B^d) \cap \text{int}(y + B^d) = \emptyset$ or equivalently $|x - y| \geq 2$. Finally, $\#S$ denotes the cardinality of a finite set S.

For infinite packings of spheres (and more generally convex bodies) there is an old and well-known concept of the density of such packings which has led to an extensive theory (see, e.g., [GL], [CS], and [FK]). As usual we denote by $\delta(d)$ the density of a densest infinite packing of spheres in E^d.

* Part of this paper was written while the first author was visiting the Technical University of Berlin. His stay in Berlin and the work of the second author was supported by the Gerhard Hess Forschungsförderpreis of the German Science Association awarded to Günter M. Ziegler (Zi 475/1-1). The paper contains some material of the Habilitationsschrift by the second author.

In contrast to this, the theory of finite packings of spheres is much younger. First results for finite packings have been obtained by Rogers [R1] for general convex planar bodies and by Groemer [Gro] for circles. They measured the size of a packing C by $V(\operatorname{conv} C)$ and some additional summands measuring the size of the boundary of $\operatorname{conv} C$. Defining the density of a finite packing as the quotient of its size and its cardinality their results showed that by taking limits with respect to the cardinality one obtains the density of the densest infinite packing. For a more detailed survey of finite packings in E^2 and finite packing in general, see [GW].

The following observation by L. Fejes Tóth [F] indicated that for higher dimensions the theory for finite packings and infinite packings should be quite different: For a finite packing $C \subset E^d$, he defined its density $\delta(C)$ by

$$\delta(C) = \frac{\#C \cdot \kappa_d}{V(\operatorname{conv} C + B^d)}.$$

This immediately leads to the definition of the maximal density $\delta(d, n)$ of packings of n-spheres in E^d by

$$\delta(d, n) = \max\{\delta(C) : C \subset E^d \text{ is packing with } \#C = n\}.$$

For $d = 2$, from Groemer's result quoted above, we have

$$\lim_{n\to\infty} \delta(2, n) = \delta(2).$$

Now L. Fejes Tóth [F] called the packing

$$S_n^d = \{2iu : u \in S^{d-1},\ i = 1, \ldots, n\}$$

a *sausage arrangement* in E^d and observed

$$\delta(S_n^d) < \delta(d)$$

for all n, provided that the dimension d is at least 5. Further, he conjectured:

Sausage Conjecture. *For $n \in \mathbb{N}$ and $d \geq 5$,*

$$\delta(d, n) = \delta(S_n^d).$$

Thus L. Fejes Tóth's observation poses two problems: The first one is to prove or disprove the sausage conjecture. The second, slightly less obvious one, is to find a common approach to the density of finite and infinite packings. To begin with, the first problem was studied by various authors, though the results were rather weak in that either n had to be small compared to d or strong additional assumptions for the packing C had to be made. For a survey on these results, see again [GW].

In fact, it turned out that the recent study of the second problem was fruitful as well for the solution of the first problem. A certain solution for the second problem was given by Betke *et al*. in [BHW1]. There a *parametric density* $\delta_\rho(C)$ of a packing C and a positive parameter ρ was introduced by

$$\delta_\rho(C) = \frac{\#C \cdot \kappa_d}{V(\operatorname{conv} C + \rho B^d)},$$

such that Fejes Tóth's definition corresponds to the special parameter $\rho = 1$. Consequently, a maximal parametric finite packing density was defined by

$$\delta_\rho(d, n) = \max\{\delta_\rho(C) : C \subset E^d \text{ is packing with } \#C = n\}.$$

Then it was shown that $\lim_{n\to\infty} \delta_\rho(d, n) = \delta(d)$ for all $\rho \geq 2$, and that $\delta_\rho(d, n) = \delta(S_n^d)$ provided that $\rho < 2/\sqrt{3}$ and d is greater than some constant depending on ρ. In [BHW2] this was improved in that $2/\sqrt{3}$ could be replaced by $\sqrt{2}$. It was further shown that $\delta_\rho(d, n) = \delta(S_n^d)$ if $\delta_{\rho_1}(d, n) = \delta(S_n^d)$ and $\rho \leq \rho_1$. This proved that asymptotically (with respect to d) a stronger result than the sausage conjecture holds and it is most interesting to prove the sausage conjecture in low dimensions. A first step in verifying the sausage conjecture was done in [BHW1]: The sausage conjecture holds for all $d \geq$ 13,387.

Here we optimize the methods developed in [BHW1] and [BHW2] for the special parameter 1 and introduce some new ideas for the study of this special parameter to prove:

Theorem. *The sausage conjecture holds for all dimensions $d \geq 42$.*

As the proof of the theorem is somewhat intricate we proceed as follows: In the second section we first introduce some quantities to measure the size of a packing. After this we state a number of results for these quantities from which we derive our theorem. We close the section by a discussion of the limits of our approach.

In the last three sections we prove the results stated in Section 2. More specifically, in Sections 3 and 4 we study sections of the Dirichlet–Voronoi cell of a fixed point of the packing with certain planes, while in the last section we examine the case that the local deviation of the packing from a sausage is not too large.

2. Proof of the Theorem

In this section we give a proof of the theorem based on several lemmas that will be proved in the next sections. First, observe that, for $n \in \mathbb{N}$,

$$V(\operatorname{conv} S_n^d + B^d) = 2(n-1)\kappa_{d-1} + \kappa_d.$$

So in order to prove the sausage conjecture we have to show that for each packing $C = \{x^1, \ldots, x^n\}$ one has

$$V(\operatorname{conv} C + B^d) \geq 2(n-1)\kappa_{d-1} + \kappa_d. \tag{2.1}$$

To this end we use a local approach, i.e., for a packing set $C = \{x^1, \ldots, x^n\}$ we consider the associated Dirichlet–Voronoi cells (DV-cells, for short) $H^i(C)$, $1 \leq i \leq n$, given by

$$\begin{aligned} H^i(C) &= \{x \in E^d : |x - x^i| \leq |x - x^j|,\ 1 \leq j \leq n\} \\ &= \{x \in E^d : 2\langle x, x^j - x^i\rangle \leq |x^j|^2 - |x^i|^2,\ 1 \leq j \leq n\} \end{aligned}$$

and the parts of $\operatorname{conv} C + B^d$ belonging to $H^i(C)$:

$$D(H^i(C)) = H^i(C) \cap (\operatorname{conv} C + B^d). \tag{2.2}$$

Obviously, we have

$$V(\operatorname{conv} C + B^d) = \sum_{i=1}^{n} V(D(H^i(C))).$$

For a sausage, we have $V(D(H^i(S_n^d))) = 2\kappa_{d-1}, i = 2, \ldots, n-1$, and $V(D(H^1(S_n^d))) = V(D(H^n(S_n^d))) = \kappa_{d-1} + \kappa_d/2$. Thus it suffices to prove

$$V(D(H^i(C))) \geq \begin{cases} 2\kappa_{d-1} & \text{for} \quad n-2 \text{ sets}, \\ \kappa_{d-1} + \kappa_d/2 & \text{for the remaining 2 sets.} \end{cases} \tag{2.3}$$

Hence for the proof of (2.3) we have to identify at most two points of C which can be compared to the ends of the sausage. This is done with the help of the following angle φ^i associated to the point x^i.

Definition 2.1. For $i = 1, \ldots, n$, let $y^{j,i} = (x^j - x^i)/|x^j - x^i|$, $1 \leq j \leq n$, $j \neq i$, and

$$\varphi^i = \max\{\arccos(|\langle y^{k,i}, y^{l,i}\rangle|) : 1 \leq k, l \leq n\},$$

where $\arccos(\cdot)$ is chosen in $[0, \pi/2]$.

We say that a point x^i is an *endpoint* of the packing C if $\varphi^i < \pi/3$ and $\langle y^{k,i}, y^{l,i}\rangle \geq 0$ for $1 \leq k, l \leq n$. Observe that a packing has at most two endpoints. Otherwise, if there were three endpoints they would form a triangle such that the sum of its angles is less than π. From now on we keep the packing C and a point x^i, say x^n, fixed. Further, we assume without loss of generality $x^n = 0$. For abbreviation, we write H, D, φ, y^k instead of $H^n(C), D(H^n(C)), \varphi^n, y^{k,n}$.

Unfortunately, it can happen that $\varphi < \pi/3$ and for the points y^k, y^l with arccos $(|\langle y^k, y^l\rangle|) = \varphi$ we have $\langle y^k, y^l\rangle \geq 0$, but the point 0 is not an endpoint. To identify in this case points in C which correspond to the "neighbors" in the sausage we define:

Definition 2.2. Let y^{j_1}, y^{j_2} be a pair such that

$$\arccos(|\langle y^{j_1}, y^{j_2}\rangle|) = \begin{cases} \varphi & \text{if} \quad \varphi \geq \pi/3 \quad \text{or} \quad \langle y^k, y^l\rangle \geq 0 \quad \text{for} \ 1 \leq k, l \leq n-1, \\ \max\limits_{1\leq k,l\leq n-1}\{\arccos(|\langle y^k, y^l\rangle|) : \langle y^k, y^l\rangle \leq 0\} & \text{otherwise.} \end{cases}$$

Without loss of generality let $y^1 = y^{j_1}$, $y^2 = y^{j_2}$, and let $L = \operatorname{lin}\{y^{j_1}, y^{j_2}\}$.

Such a pair y^1, y^2 may not be uniquely determined, but in any case the definition of φ and of y^1, y^2 gives us:

$$\begin{aligned} |\langle y^k, y^l\rangle| &\geq \cos(\varphi), && 1 \leq k, l \leq n-1, \quad \text{and} \\ |\langle y^1, y^2\rangle| &= \cos(\varphi), && \text{if} \quad \varphi \geq \pi/3 \quad \text{or} \quad \langle y^k, y^l\rangle \geq 0, \quad 1 \leq k, l \leq n-1, \\ \langle y^1, y^2\rangle &\in [-\cos(\varphi/2), -\cos(\varphi)], && \text{otherwise.} \end{aligned} \tag{2.4}$$

Moreover, we need to measure the local deviation of C at 0 from the plane L. To this end, we introduce another angle α.

Definition 2.3. Let

$$\alpha = \alpha(L) = \max\{\arccos(|y^i|L|) : 1 \le i \le n-1\},$$

where $y^i|L$ denotes the orthogonal projection of y^i onto L. Without loss of generality, let $\alpha = \arccos(|y^3|L|)$.

Clearly, the angles α, φ are not independent of each other and it is not hard to see that (see (2.4))

$$\cos(\alpha)\cos(\varphi/2) \ge \cos(\varphi). \tag{2.5}$$

We are interested in certain polytopes depending on y^1, y^2, y^3, and their faces. Therefore, we set for a polytope P

$$F_i(P) = \{F : F \text{ is an } i \text{ face of } P\}.$$

With respect to a polytope $P \subset \operatorname{conv} C$ we dissect D with the help of the nearest point map Φ: $E^d \to E^d$ which is given by (see [MS]):

$$\Phi(x) = y \in P \qquad \text{with} \quad |x-y| = \min\{|x-z| : z \in P\}.$$

Definition 2.4. For a polytope P, let

$$D^i(P) = \operatorname{cl}\{x \in D : \Phi(x) \in F,\ F \in F_i(P)\},$$

where cl denotes the closure.

Then $V(D) = \sum_{i=0}^{\dim P} V(D^i(P))$, and in the following we consider for P the polytopes

$$P^2 = \operatorname{conv}\{0, 2y^1, 2y^2\} \cap H \qquad \text{and} \qquad P^3 = \operatorname{conv}\{0, 2y^1, 2y^2, 2y^3\} \cap H. \tag{2.6}$$

Using the sets $D^i(P^2)$, $D^i(P^3)$ we shall estimate the size of $V(D)$. To this end, we use two different approaches depending on the size of φ.

A small φ means that "close to 0" the arrangement is "sausage-like." The vectors y^1, y^2 define the "direction" of the arrangement at 0 and we consider a slice of D given by sections orthogonal to this direction. Compared to a corresponding slice of a sausage this part of D is wider, but shorter. Nevertheless, in the Lemmata 2.1–2.6 we show that such a "nonsausage" slice has larger volume provided φ is not too large but the dimension is sufficiently high. For large φ we use a technique due to Rogers [R2] to compute the volume of D. Here, it turns out that the volume is large enough compared to the slice of a sausage, if φ is not too small and the dimension is sufficiently high (see Lemmas 2.7 and 2.8). Putting the results together we obtain that the sausage conjecture holds for all dimensions ≥ 42.

We start with the examination of the "sausage-like" case.

Lemma 2.1. *Let* $\varphi_L = \arccos(|\langle y^1, y^2\rangle|)$, *and for* $\delta \in [0, \pi/2]$ *let*

$$v(\delta) = \frac{\pi - \delta}{2} - \left(\arccos\left(2\sin\left(\frac{\delta}{2}\right)\right) - 2\sin\left(\frac{\delta}{2}\right)\sqrt{1 - \left(2\sin\left(\frac{\delta}{2}\right)\right)^2}\right).$$

Then

$$V(P^2 \cap B^d) \begin{cases} \geq \varphi/2 & \text{if} \quad \langle y^1, y^2\rangle \geq -\frac{1}{2}, \\ = v(\varphi_L) & \text{else}. \end{cases}$$

Proof. See the proof of Lemma 4.2 in [BHW1]. □

Lemma 2.2. *Let* $\varphi < \pi/3$ *and* $\langle y^1, y^2\rangle > 0$. *Then*

$$V(D^0(P^2)) \geq \frac{1 - \varphi/\pi}{2}\kappa_d.$$

Proof. See [BHW1, Lemma 4.5]. □

Lemma 2.3. *Let* $\varphi < \pi/3$, $\langle y^1, y^2\rangle < 0$, *and* $\tilde{D}^1(P^2) = \{x \in D^1(P^2) : \Phi(x) \in \operatorname{conv}\{2y^1, 2y^2\}\}$. *Then*

$$V(\tilde{D}^1(P^2)) \geq \frac{\cos(\varphi) - \sin(\varphi)}{\cos(\varphi/2)} \cdot \kappa_{d-1}.$$

Proof. See [BHW1, Lemma 4.6]. □

Next we define certain functions $p_1(\varphi, d)$, $p_2(\alpha, d)$, and $\tilde{p}_2(\alpha, d)$ which allow us to describe the influence of points in C outside L on the size of $D^0(P^2)$, $D^1(P^2)$, and $D^2(P^2)$.

Lemma 2.4. *Let* $\varphi_* = 1.16$, *and let*

$$p_1(\varphi, d) = \begin{cases} 1, & \varphi < \pi/4, \\ \min\left\{1, \int_0^{(1-\sin(\varphi))/\cos(\varphi)} \left(-r\frac{\cos(\varphi)}{\sin(\varphi)} + \frac{1}{\sin(\varphi)}\right)^{d-1} dr\right\}, & \pi/4 \leq \varphi \leq \varphi_*. \end{cases}$$

Then, for $d \geq 42$

$$V(D^1(P^2)) \geq V(\hat{D}^1(P^2)) \geq p_1(\varphi, d) \cdot \kappa_{d-1},$$

where $\hat{D}^1(P^2) = \{x \in D^1(P^2) : \Phi(x) \in \operatorname{conv}\{0, 2y^1\} \cup \operatorname{conv}\{0, 2y^2\}\}$.

Proof. See Section 5. □

Lemma 2.5. *Let* $\alpha_* = 1.11$, *and let*

$$p_2(\alpha, d) = \begin{cases} \frac{1}{2}, & \alpha < \pi/4, \\ \min\left\{\frac{1}{2}, \int_0^{(1-\sin(\alpha))/\cos(\alpha)} r\left(-r\frac{\cos(\alpha)}{\sin(\alpha)} + \frac{1}{\sin(\alpha)}\right)^{d-2} dr\right\}, & \pi/4 \le \alpha \le \alpha_*. \end{cases}$$

Then, for $d \ge 42$

$$V(D^2(P^2)) \ge V(P^2 \cap B^d) \cdot 2 \cdot p_2(\alpha, d)\kappa_{d-2}.$$

Proof. See Section 5. □

For certain values of α and φ it is better to consider $V(D^2(P^2))$ together with $V(D^0(P^2))$. We have

Lemma 2.6. *Let* $\alpha_* = 1.11$, *and let*

$$\tilde{p}_2(\alpha, d) = \begin{cases} \frac{1}{2}, & \alpha < \pi/4, \\ \min\left\{\frac{1}{2}, 2 \cdot \int_0^{(1-\sin(\alpha))/\cos(\alpha)} r\left(-r\frac{\cos(\alpha)}{\sin(\alpha)} + \frac{1}{\sin(\alpha)}\right)^{d-2} dr\right\}, & \pi/4 \le \alpha \le \alpha_*. \end{cases}$$

Then for $d \ge 42$ *and* $\varphi \ge \pi/3$

$$V(D^0(P^2)) + V(D^2(P^2)) \ge \frac{\varphi}{2} \cdot 2 \cdot \tilde{p}_2(\alpha, d)\kappa_{d-2}.$$

Proof. See Section 5. □

With the help of the next two lemmas we estimate $V(D)$ for large φ or α. These estimates are based on computing the size of sections of the DV-cell H with a technique due to Rogers [R2].

Lemma 2.7. *Let* $d \ge 42$. *Then*

$$V(D^1(P^2)) > 0.65019 \cdot \kappa_{d-1}.$$

Proof. See Section 3. □

For large α it becomes favorable to consider P^3 rather than P^2.

Lemma 2.8. *Let* $\alpha \ge \alpha_* = 1.11$. *Then for* $d \ge 42$

$$V(D) \ge V(D^1(P^3)) + V(D^2(P^3)) + V(D^3(P^3)) > 2\kappa_{d-1}.$$

Proof. See Section 3. □

Now, with the lemmas above we are able to give the proof of the theorem.

Proof of the Theorem. Before we start we remark that the functions $\tilde{p}_2(\alpha, d)$, $p_2(\alpha, d)$, $p_1(\varphi, d)$ (see Lemmas 2.6, 2.5, and 2.4) are monotonely decreasing in α, φ, respectively, and monotonely increasing in d. Hence, for $d \geq 42$,

$$\begin{aligned} \tilde{p}_2(\alpha, d) &\geq \tilde{p}_2(\alpha_*, 42) &\geq 0.45358, \qquad & \alpha \leq \alpha_* = 1.11, \\ p_2(\alpha, d) &= p_2\left(\frac{\pi}{3}, 42\right) &= \tfrac{1}{2}, \qquad & \alpha \leq \frac{\pi}{3}, \\ p_1(\varphi, d) &= p_1(\varphi_*, 42) &= 1, \qquad & \varphi \leq \varphi_* = 1.16. \end{aligned} \tag{2.7}$$

We recall that the quotient κ_{d-1}/κ_d is strictly monotonely increasing in d. Further, observe that we always have $\alpha \leq \varphi$ (see (2.5)). We distinguish three cases depending on the angle φ and the sign of $\langle y^1, y^2 \rangle$.

(i) $\varphi < \pi/3$ and $\langle y^1, y^2 \rangle \geq 0$.
So we have the "end of the sausage" case and by Lemmas 2.1, 2.2, 2.4, and 2.5 we get

$$\begin{aligned} V(D) &\geq V(D^0(P^2)) + V(D^1(P^2)) + V(D^2(P^2)) \\ &\geq \varphi p_2(\alpha, d)\kappa_{d-2} + p_1(\varphi, d)\kappa_{d-1} + \frac{1 - \varphi/\pi}{2}\kappa_d. \end{aligned}$$

Since $\alpha \leq \varphi < \pi/3$ we obtain by (2.7):

$$\begin{aligned} V(D) &\geq \kappa_{d-1} + \tfrac{1}{2}\kappa_d + \frac{\varphi}{2}\kappa_d\left(\frac{\kappa_{d-2}}{\kappa_d} - \frac{1}{\pi}\right) \\ &\geq \kappa_{d-1} + \tfrac{1}{2}\kappa_d + \frac{\varphi}{2}\kappa_d\left(\frac{\kappa_{40}}{\kappa_{42}} - \frac{1}{\pi}\right) \geq \kappa_{d-1} + \tfrac{1}{2}\kappa_d, \qquad d \geq 42. \end{aligned}$$

(ii) $\varphi < \pi/3$ and $\langle y^1, y^2 \rangle < 0$.
By Lemma 2.1 we have $V(P^2 \cap B^d) = v(\varphi_L)$ and the derivative of $v(\delta)$ with respect to δ is

$$\frac{\partial v(\delta)}{\partial \delta} = -\tfrac{1}{2} + 2\cos\left(\frac{\delta}{2}\right)\sqrt{1 - \left(2\sin\left(\frac{\delta}{2}\right)\right)^2}.$$

This shows that $V(P^2 \cap B^d)$ is a concave function in δ and certainly monotonely increasing for $\delta \in [0, \pi/4]$. An easy computation yields $\min\{v(\pi/8), v(\pi/3)\} = v(\pi/8)$ and so by (2.4)

$$V(P^2 \cap B^d) \geq \begin{cases} v(\varphi/2) & \text{for } \varphi \leq \pi/4, \\ v(\pi/8) & \text{for } \pi/4 \leq \varphi \leq \pi/3. \end{cases}$$

First, assume $\varphi \leq \pi/4$. Then by Lemmas 2.1, 2.3, 2.4, 2.5, and (2.7):

$$\begin{aligned} V(D) &\geq V(\tilde{D}^1(P^2)) + V(\hat{D}^1(P^2)) + V(D^2(P^2)) \\ &\geq 2 \cdot v\left(\frac{\varphi}{2}\right)p_2(\alpha, d)\kappa_{d-2} + p_1(\varphi, d)\kappa_{d-1} + \frac{\cos(\varphi) - \sin(\varphi)}{\cos(\varphi/2)}\kappa_{d-1} \\ &= 2\kappa_{d-1} + \kappa_{d-1}\left(v\left(\frac{\varphi}{2}\right)\frac{\kappa_{d-2}}{\kappa_{d-1}} + \frac{\cos(\varphi) - \sin(\varphi)}{\cos(\varphi/2)} - 1\right). \end{aligned}$$

Calculating the second derivative shows that the function in the brackets is concave with respect to φ, $\varphi \leq \pi/2$. Since $v(\pi/8) \geq 0.56373$ and $\kappa_{40}/\kappa_{41} \geq 2.57$, as a simple computation shows, we obtain for $d \geq 42$, $\varphi \in [0, \pi/4]$:

$$V(D) \geq \min\left\{2\kappa_{d-1}, 2\kappa_{d-1} + \kappa_{d-1}\left(v\left(\frac{\pi}{8}\right)\frac{\kappa_{40}}{\kappa_{41}} - 1\right)\right\} \geq 2\kappa_{d-1}. \tag{2.8}$$

Now let $\pi/4 \leq \varphi < \pi/3$. Then $V(P^2 \cap B^d) \geq v(\pi/8)$ and as above we obtain for $d \geq 42$:

$$\begin{aligned} V(D_1) &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(v\left(\frac{\varphi}{2}\right)\frac{\kappa_{d-2}}{\kappa_{d-1}} - 1\right) \geq 2\kappa_{d-1} + \kappa_{d-1}\left(v\left(\frac{\pi}{8}\right)\frac{\kappa_{40}}{\kappa_{41}} - 1\right). \\ &> 2\kappa_{d-1}. \end{aligned}$$

Together with (2.8) it implies $V(D) \geq 2\kappa_{d-1}$ for $d \geq 42$.

(iii) $\varphi \geq \pi/3$.
Here we distinguish two cases depending on the angle α.
(a) $\alpha \leq \alpha_*$.
For $d \geq 42$ and $\varphi \geq \varphi_*$ we find by Lemmas 2.6, 2.7, and (2.7)

$$\begin{aligned} V(D) &\geq V(D^0(P^2)) + V(D^2(P^2)) + V(D^1(P^2)) \\ &\geq \varphi \cdot 0.45358 \cdot \kappa_{d-2} + 0.65019 \cdot \kappa_{d-1} \\ &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(1.16 \cdot 0.45358 \cdot \frac{\kappa_{d-2}}{\kappa_{d-1}} - 1.34981\right) \\ &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(0.5261528 \cdot \frac{\kappa_{40}}{\kappa_{41}} - 1.34981\right) > 2\kappa_{d-1}. \end{aligned}$$

For $\pi/3 \leq \varphi \leq \varphi_*$ we use Lemma 2.4 instead of Lemma 2.7 and obtain

$$\begin{aligned} V(D) &\geq V(D^0(P^2)) + V(D^2(P^2)) + V(D^1(P^2)) \\ &\geq \varphi \cdot 0.45358 \cdot \kappa_{d-2} + \kappa_{d-1} \\ &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(\frac{\pi}{3} \cdot 0.45358 \cdot \frac{\kappa_{d-2}}{\kappa_{d-1}} - 1.\right) \\ &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(0.47498 \cdot \frac{\kappa_{40}}{\kappa_{41}} - 1.\right) > 2\kappa_{d-1}. \end{aligned}$$

(b) $\alpha \geq \alpha_*$.
In this case $V(D) > 2\kappa_{d-1}$, $d \geq 42$, follows immediately from Lemma 2.8.

As the first case ($\varphi < \pi/3$, $\langle y^1, y^2\rangle > 0$) can occur at most twice, the proof is finished. □

We close this section with a short discussion of our method. Since we use a local approach we have to compare for a packing $C = \{x^1, \ldots, x^n\}$ the volumes of $V(D(H^i(C)))$ to $2\kappa_{d-1}$ for at least $(n-2)$ cells (see (2.3)). Now let $\operatorname{conv} C$ be a regular triangle. In this case we have to compare $V(D(H^i(C)))$ with $2\kappa_{d-1}$ for at least one i. But

$$V(D(H^i(C))) = \frac{1}{\sqrt{3}}\kappa_{d-2} + \kappa_{d-1} + \tfrac{1}{3}\kappa_d.$$

So $V(D(H^i(C))) < 2\kappa_{d-1}$ for $d \leq 11$. Thus to prove the conjecture for $d \leq 11$ a nonlocal method has to be applied.

It is, in principle, no problem to improve several arguments in our reasoning. However, as far as we can see, such an improvement would make the proof disproportionately more technical. The dimension 42 may be considered as a compromise between a "good" dimension and complexity of the proof.

3. Sections of the Dirichlet–Voronoi Cell

Let $L^\perp$ be the orthogonal complement of the plane L, and for a parameter $\rho < \sqrt{2}$ let

$$M(\rho, L^\perp) = \{z \in S^{d-1} \cap L^\perp : \rho z \notin H\}, \qquad K(\rho, L^\perp) = \{z \in S^{d-1} \cap L^\perp : \rho z \in H\}.$$

In [BHW2] it was shown that the ratio of the spherical volumes of $M(\rho, L^\perp)$ to $K(\rho, L^\perp)$ is bounded from above by a constant c provided the dimension d is large enough (see Theorem 1.1 in [BHW2]). For $\rho < 2/\sqrt{3}$ this was already proved in [BHW1] and there it was also shown that based on such an estimate one obtains a lower bound for $V(w + (B^d \cap L^\perp))$, $w \in (P^2 \cap B^d)$, which leads to a lower bound of $V(D^2(P^2))$ (see Lemma 4.7 in [BHW1]).

Here we want to give a generalization of these results for the special parameter $\rho = 1$. To keep the paper self-contained as much as possible we first state the two basic lemmas which yield the upper bound of $V_\star(M(\rho, L^\perp))/V_\star(K(\rho, L^\perp))$ in [BHW2].

Lemma 3.1. *Let $S \subset E^d$ be a d-simplex, let F_k be a k-face of S, $k \leq d-1$, and let $\bar{F}_k$ be the $(d-k-1)$-face of S with $F_k \cap \bar{F}_k = \emptyset$. For a measurable subset $G \subset S$ and a continuous function f on S we have*

$$\int_G f\,dx = \frac{d!}{k!\,(d-1-k)!} \frac{V(S)}{V(F_k)V(\bar{F}_k)} \cdot \int_{F_k} \int_{\bar{F}_k} \int_{\mu\bar{x}+(1-\mu)x \in G} f(\mu\bar{x} + (1-\mu)x)\mu^{d-1-k}(1-\mu)^k \, d\mu \, d\bar{x} \, dx.$$

Remark. The notation $\int dx$ means integration in a space of appropriate dimension.

Proof. See Lemma 2.1 in [BHW2]. □

Lemma 3.2. *Let $k, \bar{k} \in \mathbb{N}$ with $\bar{k} \geq k+1$ and let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > \beta > 0$, $\alpha > 0$. Then for $a, b, c \in \mathbb{R}$, $d \in \mathbb{N}$, with $b, c \geq 0$, $b < c$, $a \geq \alpha$, $a^2 + c^2 \geq \gamma$, $a^2 + b^2 \leq \beta$, $d \geq \bar{k}$ the quotient*

$$\frac{\int_0^{\mu_0} (\sqrt{a^2 + (\mu c + (1-\mu)b)^2})^{-(d+1)} \mu^{d-1-k}(1-\mu)^k \, d\mu}{\int_{\mu_0}^1 (\sqrt{a^2 + (\mu c + (1-\mu)b)^2})^{-(d+1)} \mu^{d-1-k}(1-\mu)^k \, d\mu}, \tag{3.1}$$

where $\mu_0 \in [0, 1]$ is determined by $a^2 + (\mu_0 c + (1-\mu_0)b)^2 = \beta$, is maximal for $a = \alpha$, $b = 0$, $a^2 + c^2 = \gamma$, and $d = \bar{k}$.

Proof. See Lemma 2.2 in [BHW2]. □

In order to formulate our generalization we need some elementary notation from the theory of convex polytopes (see [Grü]). For a nonempty n-dimensional face F of a p-dimensional polytope $P \subset E^d$ the normal cone $N(P, F)$ is the cone generated by all vectors $v \in E^d$ with the property that there exists a $\nu \in \mathbb{R}^{\geq 0}$ with $F = P \cap \{x \in E^d : \langle v, x\rangle = \nu\}$ and $\langle v, x\rangle \leq \nu$ for all $x \in P$. The dimension of the normal cone is $d - n$. In particular, $F + N(P, F)$ is the set of all points $x \in E^d$ such that the nearest point of x with respect to P belongs to F. The ratio of the spherical volume of $N(P, F) \cap S^{d-1}$ to $V_\star(S^{d-n})$ is called the external angle of F and is denoted by $\theta(P, F)$.

Moreover, we define some functions which will be used in the forthcoming estimates:

Definition 3.1. Let $r \in \mathbb{R}$ with $0 \leq r < 1$ and let $d, k, l, m \in \mathbb{N}$, such that $k + 2 \leq d - l + m$ and $k + 2 - m > (1 + r^2)/(1 - r^2)$. Let

$$
\begin{aligned}
a(r) &= \sqrt{1 - r^2},\\
c(k, m) &= \sqrt{\frac{2(k + 2 - m)}{k + 3 - m} - r^2 - a(r)^2} = \sqrt{\frac{k + 1 - m}{k + 3 - m}},\\
\mu_0(k, m, r) &= \frac{r}{c(k, m)},\\
M(d, l, k, m, r) &= \int_0^{\mu_0(k,m,r)} \left(\sqrt{a(r)^2 + \mu^2 c(k, m)^2}\right)^{-(d-l+m)}\\
&\quad \times \mu^{d-l+m-(k+2)}(1 - \mu)^k \, d\mu,\\
K(d, l, k, m, r) &= \int_{\mu_0(k,m,r)}^1 \left(\sqrt{a(r)^2 + \mu^2 c(k, m)^2}\right)^{-(d-l+m)}\\
&\quad \times \mu^{d-l+m-(k+2)}(1 - \mu)^k \, d\mu,\\
Q(d, l, m, r) &= \left\{k \in \mathbb{N} : \frac{1 + r^2}{1 - r^2} + m < k + 2 \leq d - l + m\right\},\\
q(d, l, m, r) &= \begin{cases} \infty, & Q(d, l, m, r) = \emptyset,\\ \min\left\{\frac{M(d,l,k,m,r)}{K(d,l,k,m,r)} : k \in Q(d, l, m, r)\right\} & \text{otherwise.} \end{cases}
\end{aligned}
$$

The purpose of this section is to prove:

Lemma 3.3. *Let $\hat{L} \subset E^d$ be an l-dimensional subspace and let $P \subset \hat{L}$ be an l-dimensional polytope with vertex* 0*. Moreover, let F be an $(l - m)$-dimensional face of P with $0 \in F$ and let $w \in F$ with $|w| < 1$. Then*

$$V_\star((w + (N(P, F) \cap S^{d-1})) \cap H) \geq \theta(P, F) \cdot \frac{(d - l + m)\kappa_{d-l+m}}{1 + q(d, l, m, |w|)}. \tag{3.2}$$

Proof. Let $M_w = \{z \in N(P, F) \cap S^{d-1} : w + z \notin H\}$ and let $K_w = \{z \in N(P, F) \cap S^{d-1} : w + z \in H\}$. By the definition of the external angle, we have $V_\star(M_w) + V_\star(K_w) =$

$\theta(P, F) \cdot (d-l+m)\kappa_{d-l+m}$, and thus

$$V_\star(K_w) = \theta(P, F)\frac{(d-l+m)\kappa_{d-l+m}}{1 + V_\star(M_w)/V_\star(K_w)}.$$

It remains to show

$$\frac{V_\star(M_w)}{V_\star(K_w)} \leq q(d, l, m, |w|). \tag{3.3}$$

To this end, we may assume $Q(d, l, m, |w|) \neq \emptyset$ and let W be a d-dimensional cube with midpoint 0 and edge of length $2\sqrt{2}$. To prove (3.3) we proceed as in the proof of Theorem 1.1 in [BHW2]. First, we apply Rogers' dissection technique (see [R2]) to the $(d-l+m)$-dimensional polyhedron $P = (w + N(P, F)) \cap H$ with respect to the reference point $c^0 = w$. This means, we construct a dissection of the bounded polyhedron $P \cap W$ into simplices S of the form $S = \mathrm{conv}\{c^0, \dots, c^{d-l+m}\}$, such that c^i is contained in a $(d-l+m-i)$-face G of $P \cap W$ with $w \notin G$, G contains $\mathrm{conv}\{c^i, \dots, c^{d-l+m}\}$, and c^i is the nearest point of G to c^0.

Next we consider the distance from a point c^i, $i \geq 1$, of such a simplex to w. Obviously, if c^i belongs to a face of W, then we have $|c^i - w| \geq \sqrt{2 - |w|^2}$. Now let c^i be a point of a $(d-l+m-i)$-face G of P. As the $(d-l)$-dimensional orthogonal complement of $\hat{L}$ is contained in $N(P, F)$ we have that for $i > m$ the point c^i belongs to a $(d-(i-m))$-face of H. Clearly, for $1 \leq i \leq m$ the point c^i lies at least in 1 facet of H. In view of a result by Rogers about the distance between $(d-i)$-faces of H and the origin (see [R2]), we get

$$|c^i - w| \geq \begin{cases} \sqrt{1 - |w|^2}, & 1 \leq i \leq m, \\ \sqrt{2(i-m)/(i-m+1) - |w|^2}, & m < i. \end{cases} \tag{3.4}$$

Let $S = \mathrm{conv}\{c^0, \dots, c^{d-l+m}\}$ be an arbitrary but fixed simplex of this dissection, let C^0 be the cone generated by $c^1, \dots, c^{d-l+m}$, and let

$$\begin{aligned} M_S &= \{z \in (N(P, F) \cap S^{d-1}) \cap C^0 : w + z \notin S\}, \\ K_S &= \{z \in (N(P, F) \cap S^{d-1}) \cap C^0 : w + z \in S\}. \end{aligned}$$

Clearly, it suffices to prove (3.3) for the sets M_S, K_S. Based on Lemma 3.1, (3.4), and the definition of the set $Q(d, l, m, |w|)$ we obtain analogously to the proof of Theorem 1.1 in [BHW2] for each $k \in Q(d, l, m, |w|)$:

$$\frac{V_\star(M_S)}{V_\star(K_S)} = \frac{\int_{F_k}\int_{\bar{F}_k}\int_{|\mu\bar{x}+(1-\mu)x|_w \leq 1}[(\mu^{d-l+m-(k+2)}(1-\mu)^k)/|\mu\bar{x} + (1-\mu)x|_w^{d-l+m}]\,d\mu\,d\bar{x}\,dx}{\int_{F_k}\int_{\bar{F}_k}\int_{|\mu\bar{x}+(1-\mu)x|_w \geq 1}[(\mu^{d-l+m-(k+2)}(1-\mu)^k)/|\mu\bar{x} + (1-\mu)x|_w^{d-l+m}]\,d\mu\,d\bar{x}\,dx},$$

where $|y|_w$ denotes the distance from the point y to w and $\bar{F}_k = \mathrm{conv}\{c^{k+2}, \dots, c^{d-l+m}\}$, $F_k = \mathrm{conv}\{c^1, \dots, c^{k+1}\}$. Hence

$$\frac{V_\star(M_w)}{V_\star(K_w)} \leq \frac{\int_{|\mu\bar{x}+(1-\mu)x|_w \leq 1} |\mu\bar{x} + (1-\mu)x|_w^{-(d-l+m)}\mu^{d-l+m-(k+2)}(1-\mu)^k\,d\mu}{\int_{|\mu\bar{x}+(1-\mu)x|_w \geq 1} |\mu\bar{x} + (1-\mu)x|_w^{-(d-l+m)}\mu^{d-l+m-(k+2)}(1-\mu)^k\,d\mu}, \tag{3.5}$$

for certain points $\bar{x} \in \bar{F}_k$, $x \in F_k$. By (3.4) and the choice of k we have

$$|x|_w \geq \sqrt{1 - |w|^2}, \qquad |\bar{x}|_w \geq \sqrt{\frac{2(k+2-m)}{(k+3-m)} - |w|^2} > 1.$$

Since $|\mu\bar{x} + (1-\mu)x|_w$ is monotonely increasing in μ we may assume $|x|_w < 1$. Then (3.5) is of the form

$$\frac{V_\star(M_w)}{V_\star(K_w)} \leq \frac{\int_0^{\mu_0} \sqrt{a^2 + (\mu c + (1-\mu)b)^2}^{-(d-l+m)} \mu^{d-l+m-(k+2)} (1-\mu)^k \, d\mu}{\int_{\mu_0}^1 \sqrt{a^2 + (\mu c + (1-\mu)b)^2}^{-(d-l+m)} \mu^{d-l+m-(k+2)} (1-\mu)^k \, d\mu},$$

where $a \geq \alpha = \sqrt{1 - |w|^2}$ denotes the distance between the line through $\bar{x}, x$ to w, b is given by $a^2 + b^2 = |x|_w^2$, c is given by $a^2 + c^2 = |\bar{x}|_w^2$, and μ_0 is determined by $a^2 + (\mu_0 c + (1-\mu_0)b)^2 = 1$. But now (3.3) follows from Lemma 3.2 and Definition 3.1 with $\beta = 1$, $\gamma = 2(k+2-m)/(k+3-m) - |w|^2$, $\alpha = a(|w|)$, $b = 0$, $c = c(k, m)$, and $\mu_0 = \mu_0(k, m, |w|)$. □

Instead of the spherical volume $V_\star((w + (N(P, F) \cap S^{d-1})) \cap H)$, we are often interested in the volume $V((w + (N(P, F) \cap B^d)) \cap H)$. Since

$$V((w + (N(P, F) \cap B^d)) \cap H) = \frac{1}{d-l+m} V_\star((w + (N(P, F) \cap S^{d-1})) \cap H),$$

we have:

Corollary 3.1. *Under the assumptions of Lemma* 3.3 *one has*

$$V((w + (N(P, F) \cap B^d)) \cap H) \geq \theta(P, F) \cdot \frac{\kappa_{d-l+m}}{1 + q(d, l, m, |w|)}.$$

Furthermore, as an immediate consequence we obtain:

Corollary 3.2.

$$V(D^2(P^2)) \; \geq \; \kappa_{d-2} \int_{P^2 \cap B^d} \frac{1}{1 + q(d, 2, 0, |w|)} dw,$$
$$V(D^1(P^2)) \; \geq \; \kappa_{d-1} \int_0^1 \frac{1}{1 + q(d, 2, 1, r)} \, dr.$$

Proof. For $F = P^2$ we have $\theta(P^2, F) = 1$ and $N(P^2, F) = L^\perp$. By the definition of $D^2(P^2)$ and the normal cones, we get

$$((P^2 \cap B^d) + (N(P^2, F) \cap B^d)) \cap H \subset D^2(P^2).$$

In view of Corollary 3.1 this implies the lower bound for $V(D^2(P^2))$. For the bound of $V(D^1(P^2))$ we note that

$$(\text{conv}\{0, y^i\} + (N(P^2, \text{conv}\{0, 2y^i\}) \cap B^d)) \cap H \subset D^1(P^2)$$

and $\theta(P^2, \text{conv}\{0, 2y^i\}) = \frac{1}{2}$ for $i = 1, 2$. □

Next we collect some numerical results involving the function $q(d,l,m,r)$ which will be used in the course of our investigations. Therefore, we define

Definition 3.2. Let $\bar{h} = 0.74740141$:

$$\omega_1(d) = \int_0^1 \frac{1}{1+q(d,3,2,r)}\,dr, \qquad \omega_2(d) = \int_0^1 \frac{r}{1+q(d,3,1,r)}\,dr,$$
$$\omega_3(d) = \int_0^{\bar{h}} \frac{r^2}{1+q(d,3,0,r)}\,dr.$$

Proposition 3.1. *The functions $\omega_i(d)$ are monotonely increasing functions in d. For $d \geq 42$, we have*

$$\begin{aligned}
\omega_1(d) &\geq \omega_1(42) \geq 0.62638506,\\
\omega_2(d) &\geq \omega_2(42) \geq 0.21085103,\\
\omega_3(d) &\geq \omega_3(42) \geq 0.10145239,\\
\int_0^1 \frac{1}{1+q(d,2,1,r)}dr &\geq \int_0^1 \frac{1}{1+q(42,2,1,r)}dr \geq 0.65019115.
\end{aligned}$$

Proof. As $Q(d,l,m,r) \subset Q(d',l,m,r)$ for $d' \geq d$, we see by Lemma 3.2 that the function $q(d,l,m,r)$ is monotonely decreasing in d and thus $\omega_i(d)$ are increasing functions.

Instead of determining the exact value of $q(d,l,m,r)$ we use the following upper bound:

$$q(d,l,m,r) \leq \frac{M(d,l,k(m,r),m,r)}{K(d,l,k(m,r),m,r)},$$

where $k(m,r)$ is the smallest integer greater than $(1+r^2)/(1-r^2)+m$. If $k(m,r) \notin Q(d,l,m,r)$, then we use the trivial upper bound ∞. The numerical calculations of the integrals were carried out by the program *Mathematica*[1] with a working precision of 40 digits. □

In view of these computations, Lemma 2.7 follows from Corollary 3.2

Lemma 2.7. *Let $d \geq 42$. Then*

$$V(D^1(P^2)) > 0.65019 \cdot \kappa_{d-1}.$$

In the next section we shall apply Corollary 3.1 to the set P^3.

[1] ©1988, 1991, 1992 von Wolfram Research Inc.

4. Three-Dimensional Sections

In order to simplify the analysis we assign the following coordinates to the vectors y^1, y^2, y^3 defined by Definitions 2.2 and 2.3

$$\begin{aligned} y^1 &= (1, 0, 0, \dots, 0)^T, \\ y^2 &= (\cos(\gamma), \sin(\gamma), 0, \dots, 0)^T, \\ y^3 &= (\cos(\alpha)\cos(\beta), \cos(\alpha)\sin(\beta), \sin(\alpha), 0, \dots, 0)^T, \end{aligned}$$

where $\gamma \in [0, \pi]$ denotes the angle between y^1 and y^2 and $\beta \in [0, 2\pi]$. For $\alpha \geq \pi/3$ we clearly have $\varphi \geq \pi/3$ by (2.5) and thus $|\cos(\gamma)| = \cos(\varphi)$. Moreover, we see by (2.5)

$$\cos(\alpha) \geq \frac{\cos(\gamma)}{\cos(\gamma/2)}, \quad \gamma \leq \frac{\pi}{2}, \qquad \cos(\alpha) \geq \frac{-\cos(\gamma)}{\sin(\gamma/2)}, \quad \gamma \geq \frac{\pi}{2}. \tag{4.1}$$

Hence with

$$\Upsilon(\alpha) = \arccos\left(\tfrac{1}{4}\cos^2(\alpha) + \cos(\alpha)\sqrt{\tfrac{1}{16}\cos^2(\alpha) + \tfrac{1}{2}}\right)$$

we obtain, for $\alpha \geq \pi/3$, the following restriction on the angle γ

$$\gamma \in [\Upsilon(\alpha), \pi - \Upsilon(\alpha)]. \tag{4.2}$$

In what follows we study some geometric quantities of P^3. Let $f_{i,j}$ denote the angle between y^i and y^j, $1 \leq i < j \leq 3$. Then

$$f_{1,2} = \gamma, \quad f_{1,3} = \arccos(\cos(\alpha)\cos(\beta)) \quad \text{and} \quad f_{2,3} = \arccos(\cos(\alpha)\cos(\gamma - \beta)).$$

For $\alpha > 0$, let $u_{i,j} \in \text{lin}\{y^1, y^2, y^3\}$, $1 \leq i < j \leq 3$, be the outward unit normal vector of the 2-face $F_{i,j} = \text{conv}\{0, 2y^i, 2y^j\} \cap H$ of P^3:

$$\begin{aligned} u_{1,2} &= (0, 0, -1, 0, \dots, 0)^T, \\ u_{1,3} &= \frac{(0, -\sin(\alpha), \cos(\alpha)\sin(\beta), 0, \dots, 0)^T}{\sqrt{1 - \cos^2(\alpha)\cos^2(\beta)}}, \\ u_{2,3} &= \frac{(-\sin(\alpha)\sin(\gamma), \sin(\alpha)\cos(\gamma), \cos(\alpha)\sin(\gamma - \beta), 0, \dots, 0)^T}{\sqrt{1 - \cos^2(\alpha)\cos^2(\gamma - \beta)}}. \end{aligned}$$

Finally, let $g_{1,2}$, $g_{1,3}$, and $g_{2,3}$ denote the angle between the normal vectors $(u_{1,3}, u_{2,3})$, $(u_{1,2}, u_{2,3})$ and $(u_{1,2}, u_{1,3})$, respectively. We get

$$\begin{aligned} g_{1,2} &= \arccos\left(\frac{-\sin^2(\alpha)\cos(\gamma) + \cos^2(\alpha)\sin(\beta)\sin(\gamma - \beta)}{\sqrt{1 - \cos^2(\alpha)\cos^2(\beta)}\sqrt{1 - \cos^2(\alpha)\cos^2(\gamma - \beta)}}\right), \\ g_{1,3} &= \arccos\left(\frac{-\cos(\alpha)\sin(\gamma - \beta)}{\sqrt{1 - \cos^2(\alpha)\cos^2(\gamma - \beta)}}\right), \\ g_{2,3} &= \arccos\left(\frac{-\cos(\alpha)\sin(\beta)}{\sqrt{1 - \cos^2(\alpha)\cos^2(\beta)}}\right). \end{aligned}$$

With this notation we obtain for $V(D)$ the lower bound:

Lemma 4.1. *Let* $\alpha \geq \alpha_* = 1.11$. *Then with the notation of Definition* 3.2

$$\begin{aligned} V(D^1(P^3)) &\geq \left(\frac{g_{1,2}+g_{1,3}+g_{2,3}}{2\pi}\right)\cdot \omega_1(d)\cdot \kappa_{d-1},\\ V(D^2(P^3)) &\geq \left(\frac{f_{1,2}+f_{1,3}+f_{2,3}}{2}\right)\cdot \omega_2(d)\cdot \kappa_{d-2},\\ V(D^3(P^3)) &\geq (2\pi - g_{1,2}-g_{1,3}-g_{2,3})\cdot \omega_3(d)\cdot \kappa_{d-3}. \end{aligned}$$

Proof. From the definition of P^3 and the normal cones follows:

$$\begin{aligned} V(D^1(P^3)) &\geq \sum_{i=1}^{3}\int_{\mathrm{conv}\{0,y^i\}} V((w+(N(P^3,\mathrm{conv}\{0,2y^i\})\cap B^d))\cap H)\,dw,\\ V(D^2(P^3)) &\geq \sum_{1\leq i<j\leq 3}\int_{F_{i,j}} V((w+(N(P^3,\mathrm{conv}\{0,2y^i,2y^j\})\cap B^d))\cap H)\,dw,\\ V(D^3(P^3)) &\geq \int_{P^3} V((w+(N(P^3,P^3)\cap B^d))\cap H)\,dw. \end{aligned}$$

From Corollary 3.1 we obtain:

$$\begin{aligned} V(D^1(P^3)) &\geq \sum_{i=1}^{3}\theta(P^3,\mathrm{conv}\{0,2y^i\})\cdot\kappa_{d-1}\int_{\mathrm{conv}\{0,y^i\}}\frac{1}{1+q(d,3,2,|w|)}\,dw,\\ V(D^2(P^3)) &\geq \sum_{1\leq i<j\leq 3}\theta(P^3,\mathrm{conv}\{0,2y^i,2y^j\})\cdot\kappa_{d-2}\int_{F_{i,j}}\frac{1}{1+q(d,3,1,|w|)}\,dw,\\ V(D^3(P^3)) &\geq \theta(P^3,P^3)\cdot\kappa_{d-3}\int_{P^3}\frac{1}{1+q(d,3,0,|w|)}\,dw. \end{aligned}$$

Now $\theta(P^3,\mathrm{conv}\{0,2y^i\}) = g_{k,j}/(2\pi)$, $k, j \neq i$, $\theta(P^3,\mathrm{conv}\{0,2y^i,2y^j\}) = \frac{1}{2}$, and $\theta(P^3,P^3) = 1$. Since $\alpha \geq \pi/3$, we have $f_{1,2}, f_{1,3}, f_{2,3} \in [\pi/3, 2\pi/3]$. Thus, the intersection of the cone generated by y^i, y^j with B^d belongs to the 2-face $F_{i,j}$. Hence we get the formulas for $V(D^1(P^3))$ and $V(D^2(P^3))$.

Let h be the distance from $\mathrm{conv}\{2y^1,2y^2,2y^3\}$ to the origin. Then

$$\min\{1,h\}\cdot(\mathrm{cone}\{y^1,y^2,y^3\}\cap B^d)\subset P^3$$

and as $V_\star\left(\mathrm{cone}\{y^1,y^2,y^3\}\cap S^{d-1}\right) = (2\pi - g_{1,2}-g_{1,3}-g_{2,3})$ (see [S]), we get

$$V(D^3(P^3)) \geq (2\pi - g_{1,2}-g_{1,3}-g_{2,3})\int_0^{\min\{h,1\}}\frac{r^2}{1+q(d,3,0,r)}\,dr.$$

It remains to show that for $\alpha \geq \alpha_*$ the distance h is not less than $\bar{h}$ of Definition 3.2. A lower bound for h is given by the distance $\eta(\alpha,\beta,\gamma)$ between the affine hull of $\{2y^1,2y^2,2y^3\}$ and the origin:

$$\begin{aligned} h \geq \eta(\alpha,\beta,\gamma) &= (2\sin(\alpha)\sin(\gamma))\\ &\quad\cdot((\sin(\alpha)\sin(\gamma))^2+(\sin(\alpha)(1-\cos(\gamma)))^2\\ &\qquad+(\sin(\gamma)-\cos(\alpha)\sin(\beta)+\cos(\alpha)\sin(\beta-\gamma))^2)^{-1/2}. \end{aligned}$$

Calculating the first partial derivatives of $(\sin(\gamma) - \cos(\alpha)\sin(\beta) + \cos(\alpha)\sin(\beta - \gamma))^2$ with respect to β shows that this function becomes maximal for $\beta = \pi + \gamma/2$. Hence $\eta(\alpha, \beta, \gamma) \geq \eta(\alpha, \pi + \gamma/2, \gamma)$. Furthermore, it is easy to see that for $\gamma \in (0, \pi)$, $\alpha \in (0, \pi/2]$ the function

$$\eta\left(\alpha, \pi + \frac{\gamma}{2}, \gamma\right) = 2 \cdot \left(1 + \left(\frac{1 - \cos(\gamma)}{\sin(\gamma)}\right)^2 + \left(\frac{1}{\sin(\alpha)} + \frac{\cos(\alpha)}{\sin(\alpha)} \cdot \frac{2\sin(\gamma/2)}{\sin(\gamma)}\right)^2\right)^{-1/2}$$

is monotonely increasing in α and monotonely decreasing in γ. Since

$$\gamma \in [\Upsilon(\alpha_*), \pi - \Upsilon(\alpha_*)]$$

for $\alpha \geq \alpha_*$ (see (4.2)) we obtain

$$h \geq \eta\left(\alpha_*, \tfrac{3}{2}\pi - \frac{\Upsilon(\alpha_*)}{2}, \pi - \Upsilon(\alpha_*)\right) > 0.74740141 = \bar{h}. \quad \square \tag{4.3}$$

Based on Lemma 4.1 we give in the sequel a lower bound for $V(D)$ only depending on α. To this end, we write for abbreviation

$$f_1(\alpha, \beta, \gamma, d) = \sum g_{i,j}\left(\frac{w_1(d) \cdot \kappa_{d-1}}{2\pi} - w_3(d)\kappa_{d-3}\right) + 2\pi\, w_3(d)\kappa_{d-3} + \frac{\sum f_{i,j}}{2} w_2(d)\kappa_{d-2}, \tag{4.4}$$

where $\sum$ indicates the summation over $1 \leq i < j \leq 3$. By Lemma 4.1 we have for $\alpha \geq \alpha_*$

$$V(D) \geq f_1(\alpha, \beta, \gamma, d).$$

We claim:

Lemma 4.2. *Let $\alpha_* \leq \alpha_0 \leq \pi/2$ and let d satisfy*

$$\frac{w_1(d) \cdot \kappa_{d-1}}{2\pi} - w_3(d)\kappa_{d-3} \leq 0. \tag{4.5}$$

Then for $\alpha \geq \alpha_0$, one has

$$V(D) \geq f_1\left(\alpha_0, \frac{\Upsilon(\alpha_0)}{2}, \Upsilon(\alpha_0), d\right).$$

Proof. It suffices to show that for $\alpha \geq \alpha_0$ and based on the restriction (4.1), the function $f_1(\alpha, \beta, \gamma, d)$ is minimal for $\alpha = \alpha_0$, $\beta = \Upsilon(\alpha_0)/2$, and $\gamma = \Upsilon(\alpha_0)$. To this end we study the behavior of the partial derivatives of $\sum f_{i,j}$ and $\sum g_{i,j}$. The calculations of the derivatives were carried out with help of the program *Mathematica*, but all results can also be verified "by hand." For more details we refer to [H]. Since the trigonometric transformations are rather tedious we omit the details. With respect to γ we obtain:

$$\begin{aligned}
\frac{\partial \sum f_{i,j}}{\partial \gamma} &= \frac{\partial f_{1,2}}{\partial \gamma} + \frac{\partial f_{2,3}}{\partial \gamma} = 1 + \frac{\cos(\alpha)\sin(\gamma-\beta)}{\sqrt{1-\cos^2(\alpha)\cos^2(\gamma-\beta)}} \\
&= 1 + \frac{\cos(\alpha)\sin(\gamma-\beta)}{\sqrt{\sin^2(\alpha)+\cos^2(\alpha)\sin^2(\gamma-\beta)}} \geq 0, \\
\frac{\partial \sum g_{i,j}}{\partial \gamma} &= \frac{\partial g_{1,2}}{\partial \gamma} + \frac{\partial g_{1,3}}{\partial \gamma} \\
&= \frac{-\sin(\alpha)}{1-\cos^2(\alpha)\cos^2(\gamma-\beta)} + \frac{\sin(\alpha)\cos(\alpha)\cos(\gamma-\beta)}{1-\cos^2(\alpha)\cos^2(\gamma-\beta)} \\
&= \frac{-\sin(\alpha)}{1+\cos(\alpha)\cos(\gamma-\beta)} \leq 0.
\end{aligned}$$

So for all $\alpha \in [\alpha_0, \pi/2]$, $\beta \in [0, 2\pi]$, the function $\sum f_{i,j}$ is monotonely increasing in γ and $\sum g_{i,j}$ is monotonely decreasing in γ. By the choice of d (see (4.5)) we get that $f_1(\alpha, \beta, \gamma, d)$ is monotonely increasing in γ. In view of (4.2) and $\alpha \geq \alpha_0$ this shows

$$f_1(\alpha, \beta, \gamma) \geq f_1(\alpha, \beta, \Upsilon(\alpha_0)). \tag{4.6}$$

Next we consider the partial derivatives with respect to β and get:

$$\begin{aligned}
\frac{\partial \sum f_{i,j}}{\partial \beta} &= \frac{\partial f_{1,3}}{\partial \beta} + \frac{\partial f_{2,3}}{\partial \beta} \\
&= \frac{\cos(\alpha)\sin(\beta)}{\sqrt{1-\cos^2(\alpha)\cos^2(\beta)}} - \frac{\cos(\alpha)\sin(\gamma-\beta)}{\sqrt{1-\cos^2(\alpha)\cos^2(\gamma-\beta)}}, \\
\frac{\partial \sum g_{i,j}}{\partial \beta} &= \frac{\partial g_{1,2}}{\partial \beta} + \frac{\partial g_{1,3}}{\partial \beta} + \frac{\partial g_{2,3}}{\partial \beta} \\
&= -\frac{\sin(\alpha)\cos^2(\alpha)\sin(\gamma)\sin(\gamma-2\beta)}{(1-\cos^2(\alpha)\cos^2(\beta))(1-\cos^2(\alpha)\cos^2(\gamma-\beta))} \\
&\quad - \frac{\sin(\alpha)\cos(\alpha)\cos(\gamma-\beta)}{1-\cos^2(\alpha)\cos^2(\gamma-\beta)} + \frac{\sin(\alpha)\cos(\alpha)\cos(\beta)}{1-\cos^2(\alpha)\cos^2(\beta)} \\
&= \frac{2\sin(\alpha)\cos(\alpha)\sin(\gamma/2)\sin(\gamma/2-\beta)}{(1+\cos(\alpha)\cos(\beta))(1+\cos(\alpha)\cos(\gamma-\beta))}.
\end{aligned}$$

It is easy to see that

$$\frac{\partial \sum f_{i,j}}{\partial \beta} \begin{cases} \leq 0, & 0 \leq \beta \leq \gamma/2,\ \pi+\gamma/2 \leq \beta \leq 2\pi, \\ = 0, & \beta = \gamma/2,\ \beta = \pi+\gamma/2, \\ \geq 0, & \gamma/2 \leq \beta \leq \pi+\gamma/2, \end{cases}$$

$$\frac{\partial \sum g_{i,j}}{\partial \beta} \begin{cases} \geq 0, & 0 \leq \beta \leq \gamma/2,\ \pi+\gamma/2 \leq \beta \leq 2\pi, \\ = 0, & \beta = \gamma/2,\ \beta = \pi+\gamma/2, \\ \leq 0, & \gamma/2 \leq \beta \leq \pi+\gamma/2. \end{cases}$$

Thus by (4.6) and (4.5):

$$f_1(\alpha,\beta,\gamma,d) \geq f_1\left(\alpha, \frac{\Upsilon(\alpha_0)}{2}, \Upsilon(\alpha_0), d\right). \tag{4.7}$$

Finally, for the partial derivatives with respect to α we find:

$$\begin{aligned}
\frac{\partial \sum f_{i,j}}{\partial\alpha}\left(\alpha, \frac{\gamma}{2}, \gamma\right) &= \left(\frac{\partial f_{1,3}}{\partial\alpha} + \frac{\partial f_{2,3}}{\partial\alpha}\right)\left(\alpha, \frac{\gamma}{2}, \gamma\right) \\
&= 2\frac{\sin(\alpha)\cos(\gamma/2)}{\sqrt{1-\cos^2(\alpha)\cos^2(\gamma/2)}} \geq 0, \\
\frac{\partial \sum g_{i,j}}{\partial\alpha}\left(\alpha, \frac{\gamma}{2}, \gamma\right) &= \left(\frac{\partial g_{1,2}}{\partial\alpha} + \frac{\partial g_{1,3}}{\partial\alpha} + \frac{\partial g_{2,3}}{\partial\alpha}\right)\left(\alpha, \frac{\gamma}{2}, \gamma\right) \\
&= \frac{\cos(\alpha)\sin(\gamma)}{1-\cos^2(\alpha)\cos^2(\gamma/2)} - 2\left(\frac{\sin(\gamma/2)}{1-\cos^2(\alpha)\cos^2(\gamma/2)}\right) \\
&= \frac{2\sin(\gamma/2)\,(\cos(\gamma/2)\cos(\alpha)-1)}{1-\cos^2(\alpha)\cos^2(\gamma/2)} \leq 0.
\end{aligned}$$

Hence, the function $f_1(\alpha,\gamma/2,\gamma,d)$ is monotonely increasing in α. In view of (4.7), we obtain

$$f_1(\alpha,\beta,\gamma,d) \geq f_1\left(\alpha_0, \frac{\Upsilon(\alpha_0)}{2}, \Upsilon(\alpha_0), d\right). \qquad \square$$

Now we have all the ingredients to prove:

Lemma 2.8. *Let* $\alpha \geq \alpha_* = 1.11$. *Then for* $d \geq 42$

$$V(D) \geq V(D^1(P^3)) + V(D^2(P^3)) + V(D^3(P^3)) > 2\kappa_{d-1}.$$

Proof. First we check that for $d \geq 42$ the condition (4.5) of Lemma 4.2 is satisfied. To show this we use Proposition 3.1. Since the functions $w_i(d)$, $1 \leq i \leq 3$, are monotonely increasing in d we have $w_1(d)/w_3(d) \leq 1/w_3(42)$ for $d \geq 42$. Hence for $d \geq 42$ we have $w_1(d)/w_3(d) < 10 < 2\pi\kappa_{d-3}/\kappa_{d-1}$ and (4.5) is satisfied. Lemma 4.1 together with Lemma 4.2 yields

$$V(D) \geq V(D^1(P^3)) + V(D^2(P^3)) + V(D^3(P^3)) \geq f_1(\alpha_*, \Upsilon(\alpha_*)/2, \Upsilon(\alpha_*), d),$$

with $\Upsilon(\alpha_*) \approx 1.1942$. By (4.4) we see that $f_1(\alpha_*, \Upsilon(\alpha_*)/2, \Upsilon(\alpha_*), d)/\kappa_{d-1}$ is monotonely increasing in d and with $f_1(\alpha_*, \Upsilon(\alpha_*)/2, \Upsilon(\alpha_*), 42)/\kappa_{41} \geq 2.02124$ we get

$$\begin{aligned}
V(D) &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(\frac{f_1(\alpha_*, \Upsilon(\alpha_*)/2, \Upsilon(\alpha_*), d)}{\kappa_{d-1}} - 2\right) \\
&\geq 2\kappa_{d-1} + \kappa_{d-1}\left(\frac{f_1(\alpha_*, \Upsilon(\alpha_*)/2, \Upsilon(\alpha_*), 42)}{\kappa_{41}} - 2\right) \\
&> 2\kappa_{d-1}, \qquad d \geq 42.
\end{aligned} \qquad \square$$

5. Small Local Deviation From a Sausage Arrangement

As in the previous section, let γ be the angle between y^1 and y^2, and let $\alpha \in [0, \pi/2]$ be the maximal angle of a vector of the configuration with the two-dimensional plane L (see Definition 2.3). For $\delta \in [0, \gamma]$ let w_δ be the point of the boundary of $P^2 \cap B^d$ with $\langle w_\delta/|w_\delta|, y^1\rangle = \cos(\delta)$. Then $P^2 \cap B^d = \{\lambda w_\delta : \lambda \in [0, 1],\ \delta \in [0, \gamma]\}$ and by the definition of $D^2(P^2)$ we have

$$V(D^2(P^2)) \geq \int_0^\gamma \int_0^{|w_\delta|} r \cdot V\left(\left(r\frac{w_\delta}{|w_\delta|} + L^\perp\right) \cap D\right) dr\, d\delta,$$

where $L^\perp$ denotes the orthogonal complement of L. To evaluate the inner integral we use polar coordinates for the set $(rw_\delta/|w_\delta| + L^\perp) \cap D$ and obtain

$$V(D^2(P^2)) \geq \frac{1}{d-2} \int_0^\gamma \int_{S^{d-1}\cap L^\perp} |w_\delta|^2 \int_0^1 r \cdot h(r, w_\delta, z)^{d-2}\, dr\, dz\, d\delta,$$

where for $r \in [0, 1]$, $\delta \in [0, \gamma]$, and $z \in S^{d-1} \cap L^\perp$

$$h(r, w_\delta, z) = \max\{h \in \mathbb{R}^{\geq 0} : rw_\delta + hz \in D\},$$

denotes the "height of D" in the direction of z over rw_δ. For $\delta \in [0, \gamma]$ and $z \in S^{d-1} \cap L^\perp$ we are only interested in points rw_δ whose "height" in the direction of z is at least 1. Hence we set

$$r_{\delta,z} = \max\{r \in \mathbb{R}^{\geq 0} : h(r, w_\delta, z) \geq 1,\ r \leq 1\}.$$

With this notation, we get

$$V(D^2(P^2)) \geq \frac{1}{d-2} \int_0^\gamma \int_{S^{d-1}\cap L^\perp} |w_\delta|^2 \int_0^{r_{\delta,z}} r \cdot h(r, w_\delta, z)^{d-2}\, dr\, dz\, d\delta. \tag{5.1}$$

In general, we cannot assume that $\operatorname{conv}\{0, w_\delta\} + z \subset H$, i.e., $r_{\delta,z} = 1$, because there might be a hyperplane $M_j = \{x \in E^d : \langle x^j, x\rangle = |x^j|^2/2\}$ which separates a part of the set $\operatorname{conv}\{0, w_\delta\} + z$ from H, i.e.,

$$\langle x^j, rw_\delta + z\rangle > \frac{|x^j|^2}{2}, \qquad r > r_{\delta,z}.$$

But beside this negative influence, such a perturbing point x^j has also a positive effect: For sufficiently small values of r we find $rw_\delta + \varepsilon_r z \in \operatorname{conv}(B^d \cup x^j + B^d) \cap H$ for suitable numbers $\varepsilon_r > 1$. Hence $h(r, w_\delta, z) > 1$ for small r and in view of the exponent $(d-2)$ in (5.1) the inner integral becomes large.

In the following we discuss the relationship between perturbing points and the size of the integral $\int_0^{r_{\delta,z}} r \cdot h(r, w_\delta, z)^{d-2}\, dr$ for a fixed pair of points w_δ, z. The main result is:

Lemma 5.1. *Let $d \geq 42$, $\delta \in [0, \gamma]$, $z \in S^{d-1} \cap L^\perp$, and $p_2(\alpha, d)$ as in Lemma* 2.5. *Then for $\alpha \leq \alpha_* = 1.11$*

$$\int_0^{r_{\delta,z}} rh(r, w_\delta, z)^{d-2}\, dr \geq p_2(\alpha, d).$$

As an immediate consequence of Lemma 5.1 we obtain:

Lemma 2.5. *Let* $\alpha \leq \alpha_* = 1.11$ *and* $d \geq 42$. *Then*

$$V(D^2(P^2)) \geq V(P^2 \cap B^d) \cdot 2p_2(\alpha, d)\kappa_{d-2}.$$

Proof.

$$\begin{aligned} V(D^2(P^2)) &\geq \frac{1}{d-2}\int_0^{\gamma}\int_{S^{d-1}\cap L^{\perp}} |w_\delta|^2 p_2(\alpha, d)\, dz\, d\delta \\ &= \left(\int_0^{\gamma} \frac{|w_\delta|^2}{2} d\delta\right) \kappa_{d-2} \cdot 2 \cdot p_2(\alpha, d). \end{aligned}$$ □

At the end of this section we show that a slightly better result holds if one considers both sets $D^0(P^2)$ and $D^2(P^2)$ (see Lemma 2.6). Further, we shall show that a similar result holds for the volume of the set $\hat{D}^1(P^2)$, but with a function depending on φ instead of α (see Lemma 2.4).

For the proof of Lemma 5.1 we need the following functions:

Definition 5.1. For $\alpha \in [0, \pi/2)$ and $0 \leq \zeta \leq \min\{2\sin(\alpha), 2\cos(\alpha)\}$, let

$$\begin{aligned} \mu(\alpha, \zeta) &= \frac{\sqrt{4-\zeta^2} - 2\sin(\alpha)}{2+\zeta-2\sin(\alpha)}, \\ g_1(\alpha, \zeta, d) &= \int_0^{\mu(\alpha,\zeta)} r\left(r\frac{\zeta}{\sqrt{4-\zeta^2}} + \frac{2}{\sqrt{4-\zeta^2}}\right)^{d-2} dr, \\ g_2(\alpha, \zeta, d) &= \int_{\mu(\alpha,\zeta)}^{\sqrt{(2-\zeta)/(2+\zeta)}} r\left(r\frac{\sin(\alpha)-1}{\sin(\alpha)}\sqrt{\frac{2+\zeta}{2-\zeta}} + \frac{1}{\sin(\alpha)}\right)^{d-2} dr, \\ g_3(\alpha, \zeta, d) &= g_1(\alpha, \zeta, d) + g_2(\alpha, \zeta, d), \\ g(\alpha, d) &= \min\{g_3(\alpha, \zeta, d) : 0 \leq \zeta \leq \min\{2\sin(\alpha), 2\cos(\alpha)\}\}, \\ p(\alpha, d) &= \int_0^{(1-\sin(\alpha))/\cos(\alpha)} r\left(-r\frac{\cos(\alpha)}{\sin(\alpha)} + \frac{1}{\sin(\alpha)}\right)^{d-2} dr. \end{aligned}$$

We note that $g_3(\alpha, \zeta, d)$ is a continuous function for $\alpha \in [0, \pi/2)$ and $0 \leq \zeta \leq \min\{2\sin(\alpha), 2\cos(\alpha)\}$ with $g_3(\alpha, 0, d) = g_1(\alpha, 0, d) = \frac{1}{2}$, $\alpha \in [0, \pi/2)$. Lemma 5.1 is an easy consequence of the next two propositions.

Proposition 5.1. *Let* $\alpha \in [0, \pi/2)$, $\delta \in [0, \gamma]$, *and* $z \in S^{d-1} \cap L^{\perp}$. *Then*

$$\int_0^{r_{\delta,z}} r h(r, w_\delta, z)^{d-2} \geq \begin{cases} g(\alpha, d), & \alpha < \pi/4, \\ \min\{g(\alpha, d), p(\alpha, d)\}, & \pi/4 \leq \alpha. \end{cases}$$

Proposition 5.2. *Let $d \geq 42$ and let $\alpha \leq \alpha_* = 1.11$. Then*

$$g(\alpha, d) = \tfrac{1}{2}.$$

For the proof of these two propositions we need another result from [BHW1]

Lemma 5.2. *Let $w \in H \cap S^{d-1}$, $v \in w^{\perp} \cap S^{d-1}$, $\mu, \varepsilon > 0$ with $(\mu + \varepsilon)v \in H$. Then*

$$c_1(\mu, \varepsilon) \cdot \operatorname{conv}\{0, w\} + \mu v \subset H,$$

with $c_1(\mu, \varepsilon) = \varepsilon / \sqrt{(\mu + \varepsilon)^2 - 1}$ if $\mu \geq 1/(\mu + \varepsilon)$, else $c_1(\mu, \varepsilon) = \sqrt{1 - \mu^2}$.

Proof of Proposition 5.1. Instead of w_δ we write w for short. For the proof we replace the Dirichlet–Voronoi cell H by the "smaller" set $H_s \subset H$ given by

$$H_s = \{x \in E^d : \langle x, y^j \rangle \leq 1,\ 1 \leq j \leq n - 1\}$$

and define analogously to $h(r, w_\delta, z)$, $r_{\delta,z}$:

$$\begin{aligned} h_s(r) &= \max\{h \in \mathbb{R}^{\geq 0} : rw + hz \in H_s \cap (\operatorname{conv}(C) + B^d)\}, \\ r_s &= \max\{r \in \mathbb{R}^{\geq 0} : h_s(r) \geq 1,\ r \leq 1\}. \end{aligned}$$

As $h_s(r) \leq h(r, w, z)$ and $r_s \leq r_{\delta,z}$ it suffices to show

$$\int_0^{r_s} r h_s(r)^{d-2} \geq \begin{cases} g(\alpha, d), & \alpha < \pi/4, \\ \min\{g(\alpha, d),\ p(\alpha, d)\}, & \pi/4 \leq \alpha. \end{cases} \tag{5.2}$$

Observe that $B^d \subset H_s$ and thus $w \in P^2 \cap H_s$. In the case $r_s = 1$ there is nothing to prove because $\int_0^1 r h_s(r)^{d-2}\, dr \geq \frac{1}{2}$ and $g(\alpha, 0, d) = \frac{1}{2}$. So we may assume $r_s < 1$. Hence there exists a point $u \in \{2y^1, \ldots, 2y^{n-1}\}$ with

$$\langle u, r_s w + z \rangle = 2. \tag{5.3}$$

Let

$$u = \sigma v + \tau \frac{w}{|w|} + \zeta z,$$

with $\sigma, \tau, \zeta \in \mathbb{R}$ and $v \in \operatorname{lin}(w, z)^{\perp}$, $|v| = 1$. Then

$$\sigma^2 + \tau^2 + \zeta^2 = 4 \tag{5.4}$$

and (5.3) is equivalent to

$$\tau |w| r_s + \zeta = 2. \tag{5.5}$$

Obviously, we have $0 \leq \tau, \zeta \leq 2$. We claim that

$$\zeta \leq 2 \sin(\alpha). \tag{5.6}$$

By the definition of α we get $\langle y^j, x\rangle \le \sin(\alpha)$ for all $x \in S^{d-1} \cap L^\perp$ and $1 \le j \le n$. Since $r_s < 1$, we have $\alpha > 0$ and thus

$$\left(\frac{1}{\sin(\alpha)}\right) x \in H_s, \qquad x \in S^{d-1} \cap L^\perp. \tag{5.7}$$

As $(2/\zeta)z \notin \mathrm{int}(H_s)$, it follows $2/\zeta \ge 1/\sin(\alpha)$.

In particular, (5.6) and (5.5) imply $\tau > 0$ and we may write

$$r_s = \frac{2-\zeta}{|w|\tau}. \tag{5.8}$$

Now we study the positive effects of such a perturbing point u. For $r \in [0, 1]$, let

$$h'(r) = \max\{h \in \mathbb{R}^{\ge 0} : rw + hz \in \mathrm{conv}\{0, u\} + B^d\}.$$

The function $h'(r)$ can easily be determined by the equality

$$\left| rw + h'(r)z - \frac{\langle rw + h'(r)z, u/2\rangle}{2} u \right|^2 = 1,$$

which says that the point given by the orthogonal projection of $rw + h'(r)z$ onto the hyperplane with normal vector u has unit length. We obtain with (5.4):

$$\begin{aligned} h'(r) &= \frac{|w| r\tau\zeta + 2\sqrt{4-\zeta^2+(-4+\tau^2+\zeta^2)|w|^2 r^2}}{4-\zeta^2} \\ &= \frac{|w| r\tau\zeta + 2\sqrt{4-\zeta^2-\sigma^2|w|^2 r^2}}{4-\zeta^2}. \end{aligned}$$

We distinguish two cases.

(i) $1/\sin(\alpha) \le h'(0) = 2/\sqrt{4-\zeta^2}$.
Then $\sin(\alpha) \ge (1-(\zeta/2)^2)^{1/2}$ and by (5.6) we get $\sin(\alpha) \ge \cos(\alpha)$. Hence $\alpha \ge \pi/4$. Furthermore, since $h'(0)z \in \mathrm{conv}\, C + B^d$ we may deduce from (5.7) that

$$\frac{1}{\sin(\alpha)} z \in (\mathrm{conv}\, C + B^d) \cap H_s.$$

By Lemma 5.2 (with H_s instead of H and $c_1(1, 1/\sin(\alpha) - 1) = (1-\sin(\alpha))/\cos(\alpha)$ we obtain

$$\mathrm{conv}\left\{0, \frac{1}{\sin(\alpha)} z, \pm\frac{1-\sin(\alpha)}{\cos(\alpha)|w|} w, \pm\frac{1-\sin(\alpha)}{\cos(\alpha)|w|} w + z\right\} \subset D. \tag{5.9}$$

So

$$h_s(r) \ge \frac{1}{\sin(\alpha)} - r\frac{|w|\cos(\alpha)}{\sin(\alpha)} \qquad \text{for} \quad r \in \left[0, \frac{1-\sin(\alpha)}{|w|\cos(\alpha)}\right].$$

As $|w| \le 1$ we have

$$\int_0^{r_s} r h_s(r)^{d-2}\, dr \ge p(\alpha, d) \qquad \text{for} \quad \alpha \ge \frac{\pi}{4}. \tag{5.10}$$

(ii) $1/\sin(\alpha) \geq h'(0) = 2/\sqrt{4-\zeta^2}$.
Then $4\sin^2(\alpha) \leq 4 - \zeta^2$ which implies $\zeta \leq 2\cos(\alpha)$, and together with (5.6)

$$\zeta \leq \min\{2\sin(\alpha), 2\cos(\alpha)\}. \tag{5.11}$$

Now we determine the smallest value of r_0 such that the point $r_0 w + h'(r_0)z$ lies in the hyperplane $M = \{x \in E^d : \langle u, x\rangle = 2\}$. Such a pair $(r_0, h'(r_0))$ (if it exists) must satisfy the relations:

$$r_0|w|\tau + h'(r_0)\zeta = 2, \qquad r_0^2|w|^2 + h'(r_0)^2 = 2. \tag{5.12}$$

The first equation means that the point lies in the hyperplane M and the second one expresses the property that $r_0 w + h'(r_0)z$ belongs to the boundary of the $(d-1)$-dimensional unit ball with center $u/2$ embedded in M. By (5.12) we find

$$r_0^2|w|^2 + \left(\frac{2 - r_0|w|\tau}{\zeta}\right)^2 = 2$$

and so

$$r_0 = \frac{2\tau - \zeta\sqrt{2(\tau^2+\zeta^2) - 4}}{|w|(\tau^2+\zeta^2)}. \tag{5.13}$$

We note that r_0 is well defined, i.e., $\tau^2 + \zeta^2 \geq 2$: Since $r_s, |w| \leq 1$ we have $\tau + \zeta \geq 2$ (see (5.5)) and thus $\tau^2 + \zeta^2 \geq 2$. Moreover, from (5.11) we get $\zeta \leq \sqrt{2}$ which implies $r_0 \geq 0$. We also have $r_0 \leq r_s$. To show this we use (5.8) and obtain

$$\begin{aligned} r_0 \leq r_s \quad &\Leftrightarrow \quad \frac{2\tau - \zeta\sqrt{2(\tau^2+\zeta^2)-4}}{|w|(\tau^2+\zeta^2)} \leq \frac{2-\zeta}{|w|\tau} \\ &\Leftrightarrow \quad -\tau\zeta\sqrt{2(\tau^2+\zeta^2)-4} \leq \zeta(2\zeta - \tau^2 - \zeta^2) \\ &\Leftrightarrow \quad \tau^2 + \zeta^2 \leq 2\zeta + \tau\sqrt{2(\tau^2+\zeta^2)-4}. \end{aligned}$$

Let $h(\tau,\zeta) = \tau^2 + \zeta^2 - 2\zeta - \tau\sqrt{2(\tau^2+\zeta^2)-4}$. In order to show $h(\tau,\zeta) \leq 0$ for $0 \leq \zeta \leq \sqrt{2}$ and $\tau \in [2-\zeta, \sqrt{4-\zeta^2}]$ we calculate the first partial derivative of h with respect to τ:

$$\frac{\partial h(\tau,\zeta)}{\partial\tau} = \frac{2\tau\sqrt{2(\tau^2+\zeta^2)-4} - 4\tau^2 - 2\zeta^2 + 4}{\sqrt{2(\tau^2+\zeta^2)-4}}.$$

From this we deduce

$$\begin{aligned} \frac{\partial h(\tau,\zeta)}{\partial\tau} \leq 0 \quad &\Leftrightarrow \quad \tau\sqrt{2(\tau^2+\zeta^2)-4} \leq 2\tau^2 + \zeta^2 - 2 \\ &\Leftrightarrow \quad \tau^2\left(\frac{\zeta^2-2}{2\tau^2+\zeta^2-2} + 1\right) \leq 2\tau^2 + \zeta^2 - 2. \end{aligned}$$

Since $\zeta \leq \sqrt{2}$ and $\tau^2 + \zeta^2 \geq 2$ the function $h(\tau,\zeta)$ is monotonely decreasing in τ. Thus $h(\tau,\zeta) \leq h(2-\zeta,\zeta) = 2(2-\zeta)((1-\zeta) - \sqrt{(1-\zeta)^2}) \leq 0$. Hence $r_0 \leq r_s$.

From the right-hand side equation in (5.12) it follows $h'(r_0) > 1$ and substituting r_0 from (5.13) in the left-hand side equation of (5.12) yields

$$h'(r_0) = \frac{2\zeta + \tau\sqrt{2(\tau^2+\zeta^2)-4}}{\tau^2+\zeta^2}. \tag{5.14}$$

Now let

$$\begin{aligned} S_1 &= \operatorname{conv}\{0, h'(0)z, r_0 w, r_0 w + h'(r_0)z\},\\ S_2 &= \operatorname{conv}\{r_0 w, r_0 w + h'(r_0)z, r_s w, r_s w + z\},\\ T(\alpha) &= \operatorname{conv}\left\{0, \left(\frac{1}{\sin(\alpha)}\right)z, r_s w, r_s w + z\right\}. \end{aligned} \tag{5.15}$$

Clearly, $S_1, S_2 \subset \operatorname{conv} C + B^d$ and from the definition of r_s and (5.7) we have $T(\alpha) \subset H_s$. Hence

$$T(\alpha) \cap (S_1 \cup S_2) \subset (\operatorname{conv} C + B^d) \cap H_s.$$

In the following, we derive from the set $T(\alpha) \cap (S_1 \cup S_2)$ a lower bound for the function $h_s(r)$. To this end, we first show that we may assume $\tau^2 + \zeta^2 = 4$. Let

$$\tau_1 = r_0|w| + h'(r_0) \qquad \text{and} \qquad \zeta_1 = h'(r_0) - r_0|w|.$$

Then based on (5.12), r_0, $|w| \le 1$, and $h'(r_0) > 1$ we have

$$\tau_1, \zeta_1 > 0, \qquad \tau_1^2 + \zeta_1^2 = 4 \qquad \text{and} \qquad \tau_1 r_0|w| + \zeta_1 h'(r_0) = 2.$$

Now let $\tilde{u} = \tau_1 w/|w| + \zeta_1 z$ and let $\tilde{r}_s$, $\tilde{h}'(r)$, $\tilde{r}_0$, $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{T}(\alpha)$ be defined as above for the point u. By the choice of τ_1, ζ_1 we get $\tilde{r}_0 = r_0 = (\tau_1 - \zeta_1)/(2|w|)$ and $\tilde{h}'(\tilde{r}_0) = h'(r_0) = (\tau_1 + \zeta_1)/2$ (see (5.13) and (5.14)). Furthermore, as $\tau r_0|w| + \zeta h'(r_0) = 2$ and $\tau^2 + \zeta^2 \le 4$ we obtain $\tau_1 \ge \tau$, $\zeta_1 \le \zeta$ and (see (5.8))

$$\tilde{h}'(0) = \frac{2}{\tau_1} \le \frac{2}{\sqrt{4-\zeta^2}} = h'(0), \qquad \tilde{r}_s = \frac{2-\zeta_1}{|w|\tau_1} \le \frac{2-\zeta}{|w|\tau} = r_s.$$

Hence we have $\tilde{S}_1 \subset S_1$, $\tilde{S}_2 \subset S_2$, and $\tilde{T}(\alpha) \subset T(\alpha)$. So the sets S_1, S_2, $T(\alpha)$ become "minimal" (with respect to inclusion) for parameters $\tau, \zeta \ge 0$ which satisfy $\tau^2 + \zeta^2 = 4$ and $\zeta \le \min\{2\sin(\alpha), 2\cos(\alpha)\}$ (see (5.11)). Therefore, in the sequel we assume $\tau^2 + \zeta^2 = 4$ and thus (see (5.8), (5.13), and (5.14))

$$\begin{aligned} r_s &= \frac{\sqrt{2-\zeta}}{\sqrt{2+\zeta}\,|w|}, & r_0 &= \frac{\sqrt{4-\zeta^2}-\zeta}{2|w|},\\ h'(0) &= \frac{2}{\sqrt{4-\zeta^2}}, & h'(r_0) &= \frac{\sqrt{4-\zeta^2}+\zeta}{2}. \end{aligned} \tag{5.16}$$

Next we determine the intersection $T(\alpha) \cap (S_1 \cup S_2)$. Let $\chi_1 w + \chi_2 z$ be the point of intersection of the two segments

$$\operatorname{conv}\{(1/\sin(\alpha))z, r_s w + z\} \quad \text{and} \quad \operatorname{conv}\{h'(0)z, r_0 w + h'(r_0)w\}.$$

Observe that based on $h'(0) \le 1/\sin(\alpha) \le 2/\zeta$ such a point exists. Then we obviously have

$$\begin{aligned} T(\alpha) \cap (S_1 \cup S_2) &= \operatorname{conv}\{0, h'(0)z, \chi_1 w, \chi_1 w + \chi_2 z\} \\ &\cup \operatorname{conv}\{\chi_1 w, \chi_1 w + \chi_2 z, r_s w, r_s w + z\} \end{aligned}$$

and for χ_1, χ_2 we find (see (5.16)):

$$\begin{aligned} \chi_1 &= \frac{\mu(\alpha,\zeta)}{|w|}, \\ \chi_2 &= \frac{2}{\sqrt{4-\zeta^2}} + \mu(\alpha,\zeta)\frac{\zeta}{\sqrt{4-\zeta^2}} \\ &= \frac{1}{\sin(\alpha)} + \mu(\alpha,\zeta)\frac{\sqrt{2+\zeta}}{\sqrt{2-\zeta}}\frac{\sin(\alpha)-1}{\sin(\alpha)}. \end{aligned} \tag{5.17}$$

Hence

$$h_s(r) \ge \frac{2}{\sqrt{4-\zeta^2}} + r|w|\frac{\zeta}{\sqrt{4-\zeta^2}} \qquad \text{for} \quad 0 \le r \le \frac{\mu(\alpha,\zeta)}{|w|}$$

and

$$h_s(r) \ge \frac{1}{\sin(\alpha)} + r|w|\frac{\sqrt{2+\zeta}}{\sqrt{2-\zeta}}\frac{\sin(\alpha)-1}{\sin(\alpha)} \qquad \text{for} \quad \frac{\mu(\alpha,\zeta)}{|w|} \le r \le \frac{\sqrt{2-\zeta}}{\sqrt{2+\zeta}|w|}.$$

Together with $|w| \le 1$ and the first case (5.10) this shows (5.2). □

Proof of Proposition 5.2. First we consider the behavior of $g_3(\alpha, \zeta, d)$ with respect to α. For a given ζ the set $T(\alpha)$ in (5.15) becomes "smaller" (with respect to inclusion) if we increase the angle α. So, by construction, the function $g_3(\alpha, \zeta, d)$ is monotonely decreasing in α. In view of $\zeta \le \min\{2\sin(\alpha), 2\cos(\alpha)\}$ this means that

$$g(\alpha, d) \ge \min\left\{g_3\left(\frac{\pi}{4}, \zeta, d\right) : 0 \le \zeta \le \sqrt{2}\right\}, \qquad \alpha \le \frac{\pi}{4},$$

and for $\alpha_* \ge \alpha \ge \pi/4$:

$$g(\alpha, d) \ge \min\{g_3(\alpha, 2\cos(\alpha), d), \min\{g_3(\alpha_*, \zeta, d) : 0 \le \zeta \le 2\cos(\alpha_*)\}\}.$$

With

$$\nu(\alpha) = \left(\frac{\cos(\alpha)}{1-\sin(\alpha)}\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}\right)^2,$$

we have

$$g_3(\alpha, 2\cos(\alpha), d) = g_2(\alpha, 2\cos(\alpha), d) = \nu(\alpha) \cdot p(\alpha, d),$$

where we use the substitution $r = \cos(\alpha)/(1-\sin(\alpha)) \cdot (1-\cos(\alpha))/(1+\cos(\alpha))^{1/2} t$. Now $\nu(\alpha)$ is a monotonely increasing function with $\nu(\pi/4) = 1$ and $p(\alpha, d)$ is monotonely decreasing in α and increasing in d. Since $p(\pi/3, 42) > \frac{1}{2}$ and $\nu(\pi/3)\,p(\alpha_*, 42) > \frac{1}{2}$ we find that for $\pi/4 \le \alpha \le \alpha_*$ and $d \ge 42$

$$g_3(\alpha, 2\cos(\alpha), d) > \tfrac{1}{2}.$$

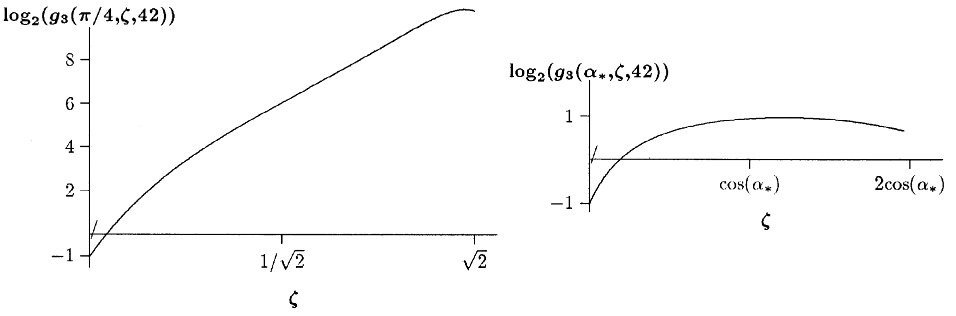

Fig. 1.

So, as $g(\alpha, d) \le g_3(\alpha, 0, d) = \frac{1}{2}$ and g_3 increases in d it suffices to prove

$$\begin{aligned} \min\{g_3(\pi/4, \zeta, 42) : 0 \le \zeta \le \sqrt{2}\} &= \tfrac{1}{2}, \\ \min\{g_3(\alpha_*, \zeta, 42) : 0 \le \zeta \le 2\cos(\alpha_*)\} &= \tfrac{1}{2}. \end{aligned} \tag{5.18}$$

Figure 1 shows a plot of the functions $\log_2(g_3(\pi/4, \zeta, 42))$ for $\zeta \in [0, \sqrt{2}]$ and $\log_2(g_3(\alpha_*, \zeta, 42))$ for $\zeta \in [0, 2\cos(\alpha_*)]$. The plots were generated by the program *Mathematica*.

We "see" that (5.18) holds. However, it is also possible to prove (5.18) "by hand." First, we check that for $d \ge 42$ and $\alpha \in \{\pi/4, \alpha_*\}$ there exists a $\zeta_*(\alpha)$ with $g_3(\alpha, \zeta, d) \ge \frac{1}{2}$ for all $\zeta \in [0, \zeta_*(\alpha)]$. By the definition of the function $g_1(\alpha, \zeta, d)$ we get with the substitution $r = \mu(\alpha, \zeta) \cdot t$

$$\begin{aligned} g_3(\alpha, \zeta, d) &\ge g_1(\alpha, \zeta, d) \\ &= \left(\frac{2}{\sqrt{4-\zeta^2}}\right)^{d-2} \mu(\alpha, \zeta)^2 \int_0^1 t \left(t \frac{\zeta}{2} \mu(\alpha, \zeta) + 1\right)^{d-2} dt \\ &\ge \left(\frac{2}{\sqrt{4-\zeta^2}}\right)^{d-2} \mu(\alpha, \zeta)^2 \frac{1}{2} \left(\frac{1}{2}\frac{\zeta}{2} \mu(\alpha, \zeta) + 1\right)^{d-2}, \end{aligned}$$

where the last inequality results from the convexity of the function $t(t\zeta\mu(\alpha, \zeta)/2 + 1)$. So, in order to prove $g_3(\alpha, \zeta, d) \ge \frac{1}{2}$ (for sufficiently small ζ) it suffices to show

$$\frac{2}{\sqrt{4-\zeta^2}} \mu(\alpha, \zeta)^{2/(d-2)} \left(\frac{\zeta}{4} \mu(\alpha, \zeta) + 1\right) \ge 1. \tag{5.19}$$

To this end, let $\psi(\alpha, \zeta)$ be defined by

$$\mu(\alpha, \zeta) = \frac{\sqrt{4-\zeta^2}/2}{1 + (\zeta/2)\psi(\alpha, \zeta)},$$

i.e.,

$$\psi(\alpha, \zeta) = \frac{\sqrt{4-\zeta^2} + 2\sin(\alpha)(2 - \sqrt{4-\zeta^2})/\zeta}{\sqrt{4-\zeta^2} + 2\sin(\alpha)}.$$

By the Bernoulli inequality $(1+x)^m \geq 1+mx$ for $x \geq -1$, $m \in \mathbb{N}$, we obtain

$$\left(1+\frac{2}{d-2}\frac{\zeta}{2}\psi(\alpha,\zeta)\right)^{(d-2)/2} \geq 1+\frac{\zeta}{2}\psi(\alpha,\zeta) = \frac{\sqrt{4-\zeta^2}/2}{\mu(\alpha,\zeta)}.$$

Hence

$$\mu(\alpha,\zeta)^{2/(d-2)} \geq \frac{(\sqrt{4-\zeta^2}/2)^{2/(d-2)}}{1+(2/(d-2))(\zeta/2)\psi(\alpha,\zeta)} \geq \frac{\sqrt{4-\zeta^2}/2}{1+(2/(d-2))(\zeta/2)\psi(\alpha,\zeta)}.$$

So (5.19) holds for all ζ with

$$\mu(\alpha,\zeta) \geq \frac{4}{d-2}\psi(\alpha,\zeta). \tag{5.20}$$

Calculating the first partial derivative with respect to ζ shows that $\psi(\alpha,\zeta)$ is monotonely increasing in ζ, $\zeta \leq \sqrt{2}$. As $\mu(\alpha,\zeta)$ is monotonely decreasing in ζ we have shown that for each $\zeta_*(\alpha)$ satisfying (5.20) and $\zeta \in [0,\zeta_*(\alpha)]$ one has

$$g_3(\alpha,\zeta,d) \geq \tfrac{1}{2}. \tag{5.21}$$

Hence a suitable $\zeta_*(\alpha)$ can easily be computed. For example, for $d = 42$ and $\alpha \in \{\pi/4, \alpha_*\}$ one may choose $\zeta_*(\alpha) = 0.008$. For $\zeta \geq \zeta_*(\alpha)$ one can find certain auxiliary functions from which (5.18) follows by evaluating these functions at finitely many points. Since the calculations are rather lengthy we omit them and refer to [H]. □

Now we come to the proof of

Lemma 2.6. *Let $\alpha_* = 1.11$ and let $\varphi \geq \pi/3$. Then for $d \geq 42$*

$$V(D^0(P^2)) + V(D^2(P^2)) \geq \frac{\varphi}{2}\cdot 2\tilde{p}_2(\alpha,d)\kappa_{d-2}.$$

Proof. Let $a^i \in L$ be the outward unit normal vector of the edge $\mathrm{conv}\{0, 2y^i\}$ with respect to the P^2, $i = 1,2$. Furthermore, let $U(\varphi)$ be the intersection of B^d with the cone generated by a^1, a^2. We set $W(\varphi) = -U(\varphi)$, $G(\varphi) = U(\varphi)$ if $\langle y^1, y^2\rangle < 0$ and $W(\varphi) = P^2 \cap B^d$, $G(\varphi) = -(P^2 \cap B^d)$ if $\langle y^1, y^2\rangle \geq 0$. Since $\varphi \geq \pi/3$ we have $W(\varphi) \subset P^2 \cap B^d$, $G(\varphi) \subset U(\varphi)$, and

$$V(W(\varphi)) = V(G(\varphi)) = \frac{\varphi}{2}.$$

For $\delta \in [0,\varphi]$ and $\langle y^1, y^2\rangle \geq 0$ ($\langle y^1, y^2\rangle < 0$) let w_δ be the point of the boundary of $W(\varphi)$ with $\langle w_\delta, y^1\rangle = \cos(\delta)$ ($\langle w_\delta, -a^2\rangle = \cos(\delta)$). Then $W(\varphi) = \{\lambda w_\delta : \lambda \in [0,1],\ \delta \in [0,\varphi]\}$ and by the definition of $D^0(P^2)$, $D^2(P^2)$ we obtain

$$\begin{aligned} V(D^0(P^2)) &\geq \int_0^{\varphi}\int_{-1}^{0} -r\cdot V((rw_\delta + L^\perp)\cap D)\,dr\,d\delta,\\ V(D^2(P^2)) &\geq \int_0^{\varphi}\int_{0}^{1} r\cdot V((rw_\delta + L^\perp)\cap D)\,dr\,d\delta. \end{aligned}$$

Now we use polar coordinates for the inner integrals and get

$$V(D^0(P^2)) \;\geq\; \frac{1}{d-2}\int_0^{\varphi}\int_{S^{d-1}\cap L^{\perp}}\int_{-1}^{0} -r\cdot h^-(r,w_\delta,z)^{d-2}\,dr\,dz\,d\delta,$$
$$V(D^2(P^2)) \;\geq\; \frac{1}{d-2}\int_0^{\varphi}\int_{S^{d-1}\cap L^{\perp}}\int_{0}^{1} r\cdot h^+(r,w_\delta,z)^{d-2}\,dr\,dz\,d\delta,$$

where for $\delta \in [0,\varphi]$ and $z \in S^{d-1}\cap L^{\perp}$

$$h^+(r,w_\delta,z) \;=\; \max\{h \in \mathbb{R}^{\geq 0} : rw_\delta + hz \in D\} \qquad \text{for} \quad r \in [0,1],$$
$$h^-(r,w_\delta,z) \;=\; \max\{h \in \mathbb{R}^{\geq 0} : rw_\delta + hz \in D\} \qquad \text{for} \quad r \in [-1,0].$$

Now, let

$$r^+_{\delta,z} \;=\; \max\{r \in \mathbb{R}^{\geq 0} : h^+(r,w_\delta,z) \geq 1,\ r \in [0,1]\},$$
$$r^-_{\delta,z} \;=\; \min\{r \in \mathbb{R}^{\geq 0} : h^-(r,w_\delta,z) \geq 1,\ r \in [-1,0]\}.$$

We claim that for $\varphi \in [\pi/3,\pi/2)$, $\delta \in [0,\varphi]$, and $z \in S^{d-1}\cap L^{\perp}$

$$\int_{r^-_{\delta,z}}^{0} -rh^-(r,w_\delta,z)^{d-2} \;+\; \int_0^{r^+_{\delta,z}} rh^+(r,w_\delta,z)^{d-2}$$
$$\geq \begin{cases} g(\alpha,d), & \alpha < \pi/4,\\ \min\{g(\alpha,d),\, 2\cdot p(\alpha,d)\}, & \pi/4 \leq \alpha.\end{cases}$$

To show this we can proceed as in the proof of Proposition 5.1. All what we have to prove is that in case (i) $1/\sin(\alpha) \leq h'(0) = 2/\sqrt{4-\zeta^2}$,

$$\int_{r^-_{\delta,z}}^{0} -rh^-(r,w_\delta,z)^{d-2} + \int_0^{r^+_{\delta,z}} rh^+(r,w_\delta,z)^{d-2} \geq 2\cdot p(\alpha,d). \tag{5.22}$$

However, this follows from (5.9) and this shows (5.22). Now the assertion is an immediate consequence of Proposition 5.2. □

Finally, it remains to prove:

Lemma 2.4. *Let* $\varphi_* = 1.16$. *Then for* $d \geq 42$

$$V(D^1(P^2)) \geq V(\hat{D}^1(P^2)) \geq p_1(\varphi,d)\cdot\kappa_{d-1},$$

where $\hat{D}^1(P^2) = \{x \in D^1(P^2) : \Phi(x) \in \mathrm{conv}\{0,2y^1\} \cup \mathrm{conv}\{0,2y^2\}\}$.

Proof. Since the proof can be done completely analogously to the proof of Lemma 2.5 we only give a brief sketch. First, observe that

$$V(\hat{D}^1(P^2)) \geq \sum_{i=1}^{2}\int_0^1 V((ry^i + N(P^2,\mathrm{conv}\{0,2y^i\}))\cap D)\,dr,$$

where $N(P^2, \text{conv}\{0, 2y^i\})$ denotes the normal cone of the edge $\text{conv}\{0, 2y^i\}$ with respect to P^2. For $i = 1, 2$ and $z \in N(P^2, \text{conv}\{0, 2y^i\}) \cap S^{d-1}$ we define $h_i(r, z) = \max\{h \in \mathbb{R}^{\geq 0} : ry^i + hz \in D\}$ and $r_{i,z} = \max\{r \in \mathbb{R}^{\geq 0} : h_i(r, z) \geq 1, r \leq 1\}$. Using polar coordinates we get (see (5.1)):

$$V(\hat{D}^1(P^2)) \geq \frac{1}{d-1} \sum_{i=1}^{2} \int_{S^{d-1} \cap N(P^2, \text{conv}\{0, 2y^i\})} \int_0^{r_{i,z}} h_i(r, z)^{d-1}\, dr\, dz.$$

For $z \in N(P^2, \text{conv}\{0, 2y^i\}) \cap S^{d-1}$ we have to estimate $\int_0^{r_{i,z}} h_i(r, z)^{d-1} dr$. To this end, we must adjust some of the functions defined in Definition 5.1 in an obvious way: for $\varphi \in [0, \pi/2)$ and $0 \leq \zeta \leq \min\{2\sin(\varphi), 2\cos(\varphi)\}$ let

$$\begin{aligned}
\tilde{g}_1(\varphi, \zeta, d) &= \int_0^{\mu(\varphi,\zeta)} \left(r \frac{\zeta}{\sqrt{4-\zeta^2}} + \frac{2}{\sqrt{4-\zeta^2}} \right)^{d-1} dr, \\
\tilde{g}_2(\varphi, \zeta, d) &= \int_{\mu(\varphi,\zeta)}^{\sqrt{(2-\zeta)/(2+\zeta)}} \left(r \frac{\sin(\varphi) - 1}{\sin(\varphi)} \sqrt{\frac{2+\zeta}{2-\zeta}} + \frac{1}{\sin(\varphi)} \right)^{d-1} dr, \\
\tilde{g}_3(\varphi, \zeta, d) &= g_1(\varphi, \zeta, d) + g_2(\varphi, \zeta, d), \\
\tilde{g}(\varphi, d) &= \min\{\tilde{g}_3(\varphi, \zeta, d) : 0 \leq \zeta \leq \min\{2\sin(\varphi), 2\cos(\varphi)\}\}, \\
\tilde{p}(\varphi, d) &= \int_0^{(1-\sin(\varphi))/\cos(\varphi)} \left(-r \frac{\cos(\varphi)}{\sin(\varphi)} + \frac{1}{\sin(\varphi)} \right)^{d-1} dr.
\end{aligned}$$

If we replace, in the proof of Proposition 5.1, α by φ, then we get that for $\varphi \in [0, \pi/2)$ and $z \in N(P^2, \text{conv}\{0, 2y^i\}) \cap S^{d-1}$

$$\int_0^{r_{i,z}} h_i(r, z)^{d-1} \geq \begin{cases} \tilde{g}(\varphi, d), & \varphi < \pi/4, \\ \min\{\tilde{g}(\varphi, d),\, \tilde{p}(\varphi, d)\}, & \pi/4 \leq \varphi. \end{cases}$$

Analogously to the proof of Lemma 5.2 we can estimate the function $\tilde{g}(\varphi, d)$ and get for $d \geq 42$ and $0 \leq \varphi \leq \varphi_*$

$$\tilde{g}(\varphi, d) = 1.$$ □

Acknowledgment

We wish to thank the referee for helpful comments and suggestions.

References

[BHW1] U. Betke, M. Henk, and J. M. Wills, Finite and infinite packings, *J. Reine Angew. Math.* **53** (1994), 165–191.

[BHW2] U. Betke, M. Henk, and J. M. Wills, Sausages are good packings, *Discrete Comput. Geom.* **13** (1995), 297–311.

[CS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 2nd edn., Springer-Verlag, New York, 1993.

[F] L. Fejes Tóth, Research problem 13, *Period. Math. Hungar.* **6** (1975), 197–199.

[FK] G. Fejes Tóth and W. Kuperberg, Packing and covering with convex sets, In: *Handbook of Convex Geometry*, vol. B (P. M. Gruber and J. M. Wills, eds.), North-Holland, Amsterdam, 1993.

[GL] P. M. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, 2nd edn., North-Holland, Amsterdam, 1987.

[Gro] H. Groemer, Über die Einlagerung von Kreisen in einem konvexen Bereich, *Math. Z.* **73** (1960), 285–294.

[Grü] B. Grünbaum, *Convex Polytopes*, Interscience, Wiley, London, 1967.

[GW] P. Gritzmann and J. M. Wills, Finite packing and covering, In: *Handbook of Convex Geometry*, vol. B (P. M. Gruber and J. M. Wills, eds.), North-Holland, Amsterdam, 1993.

[H] M. Henk, Finite and Infinite Packings, Habilitationsschrift, Universität-GH Siegen, 1995.

[MS] P. McMullen and G. C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge University Press, Cambridge, 1971.

[R1] C. A. Rogers, The closest packing of convex two-dimensional domains, *Acta Math.* **86** (1951), 309–321.

[R2] C. A. Rogers, *Packing and Covering*, Cambridge University Press, Cambridge, 1964.

[S] L. Schläfli, *Gesammelte mathematische Abhandlungen*, vol. I, Birkhäuser, Basel, 1950.

Received October 19, 1995, *and in revised form May* 28, 1996.