

## Transversals of $d$ -Intervals

T. Kaiser

Department of Applied Mathematics, Charles University,  
Malostranské nám. 25, 118 00 Praha 1, Czech Republic  
kaiser@kam.ms.mff.cuni.cz

**Abstract.** We present a method which reduces a family of problems in combinatorial geometry (concerning multiple intervals) to purely combinatorial questions about hypergraphs. The main tool is the Borsuk–Ulam theorem together with one of its extensions.

For a positive integer  $d$ , a homogeneous  $d$ -interval is a union of at most  $d$  closed intervals on a fixed line  $\ell$ . Let  $\mathcal{H}$  be a system of homogeneous  $d$ -intervals such that no  $k + 1$  of its members are pairwise disjoint. It has been known that its transversal number  $\tau(\mathcal{H})$  can then be bounded in terms of  $k$  and  $d$ . Tardos [9] proved that for  $d = 2$ , one has  $\tau(\mathcal{H}) \leq 8k$ . In particular, the bound is linear in  $k$ . We show that the latter holds for any  $d$ , and prove the tight bound  $\tau(\mathcal{H}) \leq 3k$  for  $d = 2$ .

We obtain similar results in the case of nonhomogeneous  $d$ -intervals whose definition appears below.

### 1. Introduction

Gallai was the first to observe that if  $\mathcal{H}$  is a family of closed intervals on a line such that no  $k + 1$  of its members are pairwise disjoint, then there are  $k$  points of the line that intersect all intervals from  $\mathcal{H}$ . (See [4].) Gallai's theorem answers a particular case of a more general question which we shall presently outline. Let  $\mathcal{H}$  be any system of subsets of a set  $X$ . The maximum number of pairwise disjoint elements of  $\mathcal{H}$  is called its *packing number* and denoted by  $\nu(\mathcal{H})$ . (When  $\mathcal{H}$  is viewed as a hypergraph on  $X$ , the term *matching number* is used instead.) A subset of  $X$  that intersects all members of  $\mathcal{H}$  is called a *transversal*. The minimum size of a transversal is the *transversal number*  $\tau(\mathcal{H})$ . One trivially has  $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$ , but in general there is no inequality in the other direction. Bounding  $\tau$  in terms of  $\nu$  is just the problem we had in mind. By Gallai's theorem, equality holds for systems of closed intervals. Another such example is found in graphs, where the definitions make sense since every graph is also a hypergraph. By the well-known theorem of König, every bipartite graph has  $\tau$  equal to  $\nu$ .

The objects of our investigation will be systems of multiple intervals. Let  $d$  be a positive integer and  $\ell$  a line. A *homogeneous  $d$ -interval*  $I \subset \ell$  is a nonempty union of at most  $d$  closed intervals on  $\ell$ .

For a variation on this concept, fix distinct parallel lines  $\ell^1, \dots, \ell^d$  in the plane. A  *$d$ -interval*  $J$  is a nonvoid subset of  $\ell^1 \cup \dots \cup \ell^d$  such that each  $J \cap \ell^i$  is either a closed segment or the empty set. The set  $J \cap \ell^i$  is called the  *$i$ th component* of  $J$  and denoted by  $J^i$ .

What we get for  $d = 1$  are just closed intervals. This case will not be of interest to us, and we take  $d$  to be at least 2 in the whole paper.

Families of multiple intervals have first been investigated by Gyárfás and Lehel [3], who prove that there is a function of  $\nu$  and  $d$  which bounds (from above) the transversal number  $\tau$  of any system of  $d$ -intervals with packing number equal to  $\nu$ . This function was  $O(\nu^{d1})$  for fixed  $d$ . A slightly weaker bound was established for the homogeneous case.

Tardos [9] proved that for either homogeneous and nonhomogeneous 2-intervals,  $\tau$  is bounded by a linear function of  $\nu$ .

**Theorem 1.1** (Tardos). *Every system  $\mathcal{H}$  of 2-intervals (resp. homogeneous 2-intervals) has a transversal of size  $2\nu(\mathcal{H})$  (resp. of size  $8\nu(\mathcal{H})$ ).*

The bound for (nonhomogeneous) 2-intervals is moreover tight as [3] proves that for any  $d$  and  $\nu$  there is a system of  $d$ -intervals with the packing number equal to  $\nu$  and the transversal number at least  $d\nu$ .

In the present paper, we derive results corresponding to Theorem 1.1 for all values of  $d$ . We prefer to treat the homogeneous case first. Thus in Section 2 we prove the following statement.

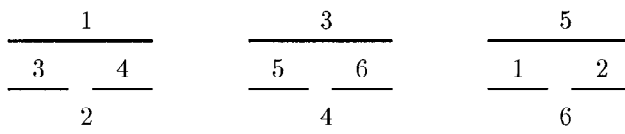
**Theorem 1.2.** *Every system  $\mathcal{H}$  of homogeneous  $d$ -intervals has a transversal of size  $(d^2 - d + 1)\nu(\mathcal{H})$ . Moreover, the bound improves to  $(d^2 - d)\nu(\mathcal{H})$  if  $d \geq 3$  and there is no projective plane of order  $d - 1$ .*

For  $d = 2$ , this specializes to  $\tau(\mathcal{H}) \leq 3\nu(\mathcal{H})$ . Consider the family in Fig. 1, taken from [3]. (Note: All segments in the figure lie on a *single* horizontal line.) Using several “disjoint copies” of this family, one can see that the above inequality is optimal.

The proof of Theorem 1.2 makes use of the following classic.

**Theorem 1.3** (Borsuk–Ulam). *Let  $f$  be a continuous map from the  $n$ -sphere  $S^n$  to  $\mathbb{R}^n$ . If  $f$  is antipodal (that is,  $f(-x) = -f(x)$  for all  $x$ ), then it has a zero.*

This is proved, e.g., in [8].



**Fig. 1.** A family of homogeneous 2-intervals with  $\nu = 1$  and  $\tau = 3$ .

In Section 3, we focus on the nonhomogeneous  $d$ -intervals. Note that any bound on homogeneous intervals carries over to the nonhomogeneous case. Indeed, if the  $d$ -intervals in  $\mathcal{H}$  lie on lines  $\ell^1, \dots, \ell^d$ , then each  $\ell^i$  can be mapped homeomorphically onto an open interval of unit length and the intersection properties of  $\mathcal{H}$  are preserved under this map. Next one can place the  $d$  unit intervals onto one line  $\ell$  so that they are pairwise disjoint. This transforms  $\mathcal{H}$  into a system of homogeneous intervals with the same  $\nu$  and the same  $\tau$ .

We can, however, improve on Theorem 1.2 a bit, obtaining

**Theorem 1.4.** *Every system  $\mathcal{H}$  of  $d$ -intervals has a transversal of size  $(d^2 - d)\nu(\mathcal{H})$ .*

To show this, we employ a generalization of the Borsuk–Ulam theorem (due to Ramos) which will be described later. This theorem, as well as two nontrivial results on graphs and hypergraphs and the Borsuk–Ulam theorem itself, will be stated without proof. However, the paragraph concluding Section 2 shows how only a slightly weaker bound  $\tau(\mathcal{H}) \leq d^2\nu(\mathcal{H})$  (certainly sufficient for most readers) can be proved in the homogeneous case, using just the Borsuk–Ulam theorem plus elementary hypergraph considerations. And according to what has been said, this bound applies to the nonhomogeneous case as well. This eliminates the need of the first three quoted statements.

## 2. Homogeneous $d$ -Intervals

Fix a system  $\mathcal{H}$  of homogeneous  $d$ -intervals on a line  $\ell$ . Considering the homeomorphism just mentioned in the paragraph following Theorem 1.3, we may assume that all members of  $\mathcal{H}$  are contained in the closed unit interval  $[0, 1]$ . It may also be assumed that all members of  $\mathcal{H}$  are *nondegenerate*, i.e., that each is the union of exactly  $d$  disjoint intervals. (For finite  $\mathcal{H}$ , one can complete each degenerate member  $I$  by extra components which do not intersect any other member of  $\mathcal{H}$ . This does not influence the relevant properties of  $\mathcal{H}$ . A similar reduction can be done in the infinite case.)

Let  $k$  denote the packing number  $\nu$  of  $\mathcal{H}$ . We search for a transversal of  $\mathcal{H}$  consisting of  $n$  points of  $\ell$ . (The number  $n$  will be specified later.) Any such  $n$ -tuple can be identified with an element  $x$  of  $I_{\leq}^n$ , where

$$I_{\leq}^n = \{y = (y_1, \dots, y_n) \mid 0 \leq y_1 \leq \dots \leq y_n \leq 1\} \subset \mathbb{R}^n.$$

For convenience, we put  $y_0 = 0$  and  $y_{n+1} = 1$ . The open unit interval is broken by  $x$  into  $n + 1$  open intervals  $L_0, \dots, L_n$ , some of which are possibly empty. Here  $L_i = (x_i, x_{i+1})$ .

There is a correspondence between  $I_{\leq}^n$  and the  $n$ -sphere  $S^n$ : for  $z = (z_0, \dots, z_n) \in S^n$ , define

$$g_i(z) = \sum_{j=0}^{i-1} z_j^2,$$

where  $i = 1, \dots, n$ . Then the vector-valued function  $g$ , restricted to one closed orthant in  $\mathbb{R}^{n+1}$ , is a homeomorphism of the corresponding part of  $S^n$  with  $I_{\leq}^n$ . (This mapping

has been used by Alon and West in the proof of the Necklace theorem, see [1].) Note that  $g_0(z) = 0$  and  $g_{n+1}(z) = 1$ , in accordance with our convention. The value of  $g(z)$  does not change on reversing the signs of any components of  $z$ .

The “space of candidates” for the transversal of  $\mathcal{H}$  can thus be naturally considered to be  $S^n$ . Its elements are mapped by  $g$  to  $n$ -tuples of points on  $\ell$ .

Consider a fixed  $z \in S^n$ . Let  $\{L_i: i \in [n]\}$  be the open intervals determined by  $g(z)$  as above, where  $[n] = \{0, \dots, n\}$ . Now  $z$  corresponds to a transversal if and only if no  $I \in \mathcal{H}$  is contained in any union  $L_{i_1} \cup \dots \cup L_{i_d}$ . For given  $i_1, \dots, i_d \in [n]$ , we define

$$w_{i_1 \dots i_d}(z) = \sup_I \text{dist}(g(z), I),$$

with  $I$  ranging over those members of  $\mathcal{H}$  that intersect each  $L_{i_m}$  for  $m = 1, \dots, d$  and are contained in the union  $\bigcup_{m=1}^d L_{i_m}$ . The symbol “dist” denotes the usual distance. Intuitively, one could say that  $w_{i_1 \dots i_d}(z)$  measures to what extent  $g(z)$  fails to be a transversal for  $\mathcal{H}$  because of members “escaping it” through  $\bigcup_{m=1}^d L_{i_m}$ .

**Observation 2.1.** *For each  $i_1, \dots, i_d$ , the following holds:*

- (i)  $w_{i_1 \dots i_d}$  is a continuous and nonnegative function from  $S^n$  to  $\mathbb{R}$ .
- (ii)  $w_{i_1 \dots i_d}(z) = 0$  iff no element of  $\mathcal{H}$  is contained in  $L_{i_1} \cup \dots \cup L_{i_d}$ .

*Proof.* (i) is clear. (ii) follows from the fact that members of  $\mathcal{H}$  are closed. □

We will shortly see how the functions just defined determine a hypergraph. First some terminology. For a hypergraph  $M = (V(M), E(M))$ , an *edge-weight function* on  $M$  is a map  $w: E(M) \rightarrow \mathbb{R}$ . Since our edge-weight functions will themselves depend on a variable  $z$ , we will write the weight of an edge  $e$  as  $w_e(z)$  or simply  $w_e$ . The *vertex-weight function induced by  $w$*  is the map  $w': V(M) \rightarrow \mathbb{R}$  defined by  $w'_v = \sum_{e \ni v} w_e$ . We use the term *weighted hypergraph* for a hypergraph equipped with an edge-weight function.

Now to any  $z \in S^n$  there corresponds a hypergraph  $M(z)$  on the vertex set  $V = [n]$ , together with an edge-weight function. Let the unordered  $d$ -tuple  $\{i_1, \dots, i_d\}$  (possibly with repetitions) be an edge of  $M(z)$  if  $w_{i_1 \dots i_d}(z) > 0$ , in which case this number will be the weight of the edge. This makes sense because the value of  $w_{i_1 \dots i_d}(z)$  does not change under a permutation of the indices. The resulting hypergraph need not be  $d$ -uniform but its *rank* (maximum size of an edge) will be at most  $d$ .

**Lemma 2.2.** *For all  $z \in S^n$ :*

- (i) *The matching number of  $M(z)$  is less than or equal to the packing number of  $\mathcal{H}$ ; in symbols,  $\nu(M(z)) \leq k$ .*
- (ii)  *$z \in S^n$  and its antipode  $-z$  determine the same weighted hypergraph.*
- (iii) *If  $z_i = 0$  for some  $i$ , then the vertex  $i$  is isolated in  $M(z)$ .*

*Proof.* (ii) and (iii) follow from the definition of  $M(z)$ . We prove (i). Let  $T$  be a set of  $k+1$  pairwise disjoint edges in  $M(z)$ . Each edge  $e = \{i_1, \dots, i_d\} \in T$  corresponds to a set  $L_{i_1} \cup \dots \cup L_{i_d}$ , disjoint edges corresponding to disjoint sets. Part (ii) of Observation 2.1

implies that there are  $k + 1$  pairwise disjoint sets of the form  $L_{i_1} \cup \dots \cup L_{i_d}$ , each containing a member of  $\mathcal{H}$ . But this contradicts the fact that  $\nu(\mathcal{H}) = k$ .  $\square$

Consider a hypergraph  $M$  with an edge-weight function  $w$ . Call  $M$  *weight regular* if  $w'_v$  is the same for all vertices  $v$ , where  $w'$  is the vertex-weight function induced by  $w$ .

**Lemma 2.3.** *There is a point  $\bar{z} \in S^n$  such that  $M(\bar{z})$  is weight regular.*

*Proof.* For  $1 \leq i \leq n$ , define functions  $h_i: S^n \rightarrow \mathbb{R}$  by

$$h_i(z) = \operatorname{sgn}(z_i)w'_i(z) - \operatorname{sgn}(z_{i-1})w'_{i-1}(z).$$

We claim that all of them are continuous. Part (iii) of Lemma 2.2 implies that for each  $i$  and  $j_2, \dots, j_d$ , the function  $\operatorname{sgn}(z_i) \cdot w_{ij_2 \dots j_d}(z)$  is continuous. And  $h_i$  is, up to signs, a sum of terms of this form.

All the  $h_i$ 's are antipodal (recall this means  $h_i(-z) = -h_i(z)$  for all  $z$ ). Taken together, they form an antipodal function  $\tilde{h}: S^n \rightarrow \mathbb{R}^n$  defined by

$$\tilde{h} = (h_1, \dots, h_k).$$

The Borsuk–Ulam theorem (Theorem 1.3) implies that  $\tilde{h}$  has a root  $\bar{z}$ . By the definition of the  $h_i$ 's, all the numbers  $w'_i(\bar{z})$  are, up to signs, equal to some number  $K$ . But the signs cannot differ since all the weights are positive. So  $M(\bar{z})$  is weight regular.  $\square$

Recall that a *fractional matching* in any hypergraph  $M$  is a nonnegative edge-weight function  $a$  for which  $a'_v \leq 1$  for every vertex  $v$ . The *size*  $|a|$  of the matching is defined as the sum of weights of all edges. The *fractional matching number* is then

$$\nu^*(M) = \sup |a|,$$

the supremum taken over all fractional matchings in  $M$ .

We will first show that the fractional matching number of  $M(\bar{z})$  is “large” and then use general theorems which bound it from above in terms of the ordinary matching number.

**Lemma 2.4.** *If  $M(\bar{z})$  has at least one edge, then*

$$\nu^*(M(\bar{z})) \geq \frac{n + 1}{d}.$$

*Proof.* Assume that  $M(\bar{z})$  has an edge. Denote by  $K$  the nonzero number  $w'_v(\bar{z})$  which is for all vertices the same. Then the edge-weight function  $\tilde{w}$  defined by  $\tilde{w}_e = w_e(\bar{z})/K$  is a fractional matching. Its size  $|\tilde{w}|$  can be estimated by double counting as follows. Let  $A = \sum_{v \in V} w'_v(\bar{z})$ . Clearly  $A = K(n + 1)$ . On the other hand, since each edge contains at most  $d$  vertices, we have  $A \leq d|w| = dK|\tilde{w}|$ . Hence  $|\tilde{w}| \geq (n + 1)/d$  which concludes the proof.  $\square$

At this point we have to separate the case  $d = 2$  from the others. Suppose that  $d = 2$  and so  $M(z)$  is just a graph for all  $z$ . We use the following theorem by Lovász (see [6] for the proof) to bound the fractional matching number of this graph in terms of its usual matching number.

**Theorem 2.5** (Lovász). *For any graph  $G$ ,*

$$v^*(G) \leq \frac{3}{2}v(G).$$

Assuming that  $M(\bar{z})$  has at least one edge and combining the theorem with Lemma 2.4, we get  $(n + 1)/2 \leq 3k/2$ , or  $n < 3k$ . So if we deliberately violate this condition by setting  $n = 3k$ , the only possibility is that  $M(\bar{z})$  has no edges, or in other words, that  $\bar{z}$  determines a transversal for  $\mathcal{H}$  consisting of  $3k$  points.

If  $d > 2$ , then a powerful analogous theorem by Füredi applies. Its proof can be found in [2].

**Theorem 2.6** (Füredi). *Let  $M$  be a hypergraph of rank  $d \geq 3$ . Suppose  $M$  does not contain  $p + 1$  pairwise disjoint copies of the projective plane of order  $d - 1$ . Then*

$$v^*(M) \leq (d - 1)v(M) + \frac{p}{d}.$$

[We remark that this theorem does not hold for  $d = 2$  (assuming that the “projective plane of order 1” is taken to be a triangle). This is demonstrated by taking the 5-cycle, for instance.]

One does not know how many pairwise disjoint copies of the projective plane there are contained in  $M(\bar{z})$ , but the number is clearly at most  $k$ . So the inequality from the theorem becomes

$$v^*(M(\bar{z})) \leq k \left( d - 1 + \frac{1}{d} \right).$$

Using Lemma 2.4 again, we see that if  $n$  is set to equal  $k(d^2 - d + 1)$ , then  $M(\bar{z})$  has no edges. Therefore, there always is a transversal consisting of  $k(d^2 - d + 1)$  points. Of course, this improves in an obvious way if the projective plane of order  $d - 1$  does not exist. So the proof of Theorem 1.2 is complete.  $\square$

Let us point out that it is possible to use arguments less sophisticated than the theorems of Lovász and Füredi, and still get a satisfactory bound which is linear in  $k$ . The simplest such argument runs as follows.  $M(\bar{z})$  is weight regular; assume again that it has at least one edge. If  $X \subset V$  is a set of vertices, then the sum  $\sum_{v \in X} w'_v(\bar{z})$  equals  $K \cdot |X|$  where  $K$  is as in the proof of Lemma 2.4. Since  $M(\bar{z})$  has  $n + 1$  vertices and its matching number is at most  $k$ , there certainly is a set  $A \subset V$  such that  $|A| = n - dk + 1$  and every edge of  $M(\bar{z})$  contains at least one vertex outside  $A$ . Summing up the weights of all vertices in  $V - A$  and in  $A$ , respectively, we get  $|V - A| \geq |A|/(d - 1)$ , because at least one of the  $\leq d$  contributions from each edge goes to the sum for  $V - A$ . The inequality can be rewritten as  $dk \geq (n - dk + 1)/(d - 1)$ , or  $n \leq d^2k - 1$ . This has been deduced from the assumption that the edge set of  $M(\bar{z})$  is nonempty. As before, we conclude that one can always find a transversal consisting of  $n = d^2k$  points.

### 3. Nonhomogeneous $d$ -Intervals

We shall now turn to families of (ordinary)  $d$ -intervals. Let  $\mathcal{H}$  denote the family in question,  $k$  stands for its packing number. The  $d$  lines containing the members of  $\mathcal{H}$  are again denoted by  $\ell^1, \dots, \ell^d$ . We make similar assumptions about  $\mathcal{H}$  as in the preceding section, i.e., that all components are nonempty and contained in a fixed unit length interval on the appropriate line  $\ell^i$ . In view of the tight bounds to the transversal number for  $d = 1$  and  $d = 2$  mentioned in the Introduction, namely,  $k$  and  $2k$ , one might be tempted to conjecture that  $dk$  points always suffice for a transversal. This is not true as shown by an example from [3]. It presents a family of ten pairwise intersecting 3-intervals which has transversal number 4.

We shall proceed in general very similarly as in the homogeneous case. Our candidate for the transversal will now consist of  $d$  independent  $n$ -tuples of points, one  $n$ -tuple on each line  $\ell^i$  (more precisely, in the unit interval of  $\ell^i$ ). So the “space of candidates” will be the product

$$(S^n)^d = S^n \times \dots \times S^n \quad (d \text{ times}).$$

If  $z = (z^1, \dots, z^d)$  is an element of this space, then the  $n$ -tuple on  $\ell^i$  is given by  $g(z^i)$  with  $g$  as in Section 2. Writing  $g(z^i)$ , we will always mean a subset of  $\ell^i$ . Hopefully this will cause no confusion. The open unit interval on each line  $\ell^i$  is again split into  $n + 1$  open intervals, denoted by  $L_0^i, \dots, L_n^i$ . We may now define functions  $w_{i_1 \dots i_d}$  for all  $d$ -element sequences of elements of  $[n]$ ; these functions will no more be symmetric with respect to the indices. Put

$$w_{i_1 \dots i_d}(z) = \sup_I \min_{j=1 \dots d} \text{dist}(g(z^j), I^j),$$

where  $I$  ranges over all members of  $\mathcal{H}$  contained in the union  $U = L_{i_1}^1 \cup \dots \cup L_{i_d}^d$ . This is again a continuous nonnegative function, and it is zero iff  $U$  contains no members of  $\mathcal{H}$ .

These functions define a hypergraph  $M(z)$  and an edge weight  $w$  again, with the difference that  $M(z)$  is now  $d$ -uniform and  $d$ -partite. Each partite is a copy of  $[n]$ . The  $i$ th vertex in the  $m$ th partite is referred to as  $(m, i)$ , where  $1 \leq m \leq d$  and  $0 \leq i \leq n$ . The whole vertex set is denoted by  $V$ . For  $i_1, \dots, i_d \in [n]$ , let  $e_{i_1 \dots i_d}$  be the set whose intersection with the  $m$ th partite consists of the vertex  $(m, i_m)$  for each  $m$ . Then  $e_{i_1 \dots i_d}$  is defined to be an edge of  $M(z)$  iff  $w_{i_1 \dots i_d}(z) > 0$ , in which case it gets this number for a weight.

We easily obtain an analogue of Lemma 2.2.

**Lemma 3.1.** *For all  $z = (z^1, \dots, z^d) \in (S^n)^d$ :*

- (i) *The matching number  $\nu(M(z))$  is less than or equal to  $k$ .*
- (ii) *If  $z'$  is obtained from  $z$  by changing the sign in any of the  $d$  components, then  $z$  and  $z'$  determine the same weighted hypergraph.*
- (iii) *If  $z_j^m = 0$  for some  $m, j$ , then the vertex  $(m, j)$  is isolated in  $M(z)$ .*

Let us now state the extension of the Borsuk–Ulam theorem to products of spheres,

established by Ramos, and refer the reader to [7] for a proof. The original version deals with products of balls rather than spheres, but it is equivalent to the one we will use.

Let  $f$  be a continuous function from the product  $S^{q_1} \times \dots \times S^{q_d}$  to  $\mathbb{R}^q$ , where  $q = \sum q_j$ . We write  $f_i$  for the  $i$ th component of  $f$ . Now let  $f$  have the property that, for some numbers  $a_{ij}$  which may be either zero or one,

$$f_i(x_1, \dots, -x_j, \dots, x_d) = (-1)^{a_{ij}} f_i(x_1, \dots, x_j, \dots, x_d)$$

for all  $1 \leq i \leq q$ ,  $1 \leq j \leq d$ , and all  $(x_1, \dots, x_d) \in S^{q_1} \times \dots \times S^{q_d}$ . (If  $a_{ij} = 0$ , then one says that  $f_i$  has positive antipodality in the  $j$ th coordinate, otherwise it has negative antipodality.) The numbers  $a_{ij}$  define a  $q$  by  $d$ , 0–1 matrix called the *antipodality matrix*  $A(f)$  of the function  $f$ . The generalized permanent of this matrix has a surprising relation to zeros of  $f$ . Define  $\text{perm}_{q_1, \dots, q_d} A(f)$  to be the sum of all products of the type  $b_1 \cdots b_q$ , where each  $b_i$  is an entry in the  $i$ th row of  $A(f)$ , and exactly  $q_j$  factors are taken from the  $j$ th column (for each  $j$ ).

**Theorem 3.2** (Ramos). *If  $\text{perm}_{q_1, \dots, q_d} A(f)$  is odd, then  $f$  has a zero.*

Using this theorem, we can convince ourselves that there again exists a point which determines a weight-regular hypergraph.

**Lemma 3.3.** *There is a point  $\bar{z} \in (S^n)^d$  such that  $M(\bar{z})$  is weight regular. In particular, the fractional matching number of  $M(\bar{z})$  is at least  $n + 1$  provided that  $M(\bar{z})$  has at least one edge.*

*Proof.* The second part follows by double counting as in Lemma 2.4, recalling that the number of vertices is now  $d(n + 1)$ . Let us prove the first part. For each element  $(m, i)$  of  $V$  which has  $i > 0$  (so  $1 \leq m \leq d$ ,  $1 \leq i \leq n$ ), define a function  $h_{(m,i)}: (S^n)^d \rightarrow \mathbb{R}$  by

$$h_{(m,i)}(z) = \text{sgn}(z_i^m) w'_{(m,i)}(z) - \text{sgn}(z_{i-1}^m) w'_{(m,i-1)}(z).$$

All the  $h_{(m,i)}$ 's are continuous. One checks easily that the antipodality matrix of the function

$$\tilde{h} = (h_{(1,1)}, \dots, h_{(1,n)}, \dots, h_{(d,1)}, \dots, h_{(d,n)})$$

has permanent (more precisely,  $\text{perm}_{n, \dots, n}$ ) equal to 1. Therefore by Theorem 3.2, it has a root  $\bar{z}$ . All the numbers  $w'_{(m,i)}(\bar{z})$  with a given  $m$  are equal, as in the proof of Lemma 2.3. Since the sum of vertex weights in a partite is for every partite the same, it follows that actually all vertex weights in  $M(\bar{z})$  coincide. □

The rest of the argument is as in the preceding section. If  $d > 2$ , we can invoke Theorem 2.6 and benefit from the fact that  $M(\bar{z})$  contains no copy of the projective plane of order  $d - 1$  as the plane is not  $d$ -partite. By Lemma 3.3, if  $M(\bar{z})$  has any edges, then  $n + 1 \leq (d - 1)k$ .

The same inequality can actually be proved for  $d = 2$ , too. For this, we have to do better than to use Theorem 2.5. An improvement is possible because the graph  $M(\bar{z})$



is bipartite now, and so it has  $\nu = \nu^* = \tau$  by König's theorem, mentioned in the Introduction. In our case this yields  $n + 1 \leq k$  as desired.

Thus for arbitrary  $d \geq 2$ , if we set  $n = (d-1)k$ , then the above inequalities are violated and  $M(\bar{z})$  has no edges. The transversal determined by  $\bar{z}$  now contains  $dn = (d^2 - d)k$  points. The proof of Theorem 1.4 is now complete.  $\square$

#### 4. Concluding Remarks

It should be mentioned that the method can be modified to use Brouwer's fixed point theorem in place of the Borsuk–Ulam theorem. This approach has been used in [5] to prove a statement related to those of the present paper.

We would like to conclude by pointing out two remaining open problems. The first one is to improve the existing lower bounds to the possible transversal number of systems of  $d$ -intervals (homogeneous or not). To our knowledge, the best one available is  $\tau \geq d\nu$ , in general (see the Introduction). We find it plausible that the upper bounds established in this paper are essentially tight, meaning that one cannot improve the  $d^2$  factor to anything subquadratic. However, we have to leave this feeling quite unsupported.

And the other open problem: Is it possible to apply a similar topological method to related problems concerning convex bodies in higher dimensions?

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