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Inclusion–Exclusion Complexes for Pseudodisk Collections*

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Abstract. Let B be a finite pseudodisk collection in the plane. By the principle of inclusion–exclusion, the area or any other measure of the union is

$$\mu\left(\bigcup B\right) = \sum_{\sigma \in 2^B - \{\emptyset\}} (-1)^{\operatorname{card}\sigma - 1} \mu\left(\bigcap \sigma\right).$$

We show the existence of a two-dimensional abstract simplicial complex, $\mathcal{X} \subseteq 2^B$, so the above relation holds even if $\mathcal{X}$ is substituted for 2^B. In addition, $\mathcal{X}$ can be embedded in $\mathbb{R}^2$ so its underlying space is homotopy equivalent to $\operatorname{int}\bigcup B$, and the frontier of $\mathcal{X}$ is isomorphic to the nerve of the set of boundary contributions.

1. Introduction

Inclusion–Exclusion Principle. Given a finite collection of measurable sets, B, the measure of the union, $\bigcup B = \bigcup_{b \in B} b$, can be expressed in terms of the measures of the common intersections of subcollections:

$$\mu\left(\bigcup B\right) = \sum_{\sigma \in 2^B - \{\emptyset\}} (-1)^{\operatorname{card}\sigma - 1} \mu\left(\bigcap \sigma\right). \tag{1}$$

This relation is known as the *inclusion–exclusion principle*. We are interested in removing redundant terms from (1). Trivially, vanishing terms can be dropped, which is the same as restricting the sum to all nonempty collections in the *nerve* of B, $\operatorname{Nrv} B = \{\sigma \subseteq B \mid \bigcap \sigma \neq \emptyset\}$. More generally, we consider *abstract simplicial complexes*, which

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are systems $\mathcal{A} \subseteq 2^B$ for which

$$\sigma \in \mathcal{A} \quad \text{and} \quad \tau \subseteq \sigma \qquad \text{implies} \quad \tau \in \mathcal{A}.$$

Clearly, Nrv B is an abstract simplicial complex. The elements in $\mathcal{A}$ are called *(abstract) simplices*, and they are subsets of the *vertex set*, $\text{vert}\,\mathcal{A} = \bigcup \mathcal{A}$. The subsets of a simplex are its *faces*, and a face is *proper* if it is proper as a subset. The *improper* faces are the empty set and the simplex itself. The *dimension* of a simplex σ is $\dim \sigma = \text{card}\,\sigma - 1$, and σ is a *k-simplex* if $k = \dim \sigma$. The 0-, 1-, and 2-simplices are also referred to as *vertices*, *edges*, and *triangles*. The *dimension* of $\mathcal{A}$ is $\dim \mathcal{A} = \max_{\sigma \in \mathcal{A}}\{\dim \sigma\}$. A *subcomplex* of $\mathcal{A}$ is an abstract simplicial complex $\mathcal{A}' \subseteq \mathcal{A}$. Finally, $\mathcal{A}$ is *connected* if there is no partition $\text{vert}\,\mathcal{A} = B_1 \dot{\cup} B_2$, both B_1, B_2 nonempty, with $\mathcal{A} \subseteq 2^{B_1} \cup 2^{B_2}$.

Geometric Balls and Simplicial Complexes. It is shown in [5] that if B is a collection of closed geometric balls in $\mathbb{R}^d$, there is a d-dimensional abstract simplicial complex $\mathcal{X} \subseteq 2^B$ so that

$$\mu\left(\bigcup B\right) = \sum_{\sigma \in \mathcal{X} - \{\emptyset\}} (-1)^{\dim \sigma} \mu\left(\bigcap \sigma\right). \tag{2}$$

This can be a considerable reduction in dimensionality since Nrv B can be of dimension up to $n - 1$. Furthermore, $\mathcal{X}$ has nice topological properties: it has a natural geometric realization in $\mathbb{R}^d$ whose underlying space is homotopy equivalent to $\text{int} \bigcup B$, and its frontier is isomorphic to the nerve of the collection of boundary contributions. For the homotopy equivalence and the isomorphism results a general position assumption on the set of balls is required. Definitions of some of the terms can be found in Section 2, and a two-dimensional example is shown in Fig. 1.1. A similar result for a generally larger complex without the above topological properties is established in [14].

Pseudoballs. The question arises whether a result similar to the one on geometric balls holds for more general sets. In particular, we consider the case where for any $k \leq d$ sets in B there is a homeomorphism from $\mathbb{R}^d$ to itself that maps the k sets to k closed geometric balls in general position. Such a collection B is referred to as a *pseudoball*

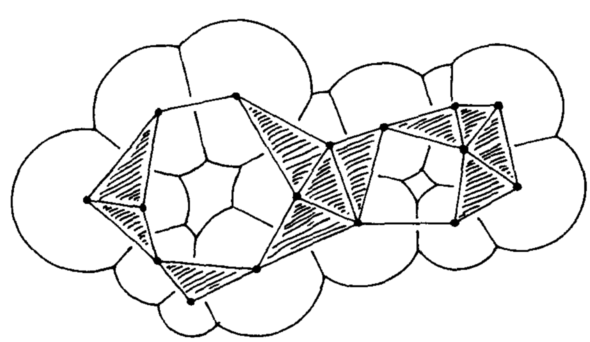

Fig. 1.1. A collection of disks in the plane and a geometric realization of the corresponding abstract simplicial complex, $\mathcal{X}$.

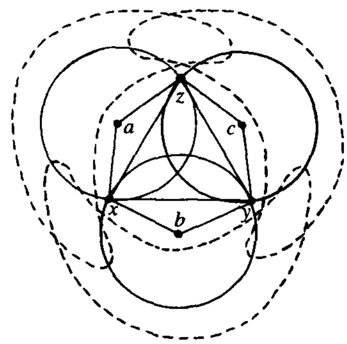

Fig. 1.2. A noncircularizable collection of six pseudodisks.

collection in $\mathbb{R}^d$; its elements are topological balls, referred to as *pseudoballs*. The main result of this paper is an affirmative answer for pseudodisks (pseudoballs in $\mathbb{R}^2$). The proof depends on the fact that it is always possible to sweep a pseudodisk collection with another pseudodisk [15]. Unfortunately, in dimensions $d > 2$ sweeping a pseudo-ball collection with another pseudoball is not always possible. A counterexample in $d = 3$ is obtained from an interesting oriented matroid due to Fukuda [3, Chapter 10.4]. Thus, our proof does not extend beyond two dimensions. Incidentally, the result in $\mathbb{R}^2$ implies a result in [11] on the number of arcs bounding a union of pseudodisks.

Noncircularizable Example. The study of pseudodisk collections begs the question whether or not there is always a set of geometric disks with the same combinatorial properties. The answer is of course negative, for otherwise the results in this paper would be implied by the geometric results in [5]. A pseudodisk collection is *circularizable* [8, p. 68] if there is a homeomorphism from the plane to itself that maps each pseudodisk to a geometric disk. By an argument in [7], the number of noncircularizable examples with n pseudodisks grows much faster, as a function of n, than the number of circularizable examples.

A particularly small example of the first kind is shown in Fig. 1.2. Its noncircularizability is argued by looking at the three inner disks, a, b, c. Denote their centers by the same labels and consider the hexagon $axbycz$ whose vertices are the three centers and the three corners of $a \cup b \cup c$. The angles at the centers add up to

$$\angle zax + \angle xby + \angle ycz > 2\pi$$

(the sum would be 2π if the three circles met at a common point in the middle). The angles at x, y, z add up to less than 2π. Consider the following three sums of two angles each:

$$\begin{aligned} A &= \angle axy + \angle xyc, \\ B &= \angle byz + \angle yza, \\ C &= \angle czx + \angle zxb. \end{aligned}$$

The total sum is $A + B + C < 3\pi$. At least one of A, B, C does not exceed the average, say $\angle czx + \angle zxb < \pi$. However, this implies no geometric disk can intersect the disks c, a, b in this sequence and leave holes between c and a and between a and b, as does the pseudodisk in Fig. 1.2.

2. Definitions and Results

We begin with the definitions necessary to give a detailed statement of the results in this paper. These consist of a technical lemma and inclusion–exclusion formulas implied by the lemma.

Abstract Concepts. Let B be a finite collection of pseudodisks in $\mathbb{R}^2$. An important notion is the *region* of a subset $\varrho \subseteq B$ defined as

$$\operatorname{reg}_B \varrho = \bigcap \varrho - \bigcup (B - \varrho).$$

The region of a subset can be empty, and if nonempty it can be disconnected and components can be multiply connected. $\varrho \subseteq B$ is *regional* if $\operatorname{reg}_B \varrho \neq \emptyset$. Different subsets of B define disjoint regions, and together they cover $\mathbb{R}^2$. The $n = \operatorname{card} B$ pseudocircles bounding pseudodisks in B decompose $\mathbb{R}^2$ into at most $2\binom{n}{2} + 2$ connected two-dimensional cells or *chambers*, and this is also an upper bound on the number of regional subsets of B. ϱ is *independent* if $\operatorname{reg}_\varrho \tau \neq \emptyset$ for all $\tau \subseteq \varrho$. Because $2\binom{n}{2} + 2 < 2^n$ whenever $n \geq 4$, we have 3 as an upper bound on the cardinality of every independent ϱ. Two independent triangles that share a common edge, $\{a, b, c\}$ and $\{a, b, d\}$, are *consistently oriented* if one point of $\operatorname{bd} a \cap \operatorname{bd} b$ belongs to c and the other to d. By assumption of independence c cannot contain both points, and neither can d.

For an abstract simplicial complex $\mathcal{A} \subseteq 2^B$, the subcomplex *induced* by $\varrho \subseteq B$ is $\mathcal{A}|^\varrho = \{\sigma \in \mathcal{A} \mid \sigma \subseteq \varrho\}$. For example, $2^B|^\varrho = 2^\varrho$. Symmetrically, the *star* of $\tau \subseteq B$ is $\mathcal{A}|_\tau = \{\sigma \in \mathcal{A} \mid \tau \subseteq \sigma\}$. The star is a subset but not a subcomplex of $\mathcal{A}$, unless $\tau = \emptyset$ in which case $\mathcal{A}|_\emptyset = \mathcal{A}$. It is, however, isomorphic to the *link*, $\operatorname{Lk} \tau = \{\sigma - \tau \mid \sigma \in \mathcal{A}|_\tau\}$, which is an abstract simplicial complex. We call $\mathcal{A}|_\tau$ *connected* if $\operatorname{Lk} \tau$ is. The star of τ in the subcomplex induced by ϱ consists of all simplices that contain τ and are contained in ϱ:

$$\mathcal{A}|^\varrho_\tau = (\mathcal{A}|^\varrho)|_\tau = \mathcal{A}|^\varrho \cap \mathcal{A}|_\tau,$$

see Fig. 2.1. For vertices $d \in B$ we use the shorter notations $\mathcal{A}|_d = \mathcal{A}|_{\{d\}}$ and $\mathcal{A}|^\varrho_d = \mathcal{A}|^\varrho_{\{d\}}$. A subcomplex of $\mathcal{A}$ is *regional* if it is induced by a regional subset of B.

The *Euler characteristic* of $\mathcal{A}$ is

$$\chi(\mathcal{A}) = \sum_{\sigma \in \mathcal{A}} (-1)^{\dim \sigma}.$$

The principle of inclusion–exclusion, relation (1), follows from $\chi(2^B|^\varrho) = 0$ for every regional $\varrho \subseteq B$. Indeed, $2^B|^\varrho = 2^\varrho$ and $\chi(2^\varrho) = (1-1)^{\operatorname{card} \varrho} = 0$. The definition of the Euler characteristic applies to any finite set system, not just to abstract simplicial complexes. Exploiting this we talk about the Euler characteristic of a star, $\chi(\mathcal{A}|_\tau) = (-1)^{\operatorname{card} \tau} \chi(\operatorname{Lk} \tau)$.

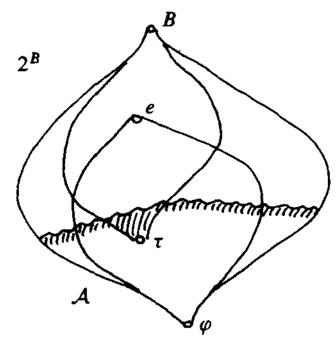

Fig. 2.1. The subcomplex induced by $\varrho \subseteq B$ consists of all subsets in $\mathcal{A}$. Symmetrically, the star of τ consists of all supersets in $\mathcal{A}$. $\mathcal{A}|_\tau^\varrho$ is the intersection of the two.

Geometric Realization and Embedding.. To state the results, we need a few additional definitions about embedding abstract complexes in space. An abstract simplicial complex, $\mathcal{A}$, is *geometrically realized* by an injective map φ: vert $\mathcal{A} \to \mathbb{R}^d$ so $\sigma, \sigma' \in \mathcal{A}$ implies the intersection of conv $\varphi(\sigma)$ and conv $\varphi(\sigma')$ is conv $\varphi(\sigma \cap \sigma')$. conv $\varphi(\sigma)$ is the geometric simplex that corresponds to $\sigma \in \mathcal{A}$, and

$$\mathcal{L} = \{\text{conv}\,\varphi(\sigma) \mid \sigma \in \mathcal{A}\}$$

is a *geometric realization* of $\mathcal{A}$. Without any further knowledge we know $d \geq \dim \mathcal{A}$ is necessary and $d \geq 2\dim \mathcal{A} + 1$ is sufficient for $\mathcal{L}$ to exist. The *underlying space* of $\mathcal{L}$ is $\bigcup \mathcal{L} = \bigcup_{\sigma\in\mathcal{A}} \text{conv}\,\varphi(\sigma)$.

$\mathcal{A}$ is *embedded* in $\mathbb{R}^2$ by a continuous injection $\varepsilon: \bigcup \mathcal{L} \to \mathbb{R}^2$ whose restriction to $\varepsilon(\bigcup \mathcal{L})$ is a homeomorphism. The image $\eta(\sigma) = \varepsilon(\text{conv}\,\varphi(\sigma))$ of every abstract k-simplex is a homeomorph of a geometric k-simplex. We use the short notation $\eta(a) = \eta(\{a\})$, $\eta(ab) = \eta(\{a, b\})$, etc. The set

$$\mathcal{K} = \{\eta(\sigma) \mid \sigma \in \mathcal{A}\}$$

is an *embedding* of $\mathcal{A}$ in $\mathbb{R}^2$. Fári's theorem [6] implies if $\mathcal{K}$ exists, then $\mathcal{A}$ can be geometrically realized in $\mathbb{R}^2$. In any case, the *underlying space* of $\mathcal{K}$ is $\bigcup \mathcal{K} = \bigcup_{\sigma\in\mathcal{A}} \eta(\sigma) = \varepsilon(\bigcup \mathcal{L})$. No geometric 3-simplex can be mapped homeomorphically into $\mathbb{R}^2$, so $\dim \mathcal{A} \leq 2$ is necessary for the existence of an embedding in $\mathbb{R}^2$.

The definitions imply that all embeddings of $\mathcal{A}$ have homeomorphic underlying spaces. The *boundary* of $\bigcup \mathcal{K}$, bd $\bigcup \mathcal{K}$, consists of all points $x \in \bigcup \mathcal{K}$ without open neighborhood contained in $\bigcup \mathcal{K}$. Since the boundaries of different embeddings are also homeomorphic, it makes sense to define the *frontier*

$$\text{Fr}\,\mathcal{A} = \left\{\sigma \in \mathcal{A} \mid \eta(\sigma) \subseteq \text{bd} \bigcup \mathcal{K}\right\}.$$

The upcoming technical lemma includes a relation between the boundary of $\bigcup B$ and the frontier of $\mathcal{A}$. The *boundary contribution* of a pseudodisk $d \in B$ is $\bar{d} = \text{bd}\,d - \bigcup (B - \{d\})$, which is a possibly disconnected set in $\mathbb{R}^2$. The set of boundary contributions is $\bar{B} = \{\bar{d} \mid d \in B\}$.

Technical Lemma. Every pseudodisk collection in $\mathbb{R}^2$ has a two-dimensional abstract simplicial complex reflecting the combinatorial properties of the collection. This complex is generally not unique. Specifically, the following is proved in this paper.

Lemma. *Every finite pseudodisk collection B in $\mathbb{R}^2$ has an abstract simplicial complex $\mathcal{X} = \mathcal{X}(B) \subseteq 2^B$ that satisfies the following properties:*

(P1) *For each regional $\varrho \subseteq B$, the induced subcomplex $\mathcal{X}|^{\varrho}$ is connected and $\chi(\mathcal{X}|^{\varrho}) = 0$.*
(P2) *For each regional $\varrho \subseteq B$ and $d \in \varrho$ for which $\varrho - \{d\}$ is also regional, $\mathcal{X}|_d^{\varrho}$ is connected and $\chi(\mathcal{X}|_d^{\varrho}) = 0$.*
(P3) *Each simplex $\sigma \in \mathcal{X}$ is independent.*
(P4) *The triangles containing a common edge in $\mathcal{X}$ are consistently oriented.*
(P5) *$\mathcal{X}$ has an embedding in $\mathbb{R}^2$, and the underlying space of this embedding is homotopy equivalent to* $\operatorname{int} \bigcup B$.
(P6) $\operatorname{Fr} \mathcal{X}$ *is isomorphic to* $\operatorname{Nrv} \bar{B}$.

The six conditions are not entirely independent, as we will see. Although it appears that some of the conditions have little to do with each other, we state them at once because they are all used concurrently to maintain the induction hypothesis in the proof of the lemma.

Measuring by Integration. Properties (P1) and (P2) imply $\mathcal{X}$ can be used as the index set of inclusion–exclusion formulas measuring the union of the pseudodisks and its boundary.

Theorem.

(i) $\mu\left(\bigcup B\right) = \sum_{\sigma \in \mathcal{X} - \{\emptyset\}} (-1)^{\dim \sigma} \mu\left(\bigcap \sigma\right)$.
(ii) $\mu\left(\operatorname{bd} \bigcup B\right) = \sum_{\sigma \in \mathcal{X}} (-1)^{\dim \sigma} \mu\left(\operatorname{bd} \bigcap \sigma\right)$.

Proof. Let $\delta(x)$ be the density at $x \in \mathbb{R}^2$, so

$$\mu\left(\bigcup B\right) = \int_{x \in \bigcup B} \delta(x)\, dx.$$

Define $\varrho(x) = \{d \in B \mid x \in d\}$. $\varrho(x)$ is regional by construction. Furthermore,

$$\chi(\mathcal{X}|^{\varrho(x)}) = \begin{cases} 0 & \text{if } x \in \bigcup B, \\ -1 & \text{if } x \notin \bigcup B. \end{cases}$$

For $x \in \bigcup B$ this is implied by property (P1), and for $x \notin \bigcup B$ we have $\varrho(x) = \emptyset$ and $\chi(\mathcal{X}|^{\varrho(x)}) = \chi(\emptyset) = -1$. $\chi(\mathcal{X}|^{\varrho(x)}) + 1$ is used as an indicator for $x \in \bigcup B$. Let

$$\iota_\sigma(x) = \begin{cases} 1 & \text{if } x \in \bigcap \sigma, \\ 0 & \text{if } x \notin \bigcap \sigma \end{cases}$$

be the indicator for x belonging to the intersection of pseudodisks in σ. Note that $\sigma \subseteq \varrho(x)$ iff $\iota_\sigma(x) = 1$. Then

$$\begin{aligned}
\mu\left(\bigcup B\right) &= \int_{x\in\mathbb{R}^2} (\chi(\mathcal{X}|^{\varrho(x)}) + 1)\delta(x)\,dx \\
&= \int_{x\in\mathbb{R}^2} \sum_{\sigma\in\mathcal{X}|^{\varrho(x)},\sigma\neq\emptyset} (-1)^{\dim\sigma}\delta(x)\,dx \\
&= \int_{x\in\mathbb{R}^2} \sum_{\sigma\in\mathcal{X}-\{\emptyset\}} (-1)^{\dim\sigma}\iota_\sigma(x)\delta(x)\,dx \\
&= \sum_{\sigma\in\mathcal{X}-\{\emptyset\}} (-1)^{\dim\sigma} \int_{x\in\mathbb{R}^2} \iota_\sigma(x)\delta(x)\,dx \\
&= \sum_{\sigma\in\mathcal{X}-\{\emptyset\}} (-1)^{\dim\sigma}\mu\left(\bigcap\sigma\right).
\end{aligned}$$

This completes the proof of (i). A similar argument is used to show (ii). The contribution of $d \in B$ to the boundary of $\bigcup B$ is measured by integrating $\delta(x)$ over $\bar{d}$. Again we use an indicator function, namely

$$\chi(\mathcal{X}|_d^{\varrho(x)}) = \begin{cases} 0 & \text{if } x \in \operatorname{bd} d \cap \bigcup(B - \{d\}), \\ 1 & \text{if } x \in \bar{d}. \end{cases}$$

The first part of the equation is implied by (P2). Indeed, if $x \in \operatorname{bd} d$ and $\varrho'(x) = \varrho(x) - \{d\}$ is nonempty, then $\varrho'(x)$ is also regional and (P2) applies. For $x \notin \bigcup(B - \{d\})$ we have $\varrho(x) = \{d\}$ and $\chi(\mathcal{X}|_d^{\varrho(x)}) = \chi(\{d\}) = 1$.

The measure of the boundary of the union is therefore

$$\begin{aligned}
\mu\left(\operatorname{bd}\bigcup B\right) &= \sum_{d\in B}\int_{x\in\bar{d}} \delta(x)\,dx \\
&= \sum_{d\in B}\int_{x\in\operatorname{bd} d} \chi(\mathcal{X}|_d^{\varrho(x)})\delta(x)\,dx \\
&= \sum_{d\in B}\int_{x\in\operatorname{bd} d} \sum_{\sigma\in\mathcal{X}|_d^{\varrho(x)}} (-1)^{\dim\sigma}\delta(x)\,dx \\
&= \sum_{d\in B}\sum_{\sigma\in\mathcal{X}|_d} (-1)^{\dim\sigma} \int_{x\in\operatorname{bd} d} \iota_\sigma(x)\delta(x)\,dx \\
&= \sum_{\sigma\in\mathcal{X}} (-1)^{\dim\sigma} \sum_{d\in\sigma} \int_{x\in\operatorname{bd} d} \iota_\sigma(x)\delta(x)\,dx \\
&= \sum_{\sigma\in\mathcal{X}} (-1)^{\dim\sigma} \sum_{d\in\sigma} \mu\left(\operatorname{bd} d \cap \bigcap\sigma\right) \\
&= \sum_{\sigma\in\mathcal{X}} (-1)^{\dim\sigma} \mu\left(\operatorname{bd}\bigcap\sigma\right).
\end{aligned}$$

□

Influence Regions. The complex $\mathcal{X}$ in the technical lemma is the abstract counterpart of the (geometric) dual complex of a collection of geometric disks [5]. The latter is defined

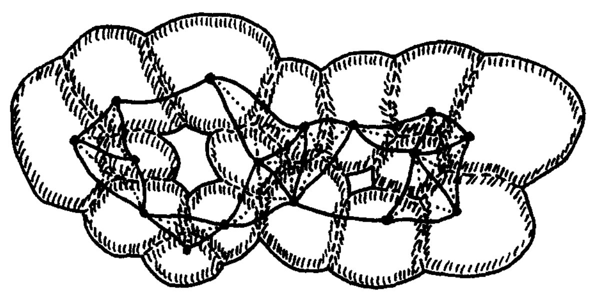

Fig. 2.2. Decomposition of $\bigcup B$ generated from the barycentric subdivision of $\mathcal{L}$.

by decomposing the disk union into convex closed regions overlapping only along the boundary, as indicated in Fig. 1.1. The complex is the geometric realization of the nerve of regions. It is natural to expect the complex $\mathcal{X}$ be isomorphic to the nerve of a similar decomposition of the union of pseudodisks into influence regions. Such a decomposition can indeed be obtained from $\mathcal{X}$, see Fig. 2.2

Let $\mathcal{L}$ be a geometric realization of $\mathcal{X}$, and let $\mathcal{K}$ be an embedding in $\mathbb{R}^2$. The existence of $\mathcal{K}$ is guaranteed by (P5). The homeomorphism $\varepsilon: \bigcup \mathcal{L} \to \bigcup \mathcal{K}$ takes each geometric simplex $\operatorname{conv} \varphi(\sigma) \in \mathcal{L}$ to $\eta(\sigma) \in \mathcal{K}$.

Let $\mathcal{L}'$ be the barycentric subdivision of $\mathcal{L}$, see, e.g., [13]. $\mathcal{L}'$ is a geometric simplicial complex with the same underlying space, $\bigcup \mathcal{L}' = \bigcup \mathcal{L}$. In short, each edge of $\mathcal{L}$ is cut into two by adding its midpoint as a new vertex, and each triangle is cut into six by adding its centroid as a new vertex and edges connecting the centroid with the vertices and edge midpoints. The *closed star* of a vertex $u \in \mathcal{L}'$ is the collection of simplices containing u and the faces of these simplices; it is a subcomplex of $\mathcal{L}'$.

To obtain the influence region of a pseudodisk $d \in B$ take the vertex $u = \varphi(d) \in \mathcal{L}'$ and the underlying space of its closed star. The image under ε of this underlying space is a closed and connected subset of $\bigcup \mathcal{K}$. The influence region of d is this subset, possibly extended outside $\bigcup \mathcal{K}$ by attaching fibers of the deformation retraction discussed below, see Fig. 2.2.

3. Proof Preparation

The proof of the lemma is simplified by assuming no pseudodisk is *redundant*, that is, $\bigcup B \neq \bigcup (B - \{d\})$ for all $d \in B$. For if d is redundant, we can take a complex $\mathcal{X}_0$ that satisfies (P1)–(P6) for $B_0 = B - \{d\}$; it also satisfies the conditions for B.

General Proof Idea. The proof is by induction on the number of pseudodisks, n. The induction basis for $n = 1$ is trivial. For the induction step, let B be a collection of $n > 1$ nonredundant pseudodisks, $d \in B$, and $B' = B - \{d\}$. We show how to modify the complex $\mathcal{X}' = \mathcal{X}(B')$ that satisfies the lemma for B' so it reflects the addition of d to B'. d is added first as a point outside $\bigcup B'$ which continuously grows until it equals d. The growth process can be understood as sweeping an arrangement of pseudocircles

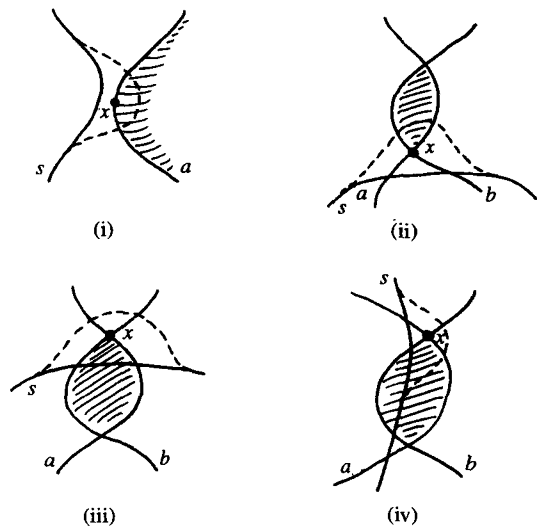

Fig. 3.1. The four possible types of sweeping steps.

with another pseudocircle. Simultaneously, the complex of B' is maintained using the operations of deleting, attaching, and shelling, so that the Euler characteristic of regional subcomplexes is preserved. A simple property used repeatedly in the proof of the lemma is the following:

(A1) Let $\mathcal{A}$ be an abstract simplicial complex and let $\mathcal{K}$ be an embedding in $\mathbb{R}^2$. $\mathcal{A}$ is connected and $\chi(\mathcal{A}) = 0$ iff $\bigcup \mathcal{K}$ is simply connected, that is, $\bigcup \mathcal{K}$ is connected and so is $\mathbb{R}^2 - \bigcup \mathcal{K}$.

Sweeping Pseudodisks. Let x be a point in d outside $\bigcup B'$. The intention is to grow a tiny pseudodisk s around x to the final shape, d, in a continuous manner so that $B_s = B' \cup \{s\}$ is always a pseudodisk collection, except at a finite number of moments. At these exceptional moments, or *sweeping steps*, s commits degeneracies of two possible types:

(1) there is an $a \in B'$ so that $\mathrm{bd}\, s$ and $\mathrm{bd}\, a$ meet in a single point, or
(2) there are $a, b \in B$ so that $\mathrm{bd}\, s$ passes through a point of $\mathrm{bd}\, a \cap \mathrm{bd}\, b$.

It is shown in [15] that such a sweep is always possible. The sweeping steps of type (2) can be further classified according to how s, a, and b intersect before and after the step. Because of the nonredundancy assumption and the start of the sweep outside $\bigcup B'$, a total of only four types (i–iv) result as illustrated in Fig. 3.1. In particular, the following cases cannot happen: type (1) in which one pseudodisk is contained in the other ($a \subseteq s$ or $s \subseteq a$), and type (2) in which one is contained in the union of the other two ($s \subseteq a \cup b$ or $a \subseteq s \cup b$ or $b \subseteq s \cup a$). The simplifications allow an argument that sweeping is always possible which is significantly shorter than that in [15]. We present it for completeness.

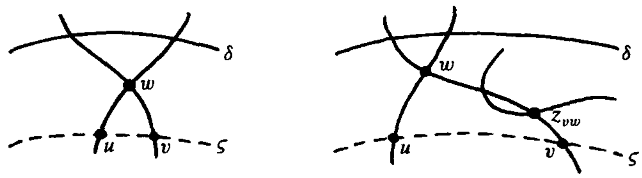

Fig. 3.2. Triangle, half-triangle, leg, and bump.

Sweeping Is Always Possible. We consider the arrangement formed by the boundaries of a nonredundant collection of pseudodisks. The current and final configurations of the sweeping curve are $\varsigma = \mathrm{bd}\, s$ and $\delta = \mathrm{bd}\, d$, with ς contained in d. The *sweep region* is the region between ς and δ. The arrangement defines vertices, edges, and chambers in the usual way. Let u, v, w be vertices of the arrangement and α and β curves so $u \in \varsigma \cap \alpha$, $v = \varsigma \cap \beta$, and $w = \alpha \cap \beta$ lies in the sweep region. uvw is a *triangle* if uv, uw, vw are edges in the arrangement, see Fig. 3.2. uvw is a *half-triangle* if uw is an edge but vw is not; vw is called the *leg* of the half-triangle. Let z_{vw} denote the vertex on the leg vw closest to ς. A half-triangle uvw has a *bump* if the curve that intersects its leg vw at z_{vw} intersects the leg twice, see Fig. 3.2.

We show that ς can make progress, that is, it can proceed to δ directly or a step of type (1) or (2) is possible. For the sake of contradiction, suppose this is not the case, and let A be a configuration with a minimum number of curves in which the sweep cannot make progress. By nonredundancy and minimality, all curves in A intersect both ς and δ. There are vertices in the sweep region, for otherwise ς could advance to δ without any sweep step. A has no triangle, for otherwise a step of type (2) would be possible. It follows A has half-triangles.

We prove by induction over the number of curves intersecting a leg that every half-triangle in A has a bump. If only one curve intersects the leg it must form a bump, or else there is a triangle in A. This forms the basis of the induction. Let uv_0w_0 be a half-triangle with leg v_0w_0. Suppose the curve γ_1 that intersects v_0w_0 at $w_1 = z_{v_0w_0}$ is not a bump, that is, γ_1 also intersects the segment uv_0 of ς, say at vertex v_1. Now, $v_0v_1w_1$ is a half-triangle, and by the induction hypothesis it has a bump, say created by γ_2, which intersects ς at v_2. $v_1v_2w_2$ with $w_2 = z_{v_1w_1}$ is again a half-triangle so it has a bump, say created by γ_3. A continuation of this argument creates an infinite sequence of half-triangles with bumps all of which are different, see Fig. 3.3. This contradiction implies the inductive step, so every half-triangle has a bump.

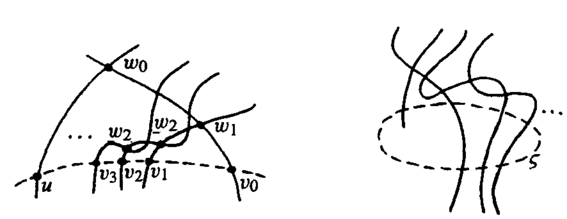

Fig. 3.3. Sequences of half-triangles and bumps.

Finally, using the fact just proved, we again find an infinite sequence of half-triangles and bumps starting with any half-triangle. Cycling around ς would require pairs of curves intersecting at more than two points, so all half-triangles are different, see Fig. 3.3. This contradicts the existence of a configuration in which the sweep cannot make progress.

Deleting and Collapsing.. We need additional definitions to describe the maintenance of $\mathcal{X}$ during the sweep. Let τ be a simplex in an abstract simplicial complex $\mathcal{A}$. The subcomplex $\mathcal{A}' = \mathcal{A} - \mathcal{A}|_\tau$ is said to be obtained from $\mathcal{A}$ by *deleting* τ. If $\mathcal{A}|_\tau$ has a unique maximal simplex $\varrho \neq \tau$, then the deletion is a *collapse*. The inverse operation of a collapse is an *anticollapse*, and the inverse of a deletion is an *attachment*. A collapse (or anticollapse) does not change the Euler characteristic, that is, $\chi(\mathcal{A}') = \chi(\mathcal{A})$. This is because $\mathcal{A}|_\tau = \mathcal{A}|_\tau^\varrho$ is isomorphic to a simplex of dimension $\dim\varrho - \dim\tau - 1 \geq 0$ and hence $\chi(\mathcal{A}|_\tau^\varrho) = 0$. A deletion (or attachment) preserves the Euler characteristic iff $\chi(\mathcal{A}|_\tau) = 0$, which is, for example, the case if it can be decomposed into a sequence of collapses (or anticollapses).

Shelling. Suppose $\mathcal{X} = \mathcal{X}(B_s)$ is an abstract simplicial complex that satisfies the properties stated in the lemma. (P2) expresses a certain shelling property of $\mathcal{X}|^\varrho$. We show it is implied by (P1) and (P5). The inductive proof of the lemma can therefore omit any further argument for (P2).

To argue the implication and describe the shelling property, let $\varrho \subseteq B_s$ be regional and $d \in \varrho$ so $\varrho' = \varrho - \{d\}$ is also regional. Then $\mathcal{X}|^{\varrho'}$ is obtained from $\mathcal{X}|^\varrho$ by deleting d, that is,

$$\mathcal{X}|^{\varrho'} = \mathcal{X}|^\varrho - \mathcal{X}|_d = \mathcal{X}|^\varrho - \mathcal{X}|_d^\varrho.$$

The operation preserves the Euler characteristic of the induced subcomplex because $\chi(\mathcal{X}|^\varrho) = \chi(\mathcal{X}|^{\varrho'}) = 0$ by (P1). $\mathcal{X}|^{\varrho'}$ and $\mathcal{X}|_d^\varrho$ are both subsets of $\mathcal{X}|^\varrho$. It follows that the Euler characteristic of $\mathcal{X}|_d^\varrho$ is

$$\chi(\mathcal{X}|_d^\varrho) = \chi(\mathcal{X}|^\varrho) - \chi(\mathcal{X}|^{\varrho'}) = 0.$$

To see that $\mathcal{X}|_d^\varrho$ is connected consider the link of d in $\mathcal{X}|^\varrho$. Its Euler characteristic is the negative of the Euler characteristic of the star and therefore also 0. Furthermore, $\mathcal{X}|^\varrho$ and $\mathcal{X}|^{\varrho'}$ are both connected by (P1). By (P5) they can be embedded in $\mathbb{R}^2$, and by (A1) their embeddings are simply connected. It follows $d \in \mathrm{Fr}(\mathcal{X}|^\varrho)$, and its link in $\mathcal{X}|^\varrho$ is a subcomplex of $\mathcal{X}|^{\varrho'}$ with Euler characteristic 0. The link of d is therefore connected, or else $\mathcal{X}|^\varrho$ is not simply connected. This finally implies the star, $\mathcal{X}|_d^\varrho$, is also connected, see Fig. 3.4.

Deformation Retraction. A brief discussion of homotopy equivalence is needed to prepare for the inductive maintenance of property (P5). The subsets $\mathbb{X} = \bigcup\mathcal{K}$ and $\mathbb{Y} = \mathrm{int}\bigcup B$ of $\mathbb{R}^2$ are *homotopy equivalent* if there are continuous maps $f: \mathbb{X} \to \mathbb{Y}$ and $g: \mathbb{Y} \to \mathbb{X}$ so $g \circ f$ is homotopic to the identity on $\mathbb{X}$ and $f \circ g$ is homotopic to the identity on $\mathbb{Y}$, see, e.g., Chapter 2 of [13]. To prove homotopy equivalence it suffices to

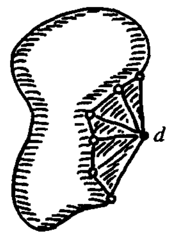

Fig. 3.4. d is attached to $\mathcal{X}|^{\varrho'}$ to form $\mathcal{X}|^{\varrho}$.

construct a *deformation retraction*, that is a continuous map $r: \mathbb{Y} \times [0, 1] \to \mathbb{Y}$ with

$$\begin{array}{rclll} r(y,0) & = & y & \text{for all } y \in \mathbb{Y}, & \\ r(y,1) & \in & \mathbb{X} & \text{for all } y \in \mathbb{Y}, & \text{and} \\ r(y,t) & = & y & \text{for all } y \in \mathbb{X} & \text{and } t \in [0,1]. \end{array}$$

Such an r exists only if $\mathbb{X} \subseteq \mathbb{Y}$, that is, every simplex $\sigma \in \mathcal{X}$ is embedded in the interior of the region covered by the pseudodisks. That this is the case will be maintained inductively. The deformation retraction is constructed by decomposing $\mathbb{Y} - \mathbb{X}$ as follows. Let $x = \text{bd}\, a \cap \text{bd}\, b$ be a point on $\text{bd}\, \mathbb{Y}$. By (P6), $\eta(a)$ and $\eta(b)$ are points on $\text{bd}\, \mathbb{X}$. Find a simple closed curve connecting x with $\eta(a)$ inside $\mathbb{Y} - \mathbb{X}$, and find another one connecting x and $\eta(b)$, see Fig. 3.5. The existence of pairwise noncrossing curves is maintained inductively.

Each region in the decomposition is homeomorphic to a half-open square. The (opposite) edges that belong to the square map to two of the curves. The other two edges do not belong to the square and can be mapped by limit considerations. We define

$$q: (0,1) \times [0,1] \to \mathbb{R}^2$$

so $0 \times [0, 1]$ maps to a vertex $\eta(a)$ or edge $\eta(ab)$ contained in the boundary of $\mathbb{X}$, and $1 \times [0, 1]$ maps to a point or an edge (a component of a boundary contribution) in $\text{bd}\, \mathbb{Y}$. The maps q can be chosen so they agree at their overlap, which are the curves decomposing $\mathbb{Y} - \mathbb{X}$.

There is one exception to this rule, namely when a pseudodisk, a, is disjoint from all

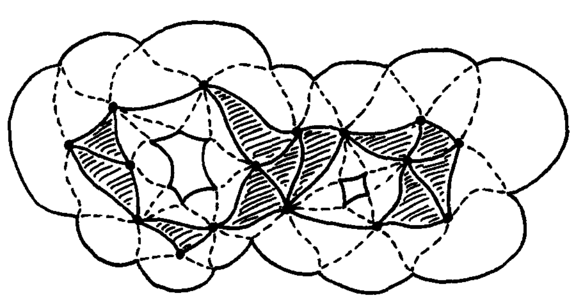

Fig. 3.5. Decomposition of $\bigcup B$ outside $\bigcup \mathcal{K}$.

other pseudodisks. In this case $\operatorname{int} a - \eta(a)$ is an open annulus and forms a region by itself. The corresponding map q glues the square at $q(u, 0) = q(u, 1)$.

The restriction of a map q to an open interval $(0, 1) \times v$, $v \in [0, 1]$,

$$f: (0, 1) \to \mathbb{R}^2,$$

is called a *fiber*. The above construction amounts to writing $\mathbb{Y} - \mathbb{X}$ as the disjoint union of fibers. The deformation retraction is defined by "moving" points along fibers toward $\mathbb{X}$. Specifically, let $y \in \mathbb{Y} - \mathbb{X}$, f the fiber covering y, and $u \in (0, 1)$ so $y = f(u)$. Then

$$r(y, t) = f((1 - t)u).$$

Inside $\mathbb{X}$, $r(y, t) = y$ for all $t \in [0, 1]$. r is continuous by construction. Furthermore, for $t = 0$ we have $r(y, 0) = f(u) = y$, and for $t = 1$ we have $r(y, 1) = f(0) \in \mathbb{X}$, as required.

4. Proof

As explained above, the proof is by induction and a pseudodisk is added by sweeping out its area. At the start of the continuous sweep, it is sufficient to add an isolated vertex, s, to the complex $\mathcal{X}(B')$ to obtain $\mathcal{X} = \mathcal{X}(B_s)$. During the sweep, no change in $\mathcal{X}$ is needed as long as the combinatorial structure of the arrangement of pseudocircles is preserved. Except for (P5) all properties of the lemma are trivially maintained. The modification of the deformation retraction is not difficult and details are omitted. Thus, we only need to verify that $\mathcal{X}$ can be updated appropriately at the sweeping steps so as to preserve (P1)–(P6). For each type of sweeping step we further distinguish whether or not the step changes the combinatorial structure of $\operatorname{bd} \bigcup B_s$. For a sweeping step, the *focus* is the point $x \in \mathbb{R}^2$ where the degeneracy occurs, and the *background* is the set of pseudodisks $\beta \subseteq B'$ that contain x in their interior. The corresponding subcomplex is denoted by $\mathcal{B} = \mathcal{X}|^{\beta}$. A sweeping step is a *boundary step* if $\beta = \emptyset$.

The homotopy equivalence of (P5) and the isomorphism of (P6) need to be verified only at boundary steps. For nonboundary steps it turns out that $\mathcal{X}$ needs to be changed only for steps of type (iii). A sweeping step creates a unique new chamber in the arrangement of pseudocircles, and we denote by $\nu \subseteq B_s$ the collection of pseudodisks that cover this chamber; $\operatorname{reg}_{B_s} \nu$ contains this new chamber. For steps of type (i), (ii), and (iv), ν has not been regional before the step and becomes regional after the step. For steps of type (iii), ν may or may not be regional before the step, and it is certainly regional after the step.

Type (i), *Boundary Case*. The growth of s is illustrated in Fig. 3.1(i). The only change in $\mathcal{X}$ is the addition of the edge $\{s, a\}$, as indicated in Fig. 4.1. (P1) holds for $\nu = \{s, a\}$ because $\mathcal{X}|^{\nu}$ contains only the new edge together with the two vertices and the empty set. All other regional subcomplexes are unchanged, so (P1) holds in general. Property (P3) is clear, and (P4) is untouched.

For (P5) we use the existence of the deformation retraction for $\operatorname{int} \bigcup B_s$ when s and a touch at point x. x has two fibers, one in s and one in a, and the union of the two, connected at x, embeds $\{s, a\}$. The change in the deformation retraction is indicated in

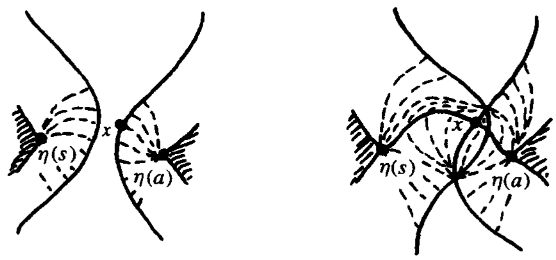

Fig. 4.1. The boundary case of a type (i) sweeping step requires the addition of an edge reflecting the overlap between s and a.

Fig. 4.1. Finally, the only new set in Nrv $\bar{B}_s$ is $\{\bar{s}, \bar{a}\}$ and the only change in Fr $\mathcal{X}$ is the addition of $\{s, a\}$, so (P6) also holds.

Type (i), *Nonboundary Case.* The growth of s is again illustrated in Fig. 3.1(i), but now the background is nonempty. No change in $\mathcal{X}$ is required, so (P1) holds for all regional subcomplexes, except possibly for $\nu = \beta \cup \{s, a\}$. To verify (P1) for ν, note that β, $\beta \cup \{s\}$, $\beta \cup \{a\}$ are all regional, both before and after the sweeping step. So $\mathcal{B}$, $\mathcal{B}_s = \mathcal{X}|^{\beta\cup\{s\}}$, and $\mathcal{B}_a = \mathcal{X}|^{\beta\cup\{a\}}$ all satisfy (P1). Furthermore, $\{s, a\} \notin \mathcal{X}$ because this would violate (P3) before the step. Hence, $\mathcal{X}|^\nu = \mathcal{B}_s \cup \mathcal{B}_a$ and $\mathcal{X}|^\nu$ is connected because $\mathcal{B}$, $\mathcal{B}_s$, and $\mathcal{B}_a$ are connected. Observe that each simplex in $\mathcal{X}|^\nu$ either belongs to all three subcomplexes, or it belongs to only one. Hence,

$$\chi(\mathcal{X}|^\nu) = \chi(\mathcal{B}_s) + \chi(\mathcal{B}_a) - \chi(\mathcal{B}) = 0,$$

and (P1) holds in general. (P3) and (P4) hold because $\{s, a\} \notin \mathcal{X}$, and (P5) and (P6) hold because $\beta \neq \emptyset$.

Type (ii), *Boundary Case.* The growth of s is illustrated in Fig. 3.1(ii). The disappearing chamber in the pseudocircle arrangement is not covered by any pseudodisk, so $\{s, a\}$, $\{a, b\}$, $\{b, s\} \in \mathcal{X}$ before the sweeping step because of (P6). It is thus possible to add $\{s, a, b\}$ to $\mathcal{X}$, and this is the only change necessary, see Fig. 4.2. The subcomplex induced by $\nu = \{s, a, b\}$ consists of this triangle and its faces, so $\chi(\mathcal{X}|^\nu) = 0$. No other regional subcomplex changes, so (P1) holds in general. (P3) holds because $\{s, a, b\}$ is independent and the step does not affect the independence of any other simplex. (P4) holds because the point of bd $s \cap$ bd a contained in b is not covered by any other pseudodisk, and ditto for bd $a \cap$ bd b and s and for bd $b \cap$ bd s and a.

The embedding of $\mathcal{X}$ before the sweeping step has a hole bounded by $\eta(sa)$, $\eta(ab)$, and $\eta(bs)$. This hole together with its boundary is a homeomorph of a geometric triangle and can be used as $\eta(sab)$ embedding $\{s, a, b\}$. The deformation retraction becomes the identity inside the former hole and remains unchanged, otherwise. Finally, (P6) still holds because $\bar{B}_s$ and Fr $\mathcal{X}$ change in unison.

Type (ii), *Nonboundary Case.* The complex $\mathcal{X}$ remains unchanged. Since the triangle $\{s, a, b\} \notin \mathcal{X}$, the validity of (P4) is not affected. Similarly, $\{s, a\}$, $\{a, b\}$, $\{b, s\}$ all

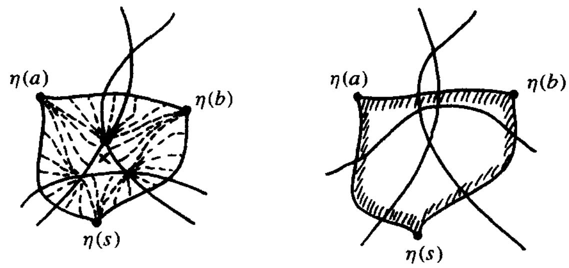

Fig. 4.2. The boundary case of a type (ii) sweeping step requires the addition of a triangle reflecting the overlap between s, a, and b.

remain independent, so (P3) still holds in case $\mathcal{X}$ contains one or more of these edges. (P1) needs to be verified only for $\nu = \beta \cup \{s, a, b\}$. For all proper faces $\tau \subset \{s, a, b\}$, $\mathcal{B}_\tau = \mathcal{X}|^{\beta \cup \tau}$ is connected with Euler characteristic 0. Therefore, $\mathcal{X}|^\nu$ is connected. To see that the Euler characteristic vanishes consider

$$X = \sum_{\tau \subseteq \{s,a,b\}} (-1)^{\dim \tau} \chi(\mathcal{B}_\tau).$$

Each simplex $\sigma \in \mathcal{X}|^\nu$ also belongs to one or more of the $\mathcal{B}_\tau$, $\tau \subset \{s, a, b\}$, and we have

$$\begin{aligned} X &= \sum_{\tau \subseteq \{s,a,b\}} \sum_{\sigma \in \mathcal{B}_\tau} (-1)^{\dim \tau} (-1)^{\dim \sigma} \\ &= \sum_{\sigma \in \mathcal{X}|^\nu} (-1)^{\dim \sigma} \sum_{\tau \subseteq \nu - \sigma} (-1)^{\dim \tau} \\ &= 0 \end{aligned}$$

because all $\nu - \sigma$ are nonempty. $\chi(\mathcal{B}_\tau) = 0$ for all proper subsets $\tau \subset \{s, a. b\}$, hence $\chi(\mathcal{X}|^\nu) = 0$. So (P1) holds in general. Finally, (P5) and (P6) are unaffected because $\beta \neq \emptyset$.

Type (iii), *Boundary Case.* The growth of s is illustrated in Fig. 3.1 (iii). Because s starts as x outside $\bigcup B'$, $s \not\subseteq a \cup b$; so $a \cap b \subseteq s$ after the sweeping step. By (P6), $\{a, b\}$ is in the frontier of $\mathcal{X}$ and thus also in $\mathcal{X}$. Furthermore, $\{s, a, b\} \in \mathcal{X}$. This is because every $\tau \subseteq \{s, a, b\}$ is regional before the sweeping step. Indeed, a violation of (P1) can be avoided only if all $\tau \subseteq \{s, a, b\}$ belong to $\mathcal{X}$. To reflect the change brought about by the sweeping step, we collapse $\{a, b\}$, which amounts to removing $\{a, b\}$ and $\{s, a, b\}$ from $\mathcal{X}$, see Fig. 4.3. There is no new regional subset. Because $a \cap b \subseteq s$ after the sweeping step, each regional $\varrho \subseteq B_s$ that contains a and b also contains s. So for each regional subcomplex, the change is either a collapse or does not affect it at all. It follows that (P1) holds after the step. (P3) would be violated only by $\{s, a, b\}$, which got removed. (P4) clearly remains valid.

To maintain the deformation retraction for (P5), we choose a fiber from x to an arbitrary interior point of $\eta(ab)$ and extend it to $\eta(s)$. The subsequent changes in the deformation retraction are easy and are illustrated in Fig. 4.3. To verify that (P6) holds after the sweeping step, note that $\{a, b\}$ is removed from $\operatorname{Fr} \mathcal{X}$ and $\{s, a\}$, $\{s, b\}$, $\{s\}$ are

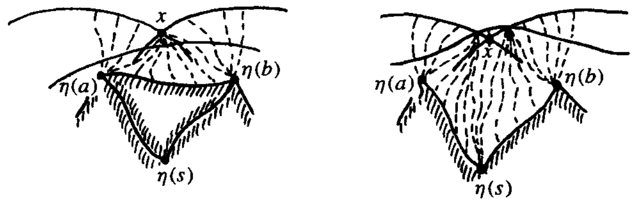

Fig. 4.3. The boundary case of a type (iii) sweeping step requires the removal of an edge and a triangle. The deformation retraction is maintained by extending the fibers from x through $\eta(sab)$ to $\eta(sa)$, $\eta(sb)$, and $\eta(s)$.

added, unless they already belong to the frontier. Similarly, $\bar{a} \cap \bar{b} = \emptyset$ after the sweeping step, and $\bar{s} \cap \bar{a}$, $\bar{s} \cap \bar{b}$, and $\bar{s}$ either become or remain non-empty. So (P6) is still valid.

Type (iii), *Nonboundary Case.* The growth of s is again shown in Fig. 3.1(iii), but now $\beta \neq \emptyset$. Consider first the case $\{s, a, b\} \in \mathcal{X}$. One point of $\operatorname{bd} a \cap \operatorname{bd} b$ is contained in s and the other is contained in each pseudodisk in β. It follows that $\bar{a} \cap \bar{b} = \emptyset$, so there are two triangles in $\mathcal{X}$ that contain $\{a, b\}$. One is $\{s, a, b\}$ and the other is $\{c, a, b\}$, with $c \in \beta$, see Fig. 4.4. Note that a, b, c, s intersect as shown in Fig. 4.5. Indeed, a contains one point of $\operatorname{bd} c \cap \operatorname{bd} s$ and b contains the other. Anything else would contradict the nonredundancy assumption. There cannot be any edge connecting s with $\mathcal{B} = \mathcal{X}|^{\beta}$. This is because $\beta \cup \{s, a\}$ and $\beta \cup \{s, b\}$ are both regional before the sweeping step. If s is connected to $\mathcal{B}$, then at least one of the two subcomplexes induced by $\beta \cup \{s, a\}$ and $\beta \cup \{s, b\}$ is not simply connected. With (A1), this would imply a violation of (P1).

After the sweeping step, (P1) fails for $\mathcal{X}|^{\nu}$, $\nu = \beta \cup \{s\}$, and (P3) fails for $\{s, a, b\}$. We claim that flipping $\{a, b\}$, that is, removing $\{a, b\}$, $\{s, a, b\}$, $\{c, a, b\}$ from $\mathcal{X}$ and adding $\{c, s\}$, $\{a, c, s\}$, $\{b, c, s\}$, fixes the situation. It fixes (P1) because the flip can be performed by attaching $\{c, s\}$ through the 3-simplex $\{a, b, c, s\}$ and then deleting $\{a, b\}$. For all regional subcomplexes, except possibly the new one, $\mathcal{X}|^{\nu}$, the corresponding operations are either void, an anticollapse, or a collapse. To see this, let $\mathcal{X}|^{\varrho}$ be such a subcomplex, and define $\tau = \varrho \cap \{a, b, c, s\}$. $\tau = \{c, s\}$ does not have to be considered because such a ϱ is regional only if $\varrho = \beta \cup \{s\} = \nu$. $\tau = \{a, b\}$ is not possible because $a \cap b \subseteq c \cup s$ after the sweeping step. For all other τ with $\operatorname{card} \tau \leq 2$ the flip has no effect. For $\operatorname{card} \tau = 3$ one operation of the flip has no effect and the other is a collapse

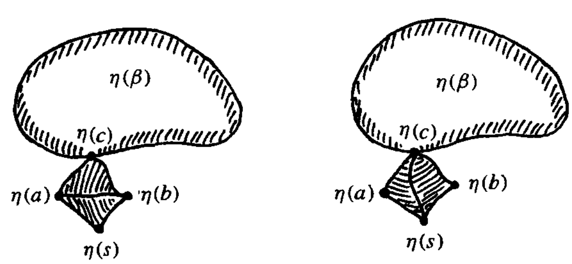

Fig. 4.4. The nonboundary case of a type (iii) sweeping step requires flipping $\{a, b\}$ if $\{s, a, b\}$ is present.

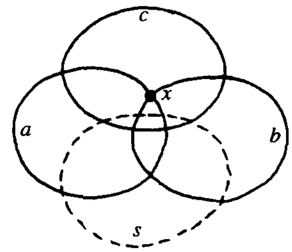

Fig. 4.5. The configuration formed by a, b, c, s immediately before bd s sweeps through the intersection point of bd a and bd b outside s.

or an anticollapse. For $\tau = \{a, b, c, s\}$ the operations are an anticollapse followed by a collapse. In any case, the Euler characteristic of $\mathcal{X}|^{\varrho}$ remains 0. For $\nu = \beta \cup \{s\}$, the effect of the flip is that s is now connected to $\mathcal{B}$ by a single edge, which is the result of an anticollapse and hence $\chi(\mathcal{X}|^{\nu}) = 0$. We conclude that (P1) holds in general. (P3) holds for the new simplices, $\{c, s\}$, $\{a, c, s\}$, $\{b, c, s\}$, see Fig. 4.5. (P4) holds for the new edge, $\{c, s\}$, and also for $\{a, c\}$, $\{c, b\}$, $\{b, s\}$, $\{s, a\}$, which are affected by the flip, see Fig. 4.5. (P5) and (P6) remain valid because $\beta \neq \emptyset$.

In the other cases, when $\{s, a, b\} \notin \mathcal{X}$, no change in the complex is necessary. We need to verify (P1) only for $\nu = \beta \cup \{s\}$. Note that $\beta \cup \tau$ is regional before the sweeping step for each $\tau \subseteq \{s, a, b\}$, except for $\tau = \{s\}$. By (P1), the corresponding subcomplex, $\mathcal{B}_\tau = \mathcal{X}|^{\beta\cup\tau}$, is connected with Euler characteristic 0. We claim s is connected by at least one edge to $\mathcal{B} = \mathcal{X}|^{\beta}$. Assume it is not. Then $\{s, a\} \in \mathcal{X}$ because $\mathcal{B}_{sa}$ is connected, and similarly $\{s, b\} \in \mathcal{X}$ because $\mathcal{B}_{sb}$ is connected. However, now s is connected within $\mathcal{B}_{sab}$ to $\mathcal{B}$ via two paths, one containing a and the other b. The two paths leave a hole in the embedding of $\mathcal{B}_{sab}$ because $\{s, a, b\} \notin \mathcal{X}$. Together with (A1) this contradicts (P1). The embedding of $\mathcal{X}|^{\nu}$ is thus connected, and it is simply connected because otherwise there is a hole already in the embeddings of $\mathcal{B}$, $\mathcal{B}_{sa}$, or $\mathcal{B}_{sb}$. (A1) now implies that (P1) holds for ν and therefore in general. (P3) and (P4) hold because $\{s, a, b\} \notin \mathcal{X}$, and (P5) and (P6) remain valid because $\beta \neq \emptyset$.

Type (iv), *Boundary Case.* The growth of s in this case is illustrated in Fig. 3.1(iv). We have $\beta = \emptyset$, so by (P6), $\{s, a\}$ and $\{a, b\}$ are both in Fr $\mathcal{X}$ and thus also in $\mathcal{X}$. The change caused by the sweeping step requires that $\{s, b\}$ and $\{s, a, b\}$ are added to $\mathcal{X}$,

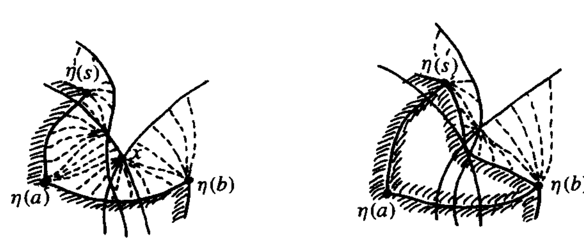

Fig. 4.6. The boundary case of a type (iv) sweeping step requires the addition of an edge and a triangle.

see Fig. 4.6. The new regional subset of B_s is $\nu = \{s, b\}$, and (P1) clearly holds for ν. All other regional $\varrho \subseteq B_s$ either do not contain $\{s, b\}$ or they also contain a, see Fig. 3.1. In the former case, $\mathcal{X}|^{\varrho}$ is not affected, and in the latter case the change is an anticollapse in $\mathcal{X}|^{\varrho}$. In both cases (P1) remains valid. (P3) still holds because $\{s, a, b\}$ is independent. (P4) holds for $\{a, b\}$ because s contains the previously uncovered point x, and for symmetric reasons (P4) holds for $\{s, a\}$ and for $\{s, b\}$.

To show that (P5) is still valid, we embed $\{s, b\}$ using the two fibers from x to $\eta(s)$ and to $\eta(b)$ and connect them at x to get $\eta(sb)$. The hole created by adding $\eta(sb)$ is bounded by $\eta(sb)$, $\eta(ba)$, and $\eta(as)$. Together with its boundary it is homeomorphic to a triangle and can thus serve as $\eta(sab)$. The deformation retraction is changed in a manner inverse to the change in type (iii), see Fig. 4.3. Finally, observe that $\mathrm{Fr}\,\mathcal{X}$ and $\mathrm{Nrv}\,\bar{B}_s$ change in the same way so (P6) follows.

Type (iv), *Nonboundary Case.* See Fig. 3.1(iv) for an illustration of the growth of s in this case. No change in $\mathcal{X}$ is required. The new regional subset of B_s is $\nu = \beta \cup \{s, b\}$. Note that $\beta \cup \tau$ is regional before the sweeping step for every $\tau \subseteq \{s, a, b\}$, except for $\tau = \{s, b\}$. By (P1), the subcomplexes induced by β, $\beta \cup \{s\}$, and $\beta \cup \{b\}$ are connected, which implies $\mathcal{X}|^{\nu}$ is connected. Furthermore, $\chi(\mathcal{B}_\tau) = 0$ for each $\{s, b\} \neq \tau \subseteq \{s, a, b\}$. Observe that $\{s, a, b\} \notin \mathcal{X}$ because it is not independent before the sweeping step. This implies

$$\sum_{\tau \subseteq \{s,a,b\}} (-1)^{\dim \tau} \chi(\mathcal{B}_\tau) = 0,$$

by the same argument as in the discussion of the type (ii), no-boundary case. It follow that $\chi(\mathcal{X}|^{\nu}) = \chi(\mathcal{B}_{sb}) = 0$. Hence, (P1) holds for ν and thus in general. All simplices in $\mathcal{X}$ are still independent, so (P3) still holds. Similarly, (P4) still holds, and (P5) and (P6) are unaffected by the change because $\beta \neq \emptyset$.

5. Concluding Remarks

This paper proves that every pseudodisk collection, B, in $\mathbb{R}^2$ has a two-dimensional abstract simplicial complex, $\mathcal{X} \subseteq 2^B$, that can be used as the index set for inclusion–exclusion formulas measuring the union of the pseudodisks and its boundary. Each term corresponds to an abstract simplex, $\sigma \subseteq B$, and measures the common intersection of the at most three pseudodisks in σ. The proof directly translates into an algorithm for constructing the complex $\mathcal{X}$. Its running time is dominated by the sweep and is linear with the number of pairs (v, a) where v is a vertex of the arrangement and $v \in a \in B$; this is $O(n^3)$ where $n = \mathrm{card}\, B$, but note that the input to the algorithm includes the arrangement whose size can be $\Theta(n^2)$.

Similar results have been established for geometric spherical balls in $\mathbb{R}^d$ [5], [14]. The geometric proofs do not seem to generalize to pseudoball collections, and the extension to $\mathbb{R}^3$ of the sweep proof in this paper hits obstacles in the form of nonsweepable arrangements. It would be interesting to decide whether the *results* extend to pseudoball collections in three and higher dimensions, or whether there are counterexamples.

More generally, it would be desirable to develop the theory started in [5] and [14] further and to study sets other than balls. The size of the largest independent subcollection determines the dimension of the complex $\mathcal{X}$ that can be expected. Let B be a finite collection of sets and let k be an upper bound on the size of every independent subcollection. Then it is fairly easy to prove the union can be measured in terms of intersections of at most k sets at a time:

$$\mu\Big(\bigcup B\Big) = \sum_{\sigma \subseteq B, \operatorname{card} \sigma \leq k} a_\sigma \cdot \mu\Big(\bigcap \sigma\Big), \tag{3}$$

where the a_σ are integers. An argument generalizing ideas of Kratky [12] is offered below. The interesting question is for which types of sets the a_σ can be chosen in $\{-1, 0, +1\}$ and for which there is an abstract simplicial complex like $\mathcal{X}$.

To verify (3) let $\sigma \subseteq B$ with $n = \operatorname{card} \sigma \geq k+1$. For each $\tau \subseteq \sigma$, $\bigcap \tau$ is the disjoint union of regions $\operatorname{reg}_\sigma(\tau \cup \upsilon)$, with $\upsilon \subseteq \sigma - \tau$. Since σ is not independent, there is a maximal τ with $\operatorname{reg}_\sigma \tau = \emptyset$. By choice of τ, $\operatorname{reg}_\sigma(\tau \cup \upsilon) \neq \emptyset$ for all nonempty υ. Using inclusion–exclusion we get

$$\sum_{\upsilon \subseteq \sigma - \tau} (-1)^{\operatorname{card} \upsilon} \mu\Big(\bigcap (\tau \cup \upsilon)\Big) = 0.$$

It follows that $\mu(\bigcap \sigma)$ can be written as a $-1, 0, +1$ combination of measures $\mu(\bigcap (\tau \cup \upsilon))$ for strict subsets $\tau \cup \upsilon \subset \sigma$. This argument applies as long as $n \geq k+1$. Starting with the straightforward inclusion–exclusion formula (1), all terms for subsets of size $k+1$ or larger can be replaced by integer combinations of terms for smaller subsets.

The number 3 for pseudodisks suggests a relation to the classical Helly theorem. There is certainly a connection but it may not be as direct as would at first be thought. The geometric version of Helly's theorem [10] deals with arbitrary convex sets, and for such collections short inclusion–exclusion formulas cannot exist. The more direct connection to Delaunay triangulations [4] and to Euler's relation for convex polytopes is apparent from the work on geometric disks and balls. Furthermore, the significance of independent subcollections suggests a connection to the theory of VC dimension [9], [16]. It is also interesting to note the complex $\mathcal{X}$ is another example of a meaningful association of a space specified through a covering with an abstract simplicial complex. The prime examples of such associations are possibly the nerve [1] and the order complex [2] reflecting overlap and enclosure among the covering sets.

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