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The Markov chain associated to a Pick function

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Abstract. To each function $\widetilde{\varphi}(\omega)$ mapping the upper complex half plane $\mathbb{H}^+$ into itself such that the coefficient of ω in the Nevanlinna integral representation is one, we associate the kernel $p(y, dx)$ of a Markov chain on $\mathbb{R}$ by

$$[\widetilde{\varphi}(\omega) - y]^{-1} = \int_{-\infty}^{\infty} (\omega - x)^{-1}\, p(y, dx).$$

The aim of this paper is to study this chain in terms of the measure μ appearing in the Nevanlinna representation of $\widetilde{\varphi}(\omega)$. We prove in particular three results. If x^2 is integrable by μ, a law of large numbers is available. If μ is singular, *i.e.* if $\widetilde{\varphi}$ is an inner function, then the operator P on $L^\infty(\mathbb{R})$ for the Lebesgue measure is the adjoint of T defined on $L^1(\mathbb{R})$ by $T(f)(\omega) = f(\varphi(\omega))$, where φ is the restriction of $\widetilde{\varphi}$ to $\mathbb{R}$. Finally, if μ is both singular and with compact support, we give a necessary and sufficient condition for recurrence of the chain.

1. Introduction

Denote by $\mathbb{H}^+$ (resp. $\mathbb{H}^-$) the set of complex numbers $\omega = a + ib$ such that $b > 0$ (resp. $b < 0$). An analytic map $\widetilde{\varphi}$ from $\mathbb{H}^+$ to itself is called a *Pick function* and is described by three parameters : a non-negative number k, a real number α and a positive bounded measure μ on the real line $\mathbb{R}$ such that :

$$\widetilde{\varphi}(\omega) = k\omega + \alpha - \int_{-\infty}^{+\infty} \frac{\omega x + 1}{\omega - x} \mu(dx) \tag{1}$$

for any ω in $\mathbb{H}^+$: see Donoghüe (1974). This is called the Nevanlinna representation and is unique. Sometimes we denote by $(k(\widetilde{\varphi}), \alpha(\widetilde{\varphi}), \mu_{\widetilde{\varphi}})$ the unique triple appearing in the representation of $\widetilde{\varphi}$. The set of Pick functions $\widetilde{\varphi}$ such that $k = k(\widetilde{\varphi})$ is denoted by $\mathscr{P}_k$.

It is a well known fact that $\widetilde{\varphi}$ in $\mathscr{P}_1$ if and only if there exists a probability measure $p(dx)$ on $\mathbb{R}$ such that

$$[\widetilde{\varphi}(\omega)]^{-1} = \int_{-\infty}^{\infty} (\omega - x)^{-1}\, p(dx).$$

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(see Lemma 2.2 page 24 in Shohat and Tamarkin (1963)). Now take a constant y in $\mathbb{R}$, and do the same with $\widetilde{\varphi}(\omega) - y$. The probability p depends on y, and we get

$$[\widetilde{\varphi}(\omega) - y]^{-1} = \int_{-\infty}^{\infty} (\omega - x)^{-1}\, p(y, dx). \tag{2}$$

The kernel $p(y, dx)$ is the transition of a Markov chain on $\mathbb{R}$.

Denote for a while $p(y, dx)$ by $p_{\widetilde{\varphi}}(y, dx)$. Such a kernel is closely related to the iteration and to the composition of functions : take $\widetilde{\varphi}$ and $\widetilde{\varphi}_1$ in the class $\mathscr{P}_1$. Then for ω in $\mathbb{H}^+$ and y in $\mathbb{R}$

$$\begin{aligned}[\widetilde{\varphi} \circ \widetilde{\varphi}_1(\omega) - y]^{-1} &= \int_{-\infty}^{\infty} (\omega - x)^{-1}\, p_{\widetilde{\varphi}\circ\widetilde{\varphi}_1}(y, dx) \\ &= \int_{-\infty}^{\infty} (\widetilde{\varphi}_1(\omega) - u)^{-1}\; p_{\widetilde{\varphi}}(y, du) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega - x)^{-1} p_{\widetilde{\varphi}}(y, du) p_{\widetilde{\varphi}_1}(y, dx)\end{aligned}$$

Therefore $\widetilde{\varphi} \circ \widetilde{\varphi}_1$ is in the class $\mathscr{P}_1$ and

$$p_{\widetilde{\varphi}\circ\widetilde{\varphi}_1}(y, dx) = \int_{-\infty}^{\infty} p_{\widetilde{\varphi}}(y, du) p_{\widetilde{\varphi}_1}(u, dx).$$

An easy consequence of this is that

$$\varphi^n(\omega) = y + \left(\int_{-\infty}^{\infty} \frac{p^{(n)}(y, dx)}{\omega - x}\right)^{-1}$$

where $p^{(n)}(y, dx)$ is the n^{th} iterate of the kernel $p(y, dx)$. Note that if $\omega \in \mathbb{H}^+$, then the sequence $(\varphi^n(\omega))_{n\in\mathbb{N}^*}$ is therefore bounded. This implies that the Julia set of φ is $\mathbb{R} \cup \infty$ (see e.g. Barnsley (1988) page 258 for definition of the Julia set of φ).

If $\omega = a + ib$ is in $\mathbb{H}^+$, denote now by μ_ω the Cauchy distribution on $\mathbb{R}$:

$$\mu_\omega(dx) = \pi^{-1}\left[(x - a)^2 + b^2\right]^{-1} b\, dx. \tag{3}$$

A trite example of the above Markov chain is given by $\widetilde{\varphi}(\omega) = \omega + \alpha + i\beta$, with $\beta > 0$, corresponding to $\mu = \beta\,\mu_i$ (defined by (3)) Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega x + 1}{x - \omega}\,\frac{dx}{1 + x^2} = i \;\text{ and }\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega - x}\,\frac{\beta dx}{x^2 + \beta^2} = \frac{1}{\omega + i\beta},$$

it follows that $p(y, dx) = \mu_{y-\alpha+i\beta}(dx)$. Thus in this case, the Markov chain is simply a random walk governed by the Cauchy distribution $\mu_{-\alpha+i\beta}$ defined by (2). A more typical situation is offered by the *Boole function* $\widetilde{\varphi}(\omega) = \omega - \omega^{-1}$, which corresponds to $\alpha = 0$ and to $\mu = \delta_0$, the Dirac mass on 0 in (1). Here, the transition probability $p(y, dx)$ is a Bernoulli distribution which is concentrated on the two roots of the equation $y = \omega - \omega^{-1}$. The weights on these roots are such that the expectation of $p(y, dx)$ is y (more generally, when $\widetilde{\varphi}$ is a rational function, we shall see in Section 2 a rather explicit form of the transition kernel $p(y, dx)$).

The aim of this paper is to study the above Markov chain in some particular cases. To describe its contents, let us recall some notations, definitions, and results.

When μ is a singular measure, $\widetilde{\varphi}$ is called an *inner function* (see Aaronson (1997) page 208). An important property of the inner function $\widetilde{\varphi}$ is that $\lim_{b\downarrow 0} \widetilde{\varphi}(a+ib)$ exists and is real for almost all a in $\mathbb{R}$. A real valued function φ will be said a *restriction* of the inner function $\widetilde{\varphi}$ on $\mathbb{R}$ if we have almost everywhere

$$\varphi(a) = \lim_{b\downarrow 0} \widetilde{\varphi}(a+ib).$$

If f is a map from $\mathbb{R}$ to $\mathbb{R}$, denote by $f_*\mu_\omega$ the image by f of the Cauchy distribution μ_ω. A striking property of a restriction φ of the inner function $\widetilde{\varphi}$ is the following:

$$\varphi_*\mu_\omega = \mu_{\widetilde{\varphi}(\omega)} \quad \text{for all } \omega \in \mathbb{H}^+. \tag{4}$$

This property implies in particular that $\widetilde{\varphi}$ is determined by a restriction φ. Furthermore, a converse of (4) is true : if $f : \mathbb{R} \to \mathbb{R}$ is such that for any ω in $\mathbb{H}^+$ there exists ω' in $\mathbb{H}^+$ such that $f_*\mu_\omega = \mu_{\omega'}$, then either f or $-f$ is a restriction φ of some inner function (see Letac (1977) for these two results). A consequence of this is the fact that the set of inner functions is a semi-group for composition.

Denote by $\mathscr{I}_1$ the class of inner functions such that $k = 1$ in the representation (1). As a corollary of (4), it is shown in Letac (1977) that a restriction φ of a function of $\mathscr{I}_1$ preserves the Lebesgue measure of $\mathbb{R}$. This means that for $-\infty < a < b < \infty$:

$$\text{measure of } \varphi^{-1}[a, b] = b - a. \tag{5}$$

This paper shows three things:

(1) Given $\widetilde{\varphi} \in \mathscr{P}_1$ such that $\int_{-\infty}^{\infty} x^2\mu(dx) < \infty$, then the Markov chain $(X_n)_{n=0}^{\infty}$ is such that $\lim_{n\to\infty} X_n/n$ is a finite constant λ (Theorem 4.1).

(2) Given $\widetilde{\varphi} \in \mathscr{I}_1$ and a restriction φ of $\widetilde{\varphi}$, then the operator P on $L^\infty(\mathbb{R})$ which is associated to the Markov chain is the adjoint of T defined on $L^1(\mathbb{R})$ by $T(f)(\omega) = f(\varphi(\omega))$ (Theorem 5.1). In the case where φ is rational, this operator P is nothing but the *Perron-Frobenius operator* associated to φ (see Katok & Hasselblatt (1995), def 5.1.7 page 187), sometimes also called the *Ruelle operator.*

(3) Given $\widetilde{\varphi} \in \mathscr{I}_1$, and assuming that the singular measure μ has compact support, we show that $(X_n)_{n=0}^{\infty}$ is recurrent when $\lambda = 0$ and is transient when $\lambda \neq 0$ (see Theorem 6.2 and 6.4). This will imply that T is conservative and φ is ergodic when $\lambda = 0$, and that T is dissipative when $\lambda \neq 0$. This extends the result of Adler and Weiss (1973), who prove that the Boole function $x \mapsto x - x^{-1}$ is ergodic on $\mathbb{R}$. Actually our argument in the proof of the Theorem (6.4) below is a natural extension of the Adler and Weiss' argument.

Section 2 is elementary and only of pedagogical value : we consider there the case where φ is rational, we compute explicitly the transition kernel and we give an elementary proof of the result (2) in this particular case. Section 3 contains some general remarks about our Markov chain generated by a Pick function φ. The three subsequent sections are devoted to the three results above.

2. An example : φ rational

Consider now the class of $\widetilde{\varphi}$ in $\mathscr{P}_1$ which are rational. In this case, μ has no continuous part and is the sum of a finite number $n \geq 1$ of atoms $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. Here φ takes the following form :

$$\varphi(x) = x - \lambda + \frac{c_1}{\gamma_1 - x} + \cdots + \frac{c_n}{\gamma_n - x} \tag{6}$$

where $c_1, \cdots, c_n$ are positive numbers (in fact $c_i = (1 + \gamma_i^2)\mu(\{\gamma_i\})$) and λ is real ($\lambda = -\alpha + \sum_{i=1}^{n} \gamma_i \mu(\{\gamma_i\})$). That such a φ satisfies (5) was known to Cayley (see Glaisher (1870) and (1879)). It was proved by Szegö that if a rational function f preserves Lebesgue measure, then either f or $-f$ has the form (6) (See Polya and Szegö (1972), page 79, part II, problem 118.1 and Szegö (1934)).

We assume that $n \in \mathbb{N}^*$. We take the conventions $\gamma_0 = -\infty$ and $\gamma_{n+1} = \infty$. Let us fix a real number y and let us denote by $a_i(y)$ the unique root of the equation $\varphi(x) = y$ such that $\gamma_i < x < \gamma_{i+1}, \ i = 0, 1, \ldots, n$. Clearly the equation $\varphi(x) = y$ is equivalent to $P(x) = 0$, where P is the $(n+1)^{\text{th}}$ degree polynomial :

$$P(x) = (x - \lambda - y) \prod_{i=1}^{n} (x - \gamma_i) - \sum_{i=1}^{n} c_i \prod_{j \neq i} (x - \gamma_i).$$

Hence $a_0(y), \ a_1(y), \ldots, \ a_n(y)$ are the only roots of P. Considering the coefficient of x^n, we get

$$a_0(y) + a_1(y) + \cdots + a_n(y) = y + \lambda + \sum_{i=1}^{n} \gamma_i. \tag{7}$$

Since $a_i(y)$ is the reciprocal of the function $\varphi(x)$ restricted to $(\gamma_i, \ \gamma_{i+1})$, $a_i'(y)$ exists and is positive, and the identity (7) implies that $\sum_{i=0}^{n} a_i'(y) = 1$. Define now the probability kernel

$$p(y, dx) = \sum_{i=0}^{n} a_i'(y) \delta_{a_i(y)}(dx) \tag{8}$$

where δ_a is the Dirac mass on a. The rational function $\omega \to [\widetilde{\varphi}(\omega) - y]^{-1}$ has only simple poles in $a_0(y), \cdots, a_n(y)$, with residues $a_i'(y) = \left[\varphi'(a_i(y))\right]^{-1}$; hence :

$$[\widetilde{\varphi}(\omega) - y]^{-1} = \sum_{i=0}^{n} \frac{a_i'(y)}{\omega - a_i(y)}. \tag{9}$$

Using (8), this can be written :

$$[\widetilde{\varphi}(\omega) - y]^{-1} = \int_{-\infty}^{\infty} (\omega - x)^{-1} p(y, dx).$$

For f in $L^1(\mathbb{R})$ let us define $Tf = f \circ \varphi$. We have

$$\int_{\gamma_i}^{\gamma_{i+1}} f \circ \varphi(x)\, dx = \int_{-\infty}^{\infty} f(y) a_i'(y)\, dy.$$

Hence $\int_{-\infty}^{\infty} (Tf)(x)\, dx = \int_{-\infty}^{\infty} f(y) dy$, and T is an isometry of $L^1(\mathbb{R})$. Similarly, for g in $L^\infty(\mathbb{R})$ let us define $(Pg)(y) = \sum_{i=0}^{n} a_i'(y)\, g(a_i(y)) = \int_{-\infty}^{\infty} g(x) p(y, dx)$. If f is in $L^1(\mathbb{R})$, we write

$$\int_{\gamma_i}^{\gamma_{i+1}} f \circ \varphi(x) g(x) dx = \int_{-\infty}^{\infty} f(y) a_i'(y) g(a_i(y)) dy.$$

Hence $\int_{-\infty}^{\infty} (Tf)\,(x) g(x) dx = \int_{-\infty}^{\infty} f(y)\,(Pg)\,(y) dy$, and P is the adjoint of T.

Some other formulas related to $p(y, dx)$ can be easily obtained : for fixed y in $\mathbb{R}$ and complex number $z \neq 0$, (9) gives

$$\left[1 - z(y+\lambda) - z^2 \sum_{i=1}^{n} (1 - z\gamma_i)^{-1} c_i\right]^{-1} = \sum_{i=0}^{n} (1 - z a_i(y))^{-1} a_i'(y).$$

Therefore we obtain the moments of p in a simple way, by considering the coefficient of z^n in both members of this equality. For instance

$$\sum_{i=0}^{n} a_i(y) a_i'(y) = y + \lambda,$$

$$\sum_{i=0}^{n} a_i^2(y) a_i'(y) = (y+\lambda)^2 + \sum_{i=1}^{n} c_i.$$

The proofs of these facts are simple, but we can observe that their extension to a function of $\mathscr{I}_1$ is not straightforward.

3. The transition kernel

The transition kernel is simple to understand outside of the closed support of the measure μ. The following proposition shows that the restriction of $p(y, dx)$ to an interval which is contiguous to this support is nothing but a Dirac mass.

Proposition 3.1. *Let us suppose that $\widetilde{\varphi}$ is defined by (1) and that $\mu\{(a, b)\} = 0$, with $-\infty \le a < b \le \infty$. Then $\widetilde{\varphi}$ has an analytic continuation in $\mathbb{H}^+ \cup (a, b) \cup \mathbb{H}^-$. Denote the inverse function of φ restricted to (a, b) by $\psi(y)$. Then the restriction φ to (a, b) of this analytic continuation is real and strictly increasing, and :*

$$\mathbb{1}_{(a,b)}(x) p(y, dx) = \begin{cases} \psi'(y) \delta_{\psi(y)}(dx) & \text{if} \quad y \in (\varphi(a), \varphi(b))\,. \\ 0 & \text{otherwise} \end{cases}$$

Here, $\delta_{\psi(y)}(dx)$ is the Dirac mass on $\psi(y)$.

Proof. The first two statements are standard (see Donoghüe (1974)). Denote $\Omega = \mathbb{H}^+ \cup (a, b) \cup \mathbb{H}^-$. Clearly we have $\widetilde{\varphi}(\overline{\omega}) = \overline{\widetilde{\varphi}(\omega)}$ for ω in $\mathbb{H}^+$. Since $\Im m\widetilde{\varphi}(\omega) > 0$ for ω in $\mathbb{H}^+$, the only zeros of $\widetilde{\varphi}(\omega) - y$ are on (a, b).

If $y \notin (\varphi(a), \varphi(b))$, the function

$$\omega \longmapsto \int_{-\infty}^{\infty} (\omega - x)^{-1} p(y, dx)$$

is analytic in Ω, and this implies (see Donoghüe (1974)) that $p(y, (a, b)) = 0$.

If $\varphi(a) < y < \varphi(b)$, then $\psi(y)$ is a simple pole of $[\widetilde{\varphi}(\omega) - y]^{-1}$ with residue $\psi'(y)$. Hence the function

$$\omega \longmapsto \int_{-\infty}^{\infty} \frac{p(y, dx) - \psi'(y)\delta_{\psi(y)}(dx)}{\omega - x}$$

is analytic in Ω and this implies that $p(y, (a, b)) = \psi'(y)$.

It is worth mentioning that φ has neither fixed points nor periodic orbits in the half plane : this comes from the fact that

$$\Im m(\varphi(\omega) - \omega) = (\Im m\omega) \int_{-\infty}^{\infty} \frac{1 + x^2}{|\omega - x|^2} \mu(dx)$$

cannot be 0. However if we take the one point compactification, $\varphi(\infty) = \infty$ with $\varphi'(\omega) = 1$, thus ∞ is a neutral fixed point. On the other hand, $\widetilde{\varphi}$ can have many fixed points and periodic orbits on $\mathbb{R}$.

4. The case $\int_{-\infty}^{\infty} x^2 \mu(dx) < \infty$

Consider an analytic map $\widetilde{\varphi} : \mathbb{H}^+ \to \mathbb{H}^+$ such that $k = 1$ and $\int_{-\infty}^{\infty} x^2 \mu(dx) < \infty$ in the representation (1). Recall that μ is not necessarily singular in this section. We denote

$$c(dx) = (1 + x^2)\mu(dx),$$

$$\lambda = -\alpha + \int_{-\infty}^{\infty} x\mu(dx).$$

Since $\int_{-\infty}^{\infty} \frac{\omega x + 1}{x - \omega} \mu(dx) = \int_{-\infty}^{\infty} \left(\frac{1 + x^2}{x - \omega} - x \right) \mu(dx)$, we can write :

$$\widetilde{\varphi}(\omega) = \omega - \lambda + \int_{-\infty}^{\infty} \frac{c(dx)}{x - \omega}. \tag{10}$$

We have the following large numbers law for the Markov chain associated to $\widetilde{\varphi}$:

Theorem 4.1. *Let $\widetilde{\varphi}$ be given by (10), where c is a positive bounded measure on $\mathbb{R}$. Then :*

$$\int_{-\infty}^{\infty} xp(y,dx) = y+\lambda \tag{11}$$

$$\int_{-\infty}^{\infty} x^2 p(y,dx) = (y+\lambda)^2 + c(\mathbb{R}). \tag{12}$$

Furthermore, the Markov chain $(X_n)_{n=0}^{\infty}$ with transition $p(y,dx)$ satisfies $X_n/n \to \lambda$ when $n \to \infty$ almost surely. If $\lambda \geq 0$, $T_a = \inf\{n\,;\ X_n \geq a\}$ is almost surely finite for any positive a. If $\lambda < 0$, T_a is finite with probability < 1.

Proof. By using the expression (10) of $\widetilde{\varphi}$ we can easily verify the two following equalities :

$$\lim_{\omega\to+i\infty} \omega^2 \left[-\frac{1}{\widetilde{\varphi}(\omega)-y} + \frac{1}{\omega} + \frac{y+\lambda}{\omega^2}\right] = 0,$$
$$\lim_{\omega\to+i\infty} \omega^3 \left[-\frac{1}{\widetilde{\varphi}(\omega)-y} + \frac{1}{\omega} + \frac{y+\lambda}{\omega^2}\right] = -\left[(y+\lambda)^2 + c(\mathbb{R})\right].$$

On the other hand, if we evaluate the left hand side of these two equalities by using the definition of $p(y,dx)$, we obtain the expressions (11) and (12).

Consider now $Y_n = X_n - n\lambda$. Equalities (11) and (12) imply

$$\mathbb{E}(X_{n+1}|X_n) = X_n + \lambda$$
$$\mathbb{E}(X_{n+1}^2|X_n) = c(\mathbb{R}) + (X_n+\lambda)^2.$$

These formulas imply that $(Y_n)_{n=1}^{\infty}$ is a martingale and that

$$\mathbb{E}\left((Y_{n+1}-Y_n)^2|X_n\right) = c(\mathbb{R}).$$

We can now use the law of large numbers for martingales (see Feller (1966), page 238, Theorem 2) to get $Y_n/n \to 0$ when $n \to \infty$, almost surely.

Let us fix $a \geq 0$. If $\lambda > 0$, $T_a < \infty$ trivially. If $\lambda < 0$, $T_a = +\infty$ with a positive probability. If $\lambda = 0$, we get from (11), (12) and from the Schwarz inequality :

$$\int_a^{\infty} xp(y,dx) \leq \sqrt{c(\mathbb{R})} + y. \tag{13}$$

Let us consider now the martingale $X'_n = X_{T_a\wedge n}$. Then

$$\begin{aligned}\mathbb{E}\left[(X'_n)^+\right] &= \mathbb{E}\left[(X'_n)^+ \bigcap T_a \leq n\right] + \mathbb{E}\left[(X'_n)^+ \bigcap T_a > n\right]\\ &\leq \sum_{k=1}^{n} \mathbb{E}\left(\mathbb{I}_{X_1<a,\cdots,X_{k-1}<a,X_k\geq a} \int_a^{\infty} xp(X_{k-1},dx)\right)\\ &\quad + \mathbb{E}\left[(X'_n)^+ \bigcap T_a > n\right]\end{aligned}$$

$$\le \left(\sup_{y\le a}\int_a^\infty xp(y,dx)\right)\sum_{k=1}^{n}\mathbb{P}(T_a=k)+a$$
$$\le \sup_{y\le a}\int_a^\infty xp(y,dx)\mathbb{P}(T_a\le n)+a$$
$$\le \sqrt{c(\mathbb{R})}+2a$$

Therefore $(X_n)_{n=0}^{\infty}$ converges, and it is easy to conclude that $\mathbb{P}[T_a=\infty]=0$.

To end this section, we mention here the following fact:

Proposition 4.2. *In the sense of weak convergence of measures, one has*

$$y^2\left[p(y,dx)-\delta_y(dx)\right]\longrightarrow c(dx),\quad |y|\to\infty$$

where δ_y is the Dirac mass on y.

Proof. Since the linear space generated by the $g_\omega(x)=[\omega-x]^{-1}$ where $\omega\in\mathbb{H}^+$ is dense in $C_0(\mathbb{R})$, enough to prove that

$$y^2\left[\int_{-\infty}^{\infty}g_\omega(x)p(y,dx)-g_\omega(y)\right]\longrightarrow\int_{-\infty}^{\infty}g_\omega(x)c(dx),\quad y\to\infty.$$

The result is now obvious :

$$y^2\left[\left[\omega-y+\int_{-\infty}^{\infty}\frac{c(dx)}{x-\omega}\right]^{-1}-[\omega-y]^{-1}\right]\longrightarrow_{y\to\infty}\int_{-\infty}^{\infty}\frac{c(dx)}{x-\omega}$$

5. The Markov chain associated to a function of $\mathscr{I}_1$

In this section, we gather easy facts about the Markov chain $(X_n)_0^\infty$ when the governing measure μ is singular.

When $\widetilde{\varphi}$ is in $\mathscr{I}_1$, $p(y,dx)$ is singular for all y in $\mathbb{R}$: the reason is that the set of inner functions is a semi group for composition. Since $\omega\mapsto-\omega^{-1}$ and $\omega\mapsto-[\widetilde{\varphi}(\omega)-y]^{-1}$ are inner functions, this is also true for $\omega\mapsto-[\widetilde{\varphi}(\omega)-y]^{-1}$. This also implies that for $\widetilde{\varphi}$ in $\mathscr{I}_1$, the Markov chain cannot be a Harris chain (see Krengel (1985) for a definition) with respect to Lebesgue measure, although the Lebesgue measure is a stationary measure for the chain (see Corollary 5.2). We shall see also that in this case $p(y,dx)$ is concentrated on the set $\varphi^{-1}(y)$ almost everywhere with respect to y (see Corollary 5.3).

Consider $\widetilde{\varphi}$ in $\mathscr{I}_1$, and the restriction φ of $\widetilde{\varphi}$. Define $T:L^1(\mathbb{R})\to L^1(\mathbb{R})$ by $(Tf)(x)=f(\varphi(x))$, and define $P:L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})$ by $(Pg)(y)=\int_{-\infty}^{\infty}g(x)p(y,dx)$. We show that P is the adjoint of T:

Theorem 5.1. *For all f in $L^1(\mathbb{R})$ and g in $L^\infty(\mathbb{R})$ we have*

$$\int_{-\infty}^{\infty}g(x)(Tf)(x)dx=\int_{-\infty}^{\infty}f(y)(Pg)(y)dy. \tag{14}$$

Proof. For ω in $\mathbb{H}^+$, denote $g_\omega(y) = (\omega - y)^{-1}$. We prove first (14) for $g = g_\omega$, that is to say

$$\int_{-\infty}^{\infty} (\omega - x)^{-1} f \circ \varphi(x) dx = \int_{-\infty}^{\infty} [\widetilde{\varphi}(\omega) - y]^{-1} f(y) dy. \tag{15}$$

To prove (15) we observe that both members are analytic functions of ω on $\mathbb{H}^+$. Suppose that f is a real function. Imaginary parts of the two members are respectively, for $\omega = a + ib$:

$$\int_{-\infty}^{\infty} b \left[(x-a)^2 + b^2 \right]^{-1} f \circ \varphi(x) dx = \pi \int_{-\infty}^{\infty} f \circ \varphi(x) \mu_\omega(dx) \tag{16}$$

$$\int_{-\infty}^{\infty} |\widetilde{\varphi}(\omega) - y|^{-2} \, \Im m \varphi(\omega) f(y) dy = \pi \int_{-\infty}^{\infty} f(y) \mu_{\widetilde{\varphi}(\omega)}(dy). \tag{17}$$

where the Cauchy measures μ_ω and $\mu_{\widetilde{\varphi}(\omega)}$ are defined by (2). Using (3), we see that (16) and (17) are equal.

Since imaginary parts are equal, real parts of the two members of (15) differ by some real constant a. Now if $\omega \to +i\infty$, since $|\widetilde{\varphi}(\omega)| \to \infty$ one can easily see that a is zero. This proves (14) for a real f, For an f with an imaginary part, one easily extends the result.

Equality (14) is true when g is a constant. Therefore (14) is true also for the g's in the closure C (in the sense of sup norm) of the space generated by 1 and the g_ω's, with ω in $\mathbb{H}^+$. The set C is the space of continuous functions f on $\mathbb{R}$ such that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and are equal.

If $-\infty \le a < b \le \infty$ and $g(x) = \mathbb{I}_{(a,b)}(x)$, we choose a positive sequence g_n of C such that $g_n(x) \uparrow g(x)$ for all x. Hence $(Pg_n)(y) \uparrow (Pg)(y)$ for all y. Therefore :

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y)(Pg)(y)dy &= \lim_{n\to\infty} \int_{-\infty}^{\infty} f(y)\left(Pg_n\right)(y)dy \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} f \circ \varphi(x) g_n(x) dx \\
&= \int_{-\infty}^{\infty} f \circ \varphi(x) g(x) dx.
\end{aligned}$$

The second equality is given by monotone convergence and the third one is given by bounded convergence. Since formula (14) is now proved for $g = \mathbb{I}_{(a,b)}$, the extension to any g in $L^\infty(\mathbb{R})$ is standard.

Corollary 5.2. *If $\widetilde{\varphi}$ is in $\mathscr{I}_1$, the Lebesgue measure on $\mathbb{R}$ is stationary for $p(y, dx)$.*

Proof. We take $g = \mathbb{I}_{(a,b)}$ with $-\infty < a < b < \infty$ and $f_n = \mathbb{I}_{(-n,n)}$ in Theorem 5.1, which gives :

$$\int_a^b \mathbb{I}_{(-n,n)}\left(\varphi(x)\right) dx = \int_{-n}^{n} p(y, (a,b)) dy.$$

Taking $n \to \infty$, we get the desired result : $b - a = \displaystyle\int_{-\infty}^{\infty} p(y, (a,b)) dy$.

Corollary 5.3. *If $\widetilde{\varphi}$ is in $\mathcal{I}_1$,*

$$p\left(y, \varphi^{-1}(y)\right) = 1 \quad \textit{for almost all } \; y.$$

Proof. For t real, denote $g_t(x) = \exp[it\varphi(x)]$. Taking f in $L^1(\mathbb{R})$, from Theorem 5.1 we get :

$$\begin{aligned}\int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} \exp[it\varphi(x)]\, p(y, dx) &= \int_{-\infty}^{\infty} f \circ \varphi(x) \exp[it\varphi(x)]\, dx \\ &= \int_{-\infty}^{\infty} f(y) \exp(ity)\, dy.\end{aligned}$$

Hence :

$$e^{ity} = \int_{-\infty}^{\infty} \exp[it\varphi(x)]\, p(y, dx) \quad \text{for almost all } \; y.$$

This implies that the image $p(y, dx)$ by φ is the Dirac mass on y (for almost all y).

6. Recurrence-transience for μ compact and singular

In this paragraph we consider the function $\widetilde{\varphi}$ of the form (10) and we suppose that the measure c is concentrated in the interval $[A, B]$. We also assume that A and B belong to the support of c. The restriction φ of $\widetilde{\varphi}$ has the explicit form for $x \notin [A, B]$

$$\varphi(x) = x - \lambda + \int_A^B \frac{c(dt)}{t - x}. \tag{18}$$

On $(-\infty, A)$ and (B, ∞), φ is strictly increasing and we write $\varphi(A) = \lim_{x \uparrow A} \varphi(x) \leq +\infty$ and $\varphi(B) = \lim_{x \downarrow B} \varphi(x) \geq -\infty$ (remark that $\varphi(A)$ and $\varphi(B)$ can be finite if φ is not rational).

We denote by $\theta(y)$ (resp. $\theta_-(y)$) the inverse function of φ restricted to (B, ∞) (resp. $(-\infty, A)$). The function θ is defined on $(\varphi(B), +\infty)$ and is valued in (B, ∞). If $\lambda = 0$, obviously $\varphi(B) < B$ and $\varphi(A) > A$. Thus iteration of θ and θ_- make sense.

The following proposition explains the importance of θ :

Proposition 6.1. *With the above hypothesis, for $y > \varphi(B)$, the restriction of $p(y, dx)$ to (B, ∞) is equal to $\theta'(y)\delta_{\theta(y)}$, where $\delta_{\theta(y)}$ is the Dirac mass on $\theta(y)$.*

Proof. Apply Proposition 3.1 to (a, b).

We choose the following definitions of recurrence and transience for a Markov chain on $\mathbb{R}$:

Definition 6.1. *A Markov chain $(X_n)_{n=0}^{\infty}$ valued in $\mathbb{R}$ is said to be recurrent if for any open interval I*

$$\mathbb{P}\left[\bigcap_{k \geq 0} \bigcup_{n \geq k} \{X_n \in I\}\right] = 1.$$

It is said to be transient if for any bounded interval I we have

$$\sum_{n=0}^{\infty} \mathbb{P}\left[X_n \in I\right] < \infty.$$

Note that the definition is compatible with the case of an aperiodic random walk in $\mathbb{R}$ (see Feller (1966)). It takes "recurrent" and "transient" in the strongest sense. One could build Markov chains on $\mathbb{R}$ which would neither recurrent nor transient in the above sense.

Theorem 6.2. *Denote by $\sigma^2 = c\left([A, B]\right)$. With the above hypothesis, for $y \longrightarrow +\infty$*

$$\theta(y) = y + \lambda + \frac{\sigma^2}{y} + \frac{1}{y^2}\left[\int_A^B xc(dx) - \lambda\sigma^2\right] + o\left(\frac{1}{y^2}\right) \tag{19}$$

$$\theta'(y) = 1 - \frac{\sigma^2}{y^2} + o\left(\frac{1}{y^2}\right). \tag{20}$$

Furthermore if $\lambda = 0$ and if $\theta^n(y)$ is the n^{th} iterate of θ, then we have :

$$\lim_{n\uparrow\infty} n^{-1/2}\theta^n(y) = \sigma\sqrt{2} \quad \textit{and} \quad \theta'\left(\theta^n(y)\right) = 1 - \frac{1}{2n} + o\left(\frac{1}{n}\right). \tag{21}$$

Proof. Since $y = \theta(y) - \lambda + \int_A^B (t - \theta(y))^{-1}\, c(dt)$ for $y > \varphi(B)$, clearly $\theta(y) = y + \lambda + o(1)$ and $\theta(y) = y + \lambda + \frac{\sigma^2}{y} + o\left(\frac{1}{y}\right)$. Now

$$\begin{aligned}\left[(\theta(y) - y - \lambda)\, y - \sigma^2\right] y &= y\int_A^B \left(\frac{y}{\theta(y) - x} - 1\right) c(dx)\\ &= -\lambda\sigma^2 + \int_{-\infty}^{\infty} xc(dx) + o(1)\end{aligned}$$

which gives (19). Thus

$$\theta'(y) = \frac{1}{\varphi'\left(\theta(y)\right)} = \left[1 + \int_{-\infty}^{\infty} \frac{c(dx)}{(t - \theta(y))^2}\right]^{-1},$$

and we get easily (20).

Suppose now that $\lambda = 0$. The definition of θ implies that $\theta(y) > y$. Clearly the increasing sequence $\theta^n(y)$ has no finite limit. From (19) we infer that there exists a constant $k > 0$ such that, for any $y > \varphi(B)$:

$$y + \frac{\sigma^2}{y} - \frac{k}{y^2} \le \theta(y) \le y + \frac{\sigma^2}{y} + \frac{k}{y^2}. \tag{22}$$

We show that $\liminf_{n\to\infty} \dfrac{\theta^n(y)}{\sqrt{2n}} \geq \sigma$. To do this, we fix ϵ and observe that there exists $N(\epsilon)$ such that, for $n \geq N(\epsilon)$:

$$\sigma(1-\epsilon)\sqrt{2n} + \frac{\sigma}{(1-\epsilon)\sqrt{2n}} - \frac{k}{(1-\epsilon)^2\sigma^2 2n} \geq \sigma(1-\epsilon)\sqrt{2n+2}. \tag{23}$$

This can be checked by Taylor expansion of both members with respect to $h = 1/\sqrt{2n}$. This gives

$$\frac{\sigma(1-\epsilon)}{h}\left[1 + \frac{h^2}{(1-\epsilon)^2} + O(h^3)\right] \geq \frac{\sigma}{h}(1-\epsilon)\left[1 + h^2 + O(h^4)\right].$$

which is trivially true for h small enough.

We take now N' such that $\theta^{N'}(y) \geq \sigma(1-\epsilon)\sqrt{2N(\epsilon)}$. (22) and (23) give for $n > N'$:

$$\theta^n(y) \geq \sigma(1-\epsilon)\sqrt{2N(\epsilon) + 2n - 2N'},$$

hence $\liminf_{n\to\infty} \dfrac{\theta^n(y)}{\sqrt{2n}} \geq \sigma(1-\epsilon)$ for any ϵ.

We show that $\limsup \dfrac{\theta^n(y)}{\sqrt{2n}} \leq \sigma$. Taking $\epsilon > 0$ we show in the same manner that there exists $N(\epsilon)$ such that for $n \geq N(\epsilon)$:

$$\sigma(1+\epsilon)\sqrt{2n} + \frac{\sigma}{(1+\epsilon)\sqrt{2n}} + \frac{k}{(1+\epsilon)^2\sigma^2 2n} \leq \sigma(1+\epsilon)\sqrt{2n+2} \tag{24}$$

We choose $N' \geq N(\epsilon)$ such that $y \leq \sigma(1+\epsilon)\sqrt{2N'}$. (22) and (24) give for all $n \in \mathbb{N}$:

$$\theta^n(y) \leq \sigma(1+\epsilon)\sqrt{2N' + 2n},$$

hence $\limsup \dfrac{\theta^n(y)}{\sqrt{2n}} \leq \sigma(1+\epsilon)$ for any ϵ, and (21) follows.

Corollary 6.3.

$$\lim_{y\to\infty} \frac{p(y,I)}{1-\theta'(y)} = \frac{c(I)}{c(\mathbb{R})}$$

for any interval $I = [a, b]$ *such that* $c(\{a\}) = c(\{b\}) = 0$.

Proof. Use Lemma 4.2 and (20).

We now state the main result of the paper:

Theorem 6.4. *If* $\lambda = 0$, *and* c *has compact support, the chain is recurrent.*

Proof. Let I an open and non empty interval of $\mathbb{R}$ and let $(X_n)_{n=0}^{\infty}$ be the Markov chain. We prove that

$$\mathbb{P}\left[X_n \in I \text{ for infinitely many } n\right] = 1.$$

We distinguish the cases $c(I) > 0$ and $c(I) = 0$.

First case, $c(I) > 0$: From Corollary 6.3 there exists $y_0 > \varphi(B)$ such that :

$$\frac{p(y,I)}{(1-\theta'(y))} \geq \frac{c(I)}{2\sigma^2} \quad \text{for } y > y_0. \tag{25}$$

Denote

$$T_1 = \inf\{n\,;\ X_n > y_0\}, \quad S_1 = \inf\{n > T_1\,;\ X_n \in I\}$$

and for $k > 1$:

$$T_k = \inf\{n > S_{k-1}\,;\ X_n > y_0\}, \quad S_k = \inf\{n > T_k\,;\ X_n \in I\}$$

with the convention T_k or $S_k = \infty$ if the defining set is empty. Theorem 4.1 implies that $T_1 < \infty$ a.s. and that $S_{k-1} < \infty$ a.s. implies $T_k < \infty$ a.s. Therefore, in order to prove that T_k and S_k are finite a.s., enough is to show that $S_1 < \infty$ a.s. The remainder of the proof comes from the Markov property.

Without loss of generality, we suppose that $X_0 = y > y_0$ with probability 1. We introduce the stopping time

$$R_y = \inf\{n\,;\ X_{n-1} > X_n\}.$$

From Proposition 3.1 we get

$$\mathbb{P}\left[R_y > n\right] = \prod_{k=0}^{n-1} \theta'\left[\theta^k(y)\right]$$

From (21), we get that $\lim_{n\to\infty} \mathbb{P}\left[R_y > n\right] = 0$ and that $R_y < \infty$ a.s. Equality (25) implies that :

$$\mathbb{P}\left[X_{R_y} \in I\right] \geq \frac{c(I)}{2\sigma^2}. \tag{26}$$

Since $\{X_n > y_0\}$ happens infinitely often, a standard argument using the Markov property shows that (26) implies $S_1 < \infty$.

Second case, $c(I) = 0$: We begin by a simple remark : By (10), if I is an open interval such that $c(I) = 0$, the function φ restricted to I is strictly increasing, and $\varphi(I)$ is an open interval. We prove now the following statement:

(*) *If I is an open interval, there exist an integer k such that $\varphi^k(I)$ is an open interval with $c\left[\varphi^k(I)\right] > 0$.*

To prove (*) we shall assume that for any integer k the measure c does not charge $\varphi^k(I)$. From the preceding remark, $\varphi^k(I)$ is an open interval for all k. We denote $I = (a,b)$ and $u_k = \varphi^k(b) - \varphi^k(a)$. There exists ξ_k in $\varphi^k(I)$ such that

$$u_{k+1} = u_k \varphi'(\xi_k) \tag{27}$$

Since

$$\varphi'(\xi) = 1 + \int_A^B \frac{c(dt)}{(t-\xi)^2} > 1, \tag{28}$$

the sequence $(u_k)_{k\in\mathbb{N}}$ is strictly increasing.

Suppose first that there exists an infinite number of k such that

$$\varphi^k(I)\cap[A,B]\neq\emptyset \tag{29}$$

For a k such that (29) holds, we have $\varphi^k(I)\subset(A,B)$, since $c\left[\varphi^k(I)\right]=0$ and since A and B are in the support of c. Thus $\xi_k\in(A,B)$. But

$$\kappa=\min_{A<\xi<B}\varphi'(\xi)\geq 1+\frac{c([A,B])}{(B-A)^2}>1$$

Thus $u_{k+1}\geq\kappa u_k$ for all k such that (29) holds and $u_k\longrightarrow+\infty$. Hence $\varphi^k(I)$ cannot be contained in (A,B) for an infinite set of k : contradiction.

A consequence is that there exists k_0 such that $\varphi^k(I)\cap[A,B]=\emptyset$ for all $k>k_0$. We prove that this also leads to a contradiction. Without loss of generality we may assume that $k_0=0$ and $A<B<a<b$. Suppose that $\varphi^k(a)>B$ for all k. This would imply that the sequence $\left(\varphi^k(a)\right)_{k\in\mathbb{N}}$ has a limit $\alpha\geq B$ since $\left(\varphi^k(a)\right)_{k\in\mathbb{N}}$ is a decreasing sequence . But $\alpha>B$ would imply $\varphi(\alpha)=\alpha$, which is impossible since

$$\varphi(\alpha)-\alpha=\int_A^B\frac{c(dt)}{\alpha-t}<0.$$

Similarly, $\alpha=B$ would imply $B=\lim_{x\downarrow B}\varphi(x)$ and $\int_A^B(t-x)^{-1}c(dt)\downarrow 0$ which is clearly impossible. Thus the sequence $\varphi^k(I)$ cannot be entirely contained in (B,∞) for all k. The same reasoning holds for $(-\infty,A)$.

We know now that there exists an infinite number of integers k such that

$$\left(\varphi^k(a),\varphi^k(b)\right)\subset(B,\infty)\ \text{ and }\ \left(\varphi^{k+1}(a),\varphi^{k+1}(b)\right)\subset(-\infty,A)$$

For such an integer k, we get $\varphi^{k+1}(b)<A\ \Rightarrow\varphi^k(b)<\theta_-(A)$ and

$$u_k=\varphi^k(b)-\varphi^k(a)<\theta_-(A)-B.$$

Hence $u=\lim_{k\to\infty}u_k\leq\theta_-(A)-B$.

There exist two infinite sequences $(t_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ such that $0\leq t_1<s_1<\cdots<t_n<s_n<t_{n+1}\cdots$ and such that

$$\begin{cases}\varphi^k(I)\subset(B,\infty) & \text{if } 0\leq k<t_1 \text{ or } s_n\leq k<t_{n+1},\\ \varphi^k(I)\subset(-\infty,A) & \text{if } t_n\leq k<s_n.\end{cases}$$

If $s_n\leq k<t_{n+1}$ (resp. $t_n\leq k<s_n$), the sequences $\varphi^k(a)$ and $\varphi^k(b)$ are decreasing (resp. increasing), because $\varphi(x)<x$ for all $x>B$. Hence if $t_n\leq k<s_n$, $\varphi^{t_n}(a)\leq\varphi^k(a)<A$. Without loss of generality we assume that $u_k\geq u/2$ for all integers k. Thus $u_{t_n-1}>u/2$. This implies that $\varphi(B+u/2)<\varphi^{t_n}(b)<A$ and $u_{t_n}<u$. Hence we get $\varphi(B+u/2)-u<\varphi^{t_n}(a)$. The same reasoning holds to show that $\varphi^k(b)\leq\varphi^{s_n}(b)$ for $s_n\leq k<t_{n+1}$ and $\varphi^{s_n}(b)<\varphi(A+u/2)+u$. Therefore we have proved that the sequences $\left(\varphi^k(a)\right)_{k=0}^{\infty}$ and $\left(\varphi^k(b)\right)_{k=0}^{\infty}$ are bounded by some constant C.

We use now (28) and denote

$$\kappa_1 = \min_{|\xi|\le C} \varphi'(\xi) \ge 1 + \frac{c([A,B])}{(B-A)^2} > 1$$

Identity (27) implies that $u_k \to \infty$. This contradicts $u_k \to u \le \theta_-(A) - B$. Hence the statement (*) is proved. We now use it to prove the recurrence of the chain.

Let k be the nonnegative integer such that $c(I) = \cdots = c\left(\varphi^{k-1}(I)\right) = 0$ and $c\left(\varphi^k(I)\right) > 0$. Let us assume that φ^k and its derivative are bounded on I (If not, I can be replaced by a smaller interval). We claim that

$$\inf_{y\in\varphi^k(I)} p^{(k)}(y, I) > 0 \tag{30}$$

Taking y in $\varphi^k(I)$, there exist a sequence $y_0,\ y_1, \cdots,\ y_{k-1},\ y_k = y$ such that $\varphi(y_i) = y_{i+1}$, $i = 0, \cdots, k-1$ and

$$p^{(k)}(y, I) \ge p\left(y, \{y_{k-1}\}\right) p\left(y_{k-1}, \{y_{k-2}\}\right) \cdots p\left(y_1, \{y_0\}\right)$$

From Proposition 3.1 : $p\left(y_i, \{y_{i-1}\}\right) = 1/\varphi'(y_{i-1})$. Hence

$$p^{(k)}(y, I) \ge \prod_{i=0}^{k-1} \left[\varphi'\left(\varphi^i(y_0)\right)\right]^{-1}$$

where $y = \varphi^k(y_0)$ and $a < y_0 < b$.

Now, to prove that $X_n \in I$ infinitely often we use the fact that $X_n \in \varphi^k(I)$ infinitely often, the first part of the proof of the theorem, and condition (30). The proof is now complete.

Theorem 6.5. *If* $\lambda \ne 0$*, the Markov chain is transient.*

Proof. We take $\lambda > 0$ and $I = (-\infty, \alpha)$, where α is any positive number, and we prove that $\sum_{n\ge 0} \mathbb{P}(X_n \in I) < \infty$ for any x_0 in $\mathbb{R}$. The function θ is defined on $(\varphi(B), \infty)$ as in Theorem 6.2. Clearly iterates θ^n of θ satisfy

$$\theta^n(y) = y + n\lambda + O\left(\frac{1}{n}\right)$$

$$\theta'\left[\theta^n(y)\right] = 1 - \frac{\sigma^2}{\lambda^2 n^2} + o\left(\frac{1}{n^2}\right) \qquad \text{for } y > B$$

From Proposition 3.1

$$\mathbb{P}\left[X_{i+1} = \theta\left(X_i\right),\ i = 0, \ldots, n-1 | X_0 = y\right] = \prod_{i=0}^{n-1} \theta'\left[\theta^i(y)\right].$$

Therefore $\mathbb{P}\left[X_{n+1} = \theta\left(X_n\right)\ \forall n | X_0 = y\right] > 0$. From the zero one law :

$$\mathbb{P}_{x_0}\left[\exists X_{n+1} = \theta\left(X_n\right)\ \forall n \ge N\right] = 1.$$

This ends the proof.

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