Probab. Theory Relat. Fields 118, 17–36 (2000)
Digital Object Identifier (DOI) 10.1007/s004400000068

M. van den Berg

Heat equation on the arithmetic von Koch snowflake

Received: 24 March 1999 / Revised version: 14 October 1999 /
Published online : 8 August 2000 –

Abstract. We investigate the asymptotic behaviour of the heat content as the time $t \to 0$ for an s-adic von Koch snowflake generated by a square. We show that the heat content satisfies a functional equation which, after appropriate transformations, takes the form of an inhomogeneous renewal equation. We obtain the structure of the solution of this equation in the arithmetic case up to an exponentially small remainder in t.

1. Introduction

Let D be an open, bounded set in euclidean space $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ with boundary ∂D, and let $u_D : \overline{D} \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \ x \in D, \ t > 0, \tag{1.1}$$

with initial condition

$$u(x; 0) = 0, \ x \in D, \tag{1.2}$$

and boundary condition

$$u(x; t) = 1, \ x \in \partial D, \ t > 0. \tag{1.3}$$

Let

$$E_D(t) = \int_D u_D(x; t)dx \tag{1.4}$$

represent the total amount of heat in D at time t.

The asymptotic behaviour of $E_D(t)$ as $t \to 0$ has been the subject of an extensive investigation, and is well understood if ∂D is smooth or piecewise smooth [1–5, 8, 15, 16]. Little is known if D is a region with a fractal boundary. In this paper we analyse the example of the s-adic arithmetic von Koch snowflake K_s in detail. The construction of K_s is as follows.

Fix $0 < s \leq \frac{1}{3}$, and let $A_1A_2A_3A_4$ be the unit square in $\mathbb{R}^2$ with vertices $A_1 = \left(-\frac{1}{2}, -\frac{1}{2}\right)$, $A_2 = \left(\frac{1}{2}, -\frac{1}{2}\right)$, $A_3 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $A_4 = \left(-\frac{1}{2}, \frac{1}{2}\right)$. We construct ∂K_s by repeatedly replacing the middle proportion s of each segment, beginning with A_1A_2, A_2A_3, A_3A_4 and A_4A_1, by the three other sides of a square.

M. van den Berg: School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom. e-mail: M.vandenBerg@bris.ac.uk

Mathematics Subject Classification (2000): 35K05, 60J65, 28A80
Key words: Heat equation – Arithmetic – Snowflake

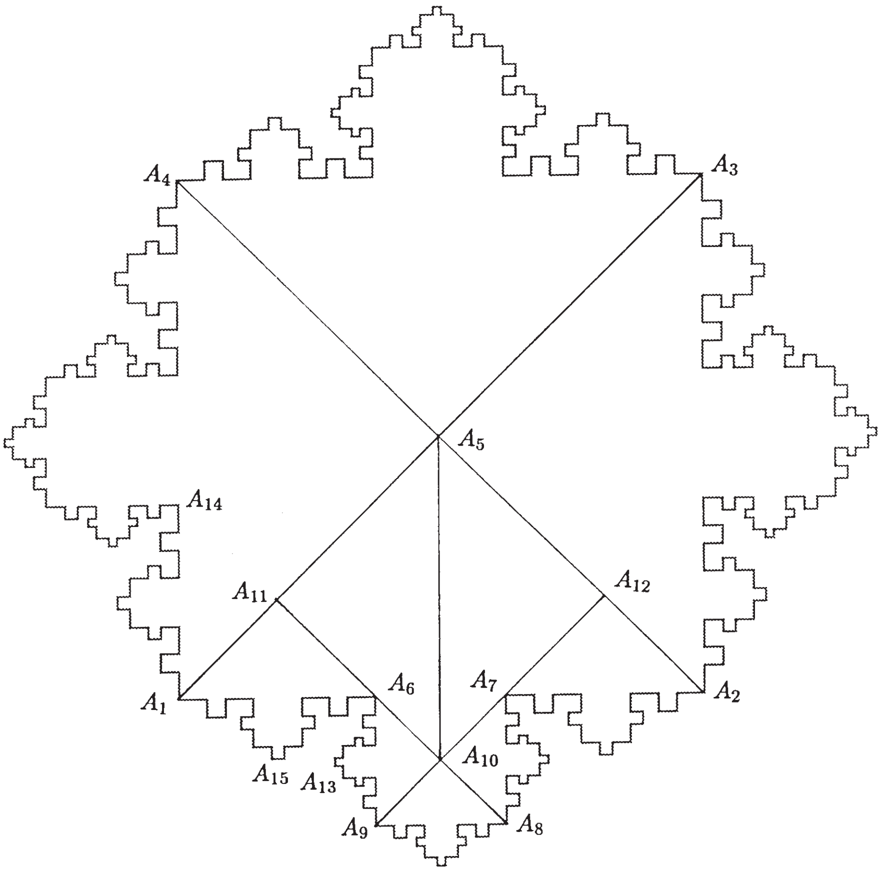

Fig. 1. A sketch of the von Koch snowflake $K_s(s = \frac{1}{4})$, together with the partition used in the proof of Proposition 2.1.

K_s is an open, bounded and simply connected set in $\mathbb{R}^2$ with volume

$$|K_s| = \frac{(1+s)^2}{1 - 7s^2 + 2s}. \tag{1.5}$$

Moreover, K_s is embeddable in $\mathbb{R}^2$ if and only if $0 < s \leq \frac{1}{3}$. One can show that the Hausdorff dimension of ∂K_s and the interior Minkowski (box) dimension of ∂K_s are equal, and are given by the unique positive root d_s of

$$3s^d + 2\left(\frac{1-s}{2}\right)^d = 1. \tag{1.6}$$

See chapter 8 in [9] and chapter 9 in [10]. Note that $1 < d_s \leq (\log 5)/\log 3$.

The heat content for the snowflake $K_{\frac{1}{3}}$ has been analysed by Fleckinger, Levitin and Vassiliev [12, 13]. They proved the existence of two strictly positive, continuous and log 9-periodic functions ψ_1 and χ such that for $t \to 0$

$$E_{K_{\frac{1}{3}}}(t) = \psi_1(-\log t)t^{1-\frac{\log 5}{\log 9}} - \chi(-\log t)t + O\left(e^{-\frac{1}{1152t}}\right). \tag{1.7}$$

In fact their proof was given for the snowflake with an equilateral triangle as a generator. Modifying their proof results in (1.7).

We say that K_s is an arithmetic snowflake if $s \in I$, where

$$I = \left\{ s \in \left(0, \frac{1}{3}\right] : \frac{\log\left(\frac{1-s}{2}\right)}{\log s} = \frac{p}{q},\ p \in \mathbb{N},\ q \in \mathbb{N},\ (p,q) = 1 \right\}. \tag{1.8}$$

The simplest example of an arithmetic snowflake is $K_{\frac{1}{3}}$ $(p = q = 1)$.

It was shown in [6] that if K_s is non-arithmetic ($s \in \left(0, \frac{1}{3}\right) \setminus I$), then there exists a positive constant C_s such that for $t \to 0$

$$E_{K_s}(t) = C_s t^{1-\frac{d_s}{2}} + o\left(t^{1-\frac{d_s}{2}}\right). \tag{1.9}$$

Moreover, if $s \in I$ then there exists a strictly positive, continuous, $\frac{2}{q}\log\frac{1}{s}$-periodic function $\tilde{\psi}$ such that for $t \to 0$,

$$E_{K_s}(t) = \tilde{\psi}(-\log t)t^{1-\frac{d_s}{2}} + o\left(t^{1-\frac{d_s}{2}}\right). \tag{1.10}$$

The main result of this paper (Theorem 1.2) is a refinement of (1.10) up to an exponential remainder, thereby completing the analysis of [13] for the remaining arithmetic snowflakes. Let $s \in I$, and let p and q be the corresponding positive integers in (1.8). Let

$$P(z) = 1 - 3s^{d_s}z^q - 2\left(\frac{1-s}{2}\right)^{d_s} z^p, \tag{1.11}$$

and let $z_1, \ldots, z_q$ be the roots of $P(z) = 0$, ordered such that

$$|z_1| \le |z_2| \le \cdots \le |z_q|. \tag{1.12}$$

The structure of the asymptotic expansion of $E_{K_s}(t)$ as $t \to 0$ and $s \in I$ (Theorem 1.2 (ii) and (iii)) depends on the geometry of $\{z_1, \ldots, z_q\}$, which is summarized in the following.

Proposition 1.1. (i) *All roots of $P(z) = 0$ are simple.*
(ii) *If z is a root with $|z| = r$, then $\overline{z}$ is the only possible other root with modulus r.*

(iii)

$$z_1 = 1, \tag{1.13}$$

$$|z_q| < s^{-\frac{d_s}{q}},\ q \text{ odd}, \tag{1.14}$$

$$z_q = -s^{-\frac{d_s}{q}},\ q \text{ even}. \tag{1.15}$$

Let

$$\sigma_j = \lim_{z \to z_j} \frac{z_j - z}{P(z)}, \tag{1.16}$$

$$z = -\log t, \tag{1.17}$$

$$\gamma = \frac{2}{q} \log \frac{1}{s}, \tag{1.18}$$

and define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(z) = \frac{1}{4} e^{z\left(1 - \frac{d_s}{2}\right)} \Bigg\{ E_{K_s}(e^{-z}) - 3s^2 E_{K_s}\left(s^{-2} e^{-z}\right) - 2\left(\frac{1-s}{2}\right)^2 E_{K_s}\left(\left(\frac{1-s}{2}\right)^{-2} e^{-z}\right) \Bigg\}. \tag{1.19}$$

Theorem 1.2. (i) *ψ is continuous and for $z \in \mathbb{R}$*

$$\sum_{m \in \mathbb{Z}} w^{-m} \psi(z - m\gamma) \tag{1.20}$$

converges absolutely on the annulus $W = \{w \in \mathbb{C} : 1 \le |w| < s^{-d_s/q}\}$. For $w \in W$ define $\psi_w : \mathbb{R} \to \mathbb{C}$ by

$$\psi_w(z) = \sum_{m \in \mathbb{Z}} w^{-m + \frac{1}{\gamma} z} \psi(z - m\gamma). \tag{1.21}$$

Then ψ_w is uniformly continuous and γ-periodic.

(ii) *Suppose $s \in I$ and q is odd. Then there exists a $p\gamma$-periodic, uniformly continuous function $\chi : \mathbb{R} \to \mathbb{R}$ such that for $t \to 0$*

$$E_{K_s}(t) = \sum_{j=1}^{q} t^{1 - \frac{d_s}{2} + \frac{1}{\gamma} \log z_j} \frac{\sigma_j}{z_j} \psi_{z_j}(-\log t) - \chi(-\log t) t + O\left(e^{-\frac{s^2}{32t}}\right). \tag{1.22}$$

(iii) *Suppose $s \in I$ and q is even. Then there exists a $2p\gamma$-periodic, uniformly continuous function $\Lambda : \mathbb{R} \to \mathbb{R}$ such that for $t \to 0$*

$$E_{K_s}(t) = \sum_{j=1}^{q-1} t^{1 - \frac{d_s}{2} + \frac{1}{\gamma} \log z_j} \frac{\sigma_j}{z_j} \psi_{z_j}(-\log t) - \Lambda(-\log t) t + O\left(e^{-\frac{s^2}{32t}}\right). \tag{1.23}$$

We see that the leading exponent of t for $t \to 0$ in (1.22) and (1.23) respectively, comes from the root with the smallest modulus i.e. $z_1 = 1$. Since K_s is simply connected, and the upper and lower d_s-dimensional interior Minkowski contents of

∂K_s are finite and strictly positive, we have by Corollary 1.5 and Proposition 1.6 in [1] that

$$E_{K_s}(t) \asymp t^{1-\frac{d_s}{2}}. \tag{1.24}$$

This implies that $\sigma_1 \psi_1 > 0$ and that the leading term in (1.22) and (1.23) respectively comes from the term with $j = 1$. Comparing (1.22) and (1.23) with (1.10), we obtain

$$\tilde{\psi}(-\log t) = \sigma_1 \sum_{m \in \mathbb{Z}} \psi(-\log t - m\gamma). \tag{1.25}$$

We also see that if $p = q = 1$, then $s = \frac{1}{3}$, $\sigma_1 = 1$ and (1.22) agrees with (1.7). For $q \geq 2$, we see (by Proposition 1.1) that all other terms in the sum over j in (1.22) and (1.23) give contributions which are much larger than t (but are $o\left(t^{1-\frac{d_s}{2}}\right)$).

The proof of Theorem 1.2 is organised as follows. In Section 2 we use the probabilistic solututION of (1.1–1.3) for $D = K_s$ to derive an approximate functional equation for E_{K_s}. This functional equation takes, after appropriate transformations, the form of an inhomogeneous renewal equation. In Section 3 we shall give the analysis of this equation. The combinatorial part is fairly standard (see for example Section A.2 in [14]). The asymptotic part is complicated by the fact that for q even, $|z_q| = s^{-\frac{d_s}{q}}$. It turns out that this root contributes $2p$ terms of logarithmic order. These terms cancel, and result in the $2p\gamma$-periodic function Λ. This complication does not occur for q odd.

As in [6, 7, 12, 13] we were unable to prove that, for example, the function $\tilde{\psi}$ in (1.25) is not a constant function. Similarly, we do not know whether Λ in (1.23) is $2p\gamma$-periodic but not $p\gamma$-periodic. These problems exist due to the fact that we only have the asymptotic properties for $t \to 0$ of the inhomogeneous term in the renewal equation.

It is possible to derive this approximate renewal equation for $p = 1$, $q \geq 1$ following Sections 1 and 2 in [13]. However, the methods used in that paper seem to break down for $p > 1$. The probabilistic methods used in this paper have the advantage that they combine both the maximum principle (or the principle of not feeling the boundary), together with standard scaling properties of functionals of brownian motion. Some of these functionals do not possess a straightforward analytic interpretation as required in [13].

The proof of Proposition 1.1 (i) and (ii) can be found, after a suitable transformation, in Lemma 6.4 of [11]. From (1.6) it is clear that $z = 1$ is a root of $P(z) = 0$, and that $z = -s^{-\frac{d_s}{q}}$ is a root of $P(z) = 0$ if q is even. It remains to prove that all roots are contained in the annulus $\left\{z \in \mathbb{C} : 1 \leq |z| \leq s^{-\frac{d_s}{q}}\right\}$. Indeed if z is a root, then

$$\begin{aligned} 1 &= 3s^{d_s} z^q + 2\left(\frac{1-s}{2}\right)^{d_s} z^p \\ &= \left|3s^{d_s} z^q + 2\left(\frac{1-s}{2}\right)^{d_s} z^p\right| \end{aligned}$$

$$\leq 3s^{d_s}|z|^q + 2\left(\frac{1-s}{2}\right)^{d_s}|z|^p. \tag{1.26}$$

Since the right hand side of (1.26) is strictly increasing in $|z|$, and equals 1 for $|z| = 1$, (by (1.6)), we obtain $|z| \geq 1$. If $q = 1$, then $p = 1$, and $z = 1$ is the only root of $P(z) = 0$. If $s \in I$, $s < \frac{1}{3}$ then $q > p$. Suppose z is a root with $|z| > s^{-\frac{d_s}{q}}$. Then

$$\begin{aligned} 3\left(s^{\frac{d_s}{q}}|z|\right)^q &= \left|1 - 2\left(\frac{1-s}{2}\right)^{d_s} z^p\right| \\ &= \left|1 - 2\left(s^{\frac{d_s}{q}} z\right)^p\right| \\ &\leq 1 + 2\left(s^{\frac{d_s}{q}}|z|\right)^p \\ &< 1 + 2\left(s^{\frac{d_s}{q}}|z|\right)^q, \end{aligned} \tag{1.27}$$

which is a contradiction.

2. The approximate functional equation for E_{K_s}

Let

$$E(t) = \frac{1}{4}E_{K_s}(t) = \frac{1}{4}\int_{K_s} u_{K_s}(x;t)dx, \tag{2.1}$$

and define $H : [0, \infty) \to \mathbb{R}$ by

$$H(t) = E(t) - 3s^2 E\left(\frac{t}{s^2}\right) - 2\left(\frac{1-s}{2}\right)^2 E\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right). \tag{2.2}$$

Note that, by (1.19), (2.1) and (2.2),

$$\psi(z) = e^{z(1-\frac{d_s}{2})}H(e^{-z}). \tag{2.3}$$

Proposition 2.1. *H is continuous and satisfies*

$$|H(t)| \leq \frac{1}{4}|K_s|, \tag{2.4}$$

$$H(t) = F(t) + O\left(e^{-\frac{s^2}{32t}}\right),\ t \to 0, \tag{2.5}$$

where $F : [0, \infty) \to \mathbb{R}$ is continuous and satisfies the functional equation

$$F(t) = \left(\frac{1-s}{2}\right)^2 F\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right), \tag{2.6}$$

and

$$|F(t)| \leq 16\pi t. \tag{2.7}$$

To prove Proposition 2.1, we let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^2)$ be a brownian motion with generator Δ. For any Borel set $C \subset \mathbb{R}^2$, we define the first hitting time by

$$\tau_C = \inf\{s \geq 0 : B(s) \in C\}. \tag{2.8}$$

For any open set $D \subset \mathbb{R}^2$, we define the first exit time T_D by

$$T_D = \tau_{\mathbb{R}^2 \setminus D}. \tag{2.9}$$

We note that if $x \in D$ then

$$T_D = \tau_{\partial D}. \tag{2.10}$$

The solution u_D of (1.1–1.3) has the following probabilistic solution:

$$u_D(x;t) = \mathbb{P}_x[T_D \leq t]. \tag{2.11}$$

The probabilistic solution (2.11) allows us to prove several comparison estimates. The following estimate is a version of Kac's principle of not feeling the boundary.

Lemma 2.2. *Let D be an open set in $\mathbb{R}^2$. Then for all $x \in D$, $t > 0$*

$$u_D(x;t) \leq 2e^{-\frac{d(x,\partial D)^2}{4t}}, \tag{2.12}$$

where

$$d(x, \partial D) = \min\{|x-y| : y \in \partial D\}. \tag{2.13}$$

Proof. See the proof of Lemma 6 in [7].

Lemma 2.3. *Let D be an open set in $\mathbb{R}^2$, and let C be a closed subset of $\mathbb{R}^2$. Then*

$$u_D(x;t) = \mathbb{P}_x\left[\tau_{(\partial D)\cup C} \leq t\right] - \mathbb{P}_x\left[\tau_{\partial D} > t,\ \tau_C \leq t\right]. \tag{2.14}$$

Proof.

$$\begin{aligned} u_D(x;t) &= \mathbb{P}_x\left[\tau_{\partial D} \leq t,\ \tau_C \leq t\right] + \mathbb{P}_x\left[\tau_{\partial D} \leq t,\ \tau_C > t\right] \\ &= \mathbb{P}_x\left[\tau_{\partial D} \leq t,\ \tau_C \leq t\right] + \mathbb{P}_x\left[\tau_{(\partial D)\cup C} \leq t,\ \tau_C > t\right] \\ &= \mathbb{P}_x\left[\tau_{\partial D} \leq t,\ \tau_C \leq t\right] + \mathbb{P}_x\left[\tau_{(\partial D)\cup C} \leq t\right] - \mathbb{P}_x\left[\tau_C \leq t\right] \\ &= \mathbb{P}_x\left[\tau_{(\partial D)\cup C} \leq t\right] - \mathbb{P}_x\left[\tau_{\partial D} > t,\ \tau_C \leq t\right]. \end{aligned} \tag{2.15}$$

The following estimate is a generalisation of Lemma 7 in [7].

Lemma 2.4. *Let D, F and G be open sets in $\mathbb{R}^2$ such that $F \subset D \cap G$. Let C be a closed subset of $\mathbb{R}^2$, and let E be a Borel subset of F with finite measure $|E|$. Then*

$$\left|\int_E \mathbb{P}_x\left[T_D \leq t,\ \tau_C \leq t\right]dx - \int_E \mathbb{P}_x\left[T_G \leq t,\ \tau_C \leq t\right]dx\right| \leq 2|E|e^{-\frac{\delta^2}{4t}}, \tag{2.16}$$

where

$$\delta = \inf\left\{|x-y| : x \in E,\ y \in \overline{(\partial F) \cap (D \cup G)}\right\}. \tag{2.17}$$

Moreover, if $C = \mathbb{R}^2$ then

$$\left|\int_E u_D(x;t)dx - \int_E u_G(x;t)dx\right| \leq 2|E|e^{-\frac{\delta^2}{4t}}. \tag{2.18}$$

Proof. The case $C = \mathbb{R}^2$ has been proved in Lemma 7 of [7]. It follows directly from (2.16). To prove (2.16), we have

$$\begin{aligned}\mathbb{P}_x [T_D \le t,\ \tau_C \le t] &= \mathbb{P}_x [T_D \le t,\ T_G \le t,\ \tau_C \le t] \\ &\quad + \mathbb{P}_x [T_D \le t,\ T_G > t,\ \tau_C \le t] \\ &\le \mathbb{P}_x [T_G \le t,\ \tau_C \le t] + \mathbb{P}_x \left[\tau_{(\partial D)\cap G} \le t,\ T_G > t\right].\end{aligned} \tag{2.19}$$

Since $E \subset F \subset D \cap G$, we have for all $x \in E$, $\left\{\tau_{(\partial D)\cap G} \le t,\ T_G > t\right\} \subseteq \left\{\tau_{\overline{(\partial F)\cap(G\cup D)}} \le t\right\}$. Hence

$$\mathbb{P}_x [T_D \le t,\ \tau_C \le t] - \mathbb{P}_x [T_G \le t,\ \tau_C \le t] \le \mathbb{P}_x \left[\tau_{\overline{(\partial F)\cap(G\cup D)}} \le t\right]. \tag{2.20}$$

By Lemma 2.2 we have for all $x \in E$

$$\mathbb{P}_x [T_D \le t,\ \tau_C \le t] - \mathbb{P}_x [T_G \le t,\ \tau_C \le t] \le 2e^{-\frac{\delta^2}{4t}}, \tag{2.21}$$

where δ is given by (2.17). Integrating (2.21) with respect to x over E gives

$$\int_E \mathbb{P}_x [T_D \le t,\ \tau_C \le t]\, dx - \int_E \mathbb{P}_x [T_G \le t,\ \tau_C \le t]\, dx \le 2|E|e^{-\frac{\delta^2}{4t}}. \tag{2.22}$$

Reversing the roles of D and G in (2.22) completes the proof of (2.16).

Lemma 2.5. *Let E be a non-empty set in $\mathbb{R}^2$ and let $\alpha > 0$. Let $E_{\alpha,A}$ denote the similitude of E by a factor α with respect to a point A, given by*

$$E_{\alpha,A} = \left\{B \in \mathbb{R}^2 : \alpha^{-1}(B - A) \in E\right\}. \tag{2.23}$$

Let C and F be closed sets in $\mathbb{R}^2$, let D be an open set in $\mathbb{R}^2$, and let $E \subset \mathbb{R}^2$ be a Borel set. Then

$$\begin{aligned}&\int_{E_{\alpha,A}} \mathbb{P}_x \left[\tau_{C_{\alpha,A}} > t,\ \tau_{F_{\alpha,A}} > t,\ T_{D_{\alpha,A}} \le t\right] dx \\ &\quad = \alpha^2 \int_E \mathbb{P}_x \left[\tau_C > \frac{t}{\alpha^2},\ \tau_F > \frac{t}{\alpha^2},\ T_D \le \frac{t}{\alpha^2}\right] dx.\end{aligned} \tag{2.24}$$

Proof. This follows directly from the scaling properties of brownian motion.

We introduce the following notation. If A and B are points in $\mathbb{R}^2$, then $\overline{AB}$ is the closed edge with endpoints A and B, and if $A \ne B$ then $l_{A,B}$ is the straight line through A and B. If A and B are points of ∂K_s and $(0,0) \notin \overline{AB}$, then A and B partition ∂K_s into a "large" and a "small" component. Let $A, B \in \partial K_s$ with $(0,0) \notin \overline{AB}$, and let $C \in K_s$. Then S_{ACB} is the open set bounded by the closed segments $\overline{CA}$ and $\overline{CB}$ and the smaller part of ∂K_s bounded by A and by B. Let $A_5 = (0,0)$, $A_6 = \left(-\frac{s}{2}, -\frac{1}{2}\right)$, $A_7 = \left(\frac{s}{2}, -\frac{1}{2}\right)$, $A_8 = \left(\frac{s}{2}, -s - \frac{1}{2}\right)$, $A_9 = \left(-\frac{s}{2}, -s - \frac{1}{2}\right)$,

$A_{10} = \left(0, -\frac{1+s}{2}\right)$, $A_{11} = \left(-\frac{1+s}{4}, -\frac{1+s}{4}\right)$ and $A_{12} = \left(\frac{1+s}{4}, -\frac{1+s}{4}\right)$. The triangle with vertices A_5, A_{10} and A_{11} is denoted by $\Delta_{A_5 A_{10} A_{11}}$.

We note that $\overline{A_1 A_3}$ and $\overline{A_2 A_4}$ partition K_s into 4 congruent sets: $S_{A_1 A_5 A_2}$, $S_{A_2 A_5 A_3}$, $S_{A_3 A_5 A_4}$ and $S_{A_4 A_5 A_1}$. Since $\overline{A_1 A_3}$ and $\overline{A_2 A_4}$ have measure zero, we have by symmetry and by (2.1)

$$E(t) = \frac{1}{4}\int_{K_s} u_{K_s}(x;t)dx = \int_{S_{A_1 A_5 A_2}} u_{K_s}(x;t)dx, \tag{2.25}$$

$$\int_{S_{A_1 A_{11} A_6}} u_{K_s}(x;t)dx = \int_{S_{A_2 A_{12} A_7}} u_{K_s}(x;t)dx, \tag{2.26}$$

$$\int_{S_{A_6 A_{10} A_9}} u_{K_s}(x;t)dx = \int_{S_{A_7 A_{10} A_8}} u_{K_s}(x;t)dx. \tag{2.27}$$

Lemma 2.6. *For $t \to 0$*

$$\int_{S_{A_9 A_{10} A_8}} u_{K_s}(x;t)dx = s^2 E\left(\frac{t}{s^2}\right) + O\left(e^{-\frac{s^2}{16t}}\right). \tag{2.28}$$

Proof. Let C be the s-adic von Koch curve constructed on $\overline{A_6 A_7}$ such that C is the reflection of the (small) part of ∂K_s bounded by A_9 and A_8. The region bounded by ∂K_s (between A_6 and A_7) and C is a similitude of K_s by a factor s. Hence

$$\int_{S_{A_9 A_{10} A_8}} \mathbb{P}_x\left[\tau_{(\partial K_s)\cup C} \le t\right] = s^2 E\left(\frac{t}{s^2}\right). \tag{2.29}$$

Since $d(x, C) \ge \frac{s}{2}$ for $x \in S_{A_9 A_{10} A_8}$, we have by Lemma 2.2

$$\begin{aligned}\int_{S_{A_9 A_{10} A_8}} \mathbb{P}_x\left[\tau_{\partial K_s} > t,\ \tau_C \le t\right] dx &\le \int_{S_{A_9 A_{10} A_8}} \mathbb{P}_x\left[\tau_C \le t\right] dx \\ &= O\left(e^{-\frac{s^2}{16t}}\right).\end{aligned} \tag{2.30}$$

Lemma 2.6 follows from Lemma 2.3 with $D = K_s$ and by (2.29–2.30).

Lemma 2.7. *For $t \to 0$,*

$$\int_{S_{A_6 A_{10} A_9}} u_{K_s}(x;t)dx = s^2 E\left(\frac{t}{s^2}\right) - F_1(t) + O\left(e^{-\frac{s^2}{32t}}\right), \tag{2.31}$$

where F_1 satisfies the functional relation (2.6)*, and*

$$0 \le F_1(t) \le 4\pi t. \tag{2.32}$$

Proof. Let C be as in the proof of Lemma 2.6 and let $\tilde{C}$ be the self-similar extended von Koch curve along the half line through A_6 and A_7 with endpoint A_6. (So $C \subset \tilde{C}$, $\tilde{C}_{\frac{1-s}{2}, A_6} = \tilde{C}$). In a similar manner we denote the extended von Koch curve constructed on the half line through A_6 and A_9 with endpoint A_6 by $\hat{C}$. (So

$\hat{C}_{\frac{1-s}{2},A_6} = \hat{C}$, and $\hat{C}$ contains the part of ∂K_s with endpoints A_6 and A_9). The extended von Koch curve along the half line through A_6 and A_1 with endpoint A_6 which contains the part of ∂K_s with endpoints A_6 and A_1 is denoted by $\check{C}$. (So $\check{C}_{\frac{1-s}{2},A_6} = \check{C}$).

Let G be the open set in $\mathbb{R}^2$, containing K_s, and with boundary $\check{C} \cup \hat{C}$. Let A_{13} be the midpoint of the part of ∂K_s bounded by A_6 and A_9. ($A_{13} = \left(-\frac{s+s^2}{2-2s}, -\frac{1+s}{2}\right)$). Put $D = K_s$ in Lemma 2.3. By scaling we obtain

$$\begin{aligned}\int_{S_{A_6A_{10}A_9}} u_{K_s}(x;t)dx &= s^2 E\left(\frac{t}{s^2}\right) - \int_{S_{A_6A_{10}A_9}} \mathbb{P}_x\left[\tau_{\partial K_s} > t,\ \tau_{\tilde{C}} \le t\right]dx \\ &= s^2 E\left(\frac{t}{s^2}\right) - \int_{S_{A_6A_{10}A_9}} \mathbb{P}_x\left[\tau_{\tilde{C}} \le t\right]dx \\ &\quad + \int_{S_{A_6A_{10}A_9}} \mathbb{P}_x\left[\tau_{\partial K_s} \le t,\ \tau_{\tilde{C}} \le t\right]dx. \end{aligned} \tag{2.33}$$

Since $d(S_{A_{13}A_{10}A_9}, \tilde{C}) \ge \frac{s}{2}$, we have by Lemma 2.3

$$\begin{aligned}\int_{S_{A_{13}A_{10}A_9}} \mathbb{P}_x\left[\tau_{\partial K_s} \le t,\ \tau_{\tilde{C}} \le t\right]dx &\le \int_{S_{A_{13}A_{10}A_9}} \mathbb{P}_x\left[\tau_{\tilde{C}} \le t\right]dx \\ &= O\left(e^{-\frac{s^2}{16t}}\right). \end{aligned} \tag{2.34}$$

Combining (2.33–2.34), we obtain

$$\begin{aligned}\int_{S_{A_6A_{10}A_9}} u_{K_s}(x;t)dx &= s^2 E\left(\frac{t}{s^2}\right) - \int_{S_{A_6A_{10}A_{13}}} \mathbb{P}_x\left[\tau_{\tilde{C}} \le t\right]dx \\ &\quad + \int_{S_{A_6A_{10}A_{13}}} \mathbb{P}_x\left[\tau_{\partial K_s} \le t,\ \tau_{\tilde{C}} \le t\right]dx + O(e^{-\frac{s^2}{16t}}). \end{aligned} \tag{2.35}$$

To estimate the third term in the right hand side of (2.35), we use Lemma 2.4 with

$$E = S_{A_6A_{10}A_{13}},\ F = \left\{x \in K_s : d\left(x, S_{A_6A_{10}A_{13}}\right) < \frac{s}{2}\right\}. \tag{2.36}$$

Then $F \subset K_s \cap G$, and the corresponding δ in (2.17) is equal to $\frac{s}{2}$. By Lemma 2.4,

$$\begin{aligned}&\int_{S_{A_6A_{10}A_{13}}} \mathbb{P}_x\left[\tau_{\partial K_s} \le t,\ \tau_{\tilde{C}} \le t\right]dx \\ &\quad = \int_{S_{A_6A_{10}A_{13}}} \mathbb{P}_x\left[\tau_{\check{C}\cup\hat{C}} \le t,\ \tau_{\tilde{C}} \le t\right]dx + O\left(e^{-\frac{s^2}{16t}}\right). \end{aligned} \tag{2.37}$$

Putting (2.35) and (2.37) together, we obtain

$$\begin{aligned}&\int_{S_{A_6A_{10}A_9}} u_{K_s}(x;t)dx \\ &\quad = s^2 E\left(\frac{t}{s^2}\right) - \int_{S_{A_6A_{10}A_{13}}} \mathbb{P}_x\left[\tau_{\check{C}\cup\hat{C}} > t,\ \tau_{\tilde{C}} \le t\right]dx + O\left(e^{-\frac{s^2}{16t}}\right). \end{aligned} \tag{2.38}$$

Let V be the von Koch sector bounded by $\hat{C}$ and the line $l_{A_6A_{10}}$, which contains $S_{A_6A_{10}A_{13}}$. For $x \in V$ we put $r = d(x, A_6)$. Then $d(x, \tilde{C}) \geq \frac{r}{\sqrt{2}}$, so that by Lemma 2.2,

$$\begin{aligned}\int_{V \setminus S_{A_6A_{10}A_{13}}} \mathbb{P}_x \left[\tau_{\check{C}\cup\hat{C}} > t,\ \tau_{\tilde{C}} \leq t\right] dx &\leq \int_{V \setminus S_{A_6A_{10}A_{13}}} \mathbb{P}_x \left[\tau_{\tilde{C}} \leq t\right] dx \\ &\leq 2 \int_{V \setminus S_{A_6A_{10}A_{13}}} e^{-\frac{d(x,\tilde{C})^2}{4t}} dx \\ &\leq 2 \int_{V \setminus S_{A_6A_{10}A_{13}}} e^{-\frac{r^2}{8t}} dx. \end{aligned} \tag{2.39}$$

In order to estimate the integral on the right hand side of (2.39), we note that V is contained in the region bounded by the line $l_{A_6A_{10}}$ and the line through A_6 perpendicular to $l_{A_6A_{10}}$. Moreover $r \geq \frac{s}{2}$ for $x \in V \setminus S_{A_6A_{10}A_{13}}$. By using polar coordinates, we obtain the following upper bound for (2.39)

$$2 \int_0^{\frac{\pi}{2}} d\varphi \int_{\frac{s}{2}}^{\infty} e^{-\frac{r^2}{8t}} r dr = O\left(e^{-\frac{s^2}{32t}}\right). \tag{2.40}$$

Similarly,

$$\int_V \mathbb{P}_x \left[\tau_{\check{C}\cup\hat{C}} > t,\ \tau_{\tilde{C}} \leq t\right] dx \leq \pi \int_0^{\infty} e^{-\frac{r^2}{8t}} r dr = 4\pi t. \tag{2.41}$$

Finally, we note that

$$\left(\check{C} \cup \hat{C}\right)_{\frac{1-s}{2}, A_6} = \check{C} \cup \hat{C}, \quad \tilde{C}_{\frac{1-s}{2}, A_6} = \tilde{C}, \quad V_{\frac{1-s}{2}, A_6} = V, \tag{2.42}$$

so that the left hand side of (2.41) (denoted by $F_1(t)$) satisfies the functional relation (2.6) by an identity similar to (2.24). The proof follows by (2.38–2.42).

Lemma 2.8. *For* $t \to 0$

$$\int_{\Delta_{A_5A_{10}A_{11}}} u_{K_s}(x;t) dx = F_2(t) + O\left(e^{-\frac{s^2}{32t}}\right), \tag{2.43}$$

where F_2 *satisfies the functional relation* (2.6)*, and*

$$0 \leq F_2(t) \leq 8\pi t. \tag{2.44}$$

Proof. Let $\check{C}$ and $\hat{C}$ be as in the proof of Lemma 2.7. We approximate ∂K_s near the point A_6 by $\check{C} \cup \hat{C}$, and use Lemma 2.4 with $D = K_s$, $C = \mathbb{R}^2$,

$$F = \left\{x \in K_s : d\left(x, \Delta_{A_5A_{10}A_{11}}\right) < \frac{s}{2}\right\}, \tag{2.45}$$

and G as the open set with boundary $\check{C} \cup \hat{C}$ which contains K_s. Then $F \subset K_s \cap G$, and the corresponding δ in (2.17) is equal to $\frac{s}{2}$. By (2.18)

$$\int_{\Delta_{A_5A_{10}A_{11}}} u_{K_s}(x;t) dx = \int_{\Delta_{A_5A_{10}A_{11}}} \mathbb{P}_x \left[\tau_{\check{C}\cup\hat{C}} \leq t\right] dx + O\left(e^{-\frac{s^2}{16t}}\right). \tag{2.46}$$

Let V be the half space bounded by $l_{A_{10}A_{11}}$ which contains $\Delta_{A_5A_{10}A_{11}}$. For $x \in V$, we put $r = d(x, A_6)$. Then $d(x, \check{C} \cup \hat{C}) \geq \frac{r}{\sqrt{2}}$, so that by Lemma 2.2,

$$\int_{V \setminus \Delta_{A_5A_{10}A_{11}}} \mathbb{P}_x \left[\tau_{\check{C} \cup \hat{C}} \leq t\right] dx \leq 2 \int_{V \setminus \Delta_{A_5A_{10}A_{11}}} e^{-\frac{r^2}{8t}} dx. \tag{2.47}$$

In order to estimate the integral in the right hand side of (2.47), we note that $r \geq \frac{s}{2}$ for $x \in V \setminus \Delta_{A_5A_{10}A_{11}}$. By using polar coordinates, we obtain that (2.47) is bounded by

$$2 \int_0^{\pi} d\varphi \int_{\frac{s}{2}}^{\infty} e^{-\frac{r^2}{8t}} r dr = O\left(e^{-\frac{s^2}{32t}}\right). \tag{2.48}$$

Similarly,

$$\int_V \mathbb{P}_x \left[\tau_{\check{C} \cup \hat{C}} \leq t\right] dx \leq 2\pi \int_0^{\infty} e^{-\frac{r^2}{8t}} r dr = 8\pi t. \tag{2.49}$$

Again, $\check{C} \cup \hat{C}$ and V satisfy the scaling identities (2.42), and the left hand side of (2.49) (denoted by $F_2(t)$) satisfies (2.6), by an identity similar to (2.24). The proof follows by (2.47–2.49).

Lemma 2.9. *For $t \to 0$*

$$\int_{S_{A_1A_{11}A_6}} u_{K_s}(x; t) dx = \left(\frac{1-s}{2}\right)^2 E\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right) - F_3(t) + O\left(e^{-\frac{s^2}{32t}}\right), \tag{2.50}$$

where F_3 satisfies (2.6) *and*

$$0 \leq F_3(t) \leq 4\pi t. \tag{2.51}$$

Proof. Let $A_{14} = \left(-\frac{1}{2}, -\frac{s}{2}\right)$, and let $\overline{C}$ denote the reflection of the part of ∂K_s bounded by A_6 and A_{14} with respect to $l_{A_6A_{14}}$. The region bounded by ∂K_s (between A_6 and A_{14}) and $\overline{C}$ is a similitude of K_s by a factor $\frac{1-s}{2}$. Hence

$$\int_{S_{A_1A_{11}A_6}} \mathbb{P}_x \left[\tau_{(\partial K_s) \cup \overline{C}} \leq t\right] dx = \left(\frac{1-s}{2}\right)^2 E\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right). \tag{2.52}$$

By using Lemma 2.3 with $D = K_s$, we obtain by (2.52)

$$\int_{S_{A_1A_{11}A_6}} u_{K_s}(x; t) dx = \left(\frac{1-s}{2}\right)^2 E\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right) - \int_{S_{A_1A_{11}A_6}} \mathbb{P}_x \left[\tau_{\partial K_s} > t, \tau_{\overline{C}} \leq t\right] dx. \tag{2.53}$$

Let $A_{15} = \left(-\frac{1+s}{4}, -\frac{1+s}{2}\right)$ be the midpoint of the part of ∂K_s bounded by A_1 and A_6. Since $d(S_{A_1 A_{11} A_{15}}, \overline{C}) \geq \frac{1-s}{4} \geq \frac{s}{2}$, we have by Lemma 2.2

$$\int_{S_{A_1 A_{11} A_{15}}} \mathbb{P}_x \left[\tau_{\partial K_s} > t, \tau_{\overline{C}} \leq t\right] dx \leq \int_{S_{A_1 A_{11} A_{15}}} \mathbb{P}_x \left[\tau_{\overline{C}} \leq t\right] dx$$
$$= O\left(e^{-\frac{s^2}{16t}}\right), \tag{2.54}$$

and so it remains to estimate the contribution from $S_{A_{15} A_{11} A_6}$ to the integral in the right hand side of (2.53). Let $\overline{C}$ intersect $l_{A_1 A_3}$ in A_{16}, and denote by $\underline{C}$ the self-similar extension of the part $\overline{C}$ between A_6 and A_{16} along the half line through these points and with endpoint A_6. (So $\underline{C}_{\frac{1-s}{2}, A_6} = \underline{C}$).

Since $d(x, \overline{C} \setminus \underline{C}) \geq \frac{1-s}{4} \geq \frac{s}{2}$ for $x \in S_{A_{15} A_{11} A_6}$ we have by Lemma 2.2 and Lemma 2.3

$$\int_{S_{A_{15} A_{11} A_6}} \mathbb{P}_x \left[\tau_{\partial K_s} > t, \tau_{\overline{C}} \leq t\right] dx = \int_{S_{A_{15} A_{11} A_6}} \mathbb{P}_x \left[\tau_{\partial K_s} > t, \tau_{\underline{C}} \leq t\right] dx$$
$$+ O\left(e^{-\frac{s^2}{16t}}\right). \tag{2.55}$$

Next we approximate ∂K_s near A_6 by $\check{C} \cup \hat{C}$, and use Lemma 2.4 with $D = K_s$, G as in the proof of Lemmas 2.7 and 2.8 and

$$F = \left\{x \in K_s : d\left(x, S_{A_{15} A_{11} A_6}\right) < \frac{s}{2}\right\}. \tag{2.56}$$

This gives

$$\int_{S_{A_{15} A_{11} A_6}} \mathbb{P}_x \left[\tau_{\partial K_s} > t, \tau_{\underline{C}} \leq t\right] dx = \int_{S_{A_{15} A_{11} A_6}} \mathbb{P}_x \left[\tau_{\check{C} \cup \hat{C}} > t, \tau_{\underline{C}} \leq t\right] dx$$
$$+ O\left(e^{-\frac{s^2}{16t}}\right). \tag{2.57}$$

Let V be the region bounded by $l_{A_6 A_{11}}$ and $\check{C}$ which contains $S_{A_{15} A_{11} A_6}$. Put $r = d(x, A_6)$. Then for $x \in V$, $d(x, \underline{C}) \geq \frac{r}{\sqrt{2}}$, and $r \geq \frac{1-s}{4} \geq \frac{s}{2}$ for $x \in V \setminus S_{A_{15} A_{11} A_6}$. Moreover V is contained in a wedge with vertex A_6 bounded by $l_{A_6 A_{11}}$ and the line perpendicular to $l_{A_6 A_{11}}$ through A_6. Following estimates similar to the ones in Lemma 2.7, we obtain

$$\int_{V \setminus S_{A_{15} A_{11} A_6}} \mathbb{P}_x \left[\tau_{\check{C} \cup \hat{C}} > t, \tau_{\underline{C}} \leq t\right] dx = O\left(e^{-\frac{s^2}{32t}}\right), \tag{2.58}$$

$$\int_{V} \mathbb{P}_x \left[\tau_{\check{C} \cup \hat{C}} > t, \tau_{\underline{C}} \leq t\right] dx \leq 4\pi t. \tag{2.59}$$

The proof follows from (2.53–2.59), where $F_3(t)$ is given by the left hand side of (2.59).

Proof of Proposition 2.1. Since

$$S_{A_1A_5A_2} = S_{A_1A_{11}A_6} \cup S_{A_6A_{10}A_9} \cup S_{A_9A_{10}A_8} \cup S_{A_8A_{10}A_7} \cup S_{A_7A_{12}A_2} \\ \cup S_{A_5A_{10}A_{11}} \cup S_{A_5A_{10}A_{12}} \cup N, \tag{2.60}$$

where N is a set of measure zero, we have by symmetry (2.25–2.27)

$$E(t) = 2\int_{S_{A_1A_{11}A_6}} u_{K_s}(x;t)dx + 2\int_{S_{A_6A_{10}A_9}} u_{K_s}(x;t)dx \\ + 2\int_{S_{A_5A_{10}A_{11}}} u_{K_s}(x;t)dx + \int_{S_{A_9A_{10}A_8}} u_{K_s}(x;t)dx. \tag{2.61}$$

We obtain by (2.61) and Lemmas 2.6–2.9 for $t \to 0$

$$E(t) = 3s^2 E\left(\frac{t}{s^2}\right) + 2\left(\frac{1-s}{2}\right)^2 E\left(\frac{t}{\left(\frac{1-s}{2}\right)^2}\right) \\ - 2F_1(t) + 2F_2(t) - 2F_3(t) + O\left(e^{-\frac{s^2}{32t}}\right). \tag{2.62}$$

The linear combination $-2F_1 + 2F_2 - 2F_3$ satisfies (2.6) since each of F_1, F_2 and F_3 satisfies (2.6). To complete the proof of the Proposition, we put

$$F(t) = -2F_1(t) + 2F_2(t) - 2F_3(t). \tag{2.63}$$

Then (2.5) follows from (2.62–2.63), and (2.7) follows from (2.32), (2.44) and (2.51). Finally, since $0 \le u_{K_s} \le 1$, we have by (2.1),

$$0 \le E(t) \le \frac{1}{4}|K_s|. \tag{2.64}$$

By (2.2) and (2.64)

$$-\frac{1}{4}\left(3s^2 + 2\left(\frac{1-s}{2}\right)^2\right)|K_s| \le H(t) \le \frac{1}{4}|K_s|, \tag{2.65}$$

which implies (2.4).

3. Proof of Theorem 1.2

To prove part (i) of Theorem 1.2 we note that ψ is continuous by definition (1.19) and the continuity of E_{K_s}. By (1.19) and (2.1–2.4),

$$|\psi(z)| \le 4^{-1} e^{z\left(1-\frac{d_s}{2}\right)} |K_s|, \; z \in \mathbb{R}. \tag{3.1}$$

Hence for $w \in W$,

$$\sum_{m=0}^{\infty} |w|^{-m} |\psi(z - m\gamma)| \le 4^{-1}|K_s| e^{z\left(1-\frac{d_s}{2}\right)} \sum_{m=0}^{\infty} e^{-m\gamma\left(1-\frac{d_s}{2}\right)} < \infty. \tag{3.2}$$

On the other hand by (2.4), (2.5) and (2.7), there exists a constant $c < \infty$ such that $|H(t)| \leq ct$ for all $t > 0$. Hence

$$|\psi(z)| \leq ce^{-z\frac{d_s}{2}},\ z \in \mathbb{R}, \tag{3.3}$$

and for $w \in W$,

$$\sum_{m=1}^{\infty} |w|^m |\psi(z+m\gamma)| \leq ce^{-z\frac{d_s}{2}} \sum_{m=1}^{\infty} \left(|w|e^{-\gamma\frac{d_s}{2}}\right)^m < \infty, \tag{3.4}$$

by definition of γ and W. The absolute convergence of the series in (1.20) for $w \in W$ follows from (3.2) and (3.4).

The function $z \mapsto w^{-m}\psi(z - m\gamma)$ tends to zero exponentially as $m \to \infty$ by (3.1), and as $m \to -\infty$ by (3.3). So the series in the right hand side of (1.21) converges uniformly on compact subsets of $\mathbb{R}$. Hence ψ_w is continuous. Substitution of $z \mapsto z + \gamma$ and $m \mapsto m + 1$ in (1.21) proves the γ-periodicity of ψ_w in z. Finally, periodic, continuous functions are uniformly continuous.

To prove parts (ii) and (iii) of Theorem 1.2, we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(z) = e^{z\left(1-\frac{d_s}{2}\right)} E(e^{-z}). \tag{3.5}$$

Substitution of (3.5) into (2.2) gives by (1.19)

$$f(z) = 3s^{d_s} f(z - q\gamma) + 2\left(\frac{1-s}{2}\right)^{d_s} f(z - p\gamma) + \psi(z). \tag{3.6}$$

From (2.64) and (3.5) we obtain the boundary condition

$$\lim_{z\to-\infty} f(z) = 0. \tag{3.7}$$

Let $C(\mathbb{R})$ be the space of bounded complex-valued continuous functions equipped with the uniform metric. Define $L : C(\mathbb{R}) \to C(\mathbb{R})$ by

$$(Lf)(z) = 3s^{d_s} f(z - q\gamma) + 2\left(\frac{1-s}{2}\right)^{d_s} f(z - p\gamma). \tag{3.8}$$

By (3.1) and (3.3),

$$|\psi(z)| \leq (c + 4^{-1}|K_s|)e^{-|z|\min\left\{\frac{d_s}{2},1-\frac{d_s}{2}\right\}},\ z \in \mathbb{R}. \tag{3.9}$$

By the renewal theorem (p. 198 in [14]), the solution of $f = Lf + \psi$, (3.7) in $C(\mathbb{R})$ is unique and is given by

$$f(z) = \sum_{n=0}^{\infty} (L^n\psi)(z). \tag{3.10}$$

Expanding the powers of L in (3.10) gives

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(3s^{d_s}\right)^k \left(2\left(\frac{1-s}{2}\right)^{d_s}\right)^{n-k} \psi(z - kq\gamma - (n-k)p\gamma). \tag{3.11}$$

Let c_m denote the coefficient of $\psi(z - m\gamma)$ in the right hand side of (3.11). A straightforward calculation yields

$$\sum_{m=0}^{\infty} c_m z^m = (P(z))^{-1}, \quad |z| < 1. \tag{3.12}$$

By Proposition 1.1 (i), all roots of $P(z) = 0$ are simple, and so by (1.16)

$$(P(z))^{-1} = \sum_{j=1}^{q} \frac{\sigma_j}{z_j - z}, \tag{3.13}$$

where

$$\sigma_j = \left(3qs^{d_s} z_j^{q-1} + 2p\left(\frac{1-s}{2}\right)^{d_s} z_j^{p-1}\right)^{-1}. \tag{3.14}$$

Expanding the right hand side of (3.13) in a power series about 0 and comparing the powers of z^m with (3.12), we obtain

$$c_m = \sum_{j=1}^{q} \sigma_j z_j^{-1-m}. \tag{3.15}$$

Hence

$$f(z) = \sum_{m=0}^{\infty} \sum_{j=1}^{q} \sigma_j z_j^{-1-m} \psi(z - m\gamma). \tag{3.16}$$

We consider the two cases:

Suppose q odd. Let

$$\alpha = |z_q| s^{\frac{d_s}{q}}. \tag{3.17}$$

Then $0 < \alpha < 1$ by (1.14), and $|z_j| \le \alpha s^{-\frac{d_s}{q}}$ for $j = 1, \dots, q$ by (1.12). Hence by (3.3), (1.12) and (1.13),

$$\sum_{m=1}^{\infty} |z_j|^{-1+m} |\psi(z + m\gamma)| \le c e^{-z\frac{d_s}{2}} \sum_{m=1}^{\infty} \alpha^m < \infty. \tag{3.18}$$

By (3.16), (3.18) and (1.21), we obtain

$$f(z) = \sum_{j=1}^{q} \sigma_j z_j^{-1-\frac{z}{\gamma}} \psi_{z_j}(z) - \sum_{m=1}^{\infty} \sum_{j=1}^{q} \sigma_j z_j^{-1+m} \psi(z + m\gamma). \tag{3.19}$$

Define $\hat{\chi} : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = t\hat{\chi}(-\log t). \tag{3.20}$$

By Proposition 2.1, $\hat{\chi}$ is continuous, $p\gamma$-periodic, and satisfies

$$|\hat{\chi}(-\log t)| \leq 16\pi. \tag{3.21}$$

By (2.5) and (3.20), there exists a constant $\hat{c}$ such that for $z \geq 0$ and $m \geq 0$,

$$\left|H(e^{-z-m\gamma}) - e^{-z-m\gamma}\hat{\chi}(z+m\gamma)\right| \leq \hat{c}e^{-\frac{s^2}{32}e^{z+m\gamma}}. \tag{3.22}$$

Hence by (2.3), (3.22) and the $p\gamma$-periodicity of $\hat{\chi}$ we obtain for $z \to \infty$

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{j=1}^{q}\sigma_j z_j^{-1+m}\psi(z+m\gamma)\\
&\quad= \sum_{m=1}^{\infty}\sum_{j=1}^{q}\sigma_j z_j^{-1+m}e^{(z+m\gamma)(1-\frac{d_s}{2})}H(e^{-z-m\gamma})\\
&\quad= \sum_{m=1}^{\infty}\sum_{j=1}^{q}\sigma_j z_j^{-1+m}e^{-\frac{d_s}{2}(z+m\gamma)}\hat{\chi}(z+m\gamma) + R(z)\\
&\quad= \sum_{m=1}^{p}\sum_{j=1}^{q}\sigma_j z_j^{-1+m}\left(1 - z_j^p s^{d_s\frac{p}{q}}\right)^{-1}e^{-\frac{d_s}{2}(z+m\gamma)}\hat{\chi}(z+m\gamma) + R(z),
\end{aligned} \tag{3.23}$$

where for $z \to \infty$

$$\begin{aligned}
|R(z)| &= \left|\hat{c}\sum_{m=1}^{\infty}\sum_{j=1}^{q}\sigma_j z_j^{-1+m}e^{(z+m\gamma)(1-\frac{d_s}{2})}e^{-\frac{s^2}{32}e^{z+m\gamma}}\right|\\
&\leq \hat{c}\sum_{j=1}^{q}|\sigma_j z_j^{-1}|\sum_{m=1}^{\infty}s^{-\frac{md_s}{q}}e^{(z+m\gamma)(1-\frac{d_s}{2})}e^{-\frac{s^2}{32}e^{z+m\gamma}}\\
&\leq \hat{c}\sum_{j=1}^{q}|\sigma_j z_j^{-1}|e^{z(1-\frac{d_s}{2})}\sum_{m=1}^{\infty}e^{m\gamma-\frac{s^2}{32}e^{z+m\gamma}}\\
&= O\left(e^{-\frac{s^2}{32}e^{z}}\right).
\end{aligned} \tag{3.24}$$

Since a finite sum of $p\gamma$-periodic, continuous functions is $p\gamma$-periodic and uniformly continuous, we conclude by (3.5), (3.19) and (3.23) that (1.22) holds with

$$\chi(z) = \sum_{m=1}^{p}\sum_{j=1}^{q}\sigma_j z_j^{-1+m}\left(1 - z_j^p s^{d_s\frac{p}{q}}\right)^{-1}s^{d_s\frac{m}{q}}\hat{\chi}(z+m\gamma). \tag{3.25}$$

Suppose q even. Since $z_q = -s^{-\frac{d_s}{q}}$, $\alpha = 1$, and the argument following (3.17) fails. Instead of (3.17) we redefine α by

$$\alpha = |z_{q-1}| s^{\frac{d_s}{q}}. \tag{3.26}$$

By Proposition 1.1, $\alpha < 1$. We estimate the contribution from the terms $j = 1, \ldots, q-1$ in (3.16) by modifying (3.18–3.25) with α given by (3.26). This gives

$$f(z) = \sum_{j=1}^{q-1} \sigma_j z_j^{-1-\frac{z}{\gamma}} \psi_{z_j}(z) - \sum_{m=1}^{p} \sum_{j=1}^{q-1} \sigma_j z_j^{-1+m} \left(1 - z_j^p s^{d_s \frac{p}{q}}\right)^{-1} \cdot e^{-\frac{d_s}{2}(z+m\gamma)} \hat{\chi}(z+m\gamma) + \sum_{m=0}^{\infty} \sigma_q z_q^{-1-m} \psi(z-m\gamma) + O\left(e^{-\frac{s^2}{32}e^z}\right). \tag{3.27}$$

By (1.15) and (3.14), σ_q is nonzero, and is given by

$$\sigma_q = (2p-3q)^{-1} s^{-\frac{d_s}{q}}. \tag{3.28}$$

Hence the third term in the right hand side of (3.27) is given by

$$(3q-2p)^{-1} e^{-z\frac{d_s}{2}} \sum_{m=0}^{\infty} (-1)^m e^{z-m\gamma} H\left(e^{-z+m\gamma}\right). \tag{3.29}$$

Let

$$K(z) = H\left(e^{-z}\right) - e^{-z}\hat{\chi}(z), \tag{3.30}$$

and denote the sum over m in (3.29) by $G(z)$. Then by (3.30),

$$\begin{aligned} G(z) - G(z+2p\gamma) &= \sum_{m=0}^{\infty} (-1)^m e^{z-m\gamma} H\left(e^{-z+m\gamma}\right) \\ &\quad - \sum_{m=0}^{\infty} (-1)^m e^{z+2p\gamma-m\gamma} H\left(e^{-z-2p\gamma+m\gamma}\right) \\ &= -\sum_{m=1}^{2p} (-1)^m e^{z+m\gamma} H\left(e^{-z-m\gamma}\right) \\ &= -\sum_{m=1}^{2p} (-1)^m e^{z+m\gamma} K(z+m\gamma) - \sum_{m=1}^{2p} (-1)^m \hat{\chi}(z+m\gamma) \\ &= -\sum_{m=1}^{2p} (-1)^m e^{z+m\gamma} K(z+m\gamma), \end{aligned} \tag{3.31}$$

by the $p\gamma$-periodicity of $\hat{\chi}$. Denoting the right hand side of (3.31) by $L(z)$, we have by (3.22) and (3.30)

$$\begin{aligned} |L(z)| &\leq 2p\hat{c}e^{2p\gamma} e^{z-\frac{s^2}{32}e^{z+\gamma}} \\ &\leq \frac{64}{\gamma s^2} e^{2p\gamma-1} e^{-\frac{s^2}{32}e^z}, \; z \geq 0. \end{aligned} \tag{3.32}$$

Following Lemma 2.5 in [13], we define

$$M(z) = \sum_{k=0}^{\infty} L(z + 2p\gamma k) - G(z). \tag{3.33}$$

The series in (3.33) converges absolutely, and it follows from the definition of $L(z)$ and (3.33) that

$$M(z + 2p\gamma) = M(z), \; z \in \mathbb{R}. \tag{3.34}$$

Moreover, by (3.32)

$$\sum_{k=0}^{\infty} |L(z + 2p\gamma k)| = O\left(e^{-\frac{s^2}{32}e^z}\right). \tag{3.35}$$

Putting (3.29) and (3.33–3.35) together, we obtain that the third term in the right hand side of (3.27) is equal to

$$-(3q - 2p)^{-1} e^{-z\frac{d_s}{2}} M(z) + O\left(e^{-\frac{s^2}{32}e^z}\right). \tag{3.36}$$

Hence (1.23) holds by (3.5), (3.27) and (3.36) with

$$\begin{aligned} \Lambda(z) = &\sum_{m=1}^{p} \sum_{j=1}^{q-1} \sigma_j z_j^{-1+m} \left(1 - z_j^p s^{d_s \frac{p}{q}}\right)^{-1} s^{d_s \frac{m}{q}} \hat{\chi}(z + m\gamma) \\ &+ (3q - 2p)^{-1} M(z). \end{aligned} \tag{3.37}$$

This completes the proof of (iii), since a finite sum of $2p\gamma$- and $p\gamma$-periodic continuous functions is $2p\gamma$-periodic and uniformly continuous.

References

[1] van den Berg, M.: Heat content and brownian motion for some regions with a fractal boundary, Probab. Theory Relat. Fields, **100**, 439–456 (1994)

[2] van den Berg, M.: Heat content asymptotics for planar regions with cusps. J. Lond. Math. Soc., **57**(2), 677–693 (1998)

[3] van den Berg, M., Davies, E.B.: Heat flow out of regions in $\mathbb{R}^m$, Math. Z., **202**, 463–482 (1989)

[4] van den Berg, M., Le Gall, J.F.: Mean curvature and the heat equation, Math. Z., **215**, 437–464 (1994)

[5] van den Berg, M., Gilkey, P.B.; Heat content asymptotics of a riemannian manifold with boundary, J. Funct. Anal., **120**, 48–71 (1994)

[6] van den Berg, M., Gilkey, P.B.: A comparison estimate for the heat equation with an application to the heat content of the s-adic von Koch snowflake, Bull. Lond. Math. Soc., **30**, 404–412 (1998)
[7] van den Berg, M., den Hollander, F.: Asymptotics for the heat content of a planar region with a fractal polygonal boundary, Proc. Lond. Math. Soc., **78**(3), 627–661 (1999)
[8] van den Berg, M., Srisatkunarajah, S.: Heat flow and brownian motion for a region in $\mathbb{R}^2$ with a polygonal boundary, Probab. Theory Relat. Fields, **86**, 41–52 (1990)
[9] Falconer, K.J.: The geometry of fractal sets, (Cambridge Univ. Press, Cambridge 1992)
[10] Falconer, K.J.: Fractal geometry, mathematical foundations and applications, (Wiley, Chichester 1990)
[11] Fawkes, J.; The behaviour of the spectral counting function for a family of sets with fractal boundaries, J. Lond. Math. Soc., **55**(2), 126–138 (1997)
[12] Fleckinger, J., Levitin, M., Vassiliev, D.: Heat content of the triadic von Koch snowflake, Internat. J. Appl. Sci. Comput., **2**, 289–305 (1995)
[13] Fleckinger, J., Levitin, M., Vassiliev, D.; Heat equation on the triadic von Koch snowflake: asymptotic and numerical analysis, Proc. Lond. Math. Soc., **71**(3), 372–396 (1995)
[14] Levitin, M., Vassiliev, D.: Spectral asymptotics, renewal theorem and the Berry conjecture for a class of fractals, Proc. Lond. Math. Soc., **72**(3), 188–214 (1996)
[15] Savo, A.: Heat content and mean curvature, Rend. Mat. Appl. **18**, 197–219 (1998)
[16] Savo, A., Uniform estimates and the whole asymptotic series of the heat content on manifolds, Geometriae Dedicata, **73**, 181–214 (1998)