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Non-Abelian supertubes

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Abstract: A supertube is a supersymmetric configuration in string theory which occurs when a pair of branes spontaneously polarizes and generates a new dipole charge extended along a closed curve. The dipole charge of a codimension-2 supertube is characterized by the U-duality monodromy as one goes around the supertube. For multiple codimension-2 supertubes, their monodromies do not commute in general. In this paper, we construct a supersymmetric solution of five-dimensional supergravity that describes two supertubes with such non-Abelian monodromies, in a certain perturbative expansion. In supergravity, the monodromies are realized as the multi-valuedness of the scalar fields, while in higher dimensions they correspond to non-geometric duality twists of the internal space. The supertubes in our solution carry NS5 and 5^2_2 dipole charges and exhibit the same monodromy structure as the SU(2) Seiberg-Witten geometry. The perturbative solution has $\mathrm{AdS}_2 \times S^2$ asymptotics and vanishing four-dimensional angular momentum. We argue that this solution represents a microstate of four-dimensional black holes with a finite horizon and that it provides a clue for the gravity realization of a pure-Higgs branch state in the dual quiver quantum mechanics.

Keywords: Black Holes in String Theory, D-branes, Spacetime Singularities, Supergravity Models

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Contents

1 Introduction and summary

1.1 Background

The fact that black holes have thermodynamical entropy means that there must be many underlying microstates that account for it. Because string theory is a microscopic theory of gravity, i.e., quantum gravity, all these microstates must be describable within string theory, at least as far as black holes that exist in string theory are concerned. A microstate must be a configuration in string theory with the same mass, angular momentum and charge as the black hole it is a microstate of, and the scattering in the microstate must be well-defined as a unitary process. The fuzzball conjecture [1–5] claims that typical microstates spread over a macroscopic distance of the would-be horizon scale. More recent arguments [6, 7] also support the view that the conventional picture of black holes must be modified at the horizon scale and replaced by some non-trivial structure.

The microstates for generic non-extremal black holes are expected to involve stringy excitations and, to describe them properly, we probably need quantum string field theory. However, for supersymmetric black holes, the situation seems much more tractable. Many microstates for BPS black holes have been explicitly constructed as regular, horizonless solutions of supergravity — the massless sector of superstring theory. It is reasonable that the massless sector plays an important role for black-hole microstates because the large-distance structure expected of the microstates can only be supported by massless fields [8]. It is then natural to ask how many microstates of BPS black holes are realized within supergravity. This has led to the so-called "microstate geometry program" (see, e.g., [9]), which is about explicitly constructing as many black-hole microstates as possible, as regular, horizonless solutions in supergravity.

A useful setup in which many supergravity microstates have been constructed is five-dimensional $\mathcal{N}=1$ ungauged supergravity with vector multiplets, for which all supersymmetric solutions have been classified [10, 11]. This theory describes the low-energy physics of M-theory compactified on a Calabi-Yau 3-fold X or, in the presence of an additional S^1 [10, 12], of type IIA string theory compactified on X. The supersymmetric solutions are completely characterized by a set of harmonic functions on a spatial $\mathbb{R}^3$ base, which we collectively denote by H. We will call these solutions *harmonic solutions*. If we assume that H has codimension-3 singularities, its general form is

$$H(\mathbf{x}) = h + \sum_{p=1}^{N} \frac{\Gamma_p}{|\mathbf{x}-\mathbf{a}_p|}. \tag{1.1}$$

The associated supergravity solution generically represents a bound state of N black-hole centers which sit at $\mathbf{x}=\mathbf{a}_p$ $(p=1,\ldots,N)$ and are made of D6, D4, D2, and D0-branes represented by the charge vectors Γ_p. In the current paper, we take $X=T^6=T^2_{45}\times T^2_{67}\times T^2_{89}$ and the D-branes wrap some of the tori directions.

By appropriately choosing the parameters in the harmonic functions, the harmonic solutions with codimension-3 centers, (1.1), can describe regular, horizonless 5D geometries that are microstates of black holes with finite horizons [13, 14]. However, although they represent a large family of microstate geometries, it has been argued that they are not sufficient for explaining the black-hole entropy [15, 16].

In fact, physical arguments naturally motivate us to generalize the codimension-3 harmonic solutions, which leads to more microstates and larger entropy. One possible way of generalization is to go to six dimensions. This is based on the CFT analysis [17] which suggests that generic black-hole microstates must have traveling waves in the sixth direction and thus depend on it. This intuition led to an ansatz for 6D solutions [18], based on which a new class of microstate geometries with traveling waves, called superstrata, was constructed [19]. For recent developments in constructing superstratum solutions, see [20–23].

The other natural way to generalize the codimension-3 harmonic solutions (1.1) is to consider *codimension-2* sources in harmonic functions. This generalization is naturally motivated by the supertube transition [24] which in the context of harmonic solutions implies that, when certain combinations of codimension-3 branes are put together, they will spontaneously polarize into a new codimension-2 brane. For example, if we bring two orthogonal D2-branes together, they polarize into an NS5-brane along an arbitrary closed curve parametrized by λ. We represent this process by the following diagram:

$$\mathrm{D2}(45) + \mathrm{D2}(67) \to \mathrm{ns5}(\lambda 4567), \tag{1.2}$$

where D2(45) denotes the D2-brane wrapped on T^2_{45} and "ns5" in lowercase means that it is a dipole charge, being along a closed curve. The original D2(45) and D2(67)-branes appeared in the harmonic functions as codimension-3 singularities, as in (1.1). The process (1.2) means that those codimension-3 singularities can transition into a codimension-2 singularity in the harmonic function along the curve λ. Another example of possible supertube transitions is

$$\mathrm{D2}(89) + \mathrm{D6}(456789) \to 5^2_2(\lambda 4567; 89), \tag{1.3}$$

where 5^2_2 is a non-geometric exotic brane [25–31] which is obtained by two transverse T-dualities of the NS5-brane [30, 31].

We emphasize that the supertube transition is not an option but a must; if two codimension-3 branes that can undergo a supertube transition are put together, they will, because the supertube is the intrinsic description of the bound state [1, Section 3.1]. This suggests that considering only codimension-3 singularities in the harmonic solutions is simply insufficient and we must include codimension-2 supertubes for a full description of the physics.

In the presence of codimension-2 branes, the harmonic functions H in general become multi-valued [32]. This is because codimension-2 branes generally have a non-trivial U-duality monodromy around them [30, 31], and H transforms in a non-trivial representation under it. For a multi-center configuration, if the i-th codimension-2 brane has U-duality monodromy represented by a matrix M_i around it, the harmonic functions will have the monodromy

$$H \to M_i H. \tag{1.4}$$

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When the matrices M_i, M_j do not commute for some i, j, we say that the configuration is *non-Abelian*.[1]

In [32], two of the authors wrote down first examples of codimension-2 harmonic solutions. They involve multiple species of codimension-2 supertubes and can have the same asymptotic charges as a four-dimensional (4D) black hole with a finite horizon area. However, the constituent branes were *unbound*; namely, by tuning parameters of the solution, we can separate the constituents of the solution infinitely far apart. This implies that the solution does not actually represent a microstate of a BPS black hole, for the following reason [1, Section 3.1]: classically, it is possible to consider a configuration in which constituents are separated by a finite fixed distance from each other. However, quantum mechanically, by the uncertainty principle, fixing the relative position of the constituents increases kinetic energy and the configuration would not exactly saturate the BPS bound. Namely, it cannot be a microstate of a BPS black hole. So, the solution constructed in [32] is not a black-hole microstate. Relatedly, the solution in [32] had Abelian monodromies. There is some kind of linearity for codimension-2 branes with commuting monodromies, and we can construct solutions with multiple codimension-2 centers basically by adding harmonic functions for each center.[2] This suggests that codimension-2 branes with Abelian monodromies do not talk to each other and are not bound.

Then the natural question is: does a configuration of supertubes with non-Abelian monodromies exist? If so, is it a bound state, and does it represent a black-hole microstate? These are precisely the questions that we address in this paper.

1.2 Main results

In this paper, we will construct a configuration of codimension-2 supertubes with *non-Abelian monodromies* within the framework of harmonic solutions, in a certain perturbative expansion. We will give evidence that, as expected, it represents a bound state, and that it corresponds to a microstate of a 4D black hole with a finite horizon.

Our configuration is made of two circular supertubes which share their axis. The two tubes are separated by distance $2|L|$ and the radii of both rings are approximately R. See figure 2 on page 22. The harmonic functions H will have a non-trivial monodromy around each of the two tubes. The monodromies for the two supertubes do not commute, namely, they are non-Abelian. Because it is technically difficult to find the solution for general R and $|L|$, we consider the "colliding limit", $|L| \ll R$, in which we can construct the harmonic functions order by order in a perturbative expansion.

Despite that the colliding limit allows us to construct the solution explicitly, it also has a drawback: we cannot determine the value of R and $|L|$ separately. If we knew the exact solution, not a perturbative one, then we would be able to constrain them by imposing

[1]This is totally different from making the gauge group non-Abelian, namely generalizing Einstein-Maxwell to Einstein-Yang-Mills. For some recent work on non-Abelian generalizations in that sense, see [33, 34].

[2]More precisely, one should include certain interaction terms as well [32]. However, it is still true in this case that one can in principle construct solutions with multiple codimension-2 centers located wherever we want.

physical conditions (the absence of closed timelike curves) on the explicit solution. In this paper we will not be able to do that. Instead, we will make use of supertube physics to argue that R and $|L|$ are fixed (section 4.4). Although the argument physically well motivated and convincing, it is not a proof; we hope to revisit this point in future work.

Because the physical parameters R and $|L|$ are fixed, it is not possible to separate apart the two supertubes and therefore the configuration represents a bound state. Moreover, it has asymptotic charges of a 4D black hole with a finite horizon. Therefore, the non-Abelian 2-supertube configuration is arguably a black-hole microstate. The geometry is not regular near the supertubes, but the singular behavior is an allowed one in string theory, just as the geometry near a 1/2-BPS brane is metrically singular but is allowed. In this sense, our solution is not a microstate geometry but a microstate solution as defined in [9]. Our solution simultaneously involves the two types of supertube, (1.2) and (1.3), and therefore is non-geometric in that the internal torus is twisted by T-duality transformations around the supertubes.

We find that the asymptotic geometry of the perturbative solution is $\mathrm{AdS}_2 \times S^2$, namely the attractor geometry [35] of the black hole with the same charge. Furthermore, we find that the 4D angular momentum of the solution is zero, $J = 0$. We will argue that this is due to cancellation between the angular momentum that the individual supertubes carry and the one coming from the electromagnetic crossing between the monopole charges carried by the supertubes.

On a more technical note, in the colliding limit $|L| \ll R$, we can split the problem of finding harmonic functions with desired monodromies into two parts. If one is at a distance $d \sim R \gg |L|$ away from the supertubes (the "far region"), the configuration is effectively considered as made of a single tube whose monodromy is the product of two individual monodromies. On the other hand, if one is at a distance $d \sim |L| \ll R$ away from the tubes (the "near region"), we can regard the tubes as infinitely long and the problem reduces to that of finding 2D harmonic functions with desired monodromies. Once we find harmonic functions in both regions, we can match them order by order in a perturbative expansion to construct the harmonic function in the entire space. This is the sense in which our solution is perturbative in nature. In the near region, the problem is to find a pair of holomorphic functions with non-trivial $\mathrm{SL}(2,\mathbb{Z})$ monodromies around two singular points on the complex z-plane. Mathematically, this problem is the same as the one encountered in the SU(2) Seiberg-Witten theory [36] and we borrow their results to construct the harmonic functions.

The solution thus constructed is perfectly consistent at the perturbative level, but it is possible that unexpected new features are encountered in the exact, full-order solution. However, constructing such an exact solution is beyond the techniques developed in this paper and left for future research.

In terms of the harmonic solutions $H = \{V, K^I, L_I, M\}$, our configuration is given by

$$\begin{aligned} V &= \operatorname{Re} G\,, \qquad K^1 = K^2 = -\operatorname{Im} G, \qquad K^3 = \operatorname{Re} F\,, \\ L_1 &= L_2 = \operatorname{Im} F\,, \qquad L_3 = \operatorname{Re} G\,, \qquad M = -\frac{1}{2}\operatorname{Re} F\,, \end{aligned} \tag{1.5}$$

where F and G are complex functions and carry the information of the monodromies.

This class of solutions describes the general configuration in which the complexified Kähler moduli of T^2_{45} and T^2_{67} are set to $\tau^{1,2} = i$ whereas the one associated with T^2_{89} is given by $\tau^3 = \frac{F}{G}$. This class is a type IIA realization of the so-called SWIP solution [37]. It is the particular choice of the pair $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ that fixes the monodromies of the configuration. In our solution, F and G are related to the defining functions of the Seiberg-Witten solution.

1.3 Implication for black-hole microstates

In the above, we argued that our codimension-2 configuration represents a black-hole microstate. Our perturbative solution is quite different from the supergravity microstates based on codimension-3 harmonic solutions [2, 13, 14] that have been extensively studied in the literature. In particular, its 4D asymptotics is the $\mathrm{AdS}_2 \times S^2$ attractor geometry of the black hole with the same asymptotic charges, because the harmonic functions cannot have constant terms. Furthermore, the 4D angular momentum of our solution vanishes, $J = 0$, because of a cancellation mechanism between the tube and crossing contributions. To better understand the possible implications of these properties, let us recall some known facts and conjectures about black-hole microstates.

For codimension-3 harmonic solutions, a well-known family of microstate geometries whose 4D asymptotics can be made arbitrarily close to $\mathrm{AdS}_2 \times S^2$ and whose angular 4D momentum J can be made arbitrarily small is the so-called scaling solutions [38–40].[3] Scaling solutions are made of three or more codimension-3 centers and exist for any value of the asymptotic moduli, provided that certain triangle inequalities are satisfied by the skew products of the charges of the centers. The defining property of the scaling solutions is that we can scale down the distance between centers in the $\mathbb{R}^3$ base so that they appear to collide. However, the actual geometry does not collapse; what is happening in this scaling process is that an AdS throat gets deeper and deeper, at the bottom of which the non-trivial 2-cycles represented by the centers sit. At the same time, the angular momentum J becomes smaller and smaller. In the infinite scaling limit where all the centers collide in the $\mathbb{R}^3$ base, the geometry becomes precisely AdS and the angular momentum J vanishes. It has been argued [43, 44] that the majority of the black-hole microstates live in this infinite scaling limit, where the branes wrapping the 2-cycles [45], called “W-branes”, become massless and condense. In the IIA picture, W-branes are fundamental strings stretching between D-brane centers. In the language of quiver quantum mechanics [38] dual to scaling solutions, the configurations with a finite throat correspond to Coulomb branch states, while the configurations with W-brane condensate would correspond to pure-Higgs branch states [46]. However, the gravity description of such condensate is unclear.[4] It cannot simply be the infinite throat limit of the scaling solution, because in that limit

[3] Note that the angular momentum here is the 4D one. In the scaling solution, the 4D angular momentum can be made arbitrarily small. If one goes to 5D, there are two angular momenta, and the 4D angular momentum is one of the two. The other 5D angular momentum, which is nothing but the D0-brane charge from the 4D viewpoint, has been quite difficult to make smaller than a certain lower limit, for the geometry to correspond to a microstate in the D1-D5 system [39–42]. This problem can be overcome by generalizing the harmonic solution to the superstratum in 6D [21]. This issue is not relevant to the current discussion.

[4] For recent attempts to construct the gravity description of W-branes, see [47–50].

the non-trivial 2-cycles disappear in the infinite depth and the entire geometry becomes just AdS, indistinguishable from the black-hole geometry. Furthermore, quantization of the solution space of the scaling solutions [51] says that the depth of the throat cannot be made arbitrarily large but is limited by quantum effects. So, it appears that, although the scaling solution is an important clue for the W-brane condensate and pure-Higgs branch states, it is not the answer itself.

Relatedly, Sen and his collaborators argued [52–54] that the contribution to black-hole microstates can be split into the "hair" part which lives away from the horizon and the "horizon" part which gives the main contribution to black-hole entropy. The horizon part has asymptotically AdS_2 geometry and vanishing angular momentum, $J = 0$. This is based on the fact that, in 4D, only $J = 0$ black holes are BPS and all extremal black holes with $J \neq 0$ are non-supersymmetric [52]. The analysis of the quiver quantum mechanics describing the worldvolume theory of a D-brane black-hole system [54] also supports the claim that all black-hole microstates in 4D have $J = 0$.

In summary, both the analysis of the scaling solutions and the arguments of Sen et al. suggest that the majority of the black-hole microstates have AdS asymptotics and vanishing angular momentum, $J = 0$. They are states with a condensate of W-branes, or equivalently fundamental strings stretching between D-branes, and correspond to the pure-Higgs branch states of the dual quiver quantum mechanics.

Now if we look at our perturbative solution, it seems to have all the above properties expected of a typical microstate of a 4D black hole. First, it has AdS_2 asymptotics. This was not done by fine-tuning of parameters but is a consequence of the non-trivial monodromy of the supertubes. Second, its angular momentum vanishes, $J = 0$. This did not require fine-tuning either, and it was due to the cancellation mechanism mentioned before between different contributions to angular momentum. Moreover, our solutions are made of supertubes generated by the supertube transition which is nothing but condensation of the strings stretching between the original D-branes. Therefore, it is natural to conjecture that our solution is giving a gravity description of the W-brane condensate and represents a state in the pure-Higgs branch. At least, it is expected to provide a clue for the gravity description of pure-Higgs branch states.

Of course, to make such a strong claim we need strong evidence, including the demonstration that non-Abelian supertube configurations do exist beyond the perturbative level, and the proof they have a huge entropy to account for the black-hole microstates. Such studies would require more sophisticated tools and techniques than developed in the current paper. At this point, we just state that it is quite non-trivial and intriguing that the perturbative non-Abelian 2-supertube solution has the properties expected of black-hole microstates, and leave further investigation as an extremely interesting direction of future research.

In [55] (see also [56]), an interesting set of solutions with $\mathrm{AdS}_2 \times S^2$ asymptotics were constructed. They belong to the so-called IWP family of solutions [57, 58] and are characterized by one complex harmonic function in three dimensions. The main differences between the solutions in [55] and ours are as follows. First, because the solutions in [55] are based on one complex harmonic function, their possible monodromies are Abelian. On the other hand, our solution has two complex harmonic functions and thus the monodromies are in general non-Abelian. Second, the solutions in [55] have two distinct $\mathrm{AdS}_2 \times S^2$ asymptotic regions. In contrast, the multiple asymptotic regions in our solutions are related by U-duality and regarded as one asymptotic region in different U-duality frames. Therefore, our solution has only one physical asymptotic region.

Let us end this section by mentioning one other difference between microstates with codimension-3 centers and ones with codimension-2 centers. One issue about the existing construction of black-hole microstates based on codimension-3 harmonic solutions is that, multi-center configurations (except for the case where there are two centers and one of them is a 1/2-BPS center) are expected to lift and disappear from the BPS spectrum once generic moduli are turned on [59]. The physical origin of this is that, if there are multiple centers, when one continuously changes the moduli to arbitrary values, the discreteness of quantized charges is incompatible with the BPS condition [60]. This is certainly an issue for codimension-3 centers but, codimension-2 supertubes may be able to avoid it by continuously deforming the tube shapes and re-distributing the monopole charge density along its worldvolume, so that the BPS condition is met even if one changes the moduli continuously. Therefore, it may be that codimension-2 solutions provide a loophole for the no-go result of [59] and represent microstates that remain supersymmetric everywhere in the moduli space.

1.4 Plan of the paper

The rest of this paper is organized as follows. In section 2 we review the BPS solutions, called harmonic solutions, which can describe a wide class of multi-center configurations in string theory in four and five dimensions. We discuss their physical properties, giving examples for cases with codimension-3 and codimension-2 centers. We also introduce the class of solutions in which only one $\mathrm{SL}(2,\mathbb{Z})$ duality is turned on and has only one modulus τ. In section 3, we explicitly construct an example of non-Abelian supertubes. We first introduce the colliding limit and the matching expansion which allow us to construct the solution order by order by connecting the far-region and near-region solutions. We then use it to perturbatively construct the solution. As the near-region solution, we use an ansatz inspired by the SU(2) Seiberg-Witten theory. In section 4, we study the physical properties of the solution. We work out the brane charge content, the asymptotic geometry and the angular momentum, and discuss the condition for the absence of closed timelike curves (CTCs). Based on the results, we argue that the solution is a bound state and thus represent a black-hole microstate. We also discuss the cancellation mechanism responsible for the vanishing of the angular momentum.

The appendices include some details of the computations carried out in the main text and some topics tangential to the content of the main text. In appendix A, we discuss some aspects of the duality transformations acting on the harmonic functions. In appendix B, we discuss some details of the matching expansion to higher order than is discussed in the main text. In the main text, we focus on the class of solutions in which only one of the three moduli of the STU model is activated. In appendix C, we discuss the class of solutions in which two of moduli are activated. In appendix D, we discuss properties of the supertubes created from a general 1/4-BPS center in the one-modulus class of solutions. In appendix E, we present the explicit harmonic functions for the D2 + D6 $\to 5^2_2$ supertube used in the main text.

2 Multi-center solutions with codimension 2 and 3

2.1 The harmonic solution

The most general supersymmetric solutions of ungauged $d = 5$, $\mathcal{N} = 1$ supergravity with vector multiplets have been classified in [61] (see also [10, 11, 62]).[5] When one applies this result to M-theory compactified on $T^6 = T^2_{45} \times T^2_{67} \times T^2_{89}$ (the so-called STU model) and further assumes a tri-holomorphic U(1) symmetry [12], the general supersymmetric solution corresponds to the following 11-dimensional fields:

$$\begin{aligned} ds^2_{11} &= -Z^{-2/3}(dt+k)^2 + Z^{1/3}ds^2_{\rm GH} + Z^{1/3}\left(Z_1^{-1}dx^2_{45} + Z_2^{-1}dx^2_{67} + Z_3^{-1}dx^2_{89}\right) , \\ \mathcal{A}_3 &= \left(B^I - Z_I^{-1}(dt+k)\right)\wedge J_I \, , \quad J_1 \equiv dx^4\wedge dx^5 \, , \; J_2 \equiv dx^6 \wedge dx^7 \, , \; J_3 \equiv dx^8 \wedge dx^9 \, , \end{aligned} \tag{2.1}$$

where $I = 1, 2, 3$; $Z \equiv Z_1Z_2Z_3$; and $dx^2_{45} \equiv (dx^4)^2 + (dx^5)^2$ etc.

Supersymmetry requires that all fields in (2.1) be written in terms of *3D harmonic functions* as follows [12]. First, the metric $ds^2_{\rm GH}$ must be a 4-dimensional metric of a Gibbons-Hawking space given by

$$ds^2_{\rm GH} = V^{-1}(d\psi + A)^2 + V d\mathbf{x}^2 \, , \qquad \psi \cong \psi + 4\pi \, , \qquad \mathbf{x} = (x^1, x^2, x^3) \, . \tag{2.2}$$

The 1-form A and the scalar V depend on the coordinates $\mathbf{x}$ of the $\mathbb{R}^3$ base and satisfy

$$dA = *_3 \, dV \, , \tag{2.3}$$

where $*_3$ is the Hodge dual operator on the $\mathbb{R}^3$. From this, we see that V has to be a harmonic function in $\mathbb{R}^3$,

$$\Delta V = 0 \, , \qquad \Delta \equiv \partial_i \partial_i \, . \tag{2.4}$$

The rest of the fields can be written in terms of additional harmonic functions K^I, L_I, M on $\mathbb{R}^3$ as follows:

$$B^I = V^{-1}K^I(d\psi + A) + \xi^I \, , \qquad d\xi^I = - *_3 \, dK^I \, , \tag{2.5}$$

$$Z_I = L_I + \frac{1}{2}C_{IJK}V^{-1}K^JK^K \, , \tag{2.6}$$

$$k = \mu(d\psi + A) + \omega \, , \tag{2.7}$$

$$\mu = M + \frac{1}{2}V^{-1}K^IL_I + \frac{1}{6}C_{IJK}V^{-2}K^IK^JK^K \, , \tag{2.8}$$

[5] Depending on whether the Killing vector constructed from the Killing spinor bilinear is timelike or null, the solutions are classified into timelike and null classes. In this paper we will consider the timelike class.

where $C_{IJK} = |\epsilon_{IJK}|$. If one replaces the internal space $T^6 = (T^2)^3$ by a Calabi-Yau 3-fold X, most of our formulas remain valid as long as we replace C_{IJK} by the triple intersection numbers of X [12]. We sometimes write eight harmonic functions collectively as $H = \{V, K^I, L_I, M\}$. For two such vectors H, H', we define the skew product $\langle H, H'\rangle$ by

$$\langle H, H'\rangle \equiv VM' - MV' + \frac{1}{2}(K^I L'_I - L_I K'^I). \tag{2.9}$$

The 1-form ω satisfies

$$*_3 d\omega = \langle H, dH\rangle \,. \tag{2.10}$$

Applying $d *_3$ on this equation implies

$$0 = \langle H, \Delta H\rangle \,. \tag{2.11}$$

This is often called the *integrability condition* [63] (see also [13]), and is a necessary requirement for the existence of ω. Harmonicity of the functions $H = \{V, K^I, L_I, M\}$ may make one think that the right-hand side identically vanishes. However, the harmonic functions generically have singularities associated with the presence of sources, which can lead to a non-vanishing contribution to the right-hand side and make ω multi-valued. Whether we must allow or disallow such contribution must be determined based on physical considerations, as we will discuss below in concrete examples.

The above represent a broad family of supersymmetric solutions characterized by eight harmonic functions, $H = \{V, K^I, L_I, M\}$. We call this set of solutions *harmonic solutions*.[6] Although we started with $d = 5$ supergravity, the existence of the isometry along ψ allows us to dimensionally reduce the solution to 4D. Therefore, the harmonic solutions can be regarded as representing configurations in 4D.

Reducing the 11D solution (2.1) along ψ, we obtain the following supersymmetric solution of type IIA supergravity:[7]

$$\begin{aligned}
ds^2_{10,\text{str}} &= -\frac{1}{\sqrt{\mathcal{Q}}}(dt+\omega)^2 + \sqrt{\mathcal{Q}}\, d\mathbf{x}^2 + \frac{\sqrt{\mathcal{Q}}}{V}\left(Z_1^{-1}dx_{45}^2 + Z_2^{-1}dx_{67}^2 + Z_3^{-1}dx_{89}^2\right) , \\
e^{2\Phi} &= \frac{\mathcal{Q}^{3/2}}{V^3 Z}\,, \qquad\qquad B_2 = \left(V^{-1}K^I - Z_I^{-1}\mu\right) J_I\,, \\
C_1 &= A - \frac{V^2\mu}{\mathcal{Q}}(dt+\omega)\,, \qquad C_3 = \left[(V^{-1}K^I - Z_I^{-1}\mu)A + \xi^I - Z_I^{-1}(dt+\omega)\right]\wedge J_I\,,
\end{aligned} \tag{2.12}$$

where $ds^2_{10,\text{str}}$ is the string-frame metric and

$$\mathcal{Q} \equiv V(Z - \mu^2 V)\,. \tag{2.13}$$

[6] These solutions were first found in [64] as solutions of $d = 4, \mathcal{N} = 2$ supergravity with vector multiplets and made more explicit in [65]. In 5D, the supersymmetric solutions in $\mathcal{N} = 2$ supergravity with vector multiplets in the timelike class were classified in [11, 61] (see also [10, 62]) and later reduced to 4D solutions in [12], which are identical to the ones in [64, 65]. In 4D, it was later shown in [66] that these solutions are the most general supersymmetric solutions in the timelike class in $d = 4, \mathcal{N} = 2$ ungauged supergravity with vector multiplets. There being no widely accepted name for these solutions, we call them harmonic solutions.

[7] For expressions for higher RR potentials, see, e.g., [32, Appendix E] and [67].

Explicitly in terms of harmonic functions,

$$\begin{aligned}\mathcal{Q} &= VL_1L_2L_3 - 2MK^1K^2K^3 - M^2V^2 \\ &\quad - \frac{1}{4}\sum_I (K^I L_I)^2 + \frac{1}{2}\sum_{I<J} K^I L_I K^J L_J - MV\sum_I K^I L_I \\ &\equiv J_4(H), \end{aligned} \tag{2.14}$$

where J_4 is the quartic invariant of the STU model; for some more discussion, see appendix A.

Let the complexified Kähler moduli for the 2-tori T^2_{45}, T^2_{67}, and T^2_{89} be τ^1, τ^2, and τ^3, respectively. The expression in terms of harmonic functions is

$$\tau^1 = B_{45} + i\sqrt{\det G_{ab}} = \left(\frac{K^1}{V} - \frac{\mu}{Z_1}\right) + i\frac{\sqrt{\mathcal{Q}}}{Z_1 V}\,, \tag{2.15}$$

where $a, b = 4, 5$ and the radii of 456789 directions have been all set to $l_s = \sqrt{\alpha'}$. The other moduli τ^2 and τ^3 are given by the same expression with 45 replaced by 67 and 89, respectively. In supergravity, these moduli parametrize the moduli space $[\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}]^3$. In string theory, this reduces to $[\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(2)}]^3$ by the $[\mathrm{SL}(2,\mathbb{Z})]^3$ duality symmetry that identifies different values of τ^I.

For other embeddings of the harmonic solutions in type IIA and IIB supergravity, see [11, 68, 69].

Duality transformations. Because we will consider codimension-2 configurations with non-trivial U-duality monodromies, it is useful to recall some facts about the U-duality group in the STU model, which is $\mathrm{SL}(2,\mathbb{Z})_1 \times \mathrm{SL}(2,\mathbb{Z})_2 \times \mathrm{SL}(2,\mathbb{Z})_3$ [70].

In particular, it is important to understand how the U-duality acts on the harmonic functions. Let us take $\mathrm{SL}(2,\mathbb{Z})_1$. This group is generated by (i) simultaneous T-duality transformations on the 45 directions and (ii) the shift symmetry $B_{45} \to B_{45}+1$. Because we know the T-duality action on 10D fields from the Buscher rule and their expression (2.12) in terms of harmonic functions, it is easy to read off how the harmonic functions transform under (i). The same is true for the B-shift symmetry (ii). The result is that (i) and (ii) are realized by the $\mathrm{SL}(2,\mathbb{Z})_1$ matrices

$$M_{\text{T-duality}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M_{B\text{-shift}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.16}$$

and that the eight harmonic functions transform as a direct sum of four doublets,

$$\begin{pmatrix} K^1 \\ V \end{pmatrix}, \quad \begin{pmatrix} 2M \\ -L_1 \end{pmatrix}, \quad \begin{pmatrix} -L_2 \\ K^3 \end{pmatrix}, \quad \begin{pmatrix} -L_3 \\ K^2 \end{pmatrix}. \tag{2.17}$$

Since (i) and (ii) generate $\mathrm{SL}(2,\mathbb{Z})_1$, we conclude that, even for general transformations $\mathrm{SL}(2,\mathbb{Z})_1$, the harmonic functions transform as a collection of doublets (2.17).

Because all three $\mathrm{SL}(2,\mathbb{Z})$'s are on the same footing, we can infer the transformation of harmonic functions under general $\mathrm{SL}(2,\mathbb{R})_I$ transformation for $I = 1, 2, 3$. Under $\mathrm{SL}(2,\mathbb{R})_I$, the eight harmonic functions transform as a direct sum of four doublets:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to M_I \begin{pmatrix} u \\ v \end{pmatrix}, \qquad M_I \equiv \begin{pmatrix} \alpha_I & \beta_I \\ \gamma_I & \delta_I \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})_I, \tag{2.18}$$

where the vector $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ represents any of the pairs

$$\begin{pmatrix} K^I \\ V \end{pmatrix}, \quad \begin{pmatrix} 2M \\ -L_I \end{pmatrix}, \quad \begin{pmatrix} -L_J \\ K^K \end{pmatrix}, \quad \begin{pmatrix} -L_K \\ K^J \end{pmatrix}, \qquad J \neq K \neq I\,. \tag{2.19}$$

One can show that the transformations (2.18) for different values of I commute, as they should because they are associated with different tori.

It is not difficult to show that the transformation (2.18) of the harmonic functions means the standard linear fractional transformation of the complexified Kähler moduli as:

$$\tau^I \to \frac{\alpha_I\,\tau^I + \beta_I}{\gamma_I\,\tau^I + \delta_I}, \qquad \tau^J \to \tau^J \quad (J \neq I)\,, \tag{2.20}$$

where there is no summation over I.

For some more aspects on the duality transformation of the harmonic solutions, see appendix A.

Conditions for the absence of closed timelike curves. (Super)gravity solutions can exhibit closed timelike curves (CTCs), signaling that the solution is not physically allowed.[8] To study their existence, let us look at the 10D metric (2.12). First, for g_{tt}, g_{ii} $(i = 1, 2, 3)$ to be real, we need $\mathcal{Q} \geq 0$. Then, for the torus directions to give no CTCs, we get $VZ_I \geq 0$, $I = 1, 2, 3$. So, we must impose the following conditions:

$$\mathcal{Q} \geq 0\,, \tag{2.21a}$$

$$VZ_I \geq 0\,. \tag{2.21b}$$

Next, let us focus on the $\mathbb{R}^3$ part of the 10D metric (2.12) which is

$$ds^2_{10,\mathrm{str}} \supset -\frac{\omega^2}{\sqrt{\mathcal{Q}}} + \sqrt{\mathcal{Q}}\, d\mathbf{x}^2. \tag{2.22}$$

It is possible that closed curve $\mathcal{C}$ in $\mathbb{R}^3$ becomes timelike under this metric, depending on the behavior of the 1-form ω. That would imply a CTC, which must be physically disallowed. We will discuss this condition in specific situations later.

2.2 Configurations with only one modulus

Thus far, we have been discussing configurations for which all moduli τ^I, $I = 1, 2, 3$ can in principle be all non-trivial. Now let us focus on configurations with

$$\tau^1 = \tau^2 = i, \qquad \tau^3 = \text{arbitrary}. \tag{2.23}$$

[8] For over-rotating supertubes, CTCs can appear along the profile of the supertube [71, 72].

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Although being particular instances of the general solution, they can still describe a wide range of physical configurations, such as ones with multiple centers with codimension 3 and 2. This class of solutions provides a particularly nice setup for our purpose of constructing codimension-2 solutions with non-Abelian monodromies. This class is nothing but a type IIA realization of the solution called the SWIP solution in the literature [37]. Here we discuss some generalities about this class.

Using the expression (2.15) for τ^I in terms of harmonic functions, we see that the condition (2.23) implies the following relations:

$$K^1 = K^2 \,, \qquad L_1 = L_2 \,, \qquad L_3 = V \,, \qquad M = -\frac{K^3}{2} \,, \tag{2.24}$$

leaving four independent harmonic functions. If we plug these expressions into (2.15), we obtain

$$\tau^3 = \frac{K^3 + iL_1}{V - iK^1} = \frac{F}{G} \,, \tag{2.25}$$

where we defined complex combinations

$$F \equiv K^3 + iL_1, \qquad G \equiv V - iK^1. \tag{2.26}$$

As we can see from (2.19), the pair $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ transforms as a (complex) doublet under $\mathrm{SL}(2,\mathbb{Z})_3$. From the expression (2.25), it is obvious that τ^3 undergoes linear fractional transformation under $\mathrm{SL}(2,\mathbb{Z})_3$ (although we already said this in (2.20) in general). The harmonic functions are written in terms of them as

$$\begin{aligned} V = \operatorname{Re} G \,, \qquad K^1 = K^2 = -\operatorname{Im} G, \qquad K^3 = \operatorname{Re} F \,, \\ L_1 = L_2 = \operatorname{Im} F \,, \qquad L_3 = \operatorname{Re} G \,, \qquad M = -\frac{1}{2} \operatorname{Re} F \,. \end{aligned} \tag{2.27}$$

In terms of the complex quantities F, G, some previous formulas become

$$\langle H, H' \rangle = \operatorname{Re}(F\bar{G}' - G\bar{F}'), \tag{2.28}$$
$$\mathcal{Q} = (\operatorname{Im} F\bar{G})^2. \tag{2.29}$$

The equation for ω, (2.10), reads

$$*_3 d\omega = \operatorname{Re}\left(F d\bar{G} - G d\bar{F}\right) \,. \tag{2.30}$$

Let us consider the general no-CTC conditions. Under the constraint (2.24), the condition (2.21a) is automatically satisfied because $\mathcal{Q} = (\operatorname{Im} F\bar{G})^2 \geq 0$. On the other hand, the condition (2.21b) gives

$$\operatorname{Im}(F\bar{G}) = |G|^2 \operatorname{Im} \tau^3 \geq 0 \,. \tag{2.31}$$

Here we have seen that switching off two moduli τ^1 and τ^2 leads to a substantial simplification. In appendix C, we discuss switching off one modulus τ^1, which also leads to interesting simplification.

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2.3 Codimension-3 solutions

The harmonic solutions are characterized by a set of 8 harmonic functions. Non-trivial harmonic functions in $\mathbb{R}^3$ must have singularities, which correspond to physical sources such as D-branes. Depending on the nature of the source, the singularity can have various codimension. Here we review some specifics about solutions with codimension-3 sources, or codimension-3 solutions for short, which have been extensively studied in the literature. In the next subsection, we will proceed to codimension-2 solutions, which is the main focus of the current paper.

If one assumes that all singularities of the harmonic functions have codimension 3, the general form of the harmonic functions is [12, 64, 65]

$$\begin{aligned} V &= h^0 + \sum_{p=1}^{N} \frac{\Gamma_p^0}{|\mathbf{x}-\mathbf{a}_p|}\,, & K^I &= h^I + \sum_{p=1}^{N} \frac{\Gamma_p^I}{|\mathbf{x}-\mathbf{a}_p|}\,, \\ L_I &= h_I + \sum_{p=1}^{N} \frac{\Gamma_I^p}{|\mathbf{x}-\mathbf{a}_p|}\,, & M &= h_0 + \sum_{p=1}^{N} \frac{\Gamma_0^p}{|\mathbf{x}-\mathbf{a}_p|}\,, \end{aligned} \tag{2.32}$$

where $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{a}_p \in \mathbb{R}^3$ $(p = 1, \ldots, N)$ specifies the location of the codimension-3 sources where the harmonic functions become singular. The charge vector $\Gamma^p \equiv \{\Gamma_p^0, \Gamma_p^I, \Gamma_I^p, \Gamma_0^p\}$ carries the charges of each source and, together with $h \equiv \{h^0, h^I, h_I, h_0\}$, fully determine the asymptotic properties of the solution, namely mass, angular momenta and the moduli at infinity.

We still have to satisfy the integrability condition (2.11). Because the Laplacian Δ acting on $|\mathbf{x}-\mathbf{a}_p|^{-1}$ gives a delta function supported at $\mathbf{x} = \mathbf{a}_p$, the right-hand side of (2.11) does not generally vanish. Mathematically, this does not pose any problem for the existence of ω, although it becomes multi-valued, having a Dirac-Misner string [73]. However, the presence of a Dirac-Misner string leads to CTCs [13]. Therefore, it is physically required that the delta-function singularities be absent on the right-hand side of (2.11). This condition implies the well-known constraint [63]

$$\sum_{q(\neq p)} \frac{\langle \Gamma_p, \Gamma_q \rangle}{a_{pq}} = \langle h, \Gamma_p \rangle \qquad \text{for each } p\,, \tag{2.33}$$

where $a_{pq} \equiv |\mathbf{a}_p - \mathbf{a}_q|$.

Let us see how this argument goes [13]. Let B^3 be a small ball containing $\mathbf{x} = \mathbf{a}_p$, and consider the integral

$$\int_{B^3} d^2\omega = \int_{B^3} d^3\mathbf{x}\, \langle H, \Delta H \rangle, \tag{2.34}$$

where we used (2.10). The integrand on the right-hand side is the same as the one in the integrability condition (2.11). If it has a delta-function source at $\mathbf{x} = \mathbf{a}_p$, the integral is nonzero. On the other hand, the left-hand side can be rewritten as

$$\int_{B^3} d^2\omega = \int_{S^2} d\omega = \int_{\partial S^2} \omega\,, \tag{2.35}$$

where $S^2 = \partial B^3$ and the boundary ∂S^2 can be taken to be an infinitesimal circle going around the north pole, through which a Dirac-Misner string passes. This being non-vanishing means that the component of ω along ∂S^2 is finite; if we take the Dirac-Miser string to be along the positive z-axis, then $\omega_\varphi \neq 0$ where φ is the azimuthal angle around the z-axis. Therefore, for this curve $\mathcal{C} = \partial S^2$, the first term in (2.22) does not vanish while the second one vanishes (note that $\mathcal{Q}$ is finite as long as we are away from $\mathbf{x} = \mathbf{a}_p$). So, curve $\mathcal{C}$ is a CTC. Therefore, the right-hand side of the integrability condition (2.11) must not even have delta-function singularities, and this is what leads to the constraint (2.33).

The interpretation of the singularities in the harmonic functions (2.32) from a string/M-theory point of view is the existence of extended objects in higher dimensions. In the string/M-theory uplift, p-form potentials are expressed in terms of the harmonic functions, which allows us to establish a dictionary between the harmonic functions and their corresponding brane configurations [65]. For example, in the type IIA picture (2.12), the dictionary between the singularities in the harmonic functions and the D-brane sources is

$$V \leftrightarrow \mathrm{D6}(456789)\,, \quad \begin{matrix} K^1 \leftrightarrow \mathrm{D4}(6789) \\ K^2 \leftrightarrow \mathrm{D4}(4589) \\ K^3 \leftrightarrow \mathrm{D4}(4567) \end{matrix}\,, \quad \begin{matrix} L_1 \leftrightarrow \mathrm{D2}(45) \\ L_2 \leftrightarrow \mathrm{D2}(67) \\ L_3 \leftrightarrow \mathrm{D2}(89) \end{matrix}\,, \quad M \leftrightarrow \mathrm{D0}\,. \tag{2.36}$$

The D-branes are partially wrapped on T^6 and appear in 4D as pointlike (codimension-3) objects sourcing the harmonic functions. The components of the charge vector $\Gamma = \{\Gamma^0, \Gamma^I, \Gamma_I, \Gamma_0\}$ are related to the quantized D-brane numbers by

$$\Gamma^0 = \frac{g_s l_s}{2} N^0, \qquad \Gamma^I = \frac{g_s l_s}{2} N^I, \qquad \Gamma_I = \frac{g_s l_s}{2} N_I, \qquad \Gamma_0 = \frac{g_s l_s}{4} N_0, \tag{2.37}$$

where $N^0, N^I, N_I, N_0 \in \mathbb{Z}$ (recall that the radii of the internal torus directions have been all set to $l_s = \sqrt{\alpha'}$). When multiple sources are present, the harmonic solution (2.32) represents a multi-center configuration of D-branes.

The harmonic solutions with codimension-3 sources have been extensively used to describe various brane systems for various purposes. Examples include a 5D 3-charge black hole made of M2(45), M2(67) and M2(89)-branes, which is dual to the Strominger-Vafa black hole [74]; the BMPV black hole [75]; the MSW black hole [76]; the supersymmetric black ring [11, 68, 77]; multi-center black hole/ring solutions [65]; and microstate geometries [13, 14].

One simple example is when (2.32) contains only one term, namely, $N = 1$. For the generic charge vector $\Gamma \equiv \Gamma^{p=1}$, this describes a single-center black hole in 4D which is made of D0, D2, D4 and D6-branes. The area-entropy of this black hole can be readily computed to be

$$S = \frac{\pi\sqrt{J_4(\Gamma)}}{G_4}\,, \tag{2.38}$$

where the 4D Newton constant is given by $G_4 = g_s^2 l_s^2/8$ and $J_4(\Gamma)$ is obtained by replacing $H = \{V, K^I, L_I, M\}$ in (2.14) by $\Gamma = \{\Gamma^0, \Gamma^I, \Gamma_I, \Gamma_0\}$. Multi-center solutions which

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have the same asymptotic moduli as this single-center solution and the same total charge $\sum_p \Gamma^p = \Gamma$ can be thought of as representing microstates/sub-ensemble of the ensemble represented by the single-center black hole.

In the one-modulus class discussed in section 2.2, the harmonic functions (2.32) can be rewritten in terms of the complex harmonic function (2.26) as

$$F = h_F + \sum_{p=1}^{N} \frac{Q_F^p}{|\mathbf{x} - \mathbf{a}_p|}, \qquad G = h_G + \sum_{p=1}^{N} \frac{Q_G^p}{|\mathbf{x} - \mathbf{a}_p|}, \tag{2.39}$$

where the complex quantities (h_F, h_G) and (Q_F^p, Q_G^p) are related to the real quantities h and Γ^p, respectively, just as (F, G) are related to H via (2.27). We will refer to (Q_F, Q_G) as complex charges. Using (2.27) and (2.37), we can see that they are related to quantized charges by

$$\begin{aligned} &Q_F = \frac{g_s l_s}{2}(N^3 + iN_1), && Q_G = \frac{g_s l_s}{2}(N^0 - iN^1), \\ &N^1 = N^2, \qquad N_1 = N_2, && N^0 = N_3, \qquad N^3 = -N_0. \end{aligned} \tag{2.40}$$

The black-hole entropy (2.38) can be written as

$$S = \frac{8\pi \, |\mathrm{Im}(Q_F \bar{Q}_G)|}{g_s^2 l_s^2} = 2\pi \, |N^3 N^1 + N_1 N^0| \, . \tag{2.41}$$

2.4 Codimension-2 solutions

Codimension-2 sources are inevitable. In addition to codimension-3 sources, the harmonic solutions can also describe codimension-2 sources. Actually, codimension-2 sources are *not an option but a must*; codimension-3 sources are insufficient because they can spontaneously polarize into codimension-2 sources by the *supertube transition* [24]. The supertube transition is a spontaneous polarization phenomenon that a certain pair of species of branes — specifically, any 1/4-BPS 2-charge system — undergo. In this transition, the original branes polarize into a new dipole charge, which has one less codimension and extends along a closed curve transverse to the worldvolume of the original branes. This new configuration represents a genuine BPS bound state of the 2-charge system [1, Section 3.1]. The supertube transition may seem similar to the Myers effect [78], but it is different; the Myers effect takes place only in the presence of an external field, whereas the supertube transition occurs spontaneously, by the dynamics of the system itself.

The system described by codimension-3 harmonic solutions involves various D-branes as we saw in (2.36). These D-branes can undergo supertube transitions into codimension-2 branes, which act as codimension-2 sources in the harmonic function. Therefore, codimension-2 solutions are in the same moduli space of physical configurations as codimension-3 solutions, and consequently *must* be considered if one wants to understand the physics of the D-brane system.

In particular, supertubes are known to be important for BPS microstate counting of black holes because of the entropy enhancement phenomenon [15, 16, 69, 79]. So, the supertubes realized as codimension-2 sources in the harmonic functions must play a crucial role in the black hole microstate geometry program, as first argued in [30, 31]. The codimension-2 brane produced by the supertube transition can generically be non-geometric, having non-geometric U-duality twists around it.

A prototypical example of the supertube transition [24] can be represented as

$$\mathrm{D0} + \mathrm{F1}(1) \to \mathrm{d2}(\lambda 1)\,. \tag{2.42}$$

This diagram means that the 2-charge system of D0-branes and F1-strings has undergone a supertube transition and polarized into a D2-brane along an arbitrary closed curve parametrized by λ. The object on the right-hand side is written in lowercase to denote that it is a dipole charge. In this case, as the D2 is along a closed curve, there is no net charge but a D2 dipole charge. The original D0 and F1 charges are dissolved into the D2 worldvolume as magnetic and electric fluxes. The Poynting momentum due to the fluxes generates the centrifugal force that prevents the arbitrary shape from collapsing.

Upon duality transformations of the process (2.42), other possible supertube transitions can be found. For example,

$$\begin{aligned}
&\mathrm{D0} && + \mathrm{D4}(4567) && \to && \mathrm{ns5}(\lambda 4567)\,,\\
&\mathrm{D4}(4589) && + \mathrm{D4}(6789) && \to && 5^2_2(\lambda 4567;89)\,,\\
&\mathrm{D2}(45) && + \mathrm{D2}(67) && \to && \mathrm{ns5}(\lambda 4567)\,,\\
&\mathrm{D2}(89) && + \mathrm{D6}(456789) && \to && 5^2_2(\lambda 4567;89)\,.
\end{aligned} \tag{2.43}$$

This means that the ordinary branes on the left-hand side can polarize into codimension-2 branes, including the exotic branes such as the 5^2_2-brane.[9] Note in particular that the D-branes appearing on the left-hand side are the ones that appear in the brane-harmonic function dictionary (2.36). So, the dictionary is insufficient and must be extended to include codimension-2 branes that the codimension-3 D-branes can polarize into. Because we solved the BPS equations and obtained harmonic solutions without specifying the co-dimensionality of the sources, the codimension-2 supertubes on the right-hand side of (2.43) must be describable in terms of the same harmonic solutions, just by allowing for codimension-2 singularities. The formulas for the M-theory/IIA uplift also remain valid.

Examples of codimension-2 solutions. Let us study some codimension-2 solutions that are given in terms of the harmonic solutions. From (2.43) let us consider the following process:

$$\mathrm{D2}(45) + \mathrm{D2}(67) \to \mathrm{ns5}(\lambda 4567)\,. \tag{2.44}$$

It was shown in [32] that the codimension-2 ns5 supertube on the right-hand side can be described within harmonic solutions by the following harmonic functions

$$\begin{aligned}
V = 1\,, \qquad & K^1 = 0\,, \qquad && K^2 = 0\,, \qquad && K^3 = \gamma\,, \\
& L_1 = f_2 \qquad && L_2 = f_1\,, \qquad && L_3 = 1\,, \qquad M = -\frac{\gamma}{2}\,,
\end{aligned} \tag{2.45}$$

[9] For a review on exotic branes and a further analysis of supertube transitions involving them, see [31]. We discuss a D2 + D6 $\to 5^2_2$ transition in appendix E.

where

$$f_1 = 1 + \frac{Q_1}{L}\int_0^L \frac{d\lambda}{|\mathbf{x}-\mathbf{F}(\lambda)|}\,, \qquad f_2 = 1 + \frac{Q_1}{L}\int_0^L \frac{|\dot{\mathbf{F}}(\lambda)|^2\, d\lambda}{|\mathbf{x}-\mathbf{F}(\lambda)|}\,. \tag{2.46}$$

The supertube lies along the closed curve $\mathbf{x} = \mathbf{F}(\lambda)$, where $F_i(\lambda)$ $(i = 1, 2, 3)$ are arbitrary functions satisfying $F_i(\lambda + L) = F_i(\lambda)$. Q_1 is the D2(67)-brane charge, while the D2(45)-brane charge is given by $Q_2 = \frac{Q_1}{L}\int_0^L |\dot{\mathbf{F}}(\lambda)|^2\, d\lambda$. The integrals in (2.46) arise as a consequence of these charges being dissolved along the worldvolume of the supertube. For expressions of L, Q_1, Q_2 in terms of microscopic quantities, see [32]. γ is a harmonic scalar function defined through the equation

$$d\alpha = *_3 d\gamma\,, \qquad \alpha = \frac{Q_1}{L}\int_0^L \frac{\dot{F}_i(\lambda)\, d\lambda}{|\mathbf{x}-\mathbf{F}(\lambda)|}\, dx^i\,. \tag{2.47}$$

Even though the 1-form α is single-valued, γ is multi-valued and has monodromy as we go once around the supertube [32]:

$$\gamma \to \gamma + 1. \tag{2.48}$$

The integrability condition (2.11) is satisfied without any delta-function singularity along the profile, because $\Delta\gamma = 0$ without any singular contribution on the profile [32]. Other data of the harmonic solutions are

$$Z_I = (f_2, f_1, 1)\,, \qquad \mu = 0\,, \qquad \omega = -\alpha\,, \qquad \xi^I = (0, 0, -\alpha)\,. \tag{2.49}$$

The charge content of the solution can be easily read off from the harmonic functions. The original codimension-3 charges for D2(45) and D2(67) are encoded in L_1 and L_2 by the dictionary (2.36). From (2.46), we see that these charges are distributed along the profile $\mathbf{x} = \mathbf{F}(\lambda)$ with densities Q_1/L and $(Q_1/L)|\dot{\mathbf{F}}|^2$, respectively. On the other hand, the NS5 charge is encoded in the monodromy. Eq. (2.48) means the following monodromy around the supertube:

$$\begin{pmatrix} K^3 \\ V \end{pmatrix} = \begin{pmatrix} \gamma \\ 1 \end{pmatrix} \to \begin{pmatrix} \gamma + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} K^3 \\ V \end{pmatrix}. \tag{2.50}$$

From (2.18), (2.19), this means that we have the following $\mathrm{SL}(2,\mathbb{Z})_3$ monodromy:

$$M_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_3. \tag{2.51}$$

One can also see this from the Kähler moduli,

$$\tau^1 = i\sqrt{\frac{f_1}{f_2}}\,, \qquad \tau^2 = i\sqrt{\frac{f_2}{f_1}}\,, \qquad \tau^3 = \gamma + i\sqrt{f_1 f_2}\,. \tag{2.52}$$

We see that, as we go once around the supertube, $\tau^{1,2}$ are single-valued whereas τ^3 has the monodromy

$$\tau^3 \to \tau^3 + 1\,. \tag{2.53}$$

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Because $\mathrm{Re}\,\tau^3 = B_{89}$, this monodromy implies that there is an NS5-brane along the closed curve.

One can consider other configurations involving codimension-2 branes. In appendix E, we discuss the D2(89)+D6(456789)$\to$ $5^2_2(\lambda 4567;89)$ supertube, which is the last entry in (2.43) and was studied in [32].

In the special case where $|\dot{\mathbf{F}}| = 1$, we have $f_1 = f_2 \equiv f$ and therefore $\tau^1 = \tau^2 = i$ as we can see from (2.52). This case belongs to the one-modulus class discussed in section 2.2, with the complex harmonic functions

$$F = \gamma + if, \qquad G = 1. \tag{2.54}$$

This setup is simple but still non-trivial enough to include interesting physical situations such as the D2(45) + D2(67) $\to$ ns5(λ4567) supertube. It can also describe the D2(89)+D6(456789)$\to$ $5^2_2(\lambda 4567;89)$ supertube discussed in appendix E. We will use this setup to construct a non-Abelian supertube configuration involving both D2 + D2 $\to$ ns5 and D2 + D6 $\to$ 5^2_2 supertubes.

In the above we discussed configurations just with codimension-3 sources or just with codimension-2 sources. One can also consider a mixed configuration in which a codimension-3 source and a codimension-2 source coexist [32].

General remarks on codimension-2 solutions. For the codimension-3 case, we could show the direct connection between the presence of delta-function sources on the right-hand side of equation (2.11) and the existence of CTCs. We can follow the same line of logic for the codimension-2 case, but the conclusion is that there is no such direct connection.

In (2.34), we had an integral over a small ball B_3 containing a point where there is a possible delta function. In the codimension-2 case, delta-function singularities are expected to be along a curve on which a source lies, and there is a Dirac-Misner "sheet" ending on that curve. Let us consider an integral over a very thin filled tube T^3 containing a piece of such a curve. Now we rewrite the integral as we did in (2.35). Instead of $S^2 = \partial B^3$, we have a cylinder $C^2 = \partial T^3$, where we can ignore the top and bottom bases for a very thin tube. As the boundary of the cylinder, ∂C^2, we take two lines that go along the curve in opposite directions. The Dirac-Misner sheet goes between the two lines. Then the integral is basically equal to the jump across the Dirac-Miner sheet in the component of ω along the curve. Let us denote it by $\Delta\omega_{\|}$. Then, the integral is equal to $l\Delta\omega_{\|}$, where l is the length of the tube. On the other hand, the same integral is equal to $l\sigma$, where σ is the local density of the delta-function source along the curve. Equating the two, we obtain

$$\Delta\omega_{\|} = \sigma. \tag{2.55}$$

Namely, the jump in ω along the curve is given by the density of delta-function sources.

However, this does not give the behavior of ω itself, which is necessary for evaluating (2.22) and study the presence of CTCs. So, the argument that worked for codimension 3 does not apply to codimension 2. It must be some other singular behavior of the harmonic functions, not just the delta-function source, that one must study to investigate the no-CTC condition. We do not pursue that in this paper. Instead, we will study (2.22) for specific explicit metrics in the presence of codimension-2 sources.

For codimension-3 sources, construction of general multi-center solutions is straightforward because of "linearity": one can simply add the poles representing different codimension-3 sources, as we did in (2.32). However, in contrast, construction of general solutions with multiple codimension-2 sources is less straightforward. This is because linearity is lost if there are multiple codimension-2 objects whose monodromy matrices do not commute, in other words, if the monodromies are non-Abelian. Indeed, the explicit construction of solutions with multiple codimension-2 supertubes thus far [32] is restricted to the case where (i) all supertubes have the same monodromy, or (ii) different supertubes have different monodromies but they all commute with each other. In either case, the monodromies are Abelian. In such cases, linearity still holds and the corresponding harmonic functions can be obtained by adding harmonic functions for each supertube.[10] In the next section, we will construct a configuration of two supertubes with non-Abelian monodromies in a certain limit.

Although we have only discussed sources with codimension 3 and 2, it is also possible to consider sources with codimension 1. Such a source represents a domain wall that connects spaces with different values of spacetime-filling fluxes, just like a D8-brane in 10D connects spacetimes with different values of the RR flux 10-form. Including codimension-1 sources should lead to a wide range of physical configurations which have been little studied. It would be very interesting to include them in the harmonic solutions and explore the physical implications of solutions with codimension 3, 2, and 1 sources.

3 Explicit construction of non-Abelian supertubes

3.1 Non-Abelian supertubes

In the previous section, we saw that harmonic solutions can describe BPS configurations of codimension-2 supertubes. A codimension-2 supertube has a non-trivial U-duality monodromy around it, which can be represented by a monodromy matrix M. If multiple codimension-2 supertubes are present and the i-th supertube has a monodromy matrix M_i then, in general, the monodromies of different supertubes do not commute, $[M_i, M_j] \neq 0$ for some pair (i, j), namely, the monodromies are *non-Abelian*. In this section, we show, for the first time, that such a non-Abelian configuration of supertubes is indeed possible.

We will focus on configurations in which only one modulus $\tau^3 \equiv \tau$ is non-trivial and has $\mathrm{SL}(2,\mathbb{Z})$ monodromies. As discussed in section 2.2, in this situation, only four harmonic functions are independent (2.24), which can be combined into two complex harmonic functions F, G. In terms of them, the modulus τ can be written as

$$\tau = \frac{F}{G}\,. \tag{3.1}$$

The simplest non-Abelian configuration is one with two supertubes. As we go around the i-th supertube, the harmonic functions transform as

$$\begin{pmatrix} F \\ G \end{pmatrix} \to M_i \begin{pmatrix} F \\ G \end{pmatrix}, \qquad M_i \in \mathrm{SL}(2,\mathbb{Z})\,, \qquad i = 1, 2. \tag{3.2}$$

[10] See footnote 2.

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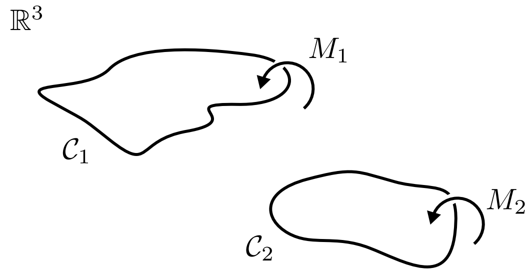

Figure 1. A non-Abelian configuration of two supertubes. The monodromy matrices M_1, M_2 of the two supertubes do not commute, $[M_1, M_2] \neq 0$.

We require that the monodromies be non-Abelian,

$$[M_1, M_2] \neq 0. \tag{3.3}$$

See figure 1 for a pictorial description of such a 2-supertube configuration.

Specifically, we will consider a two-supertube configuration with the following monodromies:

$$M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}. \tag{3.4}$$

These clearly give a non-Abelian pair of monodromies satisfying (3.3). As we will discuss later in this section, this choice is motivated by the solution to a similar monodromy problem discussed in the SU(2) Seiberg-Witten theory [36]. If we go around the two supertubes, the total monodromy is

$$M = M_2 M_1 = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

If one is far away from the supertubes, none of the monodromies of the supertubes are visible and the configuration looks like that of a single-center codimension-3 solution. From the $|\mathbf{x}| \to \infty$ behavior of the harmonic functions, we can read off the charges of the single-center solution. We will find that the charges are those of a 4-charge black hole in four dimensions with a finite horizon. In other words, seen from a large distance, our configuration looks like an ordinary 4-charge black hole without any monodromic structure. However, as one approaches it, the topology of the supertubes becomes distinguishable and discovers that the spacetime has non-trivial non-Abelian monodromies.

3.2 Strategy

The problem that we should attack in principle is the following. We first specify two closed curves $\mathcal{C}_1, \mathcal{C}_2$ in $\mathbb{R}^3$ along which the two supertubes lie, such as the ones in figure 1. Then we must find a pair of harmonic functions (F, G) which, as we go around curve $\mathcal{C}_i$, undergoes the monodromy transformation (3.2) with the monodromy matrix M_i given in (3.4). If we can find such pair (F, G), then the configuration exists.

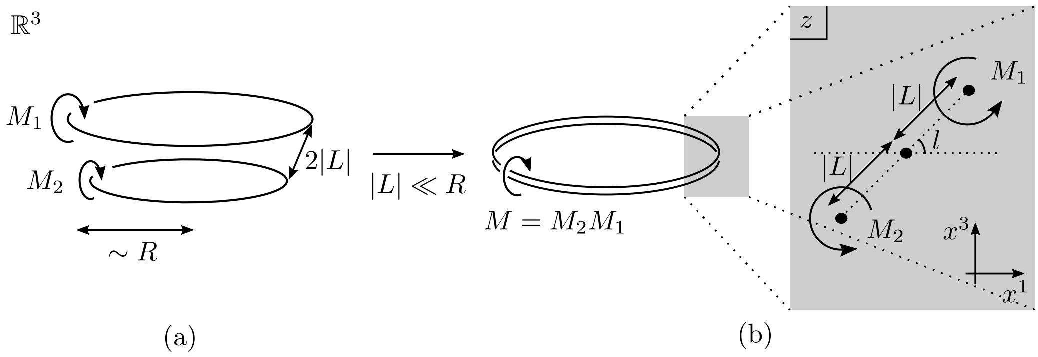

Figure 2. (a) A configuration of two circular supertubes sharing the axis. (b) The configuration in the colliding limit, $|L| \ll R$. In this limit, we can study the problem in two different regimes, the near and far regions. In the near region, the system becomes 2-dimensional but we must consider two separate monodromies M_1, M_2 of two supertubes. In the far region, the system remains 3-dimensional but there is only one tube with monodromy $M = M_2 M_1$.

Although this is a mathematically well-posed problem, explicitly carrying it out for general shapes of supertubes is technically challenging. Instead, our strategy here is to take a particularly simple configuration for the two supertubes and further take a limit in which the problem of finding the solution becomes simple and tractable but is still non-trivial. This is sufficient for the purpose of proving the existence of a configuration of non-Abelian supertubes.

Specifically, we assume that the two tubes are circular and share the axis (so that the configuration is axisymmetric). The two tubes have almost identical radius $R > 0$ and are very close to each other, separated by distance $2|L|$; see figure 2(a). More precisely, in equations, the location of supertubes 1 and 2 is specified as follows:

$$\begin{aligned} &\text{Supertube 1:} \qquad (x^1)^2 + (x^2)^2 = (R + |L|\cos l)^2, \quad x^3 = +|L|\sin l, \\ &\text{Supertube 2:} \qquad (x^1)^2 + (x^2)^2 = (R - |L|\cos l)^2, \quad x^3 = -|L|\sin l, \end{aligned} \tag{3.6}$$

where l is the angle between the two tubes relative to the x^1-x^2 plane; for example, $l = 0$ if they are concentric. We study this system in the *colliding limit*,

$$|L| \ll R. \tag{3.7}$$

In this limit, we can break down the problem into two regimes, depending on the distance d from an observer to the supertubes, as follows:

(i) *The near region, $d \sim |L| \ll R$.*

In this region, the two supertubes can be regarded as infinite straight lines and we can forget the direction along them. Therefore, the system can effectively be treated as 2-dimensional. By symmetry, we can zoom in onto the region near the point $(x^1, x^2, x^3) = (R, 0, 0)$ without loss of generality, and identify the z-plane with a small piece of the x^1-x^3 plane near that point with the relation

$$z = (x^1 - R) + ix^3, \qquad |x^1 - R|, |x^3| \sim |L| \ll R. \tag{3.8}$$

On the z-plane, the two supertubes are located at $z = L$ and $z = -L$, where we defined

$$L = |L|e^{il}. \tag{3.9}$$

So, the problem reduces to that of finding on the z-plane a pair of 2D harmonic functions (F, G) that has non-trivial monodromies M_1, M_2 given in (3.4) around $z = \pm L$. See figure 2(b).

(ii) *The far region, $|L| \ll R \sim d$.*

In this region, the two supertubes cannot be resolved and we effectively have only one supertube sitting at

$$(x^1)^2 + (x^2)^2 = R^2, \qquad x^3 = 0, , \tag{3.10}$$

with the combined monodromy $M = M_2 M_1$ given in (3.5). So, the problem reduces to that of finding 3D harmonic functions (F, G) with the monodromy M around one circular supertube.

After finding the expressions for the harmonic functions (F, G) in regions (i) and (ii), we must connect them in the intermediate region, $|L| \ll d \ll R$, in order to show the existence of (F, G) defined in the entire space. Namely, we must match the large-$|z|$ behavior of the near-region solution smoothly onto the near-ring (i.e., $(x^1, x^2, x^3) \to (R, 0, 0)$) behavior of the far-region solution.

This matching can be done order by order and the harmonic function in the entire space can be reconstructed to any order in perturbative expansion. To see exactly how this works in practice, let us study a toy example in which we can work out the matching procedure in detail.

A toy model for the matching procedure. As a simpler physical problem in which there are two very different scales $|L|$ and R with $|L| \ll R$, let us consider the following problem. In three dimensions, we would like to find the field configuration sourced by two point-like charges at $\mathbf{x} = \pm\mathbf{L} \equiv (0, 0, \pm|L|)$ with charge $Q_\pm$. Assume that the field H is governed by the Helmholtz equation

$$\left(\Delta - \frac{1}{R^2}\right) H = 0\,. \tag{3.11}$$

Of course, for this problem, we know the exact answer:

$$H = \frac{Q_+ e^{-\frac{|\mathbf{x}-\mathbf{L}|}{R}}}{|\mathbf{x}-\mathbf{L}|} + \frac{Q_- e^{-\frac{|\mathbf{x}+\mathbf{L}|}{R}}}{|\mathbf{x}+\mathbf{L}|}\,. \tag{3.12}$$

However, let us try here to recover this expression by working in the "near region" $|\mathbf{x}| \sim |L| \ll R$ and in the "far region" $|L| \ll R \sim |\mathbf{x}|$ separately, and finally matching the expressions in the intermediate region connecting the two.

In the near region $|\mathbf{x}| \sim |L| \ll R$, we can ignore the R dependence in (3.11). Therefore, the expression in the near region is

$$H = \frac{Q_+}{|\mathbf{x} - \mathbf{L}|} + \frac{Q_-}{|\mathbf{x} + \mathbf{L}|} \,. \tag{3.13}$$

Let (r, θ, φ) be the spherical polar coordinates for $\mathbb{R}^3$. If we increase r, still staying inside the near region, we can do a small $\frac{|L|}{r}$ expansion of this and obtain

$$H = \frac{Q_+ + Q_-}{r} + \frac{(Q_+ - Q_-)|L| \cos\theta}{r^2} + \mathcal{O}\left(\frac{|L|^2}{r^3}\right) , \tag{3.14}$$

which corresponds to the standard multipole expansion. We would like to find how this multipole expansion matches onto the one in the far region.

To be able to do the matching, there must be an intermediate region where the expansion (3.14) is correct. To understand what this means, let us make the scaling for the intermediate region, $|L| \ll r \ll R$, more precise by setting

$$\frac{r}{R} \sim \epsilon \,, \qquad \frac{|L|}{r} \sim \delta \,, \tag{3.15}$$

where $\epsilon, \delta \ll 1$. If we are to keep r finite, the replacement

$$R \to R\epsilon^{-1} \,, \qquad |L| \to |L|\delta \,, \tag{3.16}$$

will keep track of the order of expansion. If we do this replacement in the exact expression (3.12) and expand it in powers of ϵ and δ, we obtain

$$\begin{aligned} H = & \left[\frac{Q_+ + Q_-}{r} + \frac{(Q_+ - Q_-)|L| \cos\theta}{r^2}\delta + \mathcal{O}(\delta^2)\right] - \frac{(Q_+ + Q_-)\epsilon}{R} \\ & + \left[\frac{(Q_+ + Q_-)r}{2R^2} - \frac{(Q_+ - Q_-)|L| \cos\theta}{2R^2}\delta + \mathcal{O}(\delta^2)\right] \epsilon^2 + \mathcal{O}(\epsilon^3) \,. \end{aligned} \tag{3.17}$$

If we make ϵ small enough so that only the $\mathcal{O}(\epsilon^0)$ terms remain, then this reproduces the near-region expansion (3.14). Therefore, the correct procedure is: take $\epsilon \to 0$ first, and then match the δ expansion. In other words, take $R \to \infty$ first, and then match the small $\frac{|L|}{r}$ expansion.

With this mind, let us go to the far region. Here, the two charges cannot be resolved and the function H can be singular only at $r = 0$. The instruction is: find solutions of the Helmholtz equation such that their $R \to 0$ limit reproduces (3.14), term by term in the $\frac{|L|}{r}$ expansion. First,

$$(Q_+ + Q_-)\frac{e^{-\frac{r}{R}}}{r} \tag{3.18}$$

is clearly an exact solution with a singularity at $r = 0$. If we take $R \to \infty$, this gives r^{-1}, which reproduces the first term in (3.14). Next,

$$(Q_+ - Q_-)\, |L|\, e^{-\frac{r}{R}} \left(\frac{1}{r^2} + \frac{1}{Rr}\right) \cos\theta \tag{3.19}$$

is an exact solution and its $R \to \infty$ limit reproduces the second term in (3.14). So, up to this order, the far-region solution which reproduces (3.14) is

$$H = \frac{(Q_+ + Q_-)\, e^{-\frac{r}{R}}}{r} + (Q_+ - Q_-)\, |L|\, e^{-\frac{r}{R}} \left(\frac{1}{r^2} + \frac{1}{Rr}\right) \cos\theta + \mathcal{O}\left(\frac{|L|^2}{r^3}\right). \tag{3.20}$$

It is clear that we can keep going with this procedure to find the far-region solution that reproduces the expansion (3.14) to an arbitrarily high order, upon taking the $R \to \infty$ limit. In principle, if we can sum this expansion to all orders, we can recover the exact expression (3.12) with singular sources at $\mathbf{x} = \pm\mathbf{L}$. However, at any finite order, the perturbative expression (3.20) has a singularity only at $r = 0$; namely, some features of the exact solution can be seen only after carrying out the infinite sum, which is a limitation of the method of matching expansion.

Below, we will use the exactly same matching procedure to find the harmonic functions describing a configuration of non-Abelian supertubes.

3.3 The near region

Now with the colliding limit and the matching procedure understood, let us construct the solution starting from the near-region side.

Some general statements. As we mentioned before, in the near region, we can regard the round supertubes as parallel, infinite straight lines. Forgetting about the direction along the tubes, the problem reduces to the one on the z-plane defined in (3.8). A harmonic function in 2D can be written as the sum of holomorphic and anti-holomorphic functions. In the present case, this means that F, G are both written as a sum of holomorphic and anti-holomorphic functions.

Let us further assume that F and G are purely holomorphic:

$$F = F(z), \qquad G = G(z). \tag{3.21}$$

This is equivalent to assuming that $\tau = F/G$ is holomorphic. In this case, we can solve (2.30) to find ω explicitly. If we set

$$\omega = \omega_2 dx^2 + \omega_z dz + \omega_{\bar z} d\bar z, \tag{3.22}$$

where ω_z, $\omega_{\bar z}$ and ω_2 are independent of x^2, then

$$\omega_2 = -\operatorname{Im}(F\bar G) + C, \qquad \partial\omega_{\bar z} - \bar\partial\omega_z = 0 \tag{3.23}$$

where C is a constant.

The above ω_2 is $\mathrm{SL}(2,\mathbb{Z})$ invariant because

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \quad \operatorname{Im}(F\bar G) \to \operatorname{Im}[(\alpha F + \beta G)(\gamma \bar F + \delta \bar G)] = \operatorname{Im}[(\alpha\delta - \beta\gamma) F\bar G] = \operatorname{Im}(F\bar G), \tag{3.24}$$

for $\alpha\delta - \beta\gamma = 1$. Therefore, even if there is a singularity around which there is an $\mathrm{SL}(2,\mathbb{Z})$ monodromy and (F, G) are multi-valued, ω_2 is always single-valued. By (2.55), this means that the integrability condition (2.11) is satisfied without delta-function singularities along the supertube.

The constant C and functions $\omega_z, \omega_{\bar{z}}$ must ultimately be fixed by extending the near-region solution to the far-region solution and requiring that ω be regular everywhere and vanish at 3D infinity. In the present case, we will find that ω in the far region has a non-vanishing component only in the direction along the supertube. Therefore, we set $\omega_z = \omega_{\bar{z}} = 0$. On the other hand, the constant C cannot be fixed unless we have an exact solution (we only have a perturbative solution in the present paper).

When there is a supertube, the direction along its profile is a dangerous direction where there can be CTCs [71, 72]. This is the x^2 direction in the present case and the 22 component of the metric which is, e.g., from (2.12),

$$g_{22} \propto -\omega_2^2 + \mathcal{Q} = -[-\operatorname{Im}(F\bar{G}) + C]^2 + [\operatorname{Im}(F\bar{G})]^2 = C[2\operatorname{Im}(F\bar{G}) - C]. \tag{3.25}$$

From (2.31), $\operatorname{Im}(F\bar{G}) \geq 0$. So, for (3.25) not to be negative, the constant C must be in the following range:

$$0 \leq C \leq 2\min[\operatorname{Im}(F\bar{G})]. \tag{3.26}$$

This does not have to hold up to $z = \infty$. It only has to hold up to some value of $|z|$ above which the 2D approximation breaks down.

The solution. On the z-plane, we would like to construct a pair of harmonic functions (F, G) that has non-trivial non-Abelian monodromy (3.2) around some singular points. In doing that, we must require that the imaginary part of $\tau = F/G$ be always positive, because of the condition (2.31). There are many such possibilities, but in this paper we will take the pair of holomorphic functions that appeared in the solution of $d = 4, \mathcal{N} = 2$ supersymmetric gauge theory by Seiberg and Witten [36], because it is a fundamental example of configurations with non-Abelian monodromies.

The original work of Seiberg and Witten was about the exact determination of the low-energy effective theory of $\mathcal{N} = 2$ pure SU(2) gauge theory. At low energy, the theory has a Coulomb moduli space parametrized by the vacuum expectation value of the vector multiplet scalar, $z = \langle \operatorname{tr}\phi^2 \rangle \in \mathbb{C}$. At point z on the moduli space, one has a pair of holomorphic functions $(a_D(z), a(z))$ which represent the mass of the magnetic monopole and the electron at that point. In terms of them, the low-energy coupling constant, $\tau(z)$, is expressed as

$$\tau(z) = \frac{da_D}{da} = \frac{a'_D(z)}{a'(z)}. \tag{3.27}$$

The theory has an $SL(2, \mathbb{Z})$ duality group which changes the coupling constant τ and acts non-trivially on the spectrum of dyons. More specifically, under $SL(2, \mathbb{Z})$, the pair (a_D, a) transforms as a doublet and τ undergoes linear fractional transformation. The moduli space has three singularities at $z = \pm L, \infty$ around which there are non-trivial monodromies of the $SL(2, \mathbb{Z})$ duality. The one at $z = L$ is due to the magnetic monopole becoming massless and the monodromy around it is given by M_1 in (3.4). On the other hand, the one at $z = -L$ is due to the $(1, 1)$ dyon getting massless and the monodromy is given by M_2

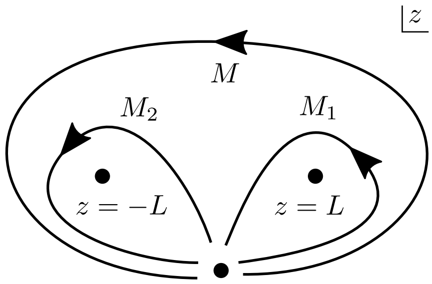

Figure 3. The monodromy structure in the near region. At $z = \pm L$ we have singularities corresponding to the position of the supertubes. When going around one of them, (F, G) gets transformed by M_i. Going around both of them induces a monodromy transformation $M = M_2 M_1$.

in (3.4). Finally, the one at $z = \infty$ is due to asymptotic freedom and the monodromy is given by M in (3.5). See figure 3 for the monodromy structure of the moduli space.

One sees that this theory has everything we need. We identify the $\mathrm{SL}(2,\mathbb{Z})$ duality group on the gauge theory side with the $\mathrm{SL}(2,\mathbb{Z})_3$ U-duality group on the supertube side, the modulus z with the z coordinate of the near region, the mass parameters (a_D, a) with the harmonic functions (F, G), and τ with the torus modulus $\tau^3 = \tau$. Furthermore, the position $z = \pm L$ of the singularities on the moduli space is identified with the position of the supertubes in the near region. The precise identification between (F, G) and (a_D, a) is

$$\begin{pmatrix} F \\ G \end{pmatrix} = c \begin{pmatrix} a'_D(z) \\ a'(z) \end{pmatrix} \tag{3.28}$$

where $c \in \mathbb{C}$ is a constant of dimension $[c] = (\text{length})^{1/2}$.[11] Now figure 3 is understood as the monodromy structure of the harmonic functions (F, G) in the near region.

One may wonder about the meaning, in the supertube context, of the singularity at $z = \infty$ of the Seiberg-Witten solution. Recall that the near-region description in terms of the z-plane is only an approximation near the tubes. In reality, the infinity of the near-region z-plane is connected to the 3D space, where the tube is not infinitely long but is finite and closed. In the context of the original Seiberg-Witten theory, which is defined in the z-plane, the monodromy at $z = \pm L$ must be canceled by the monodromy at $z = \infty$. On the other hand, in the supertube context, the z-plane is connected to a larger space, $\mathbb{R}^3$ and the monodromy is canceled by the other side of the supertube in $\mathbb{R}^3$.

The explicit expression for $a(z)$ and $a_D(z)$ is

$$\begin{aligned} a(z) &= \frac{\sqrt{2}}{\pi} \int_{-L}^{L} dx \sqrt{\frac{z-x}{(L-x)(L+x)}} = \sqrt{2(z+L)}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2L}{z+L}\right), \\ a_D(z) &= \frac{\sqrt{2}\, i}{\pi} \int_{L}^{z} dx \sqrt{\frac{z-x}{(x-L)(x+L)}} = \frac{L-z}{2i\sqrt{L}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{L-z}{2L}\right). \end{aligned} \tag{3.29}$$

[11] At this stage, c can actually be an arbitrary single-valued holomorphic function in z. However, one can show that, in order that the fields near each of the two supertube at $z = \pm L$ behave the same way as they do near ordinary supertubes, such as the $\mathrm{D2} + \mathrm{D2} \to \mathrm{ns5}$ supertube given in (2.54) or the $\mathrm{D2} + \mathrm{D6} \to 5_2^2$ supertube given in (E.6), we must take c to be constant. It must be possible to derive the behavior of c near supertubes by properly taking account of its backreaction of the brane worldvolume. See [80] for a discussion of such backreaction in F-theory configurations of 7-branes.

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Here ${}_2F_1(a,b;c;z)$ is the hypergeometric function. Note that L is a complex number (see (3.9)). The sign of the square root in the integral expression is defined to be positive for $0<L<z$ and, for complex L,z, it is defined by analytic continuation. Taking derivatives, we have

$$\begin{aligned} a'(z) &= \frac{1}{\sqrt{2}\,\pi}\int_{-L}^{L}\frac{dx}{\sqrt{(z-x)(L-x)(L+x)}} = \frac{\sqrt{2}}{\pi\sqrt{z+L}}K\left(\frac{2L}{z+L}\right)\,, \\ a'_D(z) &= \frac{i}{\sqrt{2}\,\pi}\int_{L}^{z}\frac{dx}{\sqrt{(z-x)(x-L)(x+L)}} = \frac{i}{\pi\sqrt{L}}K\left(\frac{L-z}{2L}\right)\,, \end{aligned} \tag{3.30}$$

where $K(z)=\frac{\pi}{2}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;z)$ is the complete elliptic integral of the first kind. As mentioned above, as we go around the singular points $z=L,-L$ and $z=\infty$, the pair (a_D,a) and hence (a'_D,a') undergoes $\mathrm{SL}(2,\mathbb{Z})$ transformations given by the monodromy matrices M_1,M_2 in (3.4) and M in (3.5), respectively.

Now we have (F,G) in the near region, which is related via (3.28) to (a'_D,a') given in (3.30). To match this with the far-region solution, we will later need the $|z|\to\infty$ behavior of (a'_D,a'). It is given by

$$a'(z) = \frac{1}{\sqrt{2z}}+\frac{3L^2}{4(2z)^{5/2}}+\frac{105L^4}{64(2z)^{9/2}}+\cdots\,, \tag{3.31a}$$

$$a'_D(z) = \frac{i}{\pi}\left[\frac{1}{\sqrt{2z}}\ln\frac{8z}{L}+\frac{3L^2}{4(2z)^{5/2}}\left(\ln\frac{8z}{L}-\frac{5}{3}\right)+\frac{105L^4}{64(2z)^{9/2}}\left(\ln\frac{8z}{L}-\frac{389}{210}\right)+\cdots\right]. \tag{3.31b}$$

Just from the leading terms, it is easy to check that we have the monodromy

$$\begin{pmatrix} a'_D \\ a' \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a'_D \\ a' \end{pmatrix} = M\begin{pmatrix} a'_D \\ a' \end{pmatrix}. \tag{3.32}$$

For later convenience, let us also write down the behavior near the singularities $z=\pm L$. Near $z=L$,

$$a'(z) = -\frac{1}{2\pi\sqrt{L}}\left[\ln\frac{z-L}{32L}-\frac{1}{8L}\left(\ln\frac{z-L}{32L}+2\right)(z-L)+\cdots\right]. \tag{3.33a}$$

$$a'_D(z) = \frac{i}{2\sqrt{L}}\left[1-\frac{1}{8L}(z-L)+\cdots\right] = \frac{i}{2\sqrt{L}}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}n!^2}\right)^2\left(\frac{-1}{2L}\right)^n(z-L)^n. \tag{3.33b}$$

Near $z=-L$,

$$a'(z) = \frac{i}{2\pi\sqrt{L}}\left[\ln\frac{z+L}{-32L}+\frac{1}{8L}\left(\ln\frac{z+L}{-32L}+2\right)(z+L)+\cdots\right]. \tag{3.34a}$$

$$a'_D(z) = -\frac{i}{2\pi\sqrt{L}}\left[\ln\frac{z+L}{32L}+\frac{1}{8L}\left(\ln\frac{z+L}{32L}+2\right)(z+L)+\cdots\right]. \tag{3.34b}$$

From these, it is easy to check the monodromy M_1,M_2.

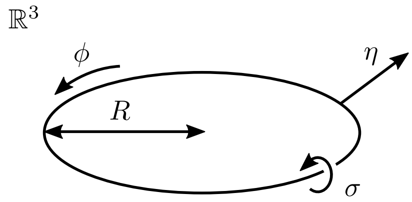

Figure 4. Toroidal coordinates (η, σ, ϕ). η is a "radial" coordinate that decreases as one goes away from the ring, σ is the angular variable around the ring and ϕ is an angular variable along the ring.

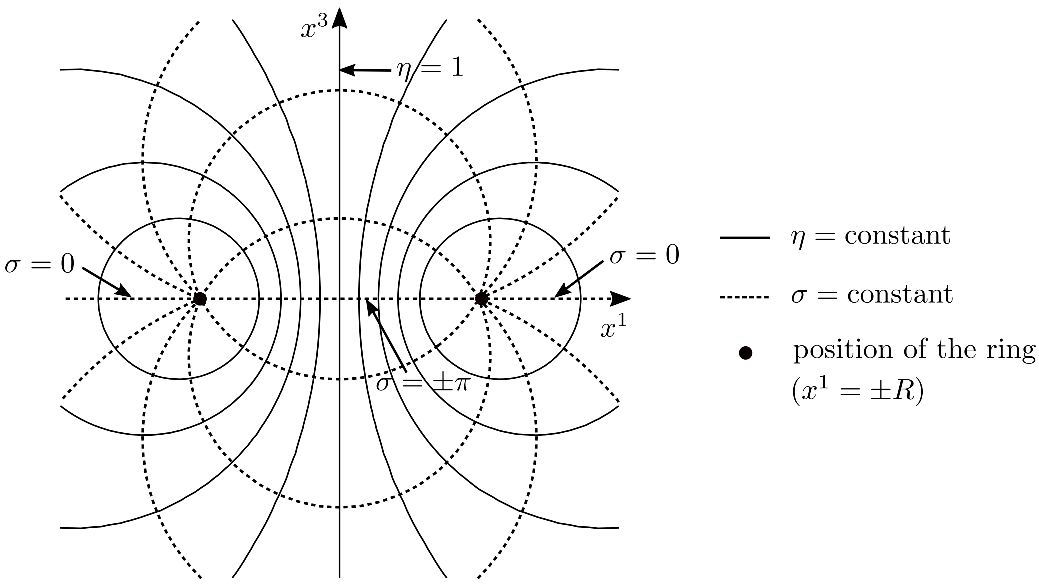

Figure 5. Toroidal coordinates in the $x^2 = 0$ section. Solid lines represent constant-η surfaces and dotted lines represent constant-σ surfaces. As $\eta \to 1$, the constant-η surface approaches the vertical (x^3) axis , while the position of the ring corresponds to the $\eta \to \infty$ limit.

3.4 The far region: coordinate system and boundary conditions

Having fixed the near-region solution, the next task is to find the far-region solution that matches onto it. For that, as preparation, let us introduce the coordinate system appropriate for our purpose and discuss the boundary conditions that the far-region solution must satisfy.

Toroidal coordinate system. As we explained, in the far region, we effectively have one supertube. To describe this configuration, we introduce the toroidal coordinate system (η, σ, ϕ) [81]; see figures 4 and 5. In terms of Cartesian coordinates (x^1, x^2, x^3), the toroidal coordinates are given by

$$x^1 = R\frac{\sqrt{\eta^2 - 1}}{\eta - \cos\sigma}\cos\phi\,, \quad x^2 = R\frac{\sqrt{\eta^2 - 1}}{\eta - \cos\sigma}\sin\phi\,, \quad x^3 = R\frac{\sin\sigma}{\eta - \cos\sigma}\,, \tag{3.35}$$

where R is the radius of the ring, σ is the angular variable around the ring and ϕ is the angular variable along the ring. The inverse relations are given by

$$\eta = \frac{\mathbf{x}^2 + R^2}{\Sigma}\,, \qquad \cos\sigma = \frac{\mathbf{x}^2 - R^2}{\Sigma}\,, \qquad \tan\phi = \frac{x^2}{x^1}\,, \tag{3.36}$$

with

$$\Sigma^2 = (\mathbf{x}^2 - R^2)^2 + 4R^2(x^3)^2\,. \tag{3.37}$$

The domain of the coordinates is $1 \le \eta < \infty$, $-\pi \le \sigma < \pi$, $0 \le \phi < 2\pi$. Then, the flat 3D metric in the toroidal coordinates is given by

$$ds^2 = \frac{R^2}{(\eta - \cos\sigma)^2}\left(\frac{d\eta^2}{\eta^2 - 1} + d\sigma^2 + (\eta^2 - 1)d\phi^2\right)\,. \tag{3.38}$$

To connect the far- and near-region solutions, we have to relate the near-region (2D) and the far-region (3D) coordinates. In the near-region limit $\eta \to \infty$, the Cartesian coordinates are given, to leading order, by

$$x^1 \simeq R + \frac{R\cos\sigma}{\eta}\,, \qquad x^2 = 0\,, \qquad x^3 \simeq \frac{R\sin\sigma}{\eta}\,. \tag{3.39}$$

Then we can relate the z coordinate defined in (3.8) to the toroidal coordinates (η, σ) as

$$z = (x^1 - R) + ix^3 = \frac{R}{\eta}e^{i\sigma}\,. \tag{3.40}$$

This is the fundamental relation to connect the near- and far-region solutions.

Boundary conditions. On the far-region solution, we have to impose boundary conditions at infinity ($\eta \to 1$ and $\sigma \to 0$ simultaneously) and near the supertube ($\eta \to \infty$).

First, let us discuss the boundary condition at infinity. We require the harmonic functions to go as

$$H = h + \frac{\Gamma}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty, \tag{3.41}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. This is the same $r \to \infty$ behavior as the codimension-3 solution, (2.32) (or (2.39)). This is because we are interested in codimension-2 branes (supertubes) which have been produced by the supertube transition out of codimension-3 branes. Very far from it, the codimension-2 brane must look like a codimension-3 object with the original monopole charge. Therefore, the harmonic function must have the $1/r$ term whose coefficient Γ is the same as the total monopole charge of the original brane configuration.

The boundary condition near the tube ($\eta \to \infty$) comes from the matching condition discussed at the end of section 3.2. Let us write the large-$|z|$ expansion of $a'(z)$ and $a'_D(z)$ as[12]

$$a'(z) = \sum_{n=0}^{\infty} a'_n(z)\,, \qquad a'_D(z) = \sum_{n=0}^{\infty} a'_{Dn}(z)\,, \tag{3.42}$$

[12] This expansion corresponds to (3.14) of the toy model in section 3.2.

where $a'_n, a'_{Dn} = \mathcal{O}(z^{-2n-1/2})$ (here it is understood that $\mathcal{O}(z^{-2n-1/2})$ includes $z^{-2n-1/2}\log z$). The first three terms of each expansion are given in (3.31a) and (3.31b). As we discussed earlier in section 3.2, we must be able to find a far-region solution that matches onto this expansion, order by order. Concretely, let us do a near-ring ($\eta \to \infty$) expansion of the far-region harmonic functions F and G and let the n-th term be F_n and G_n where their behavior as $\eta \to \infty$[13] is $F_n, G_n = \mathcal{O}(\eta^{2n+1/2})$.[14] Then, upon using the dictionary (3.40), we must have

$$F_n = ca'_{Dn} + \mathcal{O}(\eta^{2n-1/2}), \qquad G_n = ca'_n + \mathcal{O}(\eta^{2n-1/2}), \qquad \eta \to \infty. \tag{3.43}$$

Note that the lesson of the toy model in section 3.2 was that we have to take the limit $r \ll R$ first, and then match the small $\frac{|L|}{r}$ expansion. In the present case, the former corresponds to matching only the leading $\mathcal{O}(\eta^{2n+1/2})$ term in (3.43), while the latter corresponds to doing this for each value of n.

For example, for the first ($n = 0$) term, we have

$$F_0 = \frac{ic}{\pi\sqrt{2z}} \ln \frac{8z}{L} + \mathcal{O}(\eta^{-1/2})\,, \qquad G_0 = \frac{c}{\sqrt{2z}} + \mathcal{O}(\eta^{-1/2})\,. \tag{3.44}$$

In principle, we can find F_n and G_n satisfying (3.43) for n arbitrarily large. If we could carry out the infinite sum $F = \sum_n F_n$ and $G = \sum_n G_n$, it would correspond to the exact two-supertube solution defined in the entire $\mathbb{R}^3$.

3.5 The far region: the solution

In the far region, there is only one supertube (see figure 4) and we are instructed to find a pair of harmonic functions (F, G) that has the monodromy

$$\begin{pmatrix} F \\ G \end{pmatrix} \to M \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \tag{3.45}$$

as $\sigma \to \sigma + 2\pi$. In other words,

$$F \to -F + 2G\,, \tag{3.46a}$$

$$G \to -G\,, \tag{3.46b}$$

Harmonic functions in toroidal coordinates. Let us explain now how to construct F and G. We start with the ansatz for G since its monodromy (3.46b) is simpler. If we assume the following separated form,

$$G(\eta, \sigma, \phi) = \sqrt{\eta - \cos\sigma}\, T(\eta) S(\sigma) V(\phi)\,, \tag{3.47}$$

the Laplace equation becomes

$$\begin{aligned} \Delta G = {} & \frac{(\eta - \cos\sigma)^{5/2}}{R^2} T(\eta) S(\sigma) V(\phi) \\ & \times \left[\frac{1}{\eta^2 - 1} \frac{V''(\phi)}{V(\phi)} + \frac{S''(\sigma)}{S(\sigma)} + \frac{1}{T(\eta)} \left(\frac{1}{4} T(\eta) + 2\eta T'(\eta) + (\eta^2 - 1) T''(\eta) \right) \right] \\ = {} & 0\,. \end{aligned} \tag{3.48}$$

[13] The behavior will be determined in the next section 3.5 and appendix B.

[14] These n-th terms correspond to (3.20) of the toy model in section 3.2.

This can be reduced to the following three ordinary differential equations:

$$0 = V''(\phi) + m^2 V(\phi)\,, \tag{3.49a}$$

$$0 = S''(\sigma) + k^2 S(\sigma)\,, \tag{3.49b}$$

$$0 = (\eta^2-1)T''(\eta) + 2\eta T'(\eta) + \left(\frac{1}{4} - k^2 - \frac{m^2}{\eta^2-1}\right)T(\eta)\,, \tag{3.49c}$$

with arbitrary constants m and k. The general solutions for these equations are given by

$$V(\phi) = e^{im\phi}\,, \tag{3.50a}$$

$$S(\sigma) = e^{ik\sigma}\,, \tag{3.50b}$$

$$T(\eta) = P^{|m|}_{|k|-1/2}(\eta) \quad \text{and} \quad Q^{|m|}_{|k|-1/2}(\eta)\,, \tag{3.50c}$$

where $P^m_k(\eta)$ and $Q^m_k(\eta)$ are the associated Legendre functions of the first and second kind, respectively, with degree k and order m. If we require 2π periodicity along the ϕ (respectively σ) direction, the constant m (respectively k) will take integer values. Because our configuration is symmetric along ϕ (see figure 4), we should take $m = 0$. Then as we can easily see from the form of the solutions (3.50), we have to choose $k \in \mathbb{Z}+1/2$ in order for G to have the monodromy (3.46b). So the solution for G is written as

$$G = \sqrt{\eta - \cos\sigma}\, e^{ik\sigma} \left(A_{|k|-1/2} P_{|k|-1/2}(\eta) + B_{|k|-1/2} Q_{|k|-1/2}(\eta)\right)\,, \tag{3.51}$$

where $k \in \mathbb{Z}+1/2$ and $A_{|k|-1/2}, B_{|k|-1/2}$ are constants.

Let us turn to F. The monodromy (3.46a) motivates the following ansatz:

$$F(\eta,\sigma,\phi) = \sqrt{\eta - \cos\sigma}\left(U(\eta) - \frac{\sigma}{\pi}T(\eta)\right)S(\sigma)V(\phi)\,. \tag{3.52}$$

Plugging this into the Laplace equation, we obtain

$$\begin{aligned} 0 = U(\eta)\Bigg[&\frac{1}{\eta^2-1}\frac{V''(\phi)}{V(\phi)} + \frac{S''(\sigma)}{S(\sigma)} \\ &+ \frac{1}{U(\eta)}\left(\frac{1}{4}U(\eta) + 2\eta U'(\eta) + (\eta^2-1)U''(\eta)\right) - \frac{2}{\pi}\frac{T(\eta)}{U(\eta)}\frac{S'(\sigma)}{S(\sigma)}\Bigg] \\ -\frac{\sigma}{\pi}T(\eta)&\left[\frac{1}{\eta^2-1}\frac{V''(\phi)}{V(\phi)} + \frac{S''(\sigma)}{S(\sigma)} + \frac{1}{T(\eta)}\left(\frac{1}{4}T(\eta) + 2\eta T'(\eta) + (\eta^2-1)T''(\eta)\right)\right]. \end{aligned} \tag{3.53}$$

If we take T, S and V to be the solutions of (3.48) given by (3.50), then the second line of (3.53) vanishes and we are left with

$$(\eta^2-1)U''(\eta) + 2\eta U'(\eta) + \left(\frac{1}{4} - k^2 - \frac{m^2}{\eta^2-1}\right)U(\eta) = \frac{2}{\pi}T(\eta)\frac{S'(\sigma)}{S(\sigma)}\,. \tag{3.54}$$

This differential equation differs from (3.49c) in its inhomogeneous term. The solution of (3.54) for a specific choice of $T(\eta)$ and $S(\sigma)$ can be easily found. We gave a few examples in appendix B.

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Even though we have to solve (3.54) to get explicit harmonic functions, the monodromy can be easily seen without solving it. Let us assume $k \in \mathbb{Z}+1/2$ as in (3.51) to get an overall sign flip after going around the supertube ($\sigma \to \sigma + 2\pi$). We also set $m = 0$ because of the symmetry of our configuration. Then the monodromy is exactly what we want (3.46a):

$$F \to -F + 2G \quad \text{as} \quad \sigma \to \sigma + 2\pi \,. \tag{3.55}$$

If we choose a particular term in (3.42) with a specific value of n that we want to reproduce, the value of k can be determined and the equation (3.54) can be solved. Here we will focus on the first ($n = 0$) term in (3.43). The leading term in the large-$|z|$ expansion of $a'(z)$ is

$$a_0' = \frac{1}{\sqrt{2z}} = \sqrt{\frac{\eta}{2R}} e^{-i\sigma/2} \,, \tag{3.56}$$

where we have used the dictionary (3.40). Then we have to take $k = -1/2$ to reproduce this as a limit of the 3D harmonic function G. We can easily show that this is also correct choice for a'_{D0} and F. With this choice, $T(\eta)$ is also fixed and is given by a linear combination of $P_0(\eta)$ and $Q_0(\eta)$.

The resulting harmonic functions can be written as

$$F(\eta, \sigma, \phi) = \sqrt{\eta - \cos\sigma}\, e^{-i\sigma/2} U(\eta) - \frac{\sigma}{\pi} G \,, \tag{3.57}$$

$$G(\eta, \sigma, \phi) = \sqrt{\eta - \cos\sigma}\, e^{-i\sigma/2} T(\eta) \,, \tag{3.58}$$

where

$$T(\eta) = A_0 P_0(\eta) + B_0 Q_0(\eta) \tag{3.59}$$

and $U(\eta)$ is a solution of

$$(\eta^2 - 1) U''(\eta) + 2\eta U'(\eta) = -\frac{i}{\pi} T(\eta) \,. \tag{3.60}$$

A_0 and B_0 are constant of integration which should be chosen from the boundary conditions.

It is easy to write down solutions explicitly if we impose boundary conditions at infinity, (3.41), before solving (3.60). The boundary condition at infinity, (3.41), leads to the condition

$$B_0 = 0 \,, \tag{3.61}$$

since $Q_0(\eta)$ diverges at 3D infinity.[15] Then (3.60) is easily solved to give

$$U(\eta) = C_0 P_0(\eta) + D_0 Q_0(\eta) - \frac{i}{\pi} A_0 \ln \frac{\eta + 1}{2} \,. \tag{3.62}$$

By imposing the same boundary condition at infinity on $U(\eta)$, (3.41), we conclude that

$$D_0 = 0 \,. \tag{3.63}$$

[15] More precisely, $B_0 \neq 0$ would lead to divergence at 3D infinity and on the x^3-axis. If $\sigma \neq 0$, as we can see from (3.35), $\eta = 1$ corresponds to the points on the x^3-axis, $(x^1, x^2, x^3) = (0, 0, R\cot\frac{\sigma}{2})$. As $\eta \to 1$, $Q_{|k|-1/2}$ diverges as $\log(\eta - 1)$ while the prefactor is finite: $\sqrt{\eta - \cos\sigma} = \sqrt{2}\,|\sin\frac{\sigma}{2}|$. Therefore, $B_0 \neq 0$ makes the harmonic function diverge on the x^3-axis and should be avoided.

The final expression for the harmonic functions is

$$F(\eta,\sigma,\phi) = \sqrt{\eta - \cos\sigma}\, e^{-i\sigma/2}\frac{i}{\pi}A_0\left(\frac{\pi}{i}\frac{C_0}{A_0} - \ln\frac{\eta+1}{2} + i\sigma\right), \tag{3.64}$$

$$G(\eta,\sigma,\phi) = \sqrt{\eta - \cos\sigma}\, e^{-i\sigma/2}A_0\,, \tag{3.65}$$

where we used $P_0(\eta) = 1$.

Matching. We have obtained the solutions in the near and far regions. Let us fix the coefficients A_0 and C_0 by matching the two solutions in the intermediate region. This amounts to imposing the conditions (3.44). The near-ring ($\eta \to \infty$) expressions for F and G are

$$F \simeq \sqrt{\eta}\, e^{-i\sigma/2}\frac{i}{\pi}A_0\left(\frac{\pi}{i}\frac{C_0}{A_0} - \ln\frac{\eta}{2} + i\sigma\right), \qquad G \simeq \sqrt{\eta}\, e^{-i\sigma/2}A_0\,. \tag{3.66}$$

Therefore, the conditions (3.44) read

$$\begin{aligned}\frac{i}{\pi}\sqrt{\eta}\, e^{-i\sigma/2}A_0\left(\frac{\pi}{i}\frac{C_0}{A_0} - \ln\frac{\eta}{2} + i\sigma\right) &= \frac{i}{\pi}c\sqrt{\frac{\eta}{2R}}\, e^{-i\sigma/2}\left(\ln\frac{4R}{L} - \ln\frac{\eta}{2} + i\sigma\right), \\ \sqrt{\eta}\, e^{-i\sigma/2}A_0 &= c\sqrt{\frac{\eta}{2R}}\, e^{-i\sigma/2}\,.\end{aligned} \tag{3.67}$$

These determine the constants to be

$$A_0 = \frac{c}{\sqrt{2R}}\,, \qquad C_0 = \frac{i}{\pi}\frac{c}{\sqrt{2R}}\ln\frac{4R}{L}\,. \tag{3.68}$$

The final expression for the far-region solution is

$$F(\eta,\sigma,\phi) = \frac{ic}{\pi\sqrt{2R}}\sqrt{\eta - \cos\sigma}\, e^{-i\sigma/2}\left[-\ln\frac{L(\eta+1)}{8R} + i\sigma\right], \tag{3.69a}$$

$$G(\eta,\sigma,\phi) = \frac{c}{\sqrt{2R}}\sqrt{\eta - \cos\sigma}\, e^{-i\sigma/2}\,. \tag{3.69b}$$

4 Physical properties of the solution

In the previous section, we obtained the explicit expression for the harmonic functions (F, G) in (3.69) which describes the far-region behavior of a non-Abelian two-supertube configuration, at the leading order in a perturbative expansion. In terms of these complex harmonic functions, the real harmonic functions $\{V, K^I, L_I, M\}$ can be expressed via (2.27). Here we discuss some physical properties of this solution.

4.1 Geometry and charges

First, let us study the asymptotic form of the harmonic functions near 3D infinity, $r = \infty$, which corresponds to $\eta = 1, \sigma = 0$ in the toroidal coordinates. Using the relation (3.36), we find that

$$F = h_F + \frac{Q_F}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad G = h_G + \frac{Q_G}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{4.1}$$

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where

$$h_F = h_G = 0, \tag{4.2}$$

$$Q_F = ic\sqrt{R}\,\nu, \qquad Q_G = c\sqrt{R} \tag{4.3}$$

with

$$\nu \equiv \frac{1}{\pi}\log\frac{4R}{L}. \tag{4.4}$$

The asymptotic form (4.1) is the same as that of the general codimension-3 harmonic function, (2.39). Note that, under our assumption (3.7),

$$\mathrm{Re}\,\nu = \frac{1}{\pi}\log\frac{4R}{|L|} \gg 1. \tag{4.5}$$

The asymptotic monopole charges of the solution can be read off from the coefficients of the $1/r$ terms in the harmonic functions, (4.3). The corresponding D-brane numbers N^0, N^I, N_I, N_0 can be determined from the relation (2.40). Explicitly,

$$N^3 + iN_1 = \frac{2ic\sqrt{R}\,\nu}{g_s l_s}, \qquad N^0 - iN^1 = \frac{2c\sqrt{R}}{g_s l_s}. \tag{4.6}$$

The entropy of the single-center black hole with charges (4.3) can be computed using (2.41):

$$S = \frac{8\pi\,|\mathrm{Im}(Q_F\bar{Q}_G)|}{g_s^2 l_s^2} = \frac{8\pi|c|^2 R}{g_s^2 l_s^2}\,\mathrm{Re}\,\nu. \tag{4.7}$$

This is non-vanishing because of (4.5) and therefore our solution has the same asymptotic charges as a black hole with a finite horizon area.

One peculiar thing about the harmonic functions (4.1) is that the constant terms always vanish, $h_F = h_G = 0$. This fact came from the harmonic analysis in the toroidal coordinates. For example, in the ansatz for G, (3.51), the prefactor goes as $\sqrt{\eta - \cos\sigma} \sim \sqrt{2}R/r$ in the 3D infinity limit $\eta \to 1, \sigma \to 0$. On the other hand, $P_{|k|-1/2}(\eta = 1) = 1$ and therefore $G \sim 1/r$ and does not have a constant term. We do not have the option of turning on $Q_{|k|-1/2}(\eta)$, because it diverges on the x^3-axis and should not be present (see footnote 15).

This means that this solution cannot have flat asymptotics. Instead, the asymptotic geometry is always the attractor geometry [35] of a single-center black hole with D6, D4, D2 and D0 charges in the near-horizon limit. Indeed, the asymptotic form of the type IIA geometry is easily seen from (2.12) to be

$$\begin{aligned} ds^2_{10,\mathrm{str}} &= -\frac{1}{\mathrm{Im}(F\bar{G})}(dt+\omega)^2 + \mathrm{Im}(F\bar{G})\left(dr^2 + r^2 d\Omega_2^2\right) + dx^2_{4567} + \mathrm{Im}\left(\frac{F}{G}\right)dx^2_{89} \\ &\sim -\frac{r^2}{\mathrm{Im}(Q_F\bar{Q}_G)}dt^2 + \mathrm{Im}(Q_F\bar{Q}_G)\left(\frac{dr^2}{r^2} + d\Omega_2^2\right) + dx^2_{4567} + \mathrm{Im}\left(\frac{Q_F}{Q_G}\right)dx^2_{89}, \end{aligned} \tag{4.8a}$$

$$e^{2\Phi} = \mathrm{Im}\left(\frac{F}{G}\right) \sim \mathrm{Im}\left(\frac{Q_F}{Q_G}\right). \tag{4.8b}$$

We see that this is $\mathrm{AdS}_2 \times S^2 \times T^6$ with radius $\mathcal{R}_{\mathrm{AdS}_2} = \mathcal{R}_{S^2} = \sqrt{\mathrm{Im}(Q_F\bar{Q}_G)}$.

Asymptotic charge versus local charge. It is interesting to compare the asymptotic charges (4.3) with the one that we would obtain from the behavior of fields near the supertubes. From (3.33) and (3.34), we find that the behavior of the harmonic functions F, G near the supertubes is

$$\begin{aligned} z \sim +L: \qquad & F \sim \text{const.}, & G &\sim -\frac{c}{2\pi\sqrt{L}} \log(z-L), \\ z \sim -L: \qquad & F \sim -\frac{ic}{2\pi\sqrt{L}} \log(z+L), & G &\sim \frac{ic}{2\pi\sqrt{L}} \log(z+L). \end{aligned} \tag{4.9}$$

If a codimension-2 source at $|z| = 0$ has D-brane number densities n^0, n^1, n^3 and n_1 per unit length for D6(456789), D4(6789), D4(4567), and D2(45) branes, respectively, then the harmonic functions will have the following logarithmic behavior:[16]

$$\begin{aligned} V &\sim -g_s l_s n^0 \log|z|, & K^1 &\sim -g_s l_s n^1 \log|z|, \\ K^3 &\sim -g_s l_s n^3 \log|z|, & L_1 &\sim -g_s l_s n_1 \log|z|. \end{aligned} \tag{4.11}$$

Or, in terms of the complex harmonic functions F, G,

$$F \sim -g_s l_s (n^3 + i n_1) \log|z|, \qquad G \sim -g_s l_s (n^0 - i n^1) \log|z|. \tag{4.12}$$

Comparing this with (4.9), we see that the D-brane number densities are

$$\begin{aligned} z = +L: \qquad & n^3 + i n_1 = 0, & n^0 - i n^1 &= \frac{c}{2\pi g_s l_s \sqrt{L}}, \\ z = -L: \qquad & n^3 + i n_1 = \frac{ic}{2\pi g_s l_s \sqrt{L}}, & n^0 - i n^1 &= -\frac{ic}{2\pi g_s l_s \sqrt{L}}. \end{aligned} \tag{4.13}$$

Because these charges are distributed over rings of radius approximately R, the total D-brane numbers would be

$$N^3 + i N_1 \stackrel{?}{=} \frac{icR}{g_s l_s \sqrt{L}}, \qquad N^0 - i N^1 \stackrel{?}{=} \frac{(1-i)cR}{g_s l_s \sqrt{L}}. \tag{4.14}$$

These are completely different from the charge we observe at infinity, (4.6).

The reason why we obtained incorrect total charges (4.14) is that our solution is multi-valued. In normal situations, the Gaussian surface on which we integrate fluxes to obtain charges can be continuously deformed from asymptotic infinity to small surfaces enclosing local charges. However, in the present case, the fields in our solution are multi-valued because of the monodromies around the supertubes, and so are the fluxes. Another way of saying this is that there is a branch cut (or disk) inside each of the two tubes, and the fluxes are discontinuous across it. When we deform the Gaussian surface at infinity,

[16] For example, if we array D6-branes at intervals of distance a, from (2.37)

$$V \sim \frac{g_s l_s}{2} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{|z|^2 + na}} \sim \frac{g_s l_s}{2a} \int_{-\Lambda}^{\Lambda} \frac{dx}{\sqrt{|z|^2 + x^2}} \sim -\frac{g_s l_s}{a} \log\frac{|z|}{2\Lambda} + \mathcal{O}(\Lambda^{-2}) \tag{4.10}$$

where Λ is a cutoff. By replacing a with $1/n^0$, we obtain (4.11).

we cannot shrink them to enclose just the supertubes; all we can do is to deform it into two surfaces, each of which encloses one entire branch disk with the supertube on its circumference. When we evaluate the flux integral on the Gaussian surfaces, there will be contributions not just from the supertubes but also from (the discontinuity in) the fluxes on the disks. The difference between (4.6) and (4.14) is due to the contribution from the fluxes on the disks.

This situation of branch cuts carrying charge by the discontinuity in the fluxes across it is an example of the so-called Cheshire charge that appears in the presence of vortices with non-trivial monodromies called Alice strings [82–84]. For discussions on the realizations of Alice strings in string theory, see [85, 86].

When integrating fluxes on Gaussian surfaces to compute charges in the presence of Chern-Simons interactions (such as supergravity in 11, 10, and 5 dimensions), one must be careful about different definitions of charges [87]. The relevant one here is the Page charge, which is conserved, localized, quantized, and gauge-invariant under small gauge transformations. For Page charge, we can freely deform a Gaussian surface unless they cross a charge source or a branch cut for the fluxes. The discussion of charges in the paragraphs above is understood to be using the Page charge. For the explicit form of the Page fluxes for D-brane charges, see, e.g., [31, Appendix D][32, Appendix E].

Angular momentum. By solving equation (2.30) for the harmonic functions given in (3.69), we find

$$\omega = \frac{|c|^2}{2\pi}\left((\eta+1)\ln\frac{|L|(\eta+1)}{8R} + 2\ln\frac{4R}{|L|}\right) d\phi\,, \tag{4.15}$$

where the integration constant was fixed by requiring that ω vanish at $\eta = 1$ (3D infinity). In spherical polar coordinates (r, θ, φ), the asymptotic behavior of (4.15) as $r \to \infty$ is

$$\omega \simeq \frac{|c|^2 R^2}{\pi}\left(1 + \ln\frac{|L|}{4R}\right)\frac{\sin^2\theta}{r^2} d\varphi = \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{4.16}$$

In four dimensions, the angular momentum is given by the $\mathcal{O}\left(\frac{1}{r}\right)$ term in the (t, i) components of the metric, which is nothing but the 1-form ω in our case. Therefore, we conclude that the 4D angular momentum J of our configuration vanishes:

$$J = 0. \tag{4.17}$$

Note that (4.16) means that the entire angular momentum vector vanishes, not just its x^3 component.

4.2 Closed timelike curves

No-CTC conditions for the one-modulus class solutions with $\tau^1 = \tau^2 = i$ were briefly discussed in section 2.2. For the explicit harmonic functions of the far-region solution (3.69), the condition (2.31) gives

$$\mathrm{Im}(F\bar{G}) = \frac{|c|^2(\eta - \cos\sigma)}{2\pi R}\ln\frac{8R}{|L|(\eta+1)} \simeq \frac{|c|^2\eta}{2\pi R}\ln\frac{8R}{|L|\eta} \geq 0 \tag{4.18}$$

for large η (near the supertube). This means that, in order not to have CTCs, we must restrict the range of the variable η to be

$$\eta \lesssim \frac{8R}{|L|}\,. \tag{4.19}$$

Namely, the far-region solution has CTCs very near the tube.

Next, let us consider the positivity of the metric (2.22) along the supertube direction, ϕ. This gives

$$-\frac{\omega^2}{\mathcal{Q}} + \frac{R^2(\eta^2-1)}{(\eta-\cos\sigma)^2}d\phi^2 \geq 0. \tag{4.20}$$

After plugging the explicit expression for ω (4.15), we can rewrite (4.20) as

$$\begin{aligned}&\frac{R^2 d\phi^2}{(\eta-\cos\sigma)^2\left[\ln\frac{|L|(\eta+1)}{8R}\right]^2}\\ &\quad\times\left((\eta^2-1)\left[\ln\frac{|L|(\eta+1)}{8R}\right]^2-\left[(\eta+1)\ln\frac{|L|(\eta+1)}{8R}+2\ln\frac{4R}{|L|}\right]^2\right)\geq 0\,.\end{aligned} \tag{4.21}$$

Near the ring ($\eta\to\infty$), the no-CTC condition (4.21) gives

$$-2\eta\,\ln\left(\frac{2R\eta}{|L|}\right)\ln\left(\frac{|L|\eta}{8R}\right)\geq 0\,, \tag{4.22}$$

which is satisfied for

$$\frac{|L|}{2R}<1\leq\eta\leq\frac{8R}{|L|}\,. \tag{4.23}$$

The lower bound does not impose any condition on η because $\eta\geq 1$ by definition, and the upper bound is the same as (4.19).

So, we found that there are CTCs in the far-region solution very near the ring, $\eta\sim\frac{8R}{|L|}$. However, this does *not* represent a problem with our solution. It only indicates that, too much near the ring, the description in terms of the far-region solution with a single ring breaks down and we must instead switch to the near-region solution with two rings. Indeed, by the relation (3.40), $\eta\sim\frac{R}{|L|}$ corresponds to $|z|\sim|L|$ in the near region, which is the distance scale at which the single "effective" supertube must be resolved into two supertubes. This is exactly parallel to the familiar story in the context of F-theory [88, 89]. In type IIB perturbative string theory, the O7-plane has negative tension and its backreacted metric has a wrong signature very near its worldvolume. However, in F-theory, non-perturbative effects resolve the O7-plane into two (p,q) 7-branes and replace the wrong-signature metric by a new metric with the correct signature everywhere. The two (p,q) 7-branes have non-commuting monodromies of the $SL(2,\mathbb{Z})$ duality of type IIB string. We are seeing exactly the same phenomenon in a more involved situation with circular supertubes.

To rigirously show that our solution is completely free from CTCs, we must construct the exact solution by summing up the infinite perturbative series, because the perturbative solution to any finite order will have CTCs (this is related to the limitation of the matching expansion discussed below (3.20)). However, that is beyond the scope of the present paper and we will leave it as future research.

4.3 Bound or unbound?

Our 2-supertube configuration has three parameters: $c \in \mathbb{C}$ determines the overall amplitude of the harmonic functions, $L \in \mathbb{C}$ parametrizes the distance and the angle between two supertubes, and $R > 0$ is the average radius of the two supertubes. The crucial question is: does this represent a bound state or not?

In the case of codimension-3 solutions, allowed multi-center configurations are determined by imposing equation (2.33). How this works is as follows. One first fixes the value of moduli (the constant terms in H), the number of centers (say N), and the charges of each center (Γ^p, $p = 1, \ldots, N$). By plugging these data into (2.33), we can fix the inter-center distances a_{pq}. After this, some parameters will remain unfixed. They parametrize the internal degrees of freedom of the multi-center configuration, similar to the internal atomic motion inside a molecule. When it is a bound state, it is not possible to take some centers infinitely far away from the rest of the centers by tuning the parameters.

In our solution, the asymptotic moduli have already been fixed to the attractor value [35]. We have two codimension-2 supertube centers, and we know that the total monopole charges are given by (Q_F, Q_G). Actually, as we will discuss below, the monopole charges of each of the two supertubes can be also determined if we fix the complex charges Q_F, Q_G. So, the question is whether there is some free parameter left by tuning which we can make the two tubes infinitely far apart. If so, then the configuration is unbound. Otherwise, it is bound.

Our solution contains five real parameters ($R \in \mathbb{R}$; $c, L \in \mathbb{C}$) and four of them can be determined by fixing $Q_{F,G} \in \mathbb{C}$. So, we seem to be left with one free real parameter. For example, we can take it to be $|L|$, the absolute value of the inter-tube distance parameter L. If $|L|$ could take an arbitrarily large value, the two tubes could be separated infinitely far away from each other and thus the solution would be unbound. Physically, however, we expect that we can constrain this parameter by requiring the absence of CTCs [71, 72], and that the tubes cannot be infinitely separated. Such no-CTC analysis would be possible *if we knew the exact solution.* The problem is that we only have a perturbative solution in the matching expansion. As we saw in the previous section, perturbative solutions have apparent CTCs and are not suitable for such analysis.

To work around this problem, we will instead make use of supertube physics to argue that all the parameters are constrained and thus our non-Abelian solution represents a bound state. Actually, we can fix all the parameters from this argument. It is not a rigorous argument, but is robust enough to give convincing evidence that the solution represents a bound state.

4.4 An argument for a bound state

We know that the monodromy matrices of the two supertubes sitting at $z = \pm L$ are

$$M_L = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad M_{-L} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}. \tag{4.24}$$

In appendix D.2, we derived the monodromy matrix of the supertube produced by the supertube transition of a general 1/4-BPS codimension-3 center. In the one-modulus class that we are working in ($\tau^1 = \tau^2 = i$, τ^3: any), a general 1/4-BPS codimension-3 center has charge $\Gamma = \frac{g_s l_s}{2}(a, (b, b, c), (d, d, a), -\frac{c}{2})$, where $a, b, c, d \in \mathbb{Z}$, $ad + bc = 0$ and not all of a, b, c, d simultaneously vanish. Using the formulas (D.17) and (D.18), it is easy to see that the unique sets of charges that lead to supertubes with monodromy $M_{\pm L}$ are the ones with

$$M_L : c = d = 0, \qquad M_{-L} : \; a = -c, \; b = d, \tag{4.25}$$

with the dipole charge $q = 2$ for both cases. In terms of complex charges (cf. (2.40)),

$$Q_F = \frac{g_s l_s}{2}(c + id), \qquad Q_G = \frac{g_s l_s}{2}(a - ib), \tag{4.26}$$

the condition (4.25) can be written as:

$$M_L : \quad Q_F = 0, \qquad M_{-L} : \quad Q_F = -Q_G. \tag{4.27}$$

The supertubes at $z = \pm L$ must have come from two codimension-3 centers with charges satisfying this condition, respectively.[17]

From (4.3), the total charges of our two-supertube configuration is

$$\begin{pmatrix} Q_F \\ Q_G \end{pmatrix}_{\text{total}} = c\sqrt{R} \begin{pmatrix} i\nu \\ 1 \end{pmatrix}. \tag{4.28}$$

Let us split this total charge into the ones for the $z = +L$ supertube and the ones for the $z = -L$ supertube as

$$\begin{pmatrix} Q_F \\ Q_G \end{pmatrix}_{\text{total}} = \begin{pmatrix} Q_F \\ Q_G \end{pmatrix}_L + \begin{pmatrix} Q_F \\ Q_G \end{pmatrix}_{-L}, \tag{4.29}$$

and require that the individual charges satisfy the condition (4.27), namely,

$$Q_{F,L} = 0, \qquad Q_{F,-L} = -Q_{G,-L}. \tag{4.30}$$

We immediately find

$$\begin{pmatrix} Q_F \\ Q_G \end{pmatrix}_L = c\sqrt{R} \begin{pmatrix} 0 \\ 1 + i\nu \end{pmatrix}, \tag{4.31a}$$

$$\begin{pmatrix} Q_F \\ Q_G \end{pmatrix}_{-L} = c\sqrt{R} \begin{pmatrix} i\nu \\ -i\nu \end{pmatrix}. \tag{4.31b}$$

In our solution we have two codimension-2 supertubes, instead of codimension-3 centers. However, these supertubes must still carry the original monopole charges (4.31) dissolved into their worldvolume. Using the relation (2.27), we can express (4.31) in terms of charges vectors as

$$\Gamma_{\pm L} = \left(\operatorname{Re} Q_G, (-\operatorname{Im} Q_G, -\operatorname{Im} Q_G, \operatorname{Re} Q_F), (\operatorname{Im} Q_F, \operatorname{Im} Q_F, \operatorname{Re} Q_G), -\frac{1}{2} \operatorname{Re} Q_F \right)_{\pm L}. \tag{4.32}$$

[17]To be precise, by charges here, we mean Page charges discussed in section 4.1.

The radii and angular momentum of the configuration are determined by the charges of the centers. Then, we can study what the radii of the circular supertubes generated by the supertube transition of codimension-3 centers with charges (4.31) are. This has been worked out in appendix D.3 and, using the formula (D.21), it is not difficult to show that the radii of the supertubes at $z = \pm L$ are given by

$$\begin{aligned}\mathcal{R}_L^2 &= R|c(1+i\nu)|^2 = R|c|^2\left[1 + \frac{2l}{\pi} + \frac{1}{\pi^2}\left(\left(\log\frac{4R}{|L|}\right)^2 + l^2\right)\right],\\ \mathcal{R}_{-L}^2 &= R|c|^2|\nu|^2 = \frac{R|c|^2}{\pi^2}\left(\left(\log\frac{4R}{|L|}\right)^2 + l^2\right).\end{aligned} \tag{4.33}$$

In deriving this, each supertube was assumed to be in isolation; the actual radii must be corrected by the interaction between the two tubes. On the other hand, the radii squared of the two tubes in our actual solution are

$$(R \pm \mathrm{Re}\, L)^2 = (R \pm |L|\cos l)^2. \tag{4.34}$$

As a preliminary, zeroth-order approximation, let us equate (4.33) and (4.34). It is not difficult to show that, unless $l = -\frac{\pi}{2}$, there is no solution that is consistent with the colliding limit, $\frac{R}{|L|} \gg 1$. If $l = -\frac{\pi}{2}$, the two supertubes have the same radius and the condition that (4.33) equals (4.34) gives

$$|c| = \frac{\sqrt{R}}{|\nu|} = \frac{\pi\sqrt{R}}{\sqrt{\left(\log\frac{4R}{|L|}\right)^2 + \frac{\pi^2}{4}}}. \tag{4.35}$$

The total charges (4.3) are, if we set $c = |c|e^{i\gamma}$,

$$(Q_F, Q_G) = c\sqrt{R}\,(i\nu, 1) = \frac{e^{i\gamma}R}{\sqrt{\left(\log\frac{4R}{|L|}\right)^2 + \frac{\pi^2}{4}}}\left(i\log\frac{4R}{|L|} - \frac{\pi}{2}, \pi\right). \tag{4.36}$$

Fixing these charges will fix $\gamma, R, |L|$. So, everything is fixed.

In summary, consideration of supertube physics suggests that the configurational parameters of our two-supertube solution are all fixed if we fix the asymptotic charges. In particular, it is impossible to take the two tubes infinitely far apart. This is strong evidence that our solution is a bound state. Having the same asymptotic charges as a black hole with a finite horizon, it should represent a microstate of a genuine black hole. Our argument is not rigorous in the sense that, in computing the supertube radii (4.33), we ignored the interaction between the tubes. Therefore, precise values such as $l = -\frac{\pi}{2}$ may not be reliable. However, we expect that it captures the essential physics and the conclusion remains valid even for more accurate treatments.

4.5 A cancellation mechanism for angular momentum

In the last section, we pointed out the puzzling fact that the total angular momentum of our solution vanishes, even though the two constituent supertubes are expected to carry non-vanishing angular momentum. Here, we argue that this is due to cancellation between the angular momentum $J_{\pm L}$ carried by the two individual tubes and the angular momentum $J_{\rm cross}$ that comes from the electromagnetic crossing between the two tubes; namely,

$$J_{\rm total} = J_L + J_{-L} + J_{\rm cross} \approx 0. \tag{4.37}$$

Just as in section 4.4, our argument will not be rigorous; we will see that (4.37) holds only to the leading order in $\frac{|L|}{R}$. We expect that, in an exact treatment, (4.37) will hold as a precise equality. However, this study is beyond the scope of this paper.

In our solution, we have two round supertubes which were produced by the supertube effect of codimension-3 centers with charges (4.31). In appendix D.3, we computed the angular momentum carried by a round supertube created from a general 1/4-BPS codimension-3 center. Applying the formula (D.21) to the charges (4.31), it is not difficult to show that the component of angular momentum along the axis of the tubes (x^3-axis) is[18]

$$J_L = -\frac{R|c|^2(1+|\nu|^2 - 2\,{\rm Im}\,\nu)}{4G_4}, \qquad J_{-L} = -\frac{R|c|^2|\nu|^2}{4G_4}. \tag{4.38}$$

Now let us turn to $J_{\rm cross}$. For multi-center codimension-3 solutions with charge vectors Γ^p, there is non-vanishing angular momentum coming from the crossing between electric and magnetic fields given by [63]

$$\mathbf{J}_{\rm cross} = \frac{1}{2G_4}\sum_{p<q}\langle\Gamma^p,\Gamma^q\rangle\frac{\mathbf{a}_{pq}}{|\mathbf{a}_{pq}|}, \qquad \mathbf{a}_{pq} \equiv \mathbf{a}_p - \mathbf{a}_q. \tag{4.39}$$

In the present case, we have supertubes with codimension 2, not 3. However, let us still apply this formula using the tubes' monopole charges (4.31) (or (4.32)) . This is not precise, but must give a rough approximation of the crossing angular momentum for our solution. Using (4.31) and (4.32), the component of the angular momentum along the tube axis is[19]

$$J_{\rm cross} = \frac{1}{2G_4}\langle\Gamma_{-L},\Gamma_L\rangle = -\frac{R|c|^2({\rm Im}\,\nu - |\nu|^2)}{2G_4}. \tag{4.40}$$

If we add (4.40) and (4.39), we get

$$J_L + J_{-L} + J_{\rm cross} = -\frac{R|c|^2}{4G_4}. \tag{4.41}$$

This is much smaller than the individual terms:

$$J_L, J_{-L}, J_{\rm cross} \sim \frac{R|c|^2|\nu|^2}{G_4} \sim \frac{R|c|^2(\log\frac{R}{|L|})^2}{G_4} \tag{4.42}$$

because we are taking the limit $\frac{R}{|L|} \gg 1$. Therefore, we conclude that (4.37) holds to the leading order in $\frac{|L|}{R}$.

[18] The sign was determined from the sign of $\omega_2 = \omega_\phi/R$ in (3.23) near $z = \pm L$ using (3.33) and (3.34).

[19] In section 4.4, we argued that the physically allowed configuration in the limit $\frac{R}{|L|} \gg 1$ has $l = -\frac{\pi}{2}$, which means that the center of the $z = \pm L$ tubes are at $x^3 = \mp|L|$. This determines the sign of (4.40).

This is an interesting observation, suggesting that the vanishing of angular momentum in our configuration is indeed due to cancellation between the "tube" angular momentum and the "cross" angular momentum. Presumably, the nonzero reminder (4.41) gets canceled if we take into account the contribution to the angular momentum arising from the interaction between the two tubes (recall that we computed the angular momentum of supertubes as if they were in isolation).

5 Future directions

We constructed our solution by taking the configuration that appeared in the SU(2) Seiberg-Witten theory as the near-region solution. More specifically, it was a holomorphic fibration of a genus-1 Riemann surface on a base of complex dimension 1. However, this is just an example, so any other such holomorphic fibration will work. In particular, any F-theory solution can be used for the near-region solution. In the standard F-theory background, the metric only knows about the torus modulus τ, but in our case we also need the periods (a_D, a) and richer structure is expected. We can generalize this structure by replacing the torus fiber by a higher-genus Riemann surface. For example, if one considers compactification of type IIA on $T^2 \times K3$, the U-duality group becomes $\mathrm{O}(22, 6; \mathbb{Z})$, which contains the genus-2 modular group $\mathrm{Sp}(4, \mathbb{R})$. Therefore, one can construct configuration of more general supertubes using a fibration of a genus-2 Riemann surface over a base [90]. One can also consider generalizing the base. In the near region the base is complex 1-dimensional, while in the far region it is real 3-dimensional. By including an internal S^1 direction, one can extend the base to a complex 2-dimensional space, where a supertube must appear as a complex curve around which there is a monodromy of the fiber. In such a setup, one can use the power of complex analysis and it might help to construct solutions on a real 3-dimensional base as the one we encountered in the current paper.

It is known that the geometry of the Seiberg-Witten theory has a string theory realization [88, 91, 92]. If one realizes the Seiberg-Witten curve as a configuration of F-theory 7-branes, then the worldvolume theory of a probe D3-brane in that geometry is exactly the $d = 4, \mathcal{N} = 2$ theory. One may wonder if our solution also represents a moduli space of some gauge theory on a probe D-brane. However, such interpretation does not seem straightforward. The near-region geometry looks very similar to F-theory configurations, but the 7-brane in the current setup is not just a pure 7-brane but it has some worldvolume fluxes turned on to carry 5-brane and 1-brane charges. Therefore, it is not immediately obvious what probe brane one should take. Furthermore, although the near-region configuration preserves 16 supersymmetries, only 4 supersymmetries are preserved in the far region, as a 4-charge black-hole microstate. A brane probe will most likely halve the supersymmetries in each region. So, the relevant theory seems to be $d = 3, \mathcal{N} = 1$ (or $d = 2, \mathcal{N} = 2$) theory whose moduli space has a special locus, which corresponds to the near region, at which supersymmetry is enhanced to $\mathcal{N} = 4$ (or $\mathcal{N} = 8$). It is interesting to investigate what the theory can be.

We developed techniques to construct solutions in the far and near regions separately and connect them by a matching expansion. We worked out only first terms in the expansion, but one can in principle carry out this to any order. In some situations one may be able to carry out the infinite sum and obtain the exact solution in entire $\mathbb{R}^3$. Such exact solutions are important because, as discussed below (3.20), there are features of the exact solution that are not visible at any finite order. Such features include the precise structure of the monodromy and the metric near the supertubes. They are crucial to analyze the no-CTC condition near the supertubes and fix parameters of the solution, such as L and R. We hope to be able to report development in that direction in near future [93].

In this paper, we mainly considered the case where two of the three moduli are frozen. It is interesting to investigate possible solutions in the case where this assumption is relaxed. In appendix C, we discussed the case where two moduli are dynamical. For example, it is interesting to study how the solutions studied in [32] fit in the formulation developed in appendix C. Relatedly, we assumed that in the near region the modulus τ^3 is holomorphic. However, as far as supersymmetry is concerned, this is not necessary; the only requirement is that the harmonic functions be written as a sum of holomorphic and anti-holomorphic functions. It would be interesting to see if there are physically allowed solutions for which τ^3 is not holomorphic.

Our configuration has the same asymptotic charge as a 4D black hole. 4D black holes are often discussed in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality where the boundary CFT is the so-called MSW CFT [76]. However, this CFT is not as well-understood as the D1-D5 CFT which appears as the dual of black-hole systems in 5D. It is interesting to see if our solutions can be generalized to construct a microstate for 5D black holes; for recent work to relate microstates of the MSW CFT and those of the D1-D5 CFT, see [23].

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A Duality transformation of harmonic functions

In section 2, we showed that the $[\mathrm{SL}(2,\mathbb{Z})]^3$ duality of the STU model acts on harmonic functions as (2.18). Here, we discuss some aspects of the duality transformation.

In the main text, we introduced vectors such as $H = \{V, K^I, L_I, M\}$. To see the group theory structure, it is more convenient to introduce the $\mathrm{Sp}(8,\mathbb{R})$ vector [67]

$$\mathcal{H} = (\mathcal{H}^\Lambda, \mathcal{H}_\Lambda) = (\mathcal{H}^0, \mathcal{H}^I, \mathcal{H}_0, \mathcal{H}_I) = \frac{1}{\sqrt{2}}(-V, -K^I, 2M, L_I) \tag{A.1}$$

which transforms in the standard way under the four-dimensional electromagnetic $\mathrm{Sp}(8,\mathbb{R})$ duality transformation of $\mathcal{N}=2$ supergravity.

The skew product $\langle H, H'\rangle$ defined in (2.9) can be written as

$$\langle H, H'\rangle = -\mathcal{H}^\Lambda \mathcal{H}'_\Lambda + \mathcal{H}_\Lambda \mathcal{H}'^\Lambda \tag{A.2}$$

For a generic $\mathrm{Sp}(8,\mathbb{R})$ symplectic vector $\mathcal{V} = (\mathcal{V}^\Lambda, \mathcal{V}_\Lambda) = (\mathcal{V}^0, \mathcal{V}^I, \mathcal{V}_0, \mathcal{V}_I)$, the quartic invariant $\mathcal{J}_4(\mathcal{V})$ is given by

$$\mathcal{J}_4(\mathcal{V}) = -(\mathcal{V}^\Lambda \mathcal{V}_\Lambda)^2 + 4\sum_{I<J} \mathcal{V}^I \mathcal{V}_I \mathcal{V}^J \mathcal{V}_J - 4\mathcal{V}^0 \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 + 4\mathcal{V}_0 \mathcal{V}^1 \mathcal{V}^2 \mathcal{V}^3. \tag{A.3}$$

Using this, the quantity $\mathcal{Q}$ defined in (2.13) and rewritten in (2.14) can be expressed as

$$\mathcal{Q} = J_4(H) = \mathcal{J}_4(\mathcal{H}). \tag{A.4}$$

In this language, the most general U-duality transformation can be written as an 8×8 matrix $S \in [\mathrm{SU}(1,1)]^3 \cong [\mathrm{SL}(2,\mathbb{R})]^3 \subset \mathrm{Sp}(8,\mathbb{R})$ [67, 94]

$$S = \mathcal{S}\mathcal{T}\mathcal{U}\,, \tag{A.5}$$

where

$$\mathcal{S} = \begin{pmatrix} \delta_1 & \gamma_1 & & & & & & \\ \beta_1 & \alpha_1 & & & & & & \\ & & \delta_1 & & & & & \gamma_1 \\ & & & \delta_1 & & & \gamma_1 & \\ & & & & \alpha_1 & -\beta_1 & & \\ & & & & -\gamma_1 & \delta_1 & & \\ & & & \beta_1 & & & \alpha_1 & \\ & & \beta_1 & & & & & \alpha_1 \end{pmatrix}, \tag{A.6a}$$

$$\mathcal{T} = \begin{pmatrix} \delta_2 & & \gamma_2 & & & & & \\ & \delta_2 & & & & & & \gamma_2 \\ \beta_2 & & \alpha_2 & & & & & \\ & & & \delta_2 & & \gamma_2 & & \\ & & & & \alpha_2 & & -\beta_2 & \\ & & & \beta_2 & & \alpha_2 & & \\ & & & & -\gamma_1 & & \delta_2 & \\ & \beta_2 & & & & & & \alpha_2 \end{pmatrix}, \tag{A.6b}$$

$$\mathcal{U} = \begin{pmatrix} \delta_3 & & & \gamma_3 & & & & \\ & \delta_3 & & & & & \gamma_3 & \\ & & \delta_3 & & & \gamma_3 & & \\ \beta_3 & & & \alpha_3 & & & & \\ & & & & \alpha_3 & & & -\beta_3 \\ & & \beta_3 & & & \alpha_3 & & \\ & \beta_3 & & & & & \alpha_3 & \\ & & & & -\gamma_3 & & & \delta_3 \end{pmatrix}. \tag{A.6c}$$

with $\alpha_I\delta_I - \beta_I\gamma_I = 1$, $I = 1,2,3$. It is straightforward to show that the action of the matrix (A.5) on the symplectic vector $(\mathcal{H}^\Lambda, \mathcal{H}_\Lambda)$ reproduces the transformation law (2.18).

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The transformation law (2.18) means that the eight harmonic functions transform under the $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$ representation of $[\mathrm{SL}(2,\mathbb{Z})]^3$ as follows:

$$\begin{aligned}(\mathcal{H}^0,\mathcal{H}^I,\mathcal{H}_0,\mathcal{H}_I)&=\frac{1}{\sqrt{2}}(-V,-K^I,2M,L_I)\\&=(\mathcal{H}^{222};\ \mathcal{H}^{122},\mathcal{H}^{212},\mathcal{H}^{221};\ -\mathcal{H}^{111};\ \mathcal{H}^{211},\mathcal{H}^{121},\mathcal{H}^{112})\end{aligned}\tag{A.7}$$

where $\mathcal{H}^{abc}$ $(a,b,c=1,2)$ transforms as $\mathcal{H}^{abc}\to\sum_{a',b',c'}(M_1)^{aa'}(M_2)^{bb'}(M_3)^{cc'}\mathcal{H}^{a'b'c'}$. In terms of $\mathcal{H}^{abc}$,

$$-\langle H,H'\rangle=\mathcal{H}^\Lambda\mathcal{H}'_\Lambda-\mathcal{H}_\Lambda\mathcal{H}'^\Lambda=\epsilon_{a_1a_2}\epsilon_{b_1b_2}\epsilon_{c_1c_2}\mathcal{H}^{a_1b_1c_1}\mathcal{H}^{a_2b_2c_2},\tag{A.8}$$

$$J_4(H)=\mathcal{J}_4(\mathcal{H})=\frac{1}{2}\epsilon_{a_1a_2}\epsilon_{a_3a_4}\epsilon_{b_1b_2}\epsilon_{b_3b_4}\epsilon_{c_1c_3}\epsilon_{c_2c_4}\mathcal{H}^{a_1b_1c_1}\mathcal{H}^{a_2b_2c_2}\mathcal{H}^{a_3b_3c_3}\mathcal{H}^{a_4b_4c_4}.\tag{A.9}$$

A matrix M^{ab} cannot be written as a product of two vectors u^a,v^b in general but it can be written as a sum of multiple vectors, $M^{ab}=\sum_i u_i^a v_i^b$. Similarly, we must be able to decompose the tensor $\mathcal{H}^{abc}$ as

$$\mathcal{H}^{abc}=\sum_i u_i^a v_i^b w_i^c,\tag{A.10}$$

where u_i^a, v_i^b, and w_i^c are real functions transforming as doublets of $\mathrm{SL}(2,\mathbb{Z})_1$, $\mathrm{SL}(2,\mathbb{Z})_2$, and $\mathrm{SL}(2,\mathbb{Z})_3$, respectively.

Let us consider the situation considered in appendix C where we set one of the moduli to a trivial value: $\tau^1=i$. Here we will give an alternative proof that the harmonic functions in this case are given by (C.6), (C.7). As we can see in (C.3), the combinations of harmonic functions that transform nicely under the remaining $\mathrm{SL}(2,\mathbb{Z})_2\times\mathrm{SL}(2,\mathbb{Z})_3$ are $V-iK^1$, K^2+iL_3, K^3+iL_2 and $-L_1-2iM$. In terms of $\mathcal{H}^{abc}$, they are

$$\begin{aligned}V-iK^1&=\sqrt{2}\,(-\mathcal{H}^{222}+i\mathcal{H}^{122})\equiv\mathcal{H}^{22},\\K^2+iL_3&=\sqrt{2}\,(-\mathcal{H}^{212}+i\mathcal{H}^{112})\equiv\mathcal{H}^{12},\\K^3+iL_2&=\sqrt{2}\,(-\mathcal{H}^{221}+i\mathcal{H}^{121})\equiv\mathcal{H}^{21},\\-L_1-2iM&=\sqrt{2}\,(-\mathcal{H}^{211}+i\mathcal{H}^{111})\equiv\mathcal{H}^{11}.\end{aligned}\tag{A.11}$$

The components of the tensor $\mathcal{H}^{bc}$ defined here are complex functions transforming as a $\mathbf{2}\otimes\mathbf{2}$ of $\mathrm{SL}(2,\mathbb{Z})_2\times\mathrm{SL}(2,\mathbb{Z})_3$. Just as in (A.10), we can decompose it as

$$\mathcal{H}^{bc}=\sum_i V_i^b W_i^c,\tag{A.12}$$

where V_i^b,W_i^c are complex. However, this is inconsistent with the constraint (C.2), which reads in terms of $\mathcal{H}^{bc}$ as

$$\mathcal{H}^{11}\mathcal{H}^{22}=\mathcal{H}^{12}\mathcal{H}^{21},\tag{A.13}$$

unless the summation over i in (A.12) has only one term. In that case,

$$\begin{aligned}V-iK^1&=\mathcal{H}^{22}=V^2W^2, &\qquad K^2+iL_3&=\mathcal{H}^{12}=V^1W^2,\\K^3+iL_2&=\mathcal{H}^{21}=V^2W^1, &\qquad -L_1-2iM&=\mathcal{H}^{11}=V^1W^1.\end{aligned}\tag{A.14}$$

This is the same as (C.6), (C.7) with the identification $\binom{V^1}{V^2}=\binom{F_2}{G_2}$, $\binom{W^1}{W^2}=\binom{F_3}{G_3}$.

It is interesting to see how the transformations of the harmonic functions known in the literature are embedded in the general $[\mathrm{SL}(2,\mathbb{Z})]^3$ transformation (2.18). We will consider the "gauge transformation" [95] and the "spectral flow transformation" [96] as such transformations. To our knowledge, explicit $[\mathrm{SL}(2,\mathbb{Z})]^3$ matrices for these transformations have not been explicitly written down in the literature. For a discussion on how these transformations are embedded in the U-duality group of the STU model from a different perspective, see [67].

The so-called "gauge transformation" [95] is defined as the following transformation of harmonic functions:

$$\begin{aligned}
&V \to V, \\
&K^I \to K^I + c^I V, \\
&L_I \to L_I - C_{IJK} c^J K^K - \frac{1}{2} C_{IJK} c^J c^K V, \\
&M \to M - \frac{1}{2} c^I L_I + \frac{1}{4} C_{IJK} c^I c^J K^K + \frac{1}{12} C_{IJK} c^I c^J c^K V.
\end{aligned} \tag{A.15}$$

It is easy to see that this transformation is a special case of general $[\mathrm{SL}(2,\mathbb{Z})]^3$ transformations (2.18) with

$$M_I = \begin{pmatrix} 1 & c^I \\ 0 & 1 \end{pmatrix}, \qquad I = 1, 2, 3. \tag{A.16}$$

This transformation shifts the B-field as

$$B_2 \to B_2 + \frac{c^1 \alpha'}{R_4 R_5} J_1 + \frac{c^2 \alpha'}{R_6 R_7} J_2 + \frac{c^3 \alpha'}{R_8 R_9} J_3. \tag{A.17}$$

If one likes, the shift in B_2, (A.17), can be always undone by subtracting $\frac{c^1\alpha'}{R_4R_5}J_1 + \frac{c^2\alpha'}{R_6R_7}J_2 + \frac{c^3\alpha'}{R_8R_9}J_3$ from B_2 by hand, because subtracting from B_2 the closed form J_I affects none of the equations of motion or supersymmetry conditions. This is relevant especially in 5D solutions (for which $h^0 = 0$) because, changing the asymptotic value of B_2 as in (A.17) would mean to change the asymptotic value of the Wilson loop along ψ for a 5D gauge field that descends from the M-theory 3-form $A_{\mu ij}$. Such a gauge transformation would not vanish at infinity in 5D and is not allowed. So, one must always undo the shift (A.17) after doing the gauge transformation (A.15). After this procedure, no gauge-invariant fields are changed under the transformation (A.15) and it is just re-parametrization of harmonic functions $\{V, K^I, L_I, M\}$.

The "spectral flow transformation" is defined as [96]

$$\begin{aligned}
&V \to V + \gamma_I K^I - \frac{1}{2} C^{IJK} \gamma_I \gamma_J L_K + \frac{1}{3} C^{IJK} \gamma_I \gamma_J \gamma_K M, \\
&K^I \to K^I - C^{IJK} \gamma_J L_K + C^{IJK} \gamma_J \gamma_K M, \\
&L_I \to L_I - 2\gamma_I M, \\
&M \to M,
\end{aligned} \tag{A.18}$$

where $C^{IJK} = C_{IJK}$. This transformation has been used extensively to generate new solutions from known ones. It is easy to see that this transformation is a special case of general $\mathrm{SL}(2,\mathbb{Z})$ transformations with

$$M_I = \begin{pmatrix} 1 & 0 \\ \gamma_I & 1 \end{pmatrix}, \qquad I = 1,2,3. \tag{A.19}$$

B Matching to higher order

In the main text, we worked out the matching between the far- and near-region solutions to the leading order. In this appendix, we carry out the matching to higher order.

From the large-$|z|$ expansion of the near-region solution (3.31), we find that the far-region solution must have the following expansion:

$$F = \sqrt{\eta - \cos\sigma} \sum_{n=0}^{\infty} e^{-i\frac{4n+1}{2}\sigma} \left(f_n(\eta) - \frac{\sigma}{\pi} g_n(\eta) \right) , \tag{B.1a}$$

$$G = \sqrt{\eta - \cos\sigma} \sum_{n=0}^{\infty} e^{-i\frac{4n+1}{2}\sigma} g_n(\eta) . \tag{B.1b}$$

The Laplace equations for F and G lead to

$$\begin{aligned} &(1-\eta^2) f_n'' - 2\eta f_n' + 2n(2n+1) f_n = \frac{i}{\pi}(4n+1) g_n , \\ &(1-\eta^2) g_n'' - 2\eta g_n' + 2n(2n+1) g_n = 0 . \end{aligned} \tag{B.2}$$

The equation for g_n is the standard Legendre differential equation while the one for f_n is an inhomogeneous Legendre differential equation of resonant type [97].

The general solution for $g_n(\eta)$ is given by

$$g_n(\eta) = A_{2n} P_{2n}(\eta) + B_{2n} Q_{2n}(\eta) , \tag{B.3}$$

where $P_{2n}(\eta)$ is the Legendre polynomial and $Q_{2n}(\eta)$ is the Legendre function of the second kind. As $Q_{2n}(\eta)$ diverges at 3D infinity and on the x^3-axis (see footnote 15), we require $B_{2n} = 0$. The expression for $P_{2n}(\eta)$ for some small values of n is

$$P_0(\eta) = 1 , \tag{B.4a}$$

$$P_2(\eta) = \frac{1}{2}(3\eta^2 - 1) , \tag{B.4b}$$

$$P_4(\eta) = \frac{1}{8}(35\eta^4 - 30\eta^2 + 3) . \tag{B.4c}$$

$P_{2n}(\eta)$ are normalized so that $P_{2n}(1) = 1$.

Having found g_n, we can plug it into (B.2) to find f_n. We have not been able to find a simple explicit expression for f_n that works for general n. We give the following integral form:

$$\begin{aligned} f_n(\eta) &= C_{2n} P_{2n}(\eta) + D_{2n} Q_{2n}(\eta) \\ &\quad - \frac{i}{\pi} A_{2n}(4n+1) \left(P_{2n}(\eta) \int_1^\eta ds\, P_{2n}(s) Q_{2n}(s) - Q_{2n}(\eta) \int_1^\eta ds\, [P_{2n}(s)]^2 \right) . \end{aligned} \tag{B.5}$$

We have chosen the particular solution (the last term) to vanish at 3D infinity ($\eta = 1$). As before, we require $D_{2n} = 0$ so that f_n is finite at infinity. For given n, it is easy to carry out the integral and the explicit expression for a few small values of n is

$$f_0(\eta) = C_0 - \frac{i}{\pi} A_0 \ln \frac{\eta+1}{2}\,, \tag{B.6a}$$

$$f_1(\eta) = C_2 P_2(\eta) - \frac{i}{\pi} A_2 \left(P_2(\eta) \ln \frac{\eta+1}{2} + \frac{1}{4}(\eta-1)(7\eta+1) \right)\,, \tag{B.6b}$$

$$f_2(\eta) = C_4 P_4(\eta) - \frac{i}{\pi} A_4 \left(P_4(\eta) \ln \frac{\eta+1}{2} + \frac{1}{96}(\eta-1)(533\eta^3+113\eta^2-241\eta-21) \right)\,. \tag{B.6c}$$

The undetermined coefficients A_{2n} and C_{2n} are fixed by matching the expansion (B.1) order by order with the large-$|z|$ expansion of the near-region solution given in (3.31). This has been done for the leading $n = 0$ term in the main text in section 3.5; see (3.68). For $n = 1$, this determines the coefficients to be

$$A_2 = \frac{cL^2}{2(2R)^{5/2}}\,, \qquad C_2 = \frac{i}{\pi} \frac{cL^2}{2(2R)^{5/2}} \left(\ln \frac{4R}{L} - \frac{1}{2} \right)\,. \tag{B.7}$$

C Configurations with only two moduli

Let us consider configurations with one modulus set to a trivial value. Specifically, we set

$$\tau^1 = i, \qquad \tau^2, \tau^3 : \text{arbitrary}. \tag{C.1}$$

This choice fixes two harmonic functions; from eq. (2.15), we find

$$-L_1 - 2iM = \frac{(K^2+iL_3)(K^3+iL_2)}{V - iK^1}. \tag{C.2}$$

Only six harmonic functions are independent. In this case, the expression for the other moduli $\tau^{2,3}$ simplifies to

$$\tau^2 = \frac{K^2+iL_3}{V-iK^1}\,, \qquad \tau^3 = \frac{K^3+iL_2}{V-iK^1}\,. \tag{C.3}$$

Because τ^2 undergoes linear fractional transformation under $\mathrm{SL}(2,\mathbb{Z})_2$, we can set[20]

$$K^2 + iL_3 = H_2 F_2, \qquad V - iK^1 = H_2 G_2, \tag{C.4}$$

where under $\mathrm{SL}(2,\mathbb{Z})_2$ the pair $\binom{F_2}{G_2}$ transforms as a doublet while H_2 is invariant. The quantities F_2, G_2, H_2 are complex. With this choice (C.4), τ^2 is invariant under $\mathrm{SL}(2,\mathbb{Z})_3$

[20] Actually, one could more generally set $K^2 + iL_3 = \sum_i H_2^{(i)} F_2^{(i)}$, $V - iK^1 = \sum_i H_2^{(i)} G_2^{(i)}$ where $\binom{F_2^{(i)}}{G_2^{(i)}}$ transforms as a doublet under $\mathrm{SL}(2,\mathbb{Z})_2$ for all i. However, τ^1 would not be invariant under $\mathrm{SL}(2,\mathbb{Z})_3$, unless the i summation contains only one term. For a different argument for (C.6), (C.7), see appendix A.

as it should be. Similarly, because τ^3 undergoes linear fractional transformation under $\mathrm{SL}(2,\mathbb{Z})_3$, we can set

$$K^3 + iL_2 = H_3F_3, \qquad V - iK^1 = H_3G_3, \tag{C.5}$$

where under $\mathrm{SL}(2,\mathbb{Z})_3$ the pair $\binom{F_3}{G_3}$ transforms as a doublet while H_3 is invariant. F_3, G_3, H_3 are complex. Combining (C.4) and (C.5), we find that $H_2 = G_3$ and $H_3 = G_2$ and therefore

$$K^2 + iL_3 = F_2G_3, \qquad V - iK^1 = G_2G_3, \qquad K^3 + iL_2 = G_2F_3, \tag{C.6}$$

with which (C.2) becomes

$$-L_1 - 2iM = F_2F_3. \tag{C.7}$$

The moduli (C.3) can now be written as

$$\tau^2 = \frac{F_2}{G_2}, \qquad \tau^3 = \frac{F_3}{G_3}. \tag{C.8}$$

In terms of $F_{2,3}, G_{2,3}$, the harmonic functions are

$$\begin{aligned} &V = \mathrm{Re}\, G_2G_3\,, \quad K^1 = -\,\mathrm{Im}\, G_2G_3\,, \quad K^2 = \mathrm{Re}\, F_2G_3\,, \quad K^3 = \mathrm{Re}\, G_2F_3\,, \\ &L_1 = -\,\mathrm{Re}\, F_2F_3\,, \quad L_2 = \mathrm{Im}\, G_2F_3\,, \qquad L_3 = \mathrm{Im}\, F_2G_3\,, \quad M = -\frac{1}{2}\,\mathrm{Im}\, F_2F_3\,. \end{aligned} \tag{C.9}$$

Because we are parametrizing 6 real harmonic functions using 4 complex functions $F_{2,3}, G_{2,3}$, there is redundancy: the transformation $\binom{F_2}{G_2} \to H\binom{F_2}{G_2}$, $\binom{F_3}{G_3} \to H^{-1}\binom{F_3}{G_3}$, where H is a complex function, leaves the harmonic functions invariant.

Let us consider the no-CTC conditions (2.21). The condition (2.21a) is automatically satisfied because $\mathcal{Q} = (K^1K^3 + L_2V)^2(K^1K^2 + L_3V)^2/((K^1)^2 + V^2)^2 \geq 0$. The conditions $VZ_I \geq 0$, (2.21b), become

$$\begin{aligned} VZ_2 &= K^1K^3 + L_2V = |G_2|^2\, \mathrm{Im}(F_3\bar{G}_3) = |G_2G_3|^2\, \mathrm{Im}\,\tau^3 \geq 0\,, \\ VZ_3 &= K^1K^2 + L_3V = |G_3|^2\, \mathrm{Im}(F_2\bar{G}_2) = |G_2G_3|^2\, \mathrm{Im}\,\tau^2 \geq 0\,. \end{aligned} \tag{C.10}$$

D Supertubes in the one-modulus class

In section 2.2, we discussed a class of harmonic solutions for which only one modulus, $\tau^3 = \tau$, is turned on. (This class is nothing but a type IIA realization of the solution called the SWIP solution in the literature [37].) Here let us study some properties of supertubes described in this class.

D.1 Condition for a 1/4-BPS codimension-3 center

Let us consider a codimension-3 center in the harmonic solution and let the charge vector of the center be Γ. In terms of quantized charges, Γ can be written as

$$\Gamma = \frac{g_s l_s}{2}\left(a, (b,b,c), (d,d,a), -\frac{c}{2}\right), \tag{D.1}$$

where $a, b, c, d \in \mathbb{Z}$. Here, we took into account the constraint (2.24) and charge quantization (2.37). In general, this center represents a 1/8-BPS center preserving 4 supercharges, with entropy (see (2.41))

$$S = 2\pi\sqrt{j_4(\Gamma)}, \qquad j_4(\Gamma) \equiv (ad + bc)^2. \tag{D.2}$$

We would like to find the condition for the charge vector Γ to represent a 1/4-BPS center preserving 8 supercharges, which can undergo a supertube transition into a codimension-2 center. According to [98], a center with charge vector Γ represents

$$\begin{aligned}
&\text{4-charge 1/8-BPS center} \;\Leftrightarrow\; j_4(\Gamma) > 0. \\
&\text{3-charge 1/8-BPS center} \;\Leftrightarrow\; j_4(\Gamma) = 0,\ \frac{\partial j_4}{\partial x_i} \neq 0 \\
&\text{2-charge 1/4-BPS center} \;\Leftrightarrow\; j_4(\Gamma) = \frac{\partial j_4}{\partial x_i} = 0,\ \frac{\partial^2 j_4}{\partial x_i \partial x_j} \neq 0 \\
&\text{1-charge 1/2-BPS center} \;\Leftrightarrow\; j_4(\Gamma) = \frac{\partial j_4}{\partial x_i} = \frac{\partial^2 j_4}{\partial x_i \partial x_j} = 0,\ \frac{\partial^3 j_4}{\partial x_i \partial x_j \partial x_k} \neq 0,
\end{aligned} \tag{D.3}$$

where x_i represents charges of D-branes which, in the present case, are a, b, c, d. Applying this to the present case, we find that

$$\text{4-charge 1/8-BPS center} \quad \Leftrightarrow \quad ad + bc \neq 0, \tag{D.4a}$$

$$\text{2-charge 1/4-BPS center} \quad \Leftrightarrow \quad ad + bc = 0, \quad \text{but not } a = b = c = d = 0 \tag{D.4b}$$

In the present class of configurations satisfying (D.1), we cannot have a 3-charge 1/8-BPS center or a 1-charge 1/2-BPS center. For the latter, for example, even if $a = b = c = 0$ and $d \neq 0$, it still represents a D2(45)-D2(67) system which is a 2-charge 1/4-BPS system.

D.2 Puffed-up dipole charge for general 1/4-BPS codimension-3 center

If the 1/4-BPS system with charges satisfying (D.4b) polarizes into a supertube, what is its dipole charge, or more precisely, the monodromy matrix around it? From (2.19), we see that the combinations of charges that transform as doublets are

$$\begin{pmatrix} K^3 \\ V \end{pmatrix} = \begin{pmatrix} -2M \\ L_3 \end{pmatrix} \propto \begin{pmatrix} c \\ a \end{pmatrix}, \qquad \begin{pmatrix} -L_1 \\ K^2 \end{pmatrix} = \begin{pmatrix} -L_2 \\ K^1 \end{pmatrix} \propto \begin{pmatrix} -d \\ b \end{pmatrix} \tag{D.5}$$

with $ad + bc = 0$. If we act with a general $\mathrm{SL}(2, \mathbb{Z})$ matrix, the first doublet transforms as

$$\begin{pmatrix} c \\ a \end{pmatrix} \to \begin{pmatrix} c' \\ a' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} c \\ a \end{pmatrix} = \begin{pmatrix} \alpha c + \beta a \\ \gamma c + \delta a \end{pmatrix}, \tag{D.6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta - \beta\gamma = 1$. The second one transforms in the same way. Let us require that the lower component of the first doublet in (D.5) vanishes in the transformed frame, namely, $a' = \gamma c + \delta a = 0$. If we write

$$a = x\hat{a}, \qquad c = x\hat{c}, \qquad x = \gcd(a, c), \tag{D.7}$$

so that $\hat{a}$ and $\hat{c}$ are relatively prime, then it is clear that $a' = 0$ for the following choice:

$$\gamma = \hat{a}, \qquad \delta = -\hat{c}. \tag{D.8}$$

Note that the lower component of the second doublet in (D.5) also vanishes in the transformed frame:

$$b' = -\gamma d + \delta b = -\hat{a}d - \hat{c}b = -\frac{1}{x}(ad + bc) = 0 \tag{D.9}$$

by the assumption of 1/4-BPSness, (D.4b). For the matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ to be an $\mathrm{SL}(2,\mathbb{Z})$ matrix, we must satisfy

$$\alpha\delta - \beta\gamma = -\alpha\hat{c} - \beta\hat{a} = 1, \tag{D.10}$$

but there always exist $\alpha, \beta \in \mathbb{Z}$ satisfying this, for $\hat{a}, \hat{c}$ are coprime.

In the frame dualized by the $\mathrm{SL}(2,\mathbb{Z})_3$ matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \hat{a} & -\hat{c} \end{pmatrix} \tag{D.11}$$

satisfying (D.10), it is easy to show that the charges are

$$\begin{pmatrix} K^3 \\ V \end{pmatrix} = \begin{pmatrix} -2M \\ L_3 \end{pmatrix} \propto \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -L_1 \\ K^2 \end{pmatrix} = \begin{pmatrix} -L_2 \\ K^1 \end{pmatrix} \propto \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{D.12}$$

To derive this, we used the fact that, if we write b, d as

$$b = y\hat{b}, \qquad d = y\hat{d}, \qquad y = \gcd(b, d), \tag{D.13}$$

then the condition $ad + bc = 0$ implies that

$$(\hat{b}, \hat{d}) = \pm(\hat{a}, -\hat{c}). \tag{D.14}$$

Eq. (D.12) correspond to the following charges:

$$x \text{ units of D4(4567)+D0}, \qquad y \text{ units of D2(45)+D2(67)}. \tag{D.15}$$

As we can see from (2.43), both of these pairs must puff out into ns5(λ4567), where λ parametrizes a closed curve in transverse directions. The $\mathrm{SL}(2,\mathbb{Z})_3$ monodromy matrix for ns5(λ4567) is

$$M_{\mathrm{ns5}(\lambda 4567)} = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \tag{D.16}$$

where $q \in \mathbb{Z}$ is the dipole charge number (the number of NS5-branes). If we dualize this back, the monodromy of the supertube in the original frame is

$$M = U^{-1} M_{\mathrm{ns5}(\lambda 4567)} U = \begin{pmatrix} 1 - q\hat{a}\hat{c} & q\hat{c}^2 \\ -q\hat{a}^2 & 1 + q\hat{a}\hat{c} \end{pmatrix} \tag{D.17}$$

where we used (D.10). This result is symmetric under the exchange of $\binom{c}{a}$ and $\binom{-d}{b}$ as it should be because, using (D.14), we can write this as

$$M = \begin{pmatrix} 1+q\hat{b}\hat{d} & q\hat{d}^2 \\ -q\hat{b}^2 & 1-q\hat{b}\hat{d} \end{pmatrix}. \tag{D.18}$$

Even in cases where some of a, b, c, d vanish, we can use the formulas (D.17) or (D.18). If $a = c = 0$, we can use (D.18). If $b = d = 0$, we can use (D.17). If a or c vanishes, we can use the rule $\gcd(k, 0) = k$ for $k \in \mathbb{Z}_{\neq 0}$ in (D.7). For example, if $c = 0$, then $x = a$ and $\hat{a} = 1, \hat{c} = 0$.

D.3 Round supertube

Let us compute the radius and the angular momentum of the round supertube that is created from a 1/4-BPS center with general a, b, c, d satisfying $ad + bc = 0$.

If we T-dualize (D.15) along 7, S-dualize, T-dualize along 4567, and then finally S-dualize, we obtain

$$x \text{ units of F1(7)+P(7)}, \qquad y \text{ units of F1(6)+P(6)}. \tag{D.19}$$

This is the so-called FP system which is well-studied, rotated in the 67 plane. In the FP system with F1(7) and P(7) with quantized charges $N_{\rm F1}, N_{\rm P} \in \mathbb{Z}$, the radius $\mathcal{R}$ and angular momentum J of a circular configuration are given by (see, e.g., [71]):

$$\mathcal{R} = l_s \frac{\sqrt{N_{\rm F1} N_{\rm P}}}{q}, \qquad J = \frac{N_{\rm F1} N_{\rm P}}{q}, \tag{D.20}$$

where $q \in \mathbb{Z}$ is the dipole charge number. For the rotated system (D.19), this becomes

$$\mathcal{R} = l_s \frac{\sqrt{x^2 + y^2}}{q}, \qquad J = \frac{x^2 + y^2}{q}. \tag{D.21}$$

Following the duality chain back, we find this expression is again valid for the original frame with general $a, b, c, d \in \mathbb{Z}$, $ad + bc = 0$.

E Harmonic functions for the D2 + D6 → 5^2_2 supertube

In the main text, we reviewed the harmonic functions for the D2+D2→ns5 supertube (2.44). Here we recall the harmonic functions for the D2(89)+D6(456789)→ 5^2_2(λ4567;89) supertube [32], which is the last line of (2.43). This involves the exotic brane 5^2_2 with a non-geometric monodromy.

Harmonic functions which describe this supertube are [32]

$$\begin{aligned} V = f_2\,, \qquad & K^1 = \gamma\,, \quad K^2 = \gamma\,, \quad K^3 = 0\,, \\ & L_1 = 1\,, \quad L_2 = 1\,, \quad L_3 = f_1\,, \qquad M = 0\,. \end{aligned} \tag{E.1}$$

where f_1, f_2 are the same functions that appeared in (2.46). γ is defined in (2.47) and has the monodromy (2.48).

The behavior of V, L_3 shows that we do have D6(456789) and D2(89) charges distributed along the profile. On the other hand, the monodromy can be read off from

$$\begin{pmatrix} -L_1 \\ K^2 \end{pmatrix} = \begin{pmatrix} -1 \\ \gamma \end{pmatrix} \to \begin{pmatrix} -1 \\ \gamma+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -L_1 \\ K^2 \end{pmatrix}. \tag{E.2}$$

From (2.18), (2.19), this means that we have the following $\mathrm{SL}(2,\mathbb{Z})_3$ monodromy:

$$M_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_3. \tag{E.3}$$

One can also see this from the Kähler moduli,

$$\tau^1 = i\sqrt{\frac{f_1}{f_2}}, \qquad \tau^2 = i\sqrt{\frac{f_2}{f_1}}, \qquad \tau^3 = -\frac{1}{\tau'^3}, \qquad \tau'^3 = \gamma + i\sqrt{f_1 f_2}. \tag{E.4}$$

We see that, as we go once around the supertube, $\tau^{1,2}$ are single-valued whereas τ^3 has the monodromy

$$\tau^3 \to \frac{\tau^3}{-\tau^3+1}. \tag{E.5}$$

Because $\tau^3 = B_{89} + i\sqrt{\det G_{ab}}$ where $a,b = 8,9$, this monodromy implies that, every time one goes through the supertube, the radii of the torus T^2_{89} keeps changing. Namely, this spacetime is twisted by T-duality and is non-geometric. This is precisely the correct monodromy for the 5^2_2-brane [30, 31].

As in the case of the D2 + D2 $\to$ ns5 supertube discussed around (2.44), if $|\dot{\mathbf{F}}| = 1$, we have $f_1 = f_2 \equiv f$ and therefore $\tau^1 = \tau^2 = i$ as we can see from (E.4). So, the situation reduces to the one-modulus class of section 2.2, with the complex harmonic functions

$$F = i, \qquad G = -i(\gamma + if). \tag{E.6}$$

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