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Holographic interpolation between a and F

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Abstract: An interpolating function $\tilde{F}$ between the a-anomaly coefficient in even dimensions and the free energy on an odd-dimensional sphere has been proposed recently and is conjectured to monotonically decrease along any renormalization group flow in continuous dimension d. We examine $\tilde{F}$ in the large-N CFT's in d dimensions holographically described by the Einstein-Hilbert gravity in the AdS$_{d+1}$ space. We show that $\tilde{F}$ is a smooth function of d and correctly interpolates the a coefficients and the free energies. The monotonicity of $\tilde{F}$ along an RG flow follows from the analytic continuation of the holographic c-theorem to continuous d, which completes the proof of the conjecture.

Keywords: AdS-CFT Correspondence, Renormalization Group

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Contents

1 Introduction

A measure of degrees of freedom in a quantum field theory (QFT) remains to be elucidated in arbitrary d dimensions. Physically, it decreases monotonically as the energy scale is lowered because of the decoupling of massive particles. Implementation of such a measure in any QFT in diverse dimensions is intriguing and desirable to characterize the behavior under a renormalization group (RG) flow.

For even d, the conformal anomaly in the stress-energy tensor[1]

$$\langle T_\mu{}^\mu \rangle = \frac{(-1)^{\frac{d}{2}+1}}{2} a\, E_d + \sum_i b_i\, I_i\,, \tag{1.1}$$

defines the unique a coefficient for the Euler density E_d and several b_i coefficients for the Weyl invariants I_i labeled by an integer i. The a coefficients are believed to be monotonically decreasing along any RG flow, namely the value $a_{\rm UV}$ at the ultra-violet (UV) fixed point is equal or greater than that $a_{\rm IR}$ at the infra-red (IR) fixed point, $a_{\rm UV} \ge a_{\rm IR}$. This statement was established in two dimensions by the Zamolodchikov's c-theorem [1] and in four dimensions by the a-theorem [2–4]. On the other hand, the F-theorem asserts that the free energy, $F \equiv (-1)^{\frac{d-1}{2}} \log Z_{S^d}$, defined by the conformal invariant partition function Z_{S^d} on S^d of radius R, decreases under any RG flow in odd dimensions [5, 6]. A proof for $d = 3$ was presented by [7] through the relation of the free energy to the entanglement entropy S across an entangling surface S^{d-2} of radius R in $\mathbb{R}^{1,d-1}$ [8]

$$F = (-1)^{\frac{d-1}{2}} S\,, \tag{1.2}$$

that holds for odd d up to UV divergences.

These two proposals look quite different at first sight, but share the fact that both the a coefficient and the free energy can be read off on S^d; the former arises from the integration of the trace of the stress-energy tensor (1.1) and the latter from the partition function. To interpolate between the a coefficient and the free energy, Giombi and Klebanov define a new function [9]

$$\tilde{F} \equiv \sin\left(\frac{\pi d}{2}\right) \log Z_{S^d}\,, \tag{1.3}$$

[1] We define the stress-energy tensor by $T_{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta I}{\delta g^{\mu\nu}}$ for an action I. The Euler density is normalized to be $\int_{S^d} d^dx \sqrt{g}\, E_d = 2$.

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which correctly reduces to the free energies for odd d. They show as d approaches to even integers[2] (see also [10] as a related work)

$$\tilde{F} = \frac{\pi}{2}\, a \ . \tag{1.4}$$

Note that the partition function Z_{S^d} used in (1.3) is conformal invariant and UV divergent for even d. The relation (1.4) follows from the fact that the conformal invariant partition function in $d = 2n + \epsilon$ dimensions behaves as $\log Z_{S^d} = (-1)^{\frac{d}{2}} \frac{a}{2\epsilon} + O(1)$ for small ϵ. This is because one has to add a local counter term

$$I_{\text{c.t.}} = (-1)^{\frac{d}{2}+1} \frac{a}{2\epsilon} \int_{S^d} d^d x \sqrt{g}\, E_{2n} \,, \tag{1.5}$$

to the partition function to obtain the renormalized partition function $\log Z^{(\text{ren})}_{S^d} = \log Z_{S^d} + I_{\text{c.t}}$, reproducing the conformal anomaly $\log Z^{(\text{ren})}_{S^{2n}} = (-1)^{n+1} a \log R$ on S^{2n} of radius R in $\epsilon \to 0$ limit.

The function $\tilde{F}$ is also defined for non-integer d and therefore smoothly interpolates between the a coefficients in even dimensions and the free energies in odd dimensions. They conjecture that $\tilde{F}$ is positive and decreases along any RG flow in arbitrary d dimensions, based on several examples including a double-trace deformation of the large-N conformal field theory (CFT). We will call their proposal the $\tilde{F}$-theorem.

In this letter, we provide a further evidence to the $\tilde{F}$-theorem from the holographic viewpoint. To this end, we take advantage of the relation (1.2) and calculate the holographic entanglement entropy [11, 12] across a sphere S^{d-2} in the Einstein-Hilbert gravity on the AdS$_{d+1}$ space. We perform the dimensional regularization in the bulk and obtain the analytic result of $\tilde{F}$ that is a positive and smooth function of dimension d. We show that the equality (1.4) holds for even d and furthermore prove the $\tilde{F}$-theorem that follows from the holographic c-theorem [13–16] assuming the dimensional continuation of the null energy condition.

2 Holographic proof of the $\tilde{F}$-theorem

We will evaluate $\tilde{F}$ with the relation (1.2) between the free energy on S^d and the entanglement entropy across S^{d-2}. The latter can be holographically calculated by the Ryu-Takayanagi formula in the Einstein-Hilbert gravity [11, 12]

$$S = \frac{\text{Area}(\gamma)}{4G_N^{(d+1)}} \,, \tag{2.1}$$

where $G_N^{(d+1)}$ is the Newton constant, and γ stands for the $(d-1)$-dimensional minimal surface in the AdS$_{d+1}$ space, whose boundary is the entangling surface S^{d-2}. Since the boundary of the AdS$_{d+1}$ space is the flat space $\mathbb{R}^{1,d-1}$, we will use the Poincaré coordinates

$$ds^2 = L^2 \frac{dz^2 - dt^2 + dr^2 + r^2 d\Omega_{d-2}^2}{z^2} \,, \tag{2.2}$$

[2]There is no sign factor $(-1)^{d/2}$ in the right hand side because our convention of the a-anomaly (1.1) differs from theirs in [9].

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where L is the AdS radius. The entangling surface is located at $t = 0$ and $r = R$ at the boundary $z = 0$. In these coordinates, the minimal surface γ in the bulk is a hemi-hypersphere satisfying $r^2 + z^2 = R^2$ [11, 12]. This solution leads the entanglement entropy across S^{d-2}

$$S = \frac{1}{4G_N^{(d+1)}} L^{d-1} \mathrm{Vol}(S^{d-2}) \int_{\epsilon/R}^{1} dy \frac{(1-y^2)^{\frac{d-3}{2}}}{y^{d-1}} , \tag{2.3}$$

where we introduced a small cutoff at $z = \epsilon$ to regularize the UV divergence and $\mathrm{Vol}(S^{d-2})$ is the volume of a unit $(d-2)$-dimensional round sphere. Expanding the integrand with respect to y and performing the integration, one obtains the UV divergent parts of the entanglement entropy. We, however, want to employ the dimensional regularization instead of putting the UV cutoff at $z = \epsilon$ for our purpose. So we take $\epsilon = 0$ and carry out the integral in the range $1 < d < 2$, that yields

$$S = \frac{L^{d-1}}{4G_N^{(d+1)}} \pi^{\frac{d}{2}-1} \Gamma\left(1 - \frac{d}{2}\right) . \tag{2.4}$$

Then we analytically continue d to any real value. It is clear that there are poles at even d in the entanglement entropy (2.4) corresponding to the conformal anomalies. Finally, using the relations (1.2) and (1.3), and the formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we obtain $\tilde{F}$ in the holographic theories

$$\tilde{F} = \frac{L^{d-1}}{4G_N^{(d+1)}} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} . \tag{2.5}$$

This is manifestly a positive and smooth function of dimension d without poles at even d.

Now let us extrapolate the holographic values of $\tilde{F}$ to even dimensions and see if the relation (1.4) holds. The a coefficients holographically computed in the Einstein-Hilbert gravity are known to be [15–18]

$$a = \frac{L^{d-1}}{2\pi G_N^{(d+1)}} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} . \tag{2.6}$$

Combining it with (2.5), we confirm the relation (1.4) between $\tilde{F}$ and a. Moreover, imposing the null energy condition in the bulk, the holographic c-theorem states that the a coefficient given by (2.6) satisfies the monotonicity, $a_{\mathrm{UV}} \geq a_{\mathrm{IR}}$, for positive integer d [13–16]. Assuming the analytic continuation of dimension d in the gravity, the holographic c-theorem holds for $d \geq 1$,[3] which assures the $\tilde{F}$-theorem due to the relation (1.4).

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[3]The null energy condition $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ is crucial in the proof of the holographic c-theorem [14–16] where the d-dimensional null vector ξ has only two non-zero components ξ^z and ξ^t. Thus defining a formal null vector $\xi = (\xi^z, \xi^t, 0, \cdots, 0)$ in continuous d dimensions, the proof can be carried over for $d \geq 1$.

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