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Deformations of $T^{1,1}$ as Yang-Baxter sigma models

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Abstract: We consider a family of deformations of $T^{1,1}$ in the Yang-Baxter sigma model approach. We first discuss a supercoset description of $T^{1,1}$, which makes manifest the full symmetry of the space and leads to the standard Sasaki-Einstein metric. Next, we consider three-parameter deformations of $T^{1,1}$ by using classical r-matrices satisfying the classical Yang-Baxter equation (CYBE). The resulting metric and NS-NS two-form agree exactly with the ones obtained via TsT transformations, and contain the Lunin-Maldacena background as a special case. It is worth noting that for $\mathrm{AdS}_5 \times T^{1,1}$, classical integrability for the full sector has been argued to be lost. Hence our result indicates that the Yang-Baxter sigma model approach is applicable even for non-integrable cosets. This observation suggests that the gravity/CYBE correspondence can be extended beyond integrable cases.

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Contents

1 Introduction

A fascinating subject in string theory is dualities between gravitational theories and gauge theories. The original form proposed in [1] is the AdS/CFT correspondence, stating a duality between type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory in four dimensions. The integrable structure behind AdS/CFT plays a significant role in this duality [2]. It enables one to exactly compute some physical quantities such as anomalous dimensions and scattering amplitudes, even at finite coupling without supersymmetries.

Here we are concerned with the string theory side of the correspondence. In the Green-Schwarz formalism, the classical action for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring is given by a 2d σ-model on the coset superspace [3],

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)} . \tag{1.1}$$

Classical integrability for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring is closely related to the existence of a $\mathbb{Z}_4$-grading [4]. For an argument of integrability based on the Roiban-Siegel formalism [5], see [6, 7]. A classification of possible integrable cosets is given in [8, 9].

Recently, there has been progress in the study of integrable deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. The Yang-Baxter sigma model approach [10–13] (generalized to the coset case in [14]) plays an important role in this direction.

A q-deformed action for the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring has been constructed in [15]. Since a bosonic subsector of this action exhibits a q-deformed $\mathfrak{su}(2)$, the full symmetry algebra is expected to be a q-deformed $\mathfrak{psu}(2,2|4)$ [14, 16, 17].[1] In the end, the deformation used in [15] is the standard one with the classical r-matrix of Drinfeld-Jimbo type [23–25]. The metric in the string frame and NS-NS two-form were obtained in [26], though the complete supergravity solutions have not been found yet. Some limits of the deformed background are considered in [27, 28]. A mirror TBA is discussed in [29]. A non-relativistic limit on the world-sheet is considered in [30, 31]. Notably, the singularity of the metric disappears in this limit. Giant magnons are constructed in [29, 32, 33].

One may consider non-standard q-deformations (often called Jordanian deformations) [34, 35] as well. Jordanian-deformed actions for $\mathrm{AdS}_5\times\mathrm{S}^5$ have been constructed in [36]. The deformations are characterized by classical r-matrices satisfying the classical Yang-Baxter equation (CYBE). So far, some r-matrices, corresponding to well-known string backgrounds such as Lunin-Maldacena-Frolov backgrounds [37, 38], and the gravity duals of noncommutative gauge theories [39, 40], have been found in [41] and [42], respectively.[2] A new gravitational solution[3] was also constructed from an r-matrix in [43]. The relation between gravitational solutions and classical r-matrices may be referred to as the gravity/CYBE correspondence, as proposed in [41]. This correspondence surely contains the relation between r-matrices and TsT transformations on coset spaces, but these are not all. Indeed, some examples presented in [43] exhibit a curvature singularity in the middle of the bulk, but TsT transformations change only the asymptotic boundary behavior and would not lead to such a singularity.[4] At the present moment, to what degree the gravity/CYBE correspondence can be extended is unknown. One of the motivations of this paper is to give a new example of the correspondence, which goes beyond the class of known cases and discuss possible further extensions.

In this paper we consider type IIB superstrings on $\mathrm{AdS}_5\times T^{1,1}$. This geometry is realized by taking the near-horizon limit of a stack of N D3-branes sitting at the tip of a conifold [47]. The internal manifold $T^{1,1}$ is a Sasaki-Einstein manifold with $\mathrm{S}^2\times\mathrm{S}^3$ topology and a $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_R$ symmetry (for details on the conifold see [48], and for a review on aspects of AdS/CFT on this background see [49]). At the present moment, the Green-Schwarz string action on this background has not been constructed. Thus, we will focus only on the bosonic sector.

[1] It would be nice to show an affine extension of $\mathfrak{psu}(2,2|4)$ by following the procedure [18–22].

[2] The fermionic sector has not been studied yet, simply due to some technical complications. To do so, one would have to perform a supercoset construction in the supermatrix notation to evaluate the R-operator. It would be an important task to complete the analysis.

[3] It contains 3D Schrödinger spacetime. The related integrable structure is studied in [44–46].

[4] Actually, the appearance of singularity may depend on the parent geometry as argued in [52]. For example, for TsT transformations of $Y^{p,q}$ with three parameters, the resulting geometry may be singular. However, note that TsT transformations of $\mathrm{AdS}_5\times\mathrm{S}^5$ lead to no singularity. The singular geometries in [43] cannot be explained as TsT transformations, because those are derived as deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$.

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The usual description of $T^{1,1}$ as a coset is given by

$$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)}\,. \tag{1.2}$$

However, in this coset description one encounters a difficulty in applying the Yang-Baxter deformation to the usual coset decription of $T^{1,1}$, as we discuss now. Although (1.2) describes the space topologically, the coset metric is not the Sasaki-Einstein metric that the space admits,[5] and the one which is required as a proper string background. Since the class of deformations we are interested in are based on the coset description of the undeformed metric, before discussing deformations of $T^{1,1}$ we must develop a coset description that automatically leads to the Sasaki-Einstein metric.

Our proposal is to describe $T^{1,1}$ as the bosonic part of the *supercoset*:[6]

$$T^{1,1}=\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_R}{\mathrm{U}(1)_1\times\mathrm{U}(1)_2}\,. \tag{1.3}$$

As we shall show, it is possible to choose an embedding of the U(1)'s in the denominator that directly leads to the standard Sasaki-Einstein metric on $T^{1,1}$. In addition to leading to the correct undeformed metric, the description (1.3) has the advantage that one can easily describe the general (three-parameter) deformation of this space, as a consequence of the explicit appearance of the $\mathrm{U}(1)_R$ symmetry in the numerator. This is rather natural given that $\mathrm{U}(1)_R$ is part of the full global symmetry, and the grading of the matrices is rather natural from the point of view of the $\mathcal{N}=1$ superconformal symmetry of the dual gauge theory. It would be interesting to study whether this supercoset is relevant to the construction of the Green-Schwarz action on this background. The first step in this direction would be to find an appropriate supersymmetric extension by including fermions. However, the simplest extension (discussed below) will not contain 32 fermionic degrees of freedom and it may be difficult to construct the full Green-Schwarz action, as is usually the case in theories with reduced supersymmetry.

Next, we consider a family of three-parameter deformations of $T^{1,1}$ as Yang-Baxter sigma models with classical r-matrices satisfying the CYBE. This is analogous to the three-parameter real γ-deformations of S^5 as discussed in [38]. The resulting metric and NS-NS two form exactly agree with the ones obtained via TsT transformations in [52] and it contains the Lunin-Maldacena background [37] as a special case. This agreement indirectly supports that the proposed supercoset description is the appropriate description of bosonic strings on $\mathrm{AdS}_5\times T^{1,1}$.

It is worth making a comment regarding the issue of integrability for $T^{1,1}$. Although it is generally believed that an integrability structure is present in some sectors, it was argued in [53] that integrability for the full theory is lost due to the appearance of chaos in a certain

[5] This is well known and has been discussed in [50, 51], where a general method for obtaining Einstein metrics on cosets was developed. However, this method does not seem suited for the study of the deformations we discuss here — see appendix A for a discussion on this issue.

[6] Although the groups appearing below are bosonic, we refer to this as a supercoset due to a particular grading which is chosen. This will be discussed in detail in the main text.

subsector. Assuming that this conclusion is correct, our result indicates that the Yang-Baxter sigma model approach is applicable even for non-integrable cosets. This observation suggests that the gravity/CYBE correspondence can be extended beyond integrable cases; integrability is not essential for the correspondence and it is just the tip of an iceberg.

This paper is organized as follows. Section 2 considers a coset construction of $T^{1,1}$. A supercoset description is proposed. In section 3, we consider a family of deformations of $T^{1,1}$ as Yang-Baxter sigma model approach. We first give a short introduction to the Yang-Baxter sigma model approach. Then, the one-parameter deformation of $T^{1,1}$ is presented. Finally, three-parameter deformations are considered. Section 4 is devoted to conclusion and discussion. Appendix A reviews an alternative way to derive the $T^{1,1}$ metric. In appendix B, we give the detailed derivation of three-parameter deformation of $T^{1,1}$.

2 A coset construction of $T^{1,1}$

In this section, we consider a coset construction of the $T^{1,1}$ metric. Instead of the conventional coset (1.2), we describe the supercoset (1.3).[7]

2.1 The $T^{1,1}$ metric

The internal manifold $T^{1,1}$ is a five-dimensional Sasaki-Einstein manifold with global isometry $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_R$. The standard metric on $T^{1,1}$ is given by [48]

$$
\begin{aligned}
ds^2_{T^{1,1}} = &\frac{1}{6}(d\theta_1^2+\sin^2\theta_1 d\phi_1^2)+\frac{1}{6}(d\theta_2^2+\sin^2\theta_2 d\phi_2^2)\\
&+\frac{1}{9}(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2)^2\,.
\end{aligned}
\tag{2.1}
$$

This geometry may be regarded as a U(1)-fibration over $\mathrm{S}^2\times\mathrm{S}^2$. Here $0\le\theta_i<\pi$ and $0\le\phi_i<2\pi$ $(i=1,2)$ are the angle variables on two two-spheres. Then $0\le\psi<4\pi$ is the coordinate along the U(1)-fiber.

2.2 A supercoset representation of $T^{1,1}$

As we have discussed, the coset representation (1.2) does not lead to the metric (2.1). Consider instead the following coset:

$$
T^{1,1}=\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_R}{\mathrm{U}(1)_1\times\mathrm{U}(1)_2}\,. \tag{2.2}
$$

The generators of the two $\mathfrak{su}(2)$'s and the $\mathfrak{u}(1)_R$ in the numerator of (2.2) are denoted by K_i, L_i $(i=1,2,3)$ and M, respectively. Rather than 5×5 bosonic matrices, we choose a fundamental representation in terms of $(4|1)\times(4|1)$ *supermatrices*, i.e.,

$$
K_i=-\frac{i}{2}\left(\begin{array}{cc|c}\sigma_i & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right),\qquad L_i=-\frac{i}{2}\left(\begin{array}{cc|c}0 & 0 & 0\\ 0 & \sigma_i & 0\\ \hline 0 & 0 & 0\end{array}\right),\qquad M=-\frac{i}{2}\left(\begin{array}{cc|c}0 & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & 1\end{array}\right). \tag{2.3}
$$

[7]P.M.C. would like to thank Martin Roček for discussions on a related issue that inspired this construction.

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Here σ_i $(i=1,2,3)$ are the standard Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.4}$$

As we shall discuss below, the appearance of supermatrices — rather than bosonic matrices — is in fact natural from the perspective of the full $\mathrm{AdS}_5 \times T^{1,1}$ coset space.

It is easy to see that the generators satisfy the following relations:

$$\begin{aligned} [K_a, K_b] &= \epsilon_{ab}{}^{c} K_c\,, & [L_a, L_b] &= \epsilon_{ab}{}^{c} L_c\,, \\ \mathrm{STr}(K_a K_b) = \mathrm{STr}(L_a L_b) &= -\frac{1}{2}\delta_{ab}\,, & \mathrm{STr}(MM) &= \frac{1}{4}\,. \end{aligned}$$

Here the structure constant is normalized as $\epsilon_{123} = +1$ and the $\mathfrak{su}(2)$ indices are raised and lowered by the Killing form δ_{ab}. As usual, the supertrace of a supermatrix is defined as

$$\mathrm{STr}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \equiv \mathrm{Tr}(A) - \mathrm{Tr}(D)\,, \tag{2.5}$$

where A, D are bosonic block matrices and B, C are fermionic blocks. We denote the generators of the two $\mathfrak{u}(1)$'s in the denominator of (2.2) by $T_{1,2}$ and we choose to embed them into the numerator by

$$T_1 = K_3 + L_3\,, \qquad T_2 = K_3 - L_3 + 4M\,. \tag{2.6}$$

Note that T_1 denotes the U(1) in the usual description (1.2). The final coset metric depends on the embedding of T_2 in the numerator, and we have chosen it such to obtain the Sasaki-Einstein metric (2.1).

2.3 The $T^{1,1}$ metric from a supercoset

Let us first show that the supercoset (2.2) indeed leads to the metric (2.1).

It is convenient to introduce the orthogonal basis of the quotient vector space as follows:

$$\frac{\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)_R}{\mathfrak{u}(1)_1 \oplus \mathfrak{u}(1)_2} = \mathrm{span}_{\mathbb{R}}\{K_1, K_2, L_1, L_2, H\}\,. \tag{2.7}$$

Here the diagonal element H is defined as

$$H \equiv K_3 - L_3 + M\,. \tag{2.8}$$

With this basis, one may introduce a group element parametrized by

$$g = \exp\bigl(\phi_1 K_3 + \phi_2 L_3 + 2\psi M\bigr) \exp\bigl(\theta_1 K_2 + (\theta_2 + \pi) L_2\bigr)\,. \tag{2.9}$$

Then the left-invariant one-form

$$A \equiv g^{-1} dg \tag{2.10}$$

can be written in terms of the coordinates ψ, θ_i and ϕ_i $(i = 1, 2)$.

The coset metric is given by the simple expression,

$$ds^2_{T^{1,1}} = -\frac{1}{3}\,\mathrm{STr}\left[AP(A)\right]\,, \tag{2.11}$$

where P is a projector to the coset space (2.7) and the associated projected current reads

$$\begin{aligned} P(A) &= A + T_1 \mathrm{STr}[T_1 A] - \frac{1}{3} T_2 \mathrm{STr}[T_2 A] \\ &= -\sin\theta_1 d\phi_1\, K_1 + d\theta_1\, K_2 + \sin\theta_2 d\phi_2\, L_1 + d\theta_2\, L_2 \\ &\quad + \frac{2}{3}\big(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\big)\, H\,. \end{aligned} \tag{2.12}$$

From this expression, it is direct to see that (2.11) leads to the metric (2.1).

2.4 What is the origin of the supercoset?

Before discussing deformations of this space, it is worth discussing the origin of the supermatrix representations in (2.3). A possible explanation is the following. It is believed that string theory on $\mathrm{AdS}_5 \times T^{1,1}$ is dual to an $\mathcal{N} = 1$ superconformal field theory in four dimensions [47]. The $\mathcal{N} = 1$ superconformal group is composed of the conformal group $\mathrm{SU}(2,2)$, two sets of four real fermionic generators $\overline{F}^A$, F_A, and the $\mathrm{U}(1)_R$ symmetry. These generators can be organized into the supermatrix,

$$\left(\begin{array}{c|c} \mathrm{SU}(2,2) & \overline{F}^A \\ \hline F_A & \mathrm{U}(1)_R \end{array}\right)\,. \tag{2.13}$$

Note that this supermatrix describes only the superconformal group $\mathrm{PSU}(2,2|1)$, and does not contain the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ flavor symmetry, unlike the case of $\mathrm{PSU}(2,2|4)$ which includes the full flavor symmetry.

Thus, to include flavor symmetry it is necessary to consider an embedding of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$ into a bigger supermatrix. A natural candidate is the following $(8|1) \times (8|1)$ supermatrix:

$$\left(\begin{array}{cc|c} \mathrm{SU}(2) & 0 & 0 \\ 0 & \mathrm{SU}(2) & 0 \\ \hline 0 & 0 & \mathrm{U}(1)_R \end{array}\right) \hookrightarrow \left(\begin{array}{ccc|c} \mathrm{SU}(2,2) & 0 & 0 & \overline{F}^A \\ 0 & \mathrm{SU}(2) & 0 & 0 \\ 0 & 0 & \mathrm{SU}(2) & 0 \\ \hline F_A & 0 & 0 & \mathrm{U}(1)_R \end{array}\right)\,. \tag{2.14}$$

Here $\mathrm{PSU}(2,2|1)$ is located at the four corners of (2.14). Thus, the bosonic sector of the supercoset

$$\frac{\mathrm{PSU}(2,2|1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{SO}(1,4) \times \mathrm{U}(1) \times \mathrm{U}(1)} \tag{2.15}$$

describes the bosonic sector of type IIB strings on $AdS_5 \times T^{1,1}$. This is indeed a rather natural description of the full $\mathrm{PSU}(2,2|1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry group and it may explain the origin of the supermatrix representation (2.3).[8] As we shall discuss in

[8] It would be interesting to study whether turning on the fermions in this supercoset sigma model is relevant for the construction of the Green-Schwarz action in this background, but we do not discuss this here.

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section 3, the Yang-Baxter deformation of this supercoset leads to a family of deformations of the metric and NS-NS two-form that exactly agree with the ones obtained in [52]. The Lunin-Maldacena deformation [37] is contained as a special case. We consider this fact as further support for the supermatrix description. It would be quite interesting to find further support for this interpretation from other points of view.

3 Deformations of $T^{1,1}$ as Yang-Baxter sigma models

Thus far, we have presented a supercoset construction of the Sasaki-Einstein metric on $T^{1,1}$. In this section we use this description to study Yang-Baxter deformations.

By specifying classical r-matrices, we first discuss a one-parameter deformation in subsection 3.2 and then a three-parameter deformation in subsection 3.3.

3.1 The action of Yang-Baxter sigma models on $T^{1,1}$

An interesting class of deformations of nonlinear sigma models is given by Yang-Baxter sigma models [10–14]. The original procedure depends on the classical r-matrix of Drinfeld-Jimbo type, which satisfies the modified CYBE (mCYBE). However, in this approach, it seems difficult to perform partial deformations (for instance, deformations of the internal manifold only and not of the AdS factor).[9] Since here we are interested in deformations of the internal manifold $T^{1,1}$ only, we apply the formalism of Yang-Baxter sigma models based on the CYBE [36] instead.

Our original motivation is to study type IIB superstrings on $\mathrm{AdS}_5 \times T^{1,1}$, and its deformations. However, since the Green-Schwarz action for these backgrounds have not been constructed, we restrict ourselves to the bosonic sector. For simplicity, we consider deformations of the internal manifold $T^{1,1}$ only (the AdS_5 part is untouched) and therefore we focus on this part of the action.

The action is given by

$$S = \frac{1}{3}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \, \mathrm{STr}\left(A_\alpha P \circ \frac{1}{1 - 2\eta R_g \circ P} A_\beta \right), \tag{3.1}$$

where the flat metric $\gamma^{\alpha\beta}$ and the anti-symmetric tensor $\epsilon^{\alpha\beta}$ on the string world-sheet are normalized as $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$ and $\epsilon^{\tau\sigma} = 1$. The projector P to the coset space is given in (2.12). Here η is a parameter that measures deformations from $T^{1,1}$. In the $\eta \to 0$ limit, the action (3.1) reduces to the undeformed $T^{1,1}$, as shown in section 2.

[9]This point is explained as follows. The mCYBE for a Lie algebra $\mathfrak{g}$ takes the form,

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = c^2[x, y] \qquad \text{for} \qquad {}^\forall x, y \in \mathfrak{g}$$

with a parameter c. To consider a partial deformation of a certain subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the R-operator needs to satisfy $R(\mathfrak{h}) \subset \mathfrak{h}$ and $R(\mathfrak{m}) = 0$, where $\mathfrak{m}$ is defined as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. From the mCYBE, this demands that the following two conditions are satisfied; (i) either $c = 0$ or $\mathfrak{m}$ is abelian, and (ii) $R([R(x), y]) = -c^2[x, y]$ for any $x \in \mathfrak{h}$ and $y \in \mathfrak{m}$. Note that, when $x \in \mathfrak{m}$ and $y \in \mathfrak{h}$, the mCYBE requires the same condition (ii) since it is invariant by exchanging x and y. Obviously, the R-operator of Drinfeld-Jimbo type does not satisfy these conditions. For $c \neq 0$, these conditions appear hard to satisfy.

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The left-invariant one-form is defined as usual by

$$A_\alpha \equiv g^{-1}\partial_\alpha g\,, \qquad g \in \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_R\,. \tag{3.2}$$

The group element g is parameterized as (2.9). Note that the supertrace appears in the action (3.1), even though all the fermions are set to zero in the present case.

The most important ingredient in (3.1) is a linear R-operator. The symbol R_g denotes a dressed R-operator, given by the adjoint operation of the group, as:

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g\,. \tag{3.3}$$

It is easy to see that if R satisfies the CYBE, so does R_g. This R-operator is related to the tensorial notation of a classical r-matrix through

$$R(X) = \mathrm{STr}_2[r(1\otimes X)] = \sum_i \big(a_i \mathrm{STr}(b_i X) - b_i \mathrm{STr}(a_i X)\big) \tag{3.4}$$
$$\text{with} \quad r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i\otimes b_i - b_i \otimes a_i)\,.$$

In our case, a_i and b_i are generators in $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)_R$.

3.2 One-parameter deformation

We now consider examples of r-matrices describing deformations of $T^{1,1}$.

Let us begin with the simplest example. This is provided by the abelian r-matrix,

$$r^{(\mu)}_{\mathrm{Abe}} = \mu K_3 \wedge L_3\,, \tag{3.5}$$

with deformation parameter μ. Here K_3 and L_3 are the Cartan generators of two $\mathfrak{su}(2)$'s, respectively. The fundamental representation is given in (2.3).

Then the Lagrangian (3.1) is given by

$$L = \frac{1}{3}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{STr}\left[A_\alpha P(J_\beta)\right] \tag{3.6}$$

$$\text{with} \qquad J_\beta \equiv \frac{1}{1-2\big[R^{(\mu)}_{\mathrm{Abe}}\big]_g \circ P} A_\beta\,, \tag{3.7}$$

where we have set the scaling factor $\eta = 1$ in the deformed action.[10] The operator $R^{(\mu)}_{\mathrm{Abe}}$ associated with (3.5) is determined by the relation (3.4). It is convenient to separate the Lagrangian into the two parts $L = L_G + L_B$, where L_G is the metric part and L_B is the coupling to the NS-NS two-form:

$$\begin{aligned} L_G &\equiv -\frac{1}{3}[\mathrm{STr}(A_\tau P(J_\tau)) - \mathrm{STr}(A_\sigma P(J_\sigma))]\,, \\ L_B &\equiv -\frac{1}{3}[\mathrm{STr}(A_\tau P(J_\sigma)) - \mathrm{STr}(A_\sigma P(J_\tau))]\,. \end{aligned} \tag{3.8}$$

[10] In fact, η can be absorbed into the normalization of the r-matrices satisfying the CYBE.

To evaluate the Lagrangian explicitly, it is sufficient to compute the projected current $P(J_\alpha)$ rather than J_α itself. Hence the computation is reduced to solving the following set of equations,

$$\left(1-2P\circ\left[R^{(\mu)}_{\text{Abe}}\right]_g\right)P(J_\alpha)=P(A_\alpha)\,. \tag{3.9}$$

Plugging the expression for $P(A_\alpha)$ given in (2.12) into (3.9), one can solve for the deformed projected current, finding

$$P(J_\alpha)=j^1_\alpha\,K_1+j^2_\alpha\,K_2+j^3_\alpha\,L_1+j^4_\alpha\,L_2+j^5_\alpha\,H\,, \tag{3.10}$$

with the coefficients

$$\begin{aligned}
j^1_\alpha&=\frac{G(6\mu)}{6}\sin\theta_1\big[(-6+4\mu\cos\theta_1\cos\theta_2)\partial_\alpha\phi_1+\mu(5-\cos 2\theta_2)\partial_\alpha\phi_2\\
&\qquad\qquad+4\mu(\cos\theta_2+\mu\cos\theta_1\sin^2\theta_2)\partial_\alpha\psi\big]\,,\\
j^2_\alpha&=\partial_\alpha\theta_1\,,\\
j^3_\alpha&=\frac{G(6\mu)}{6}\sin\theta_2\big[(6+4\mu\cos\theta_1\cos\theta_2)\partial_\alpha\phi_2+\mu(5-\cos 2\theta_1)\partial_\alpha\phi_1\\
&\qquad\qquad+4\mu(\cos\theta_1-\mu\cos\theta_2\sin^2\theta_1)\partial_\alpha\psi\big]\,,\\
j^4_\alpha&=\partial_\alpha\theta_2\,,\\
j^5_\alpha&=\frac{2G(6\mu)}{3}\big[(\cos\theta_1+\mu\sin^2\theta_1\cos\theta_2)\partial_\alpha\phi_1+(\cos\theta_2-\mu\sin^2\theta_2\cos\theta_1)\partial_\alpha\phi_2\\
&\qquad\qquad+(1+\mu^2\sin^2\theta_1\sin^2\theta_2)\partial_\alpha\psi\big]\,,
\end{aligned} \tag{3.11}$$

where the scalar function $G(x)$ is defined as

$$G(x)^{-1}\equiv 1+x^2\left(\frac{\sin^2\theta_1\sin^2\theta_2}{36}+\frac{\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1}{54}\right)\,. \tag{3.12}$$

The resulting L_G and L_B are given by

$$\begin{aligned}
L_G=&-\gamma^{\alpha\beta}G(\hat\gamma)\Bigg[\frac{1}{6}\sum_{i=1,2}\left(G(\hat\gamma)^{-1}\partial_\alpha\theta_i\partial_\beta\theta_i+\sin^2\theta_i\partial_\alpha\phi_i\partial_\beta\phi_i\right)+\hat\gamma^2\frac{\sin^2\theta_1\sin^2\theta_2}{324}\partial_\alpha\psi\partial_\beta\psi\\
&+\frac{1}{9}(\partial_\alpha\psi+\cos\theta_1\partial_\alpha\phi_1+\cos\theta_2\partial_\alpha\phi_2)(\partial_\beta\psi+\cos\theta_1\partial_\beta\phi_1+\cos\theta_2\partial_\beta\phi_2)\Bigg]\,,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
L_B=&\,2\epsilon^{\alpha\beta}\hat\gamma G(\hat\gamma)\Bigg[\frac{\cos\theta_2\sin^2\theta_1}{54}\partial_\alpha\phi_1\partial_\beta\psi-\frac{\cos\theta_1\sin^2\theta_2}{54}\partial_\alpha\phi_2\partial_\beta\psi\\
&+\left(\frac{\sin^2\theta_1\sin^2\theta_2}{36}+\frac{\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1}{54}\right)\partial_\alpha\phi_1\partial_\beta\phi_2\Bigg]\,,
\end{aligned} \tag{3.14}$$

where the new quantity $\hat\gamma$ is defined as

$$\hat\gamma\equiv-6\mu\,. \tag{3.15}$$

Thus, the deformed metric and NS-NS two-form are given by

$$\begin{aligned}
ds^2=&\,G(\hat\gamma)\Bigg[\frac{1}{6}\sum_{i=1,2}\left(G(\hat\gamma)^{-1}d\theta_i^2+\sin^2\theta_i d\phi_i^2\right)+\hat\gamma^2\frac{\sin^2\theta_1\sin^2\theta_2}{324}d\psi^2\\
&+\frac{1}{9}(d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2)^2\Bigg]\,,
\end{aligned} \tag{3.16}$$

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$$
\begin{aligned}
B_2 = \hat{\gamma} G(\hat{\gamma}) \Bigg[& \frac{\cos\theta_2 \sin^2\theta_1}{54} d\phi_1 \wedge d\psi - \frac{\cos\theta_1 \sin^2\theta_2}{54} d\phi_2 \wedge d\psi \\
& + \left(\frac{\sin^2\theta_1 \sin^2\theta_2}{36} + \frac{\cos^2\theta_1 \sin^2\theta_2 + \cos^2\theta_2 \sin^2\theta_1}{54} \right) d\phi_1 \wedge d\phi_2 \Bigg] . \qquad (3.17)
\end{aligned}
$$

These expressions agree exactly with the one-parameter γ-deformed backgrounds presented by Lunin and Maldacena [37]. Thus, the abelian r-matrix (3.5) is the algebraic origin of the γ-deformation of $\mathrm{AdS}_5 \times T^{1,1}$.

3.3 Three-parameter deformation

It is straightforward to generalize the one-parameter case to the three-parameter case. Since there are three Cartan generators L_3, K_3 and M, the most generic form for the abelian r-matrix is given by

$$
r_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)} = \mu_1 L_3 \wedge M + \mu_2 M \wedge K_3 + \mu_3 K_3 \wedge L_3 \,, \qquad (3.18)
$$

with three deformation parameters μ_1, μ_2 and μ_3. Note that the explicit appearance of the $\mathrm{U}(1)_R$ symmetry — generated by M — in the supercoset (2.2) allows us to consider this three-parameter deformation.

The computation is completely parallel to the one-parameter case. Thus, we do not repeat it here but simply give the final result. For details, see appendix B.

With parameter identifications[11]

$$
3\mu_1 = \hat{\gamma}_1 \,, \qquad 3\mu_2 = \hat{\gamma}_2 \,, \qquad -6\mu_3 = \hat{\gamma}_3 \,, \qquad (3.19)
$$

we obtain the following deformed metric and NS-NS two-form:

$$
\begin{aligned}
ds^2 = G(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) \Bigg[& \frac{1}{6} \sum_{i=1,2} \left(G(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)^{-1} d\theta_i^2 + \sin^2\theta_i d\phi_i^2 \right) \\
& + \frac{1}{9} (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{\sin^2\theta_1 \sin^2\theta_2}{324} (\hat{\gamma}_3 d\psi + \hat{\gamma}_1 d\phi_1 + \hat{\gamma}_2 d\phi_2)^2 \Bigg] , \qquad (3.20) \\
B_2 = G(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) \Bigg[& \Bigg\{ \hat{\gamma}_3 \left(\frac{\sin^2\theta_1 \sin^2\theta_2}{36} + \frac{\cos^2\theta_1 \sin^2\theta_2 + \cos^2\theta_2 \sin^2\theta_1}{54} \right) \\
& - \hat{\gamma}_2 \frac{\cos\theta_2 \sin^2\theta_1}{54} - \hat{\gamma}_1 \frac{\cos\theta_1 \sin^2\theta_2}{54} \Bigg\} d\phi_1 \wedge d\phi_2 \\
& + \frac{(\hat{\gamma}_3 \cos\theta_2 - \hat{\gamma}_2) \sin^2\theta_1}{54} d\phi_1 \wedge d\psi - \frac{(\hat{\gamma}_3 \cos\theta_1 - \hat{\gamma}_1) \sin^2\theta_2}{54} d\phi_2 \wedge d\psi \Bigg] , \qquad (3.21)
\end{aligned}
$$

where the scalar function is defined as

$$
\begin{aligned}
G(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)^{-1} \equiv 1 + \hat{\gamma}_3^2 & \left(\frac{\sin^2\theta_1 \sin^2\theta_2}{36} + \frac{\cos^2\theta_1 \sin^2\theta_2 + \cos^2\theta_2 \sin^2\theta_1}{54} \right) + \hat{\gamma}_2^2 \frac{\sin^2\theta_1}{54} \\
& + \hat{\gamma}_1^2 \frac{\sin^2\theta_2}{54} - \hat{\gamma}_2 \hat{\gamma}_3 \frac{\sin^2\theta_1 \cos\theta_2}{27} - \hat{\gamma}_3 \hat{\gamma}_1 \frac{\sin^2\theta_2 \cos\theta_1}{27} . \qquad (3.22)
\end{aligned}
$$

[11] Here we also normalize the scaling factor in (3.1) as $\eta = 1$.

These expressions are rather complicated but agree perfectly with the ones obtained in [52] . Thus, the abelian r-matrix (3.18) corresponds to the three-parameter γ-deformation. The previous one-parameter deformation is reproduced by simply setting $\hat{\gamma}_1 = \hat{\gamma}_2 = 0$ and $\hat{\gamma}_3 = \hat{\gamma}$.

Finally, let us comment on the amount of supersymmetry remaining in the three-parameter deformation. Recall that in the undeformed $T^{1,1}$ case there is an $\mathcal{N}$=1 superconformal symmetry. Without studying the Killing spinor equations, we can understand the remaining supersymmetry by considering the $\mathrm{U}(1)_\mathrm{R}$ symmetry. In the classical r-matrix (3.18), the generator M is associated with the U(1) R-symmetry, while K_3 and L_3 are associated to the non-R symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2)$. In the Lunin-Maldacena case of $T^{1,1}$ [37] with $\mu_3 \neq 0$ and $\mu_1 = \mu_2 = 0$, the $\mathcal{N}$=1 superconformal symmetry is preserved because the U(1) R-symmetry is not affected by the TsT transformation. However, if either μ_1 or μ_2 is non-zero, the $\mathrm{U}(1)_\mathrm{R}$ symmetry is broken due to the shift of the period and hence the solution is non-supersymmetric.[12]

4 Conclusion and discussion

In this paper we have considered a family of deformations of $T^{1,1}$ as Yang-Baxter sigma models.

We first provided a new coset description of $T^{1,1}$ which directly leads to the standard Sasaki-Einstein metric. This is necessary to study deformations of this space as Yang-Baxter sigma models. The coset description we presented is a rather natural description from the point of view of the $\mathcal{N} = 1$ superconformal symmetry of the dual gauge theory. However, to the best of our knowledge this description has not appeared in the literature.

Next, we considered three-parameter deformations of $T^{1,1}$ by using classical r-matrices satisfying the CYBE. The resulting metric and NS-NS two-form perfectly agree with the ones obtained via TsT transformations [37, 52].

It was shown in [41] that three-parameter real γ-deformations $\mathrm{AdS}_5\times\mathrm{S}^5$ [37, 38] are realized by the Yang-Baxter sigma model approach with abelian classical r-matrices. Thus, the results obtained here may be regarded as a generalization of the work [41], giving further support for the gravity/CYBE correspondence. However, it should be stressed that there is a significant difference between S^5 and $T^{1,1}$. The former is represented by a symmetric coset and therefore corresponds to an *integrable* nonlinear sigma model. In the case of $T^{1,1}$, however, this is not the case and the claim that it is not integrable was made in [53], by showing the appearance of chaos in a subsector of the theory. Assuming that this result is correct, the class of deformations considered here are not regarded as integrable deformations. However, this would lead to the stronger statement that the gravity/CYBE correspondence would hold independently of integrability and that it captures a much wider class of gravitational solutions.

[12] Note that the background still seems to preserve the U(1) R-symmetry. However, one should be careful with the periodicity of the angle variables and note that the Killing spinors cannot survive for generic values of μ_1 and μ_2 . This is a global property and cannot be seen from a local quantity like the metric.

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Let us make a few comments on possible further generalizations. An interesting class of metrics on $S^2 \times S^3$ is given by the well-known $Y^{p,q}$ metrics [54]. However, since these have not been explicitly constructed as coset metrics, it would be difficult to consider deformations in this approach. It would also be interesting to study additional coset spaces which may or may not be integrable, a possible candidate being the Lifshitz spacetime. The coset description was given in [55], and it has been argued to be non-integrable in [56]. Other important supercosets appear in descriptions of type IIA compactifications on AdS_4, such as ABJM theory [57]. The supercoset description has been given in [58, 59].

What is the general class of gravitational solutions included in the gravity/CYBE correspondence? As we have discussed above, it has already been shown that the correspondence includes deformations which cannot be obtained by TsT transformations. The result obtained in this paper indicates that the integrability of the parent theory is not an essential feature. Thus, we see that the class of gravitational solutions captured by the correspondence is much wider than the examples that were first discovered. What the full moduli space of gravity solutions captured by the gravity/CYBE correspondence is remains an open problem at the present moment.

As we have seen, at this point there are various examples of coset supergravity backgrounds, integrable and non-integrable, such that its Yang-Baxter deformations remain as supergravity solutions. The non-trivial question is whether this is the case for a generic coset supergravity background and a generic r-matrix. Although a counter-example has not been found so far, there is no proof that this is true in general. One possible approach to studying this would be to exploit kappa-symmetry. Answering this question could lead to new insights into the structure of the moduli space of possible gravity solutions, and the action of classical r-matrices on this space. This issue deserves to be studied as a fundamental problem.

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A $T^{1,1}$ metric from the rescaling of vielbeins

As we have discussed, the $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{U}(1)$ coset description of $T^{1,1}$ does not lead to the Sasaki-Einstein metric (2.1) that the space admits. This comes as no surprise, since it is well known that coset spaces are not typically Einstein spaces. However, it was shown in [50] that given a coset space G/H it may be possible to rescale the vielbeins to obtain

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an Einstein space, without loosing the original symmetry of the coset space. This is in fact the case for $T^{1,1}$, as discussed in [51]. Take the left-invariant current $A = g^{-1}dg$ with $g \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ and rescale the coset space directions by three parameters α, β, γ, as

$$A_{\text{resc.}} = \alpha \sum_{i=1,2} A^i K_i + \beta \sum_{i=1,2} A^i L_i + \gamma\, A^-(L_3 - K_3) + A^+(L_3 + K_3)\,. \tag{A.1}$$

The term proportional to A^+ is the one projected out by the coset and is not rescaled. For $\alpha = \beta = \gamma = 1$, this current describes a natural metric on the coset space $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{U}(1)$ but not the Sasaki-Einstein metric. However, for arbitrary values of the parameters one finds[13]

$$ds^2 = \alpha^2(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \beta^2(d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2) + \frac{\gamma^2}{2}\,(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2)^2\,. \tag{A.2}$$

Imposing the Einstein condition on this metric one finds

$$\alpha^2 = \beta^2 = \frac{1}{6}\,, \quad \gamma^2 = \frac{2}{9}\,, \tag{A.3}$$

corresponding to (2.1). Thus, a possible starting point to study deformations of the $T^{1,1}$ sigma model would be to study deformations of the sigma model defined by the rescaled current (A.1). However, since this approach is based on a rescaling of the *current*, rather than the group elements g, is not clear how to implement the Yang-Baxter deformation (defined by the action of the group elements in (3.3)) in this formulation. Thus, one of the advantages of the supercoset description (2.2) is that the Yang-Baxter deformation can be applied directly, as we have shown. Another advantage is that by making manifest the $\mathrm{U}(1)_R$ symmetry, it is clear how to implement the three-parameter deformation discussed in section 3.3.

As a final comment, we would like to point out that a related issue arises in the description of the conifold as a classical Kähler quotient. It is well known that this can be realized as an $\mathcal{N} = (2,2)$ gauged linear sigma model (GLSM) for four chiral fields with charges $(1,1,-1,-1)$ under a U(1) [47]. It is easy to see that the classical quotient metric is not the Calabi-Yau metric, i.e., the metric of the base is not the Sasaki-Einstein metric (in fact, it coincides with the coset metric). Again, this comes as no surprise since the classical quotient metric describes the UV behavior of the GLSM, while the Calabi-Yau metric describes the IR behavior, at the endpoint of the RG flow. It would be interesting to study whether it is possible to formulate the supercoset description of the conifold that we have given here in terms of a GLSM.[14]

B Derivation of three-parameter deformations

It would be useful to present here the detailed derivation of the three-parameter deformed metric (3.20) and NS-NS two-form (3.21).

[13] A more general metric is obtained by taking the general invariant two-form into account [60].

[14] We would like to thank Martin Roček for discussions on this.

The classical r-matrix is composed of three Cartan generators L_3, K_3 and M as follows:

$$r_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)} = \mu_1 L_3 \wedge M + \mu_2 M \wedge K_3 + \mu_3 K_3 \wedge L_3\,. \tag{B.1}$$

Here μ_1, μ_2 and μ_3 are deformation parameters. Then the associated linear R-operator is written in terms of L_3, K_3 and M like

$$\begin{aligned} R_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)}(K_3) &= \frac{1}{2}(\mu_3 L_3 - \mu_2 M)\,, & R_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)}(L_3) &= \frac{1}{2}(\mu_1 M - \mu_3 K_3)\,, \\ R_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)}(M) &= \frac{1}{4}(\mu_1 L_3 - \mu_2 K_3)\,, & R_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)}(\text{others}) &= 0\,. \end{aligned} \tag{B.2}$$

These transformation laws are utilized to rewrite the Lagrangian (3.8).

First of all, let us evaluate the projected deformed current $P(J_\alpha)$. It can be done by solving the relation,

$$\left(1 - 2P \circ \left[R_{\text{Abe}}^{(\mu_1,\mu_2,\mu_3)}\right]_g\right) P(J_\alpha) = P(A_\alpha)\,. \tag{B.3}$$

Plugging the expression of $P(A_\alpha)$ in (2.12) with the above equation, the deformed projected current is obtained as

$$P(J_\alpha) = j_\alpha^1\, K_1 + j_\alpha^2\, K_2 + j_\alpha^3\, L_1 + j_\alpha^4\, L_2 + j_\alpha^5\, H\,, \tag{B.4}$$

with the coefficients

$$\begin{aligned} j_\alpha^1 &= \frac{1}{6} G(3\mu_1, 3\mu_2, -6\mu_3) \sin\theta_1 \\ &\quad \times \big[-\left(6 + \mu_1 \sin^2\theta_2(\mu_1 + 2\mu_3\cos\theta_1) - 2\cos\theta_1(\mu_2 + 2\mu_3\cos\theta_2)\right)\partial_\alpha\phi_1 \\ &\qquad + \left(2\cos\theta_2(\mu_2 + 2\mu_3\cos\theta_2) - \sin^2\theta_2(\mu_1\mu_2 - 6\mu_3 + 2\mu_2\mu_3\cos\theta_1)\right)\partial_\alpha\phi_2 \\ &\qquad + 2\left(\mu_3\sin^2\theta_2(\mu_1 + 2\mu_3\cos\theta_1) + \mu_2 + 2\mu_3\cos\theta_2\right)\partial_\alpha\psi\big]\,, \\ j_\alpha^2 &= \partial_\alpha\theta_1\,, \\ j_\alpha^3 &= \frac{1}{6} G(3\mu_1, 3\mu_2, -6\mu_3) \sin\theta_2 \\ &\quad \times \big[\left(6 + \mu_2 \sin^2\theta_1(2\mu_3\cos\theta_2 + \mu_2) + 2\eta\cos\theta_2(2\mu_3\cos\theta_1 + \mu_1)\right)\partial_\alpha\phi_2 \\ &\qquad + \left(2\cos\theta_1(\mu_1 + 2\mu_3\cos\theta_1) + \sin^2\theta_1(\mu_1\mu_2 + 6\mu_3 + 2\mu_1\mu_3\cos\theta_2)\right)\partial_\alpha\phi_1 \\ &\qquad + 2\left(-\mu_3\sin^2\theta_1(2\mu_3\cos\theta_2 + \mu_2) + \mu_1 + 2\mu_3\cos\theta_1\right)\partial_\alpha\psi\big]\,, \\ j_\alpha^4 &= \partial_\alpha\theta_2\,, \\ j_\alpha^5 &= \frac{1}{3} G(3\mu_1, 3\mu_2, -6\mu_3) \\ &\quad \times \big[\left(2\cos\theta_1 + \sin^2\theta_1\left(\mu_2 - \mu_1\mu_3\sin^2\theta_2 + 2\mu_3\cos\theta_2\right)\right)\partial_\alpha\phi_1 \\ &\qquad + \left(2\cos\theta_2 - \eta\sin^2\theta_2\left(\mu_1 + \mu_2\mu_3\sin^2\theta_1 + 2\mu_3\cos\theta_1\right)\right)\partial_\alpha\phi_2 \\ &\qquad + 2\left(1 + \mu_3^2\sin^2\theta_1\sin^2\theta_2\right)\partial_\alpha\psi\big]\,. \end{aligned} \tag{B.5}$$

Here the scalar function $G(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)$ is defined in (3.22).

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As a result, L_G and L_B are given by, respectively,

$$\begin{aligned}
L_G = -\gamma^{\alpha\beta}G(\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3)\Bigg[&\frac{1}{6}\sum_{i=1,2}(G(\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3)^{-1}\partial_\alpha\theta_i\partial_\beta\theta_i+\sin^2\theta_i\partial_\alpha\phi_i\partial_\beta\phi_i) \\
&+\frac{1}{9}(\partial_\alpha\psi+\cos\theta_1\partial_\alpha\phi_1+\cos\theta_2\partial_\alpha\phi_2)(\partial_\beta\psi+\cos\theta_1\partial_\beta\phi_1+\cos\theta_2\partial_\beta\phi_2) \\
&+\frac{\sin^2\theta_1\sin^2\theta_2}{324}(\hat{\gamma}_3\partial_\alpha\psi+\hat{\gamma}_1\partial_\alpha\phi_1+\hat{\gamma}_2\partial_\alpha\phi_2)(\hat{\gamma}_3\partial_\beta\psi+\hat{\gamma}_1\partial_\beta\phi_1+\hat{\gamma}_2\partial_\beta\phi_2)\Bigg]\,, \qquad \text{(B.6)}\\
L_B = 2\epsilon^{\alpha\beta}G(\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3)\Bigg[&\Bigg\{\hat{\gamma}_3\left(\frac{\sin^2\theta_1\sin^2\theta_2}{36}+\frac{\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1}{54}\right) \\
&\qquad-\hat{\gamma}_2\frac{\cos\theta_2\sin^2\theta_1}{54}-\hat{\gamma}_1\frac{\cos\theta_1\sin^2\theta_2}{54}\Bigg\}\partial_\alpha\phi_1\partial_\beta\phi_2 \\
&+\frac{(\hat{\gamma}_3\cos\theta_2-\hat{\gamma}_2)\sin^2\theta_1}{54}\partial_\alpha\phi_1\partial_\beta\psi-\frac{(\hat{\gamma}_3\cos\theta_1-\hat{\gamma}_1)\sin^2\theta_2}{54}\partial_\alpha\phi_2\partial_\beta\psi\Bigg]\,, \qquad \text{(B.7)}
\end{aligned}$$

with the following parameter identifications:

$$3\mu_1=\hat{\gamma}_1\,,\qquad 3\mu_2=\hat{\gamma}_2\,,\qquad -6\mu_3=\hat{\gamma}_3\,. \qquad \text{(B.8)}$$

Thus, the resulting metric and NS-NS two-form turn out to be (3.20) and (3.21), respectively.

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