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Reevaluating uncertainties in $\bar{B} \rightarrow X_s \gamma$ decay

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ABSTRACT: The rare decay $\bar{B} \rightarrow X_s \gamma$ is an important probe of physics beyond the standard model. The largest uncertainty on the total rate and the CP asymmetry arises from resolved photon contributions. These appear first at order $1/m_b$ and are related to operators other than $Q_{7\gamma}$ in the effective weak Hamiltonian. One of the three leading contributions, $Q_1^q - Q_{7\gamma}$, is described by a non-local function whose moments are related to HQET parameters. We use recent progress in our knowledge of these parameters to reevaluate the resolved photon contribution to $\bar{B} \rightarrow X_s \gamma$ total rate and CP asymmetry.

KEYWORDS: CP violation, Heavy Quark Physics, Nonperturbative Effects

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Contents

1	Introduction	1
2	Moments of g_{17}	3
2.1	Moments in ω_1 alone	3
2.2	Moments in ω alone	5
2.3	Moments in both ω_1 and ω	6
3	Applications	7
3.1	Current numerical values of moments	7
3.2	Resolved photon contributions for $Q_1^q - Q_{7\gamma}$	8
3.3	Λ_{17} estimates based on expanded penguin function	10
3.4	Modeling of h_{17}	11
3.4.1	One Hermite polynomial model	12
3.4.2	Sum of two Hermite polynomials model	12
3.4.3	Sum of three Hermite polynomials model	13
3.4.4	Sum of four Hermite polynomials model	14
3.4.5	Sum of five and six Hermite polynomials model	15
3.4.6	Summary	15
3.5	Phenomenological estimates	15
4	Conclusions and outlook	16
A	A useful identity	18

1 Introduction

The rare decay $\bar{B} \rightarrow X_s \gamma$ is an important probe of physics beyond the standard model. Both the CP averaged branching ratio and the CP asymmetry are used to constrain many models of new physics. The latest theoretical prediction for the branching ratio within the Standard Model (SM) is $(3.36 \pm 0.23) \times 10^{-4}$ [1] where a cut on the photon energy of $E_\gamma > 1.6 \text{ GeV}$ is assumed. This can be compared to the 2019 update of the 2018 PDG (Particle Data Group) experimental value of $(3.49 \pm 0.19) \times 10^{-4}$ [2]. See also the HFLAV (Heavy Flavor Averaging Group) values in [3]. For the Belle II experiment the uncertainty is expected to be reduced [4]. This motivates an effort to reduce the uncertainty on the theoretical prediction.

The largest source of uncertainty in the theoretical prediction, $\sim 5\%$, is non-perturbative effects in the form of resolved photon contributions. These arise when the

photon couples to light partons instead of coupling directly to the effective weak interaction. These effects were systematically studied in [5]. They first appear at power Λ_{QCD}/m_b and arise from the pairs of the weak hamiltonian operators $Q_1^q - Q_{7\gamma}$ (where $q = u, c$), $Q_{8g} - Q_{8g}$, and $Q_{7\gamma} - Q_{8g}$ [5]. While not identified as resolved photon contributions, non-perturbative effects from $Q_{8g} - Q_{8g}$ [6] and $Q_1^q - Q_{7\gamma}$ [7–10] were considered in the literature in the 1990's. Resolved photon contributions from $Q_{7\gamma} - Q_{8g}$ were only considered¹ in 2006 [11].

Resolved photon contributions have a more dramatic effect for the $\bar{B} \rightarrow X_s \gamma$ CP asymmetry due to the suppression of direct photon contributions in the SM [13]. In the SM the CP asymmetry is dominated by the resolved photon contributions from $Q_1^q - Q_{7\gamma}$. Including both direct and resolved photon effects the SM prediction of [13] is $-0.6\% < \mathcal{A}_{X_s \gamma}^{\text{SM}} < 2.8\%$. This can be compared to the 2019 update of the 2018 PDG experimental value of $1.5\% \pm 1.1\%$ [2]. See also the HFLAV values in [3].

In extensions of the SM in which there is a relative phase between the Wilson coefficients $C_{7\gamma}$ and C_1 or $C_{7\gamma}$ and C_{8g} , resolved photon contributions lead to new CP-violating effects [13]. In particular, effects from $Q_{7\gamma} - Q_{8g}$ depends on the flavor of the spectator quark inside the B meson. Such effects can be isolated by considering the difference between the CP asymmetries of charged and neutral B mesons: $\Delta \mathcal{A}_{X_s \gamma} \equiv \mathcal{A}_{X_s^- \gamma} - \mathcal{A}_{X_s^0 \gamma}$. In [13] it was shown that $\Delta \mathcal{A}_{X_s \gamma}$ can reach the level of 10% in magnitude for such models. This quantity was first measured by BaBar to be $\Delta \mathcal{A}_{X_s \gamma} = (5.0 \pm 3.9 \pm 1.5)\%$ [14]. Recently it was also measured by Belle to be $\Delta \mathcal{A}_{X_s \gamma} = (3.69 \pm 2.65 \pm 0.76)\%$ [15]. The PDG average of these results is $\Delta \mathcal{A}_{X_s \gamma} = (4.1 \pm 2.3)\%$ [2]. The measurements are dominated by the statistical uncertainty. Therefore, the upcoming Belle II experiment is expected to reduce the uncertainty [15].

The analysis of [5] for the total rate gives the following contributions to the uncertainty from resolved photon contributions. For $Q_1^c - Q_{7\gamma}$ it was $[-1.7, +4.0]\%$. For $Q_{8g} - Q_{8g}$ it was $[-0.3, +1.9]\%$. For $Q_{7\gamma} - Q_{8g}$ two values were given. One based on vacuum insertion approximation (VIA) $[-2.8, -0.3]\%$ and the other based on experimental data available at the time $[-4.4, +5.6]\%$. In particular the latter is related to Δ_{0-} , the isospin asymmetry of inclusive neutral and charged B decay to $X_s \gamma$. In 2010 only values from BaBar were available in [16, 17].

Recently Belle has also published a value for Δ_{0-} [15]. They obtain $\Delta_{0-} = [-0.48 \pm 1.49(\text{stat}) \pm 0.97(\text{syst}) \pm 1.15(f_{+-}/f_{00})]\%$, where the last uncertainty is due to the uncertainty on the production ratio of $B^+ B^-$ to $B^0 \bar{B}^0$ in $\Upsilon(4S)$ decays. The PDG average of [15–17] is $\Delta_{0-} = (-0.6 \pm 2.0)\%$. If we take the 95% confidence level experimental range of this average, as was done in [5], and translate it to the relative uncertainty via $-(1 \pm 0.3)\Delta_{0-}/3$ [5, 18] we obtain that the $Q_{7\gamma} - Q_{8g}$ uncertainty is $[-1.4, +2]\%$. This is more than a factor of two reduction compared to the 2010 analysis [5].

Can we better constrain other resolved photon contributions? For $Q_{8g} - Q_{8g}$ this seems difficult as little is known about the soft function it depends on. For $Q_1^q - Q_{7\gamma}$, the contribution depends on a soft function whose moments are related to Heavy Quark

¹See also [12] for a model-dependent treatment.

Effective Theory (HQET) parameters. These HQET parameters appear also for inclusive semileptonic B decays. The contributions to inclusive semileptonic B decays of HQET parameters corresponding to dimension 7 and 8 HQET operators with tree level coefficients² were classified in [21]. In 2016 the authors of [22] used moments of semileptonic B decay spectra and information based on the lowest-lying state saturation approximation in [21, 23] to perform a global fit of these HQET parameters. Using the results of [19] one can relate higher moments of the soft function to these HQET parameters. The goal of this paper is to use this new information to better constrain the $Q_1^q - Q_{7\gamma}$ resolved photon contribution to $\bar{B} \rightarrow X_s \gamma$ total rate and CP asymmetry.

The paper is structured as follows. In section 2 we derive general expressions for the moments of soft function in terms of HQET matrix elements. We relate such matrix elements of dimension 7 and 8 HQET operators to the HQET parameters of [21]. In section 3 we apply these relations to construct a systematic and improvable model of the soft function. We use the existing information about the HQET parameters to estimate the $Q_1^q - Q_{7\gamma}$ resolved photon contribution to $\bar{B} \rightarrow X_s \gamma$ total rate and CP asymmetry. We present our conclusions and outlook in section 4. In the appendix we derive a useful identity that is used in section 2.

2 Moments of g_{17}

The resolved photon contribution of $Q_1^q - Q_{7\gamma}$ to the $\bar{B} \rightarrow X_s \gamma$ spectrum depends on a non-perturbative soft function $g_{17}(\omega, \omega_1, \mu)$, see [5]. It is defined as

$$g_{17}(\omega, \omega_1, \mu) = \int \frac{dr}{2\pi} e^{-i\omega_1 r} \int \frac{dt}{2\pi} e^{-i\omega t} \quad (2.1)$$

$$\times \frac{\langle \bar{B} | (\bar{h} S_n)(tn) \not{n}(1 + \gamma_5) (S_n^\dagger S_{\bar{n}})(0) i\gamma_\alpha^\perp \bar{n}_\beta (S_{\bar{n}}^\dagger g G_s^{\alpha\beta} S_{\bar{n}})(r\bar{n}) (S_{\bar{n}}^\dagger h)(0) | \bar{B} \rangle}{2M_B},$$

where S_n and $S_{\bar{n}}$ are Wilson line defined in [5] and in the appendix. Intuitively we can think of ω as the momentum carried by the heavy quark and ω_1 as the momentum carried by the soft gluon. Moments of g_{17} can be expressed in terms of HQET matrix elements and related to HQET parameters extracted from inclusive semileptonic B decays. We consider three types of moments: moments in ω_1 alone, moments in ω alone, and moments in both ω_1 and ω . We derive general expressions for each type of moments. We then relate matrix elements of HQET operators up to and including dimension 8 to known HQET parameters.

2.1 Moments in ω_1 alone

We look at moments of the form

$$\langle \omega^0 \omega_1^k g_{17} \rangle \equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 \omega_1^k g_{17}(\omega, \omega_1, \mu) = \quad (2.2)$$

$$= (-1)^k \frac{1}{2M_B} \langle \bar{B} | (\bar{h} S_{\bar{n}})(0) \not{n}(1 + \gamma_5) i\gamma_\alpha^\perp \bar{n}_\beta (i\bar{n} \cdot \partial)^k (S_{\bar{n}}^\dagger g G_s^{\alpha\beta} S_{\bar{n}})(r\bar{n}) (S_{\bar{n}}^\dagger h)(0) | \bar{B} \rangle \Big|_{r=0}.$$

²When considering $\mathcal{O}(\alpha_s)$ coefficients, more operators appear [19, 20].

Using the identity, proven in the appendix, $i\bar{n} \cdot \partial \left(S_{\bar{n}}^\dagger(x) O(x) S_{\bar{n}}(x) \right) = S_{\bar{n}}^\dagger(x) [i\bar{n} \cdot D, O(x)] S_{\bar{n}}(x)$, we can express the moments as matrix elements of the local operators

$$\begin{aligned}
 \langle \omega^0 \omega_1^k g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 \omega_1^k g_{17}(\omega, \omega_1, \mu) = \\
 &= (-1)^k \frac{1}{2M_B} \langle \bar{B} | \bar{h} \not{n} (1 + \gamma_5) i\gamma_\alpha^\perp \bar{n}_\beta \underbrace{[i\bar{n} \cdot D, [i\bar{n} \cdot D, \dots [i\bar{n} \cdot D, g_s^{\alpha\beta}] \dots]]}_{k \text{ times}} h | \bar{B} \rangle = \\
 &= (-1)^k \frac{1}{2M_B} \langle \bar{B} | \bar{h} \not{n} (1 + \gamma_5) \gamma_\alpha^\perp \underbrace{[i\bar{n} \cdot D, [i\bar{n} \cdot D, \dots [i\bar{n} \cdot D, [iD^\alpha, i\bar{n} \cdot D] \dots]]}_{k \text{ times}} h | \bar{B} \rangle.
 \end{aligned} \tag{2.3}$$

In the last line we have used the identity $[iD^\mu, iD^\nu] = igG^{\mu\nu}$.

The Dirac structure of HQET matrix elements is simplified by using that $P_+ h = h$, where $P_+ \equiv (1 + \not{\psi})/2$. As was shown in [24], between two P_+ 's the Dirac basis reduces to four matrices: P_+ and $s^\lambda = P_+ \gamma^\lambda \gamma^5 P_+$. The matrices s^λ are a generalization of the Pauli spin matrices that satisfy $v \cdot s = 0$. This allows to simplify the Dirac structure.

Consider $\not{n} \gamma_\alpha^\perp$ first. Since \bar{n} and α are orthogonal, $\not{n} \gamma_\alpha^\perp = -i\sigma_{\mu\alpha\perp} \bar{n}^\mu$. The matrix $(-i\sigma_{\mu\nu})$ is related to s^λ via [24]

$$(-i\sigma_{\mu\nu}) \rightarrow \frac{1 + \not{\psi}}{2} (-i\sigma^{\mu\nu}) \frac{1 + \not{\psi}}{2} = iv^\rho \epsilon_{\rho\mu\nu\lambda} s^\lambda. \tag{2.4}$$

Note that this equation uses the convention $\epsilon_{0123} = -1$. Thus $\not{n} \gamma_\alpha^\perp \rightarrow iv^\rho \epsilon_{\rho\mu\alpha\perp\lambda} s^\lambda \bar{n}^\mu$. The Dirac structure $\not{n} \gamma^5 \gamma_\alpha^\perp$ can be simplified using the identity [24]

$$P_+ \Gamma P_+ = \frac{1}{2} P_+ \text{Tr} [P_+ \Gamma] - \frac{1}{2} s^\lambda \text{Tr} [P_+ s_\lambda P_+ \Gamma], \tag{2.5}$$

which gives $P_+ \not{n} \gamma^5 \gamma_\alpha^\perp P_+ \rightarrow -s^{\alpha\perp}$. We thus have

$$\begin{aligned}
 \langle \omega^0 \omega_1^k g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 \omega_1^k g_{17}(\omega, \omega_1, \mu) = \\
 &= (iv^\rho \epsilon_{\rho\mu\alpha\perp\lambda} \bar{n}^\mu - g_{\alpha\perp\lambda}) (-1)^k \frac{1}{2M_B} \langle \bar{B} | \bar{h} \underbrace{[i\bar{n} \cdot D, [i\bar{n} \cdot D, \dots [i\bar{n} \cdot D, [iD^\alpha, i\bar{n} \cdot D] \dots]]}_{k \text{ times}} s^\lambda h | \bar{B} \rangle.
 \end{aligned} \tag{2.6}$$

The tensors in the last line can be related to the $g_\perp^{\mu\nu}$ and $\epsilon_\perp^{\mu\nu}$ defined in [25], although we will not need such relations.

The nested commutator structure implies that odd moments vanish. The covariant derivative is a Hermitian operator. Odd number of commutators of Hermitian operators is a Hermitian operator whose forward matrix element is real. Since the spin-dependent matrix elements are imaginary, see [19], such matrix elements are zero. As was shown in [5], the integral over ω of $g_{17}(\omega, \omega_1, \mu)$ is symmetric in ω_1 which also implies that odd moments in ω_1 must vanish.

We use the general decomposition of HQET matrix elements presented in [19] to find the moments in ω_1 up to the third moment. We find no contribution from the structure

$\not{n}\gamma^5\gamma_\alpha^\perp$, as expected from [5], and that odd moments in ω_1 vanish. We have

$$\begin{aligned}
\langle \omega^0 \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = 4\tilde{a}^{(5)} = 2\lambda_2 = 2\mu_G^2/3 \\
\langle \omega^0 \omega_1^1 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 \omega_1 g_{17}(\omega, \omega_1, \mu) = 0 \\
\langle \omega^0 \omega_1^2 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 \omega_1^2 g_{17}(\omega, \omega_1, \mu) = 4 \left(-4\tilde{a}_{12}^{(7)} + 2\tilde{a}_{13}^{(7)} + 3\tilde{a}_{14}^{(7)} - \tilde{a}_{23}^{(7)} + \tilde{b}^{(7)} \right) = \\
&= \frac{2}{15} (5m_5 + 3m_6 - 2m_9) \\
\langle \omega^0 \omega_1^3 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 \omega_1^3 g_{17}(\omega, \omega_1, \mu) = 0.
\end{aligned} \tag{2.7}$$

The zeroth moment is a known result. The result for the second moment is new.

Here and in the following we are expressing the matrix elements in terms of the parameters $\tilde{a}_{ij}^{(k)}, \tilde{b}_{ij}^{(k)}, \tilde{c}_{ij}^{(k)}$ of [19], λ_2, ρ_2 of [24], and $\mu_G^2, \rho_{LS}^3, m_i, r_i$ of [21]. See [19] for definitions and relations between these parameters. Since $\tilde{a}_{ij}^{(k)}, \tilde{b}_{ij}^{(k)}, \tilde{c}_{ij}^{(k)}$ and λ_2, ρ_2 are defined in the heavy quark limit while $\mu_G^2, \rho_{LS}^3, m_i, r_i$ are not, there are $1/m_b$ differences between, e.g., λ_2 and $\mu_G^2/3$. We discuss these differences in section 3.1.

2.2 Moments in ω alone

We look at moments of the form

$$\begin{aligned}
\langle \omega^k \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^k \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = \\
&= \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^k \int \frac{dt}{2\pi} e^{-i\omega t} \frac{1}{2M_B} \langle \bar{B} | (\bar{h} S_n)(tn) \not{n}(1 + \gamma_5) S_n^\dagger(0) i\gamma_\alpha^\perp \bar{n}_\beta g G_s^{\alpha\beta}(0) h(0) | \bar{B} \rangle = \\
&= \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^k \int \frac{dt}{2\pi} e^{i\omega t} \frac{1}{2M_B} \langle \bar{B} | (\bar{h} S_n)(0) \not{n}(1 + \gamma_5) S_n^\dagger(tn) i\gamma_\alpha^\perp \bar{n}_\beta g G_s^{\alpha\beta}(tn) h(tn) | \bar{B} \rangle = \\
&= \int dt \delta(t) \frac{1}{2M_B} \langle \bar{B} | \bar{h}(0) S_n(0) (in \cdot \partial)^k S_n^\dagger(tn) \not{n}(1 + \gamma_5) i\gamma_\alpha^\perp \bar{n}_\beta g G_s^{\alpha\beta}(tn) h(tn) | \bar{B} \rangle.
\end{aligned} \tag{2.8}$$

where we have used the translation invariance of forward matrix elements of non-local operators. The identity $S_n^\dagger(x) in \cdot D S_n(x) = in \cdot \partial$ which follows from $in \cdot D S_n(x) = 0$ implies that $S_n^\dagger(tn) in \cdot D = in \cdot \partial S_n^\dagger(tn)$. This allows us to express the moments as

$$\begin{aligned}
\langle \omega^k \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^k \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = \\
&= (iv^\rho \epsilon_{\rho\mu\alpha\lambda} \bar{n}^\mu - g_{\alpha\lambda}) \frac{1}{2M_B} \langle \bar{B} | \bar{h} (in \cdot D)^k [iD^\alpha, in \cdot D] s^\lambda h | \bar{B} \rangle.
\end{aligned} \tag{2.9}$$

Notice that the location of $(in \cdot D)^k$ is determined by the Wilson lines in the n direction.

We use the general decomposition of HQET matrix elements presented in [19] to find the moments in ω up to the third moment. As before there is no contribution from the

structure $\not{n}\gamma^5\gamma_\alpha^\perp$. We have

$$\begin{aligned}
 \langle \omega^0 \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = 4\tilde{a}^{(5)} = 2\lambda_2 = 2\mu_G^2/3 \\
 \langle \omega^1 \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = -2\tilde{a}^{(6)} = -\rho_2 = -\rho_{LS}^3/3 \\
 \langle \omega^2 \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^2 \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = -2 \left(2\tilde{a}_{12}^{(7)} - \tilde{a}_{14}^{(7)} + \tilde{a}_{23}^{(7)} + \tilde{b}^{(7)} \right) = \\
 &= -\frac{1}{60} (20m_5 + 2m_7 + m_8) \\
 \langle \omega^3 \omega_1^0 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^3 \int_{-\infty}^{\infty} d\omega_1 g_{17}(\omega, \omega_1, \mu) = \\
 &= -2 \left(2\tilde{a}_{12}^{(8)} - \tilde{a}_{15}^{(8)} + \tilde{a}_{24}^{(8)} + 2\tilde{b}_{13}^{(8)} + \tilde{b}_{14}^{(8)} - \tilde{b}_{15}^{(8)} - 2\tilde{b}_{35}^{(8)} - \tilde{b}_{45}^{(8)} + \tilde{c}^{(8)} \right) = \\
 &= -\frac{1}{15} (5r_8 - r_9 + 2r_{10} + r_{11} - 2r_{12} - r_{13} + 2r_{15} - r_{16} + r_{17}) \quad (2.10)
 \end{aligned}$$

The first moment was derived in [5]. The results for the second and third moments are new.

2.3 Moments in both ω_1 and ω

Combining the derivations in the previous subsections, the expression for the mixed moments in ω_1 and ω is

$$\begin{aligned}
 \langle \omega^l \omega_1^k g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^l \int_{-\infty}^{\infty} d\omega_1 \omega_1^k g_{17}(\omega, \omega_1, \mu) = (iv^\rho \epsilon_{\rho\mu\alpha\lambda} \bar{n}^\mu - g_{\alpha\lambda}) (-1)^k \times \\
 &\times \frac{1}{2M_B} \langle \bar{B} | \bar{h} (i\bar{n} \cdot D)^l \underbrace{[i\bar{n} \cdot D, [i\bar{n} \cdot D, \dots [i\bar{n} \cdot D, [iD^\alpha, i\bar{n} \cdot D] \dots]]}_{k \text{ times}} s^\lambda h | \bar{B} \rangle. \quad (2.11)
 \end{aligned}$$

We use the general decomposition of HQET matrix elements presented in [19] to find the mixed moments in ω up to operators of dimension 8. These are

$$\begin{aligned}
 \langle \omega^1 \omega_1^1 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega \int_{-\infty}^{\infty} d\omega_1 \omega_1 g_{17}(\omega, \omega_1, \mu) = 2 \left(-4\tilde{a}_{12}^{(7)} + 2\tilde{a}_{13}^{(7)} + 3\tilde{a}_{14}^{(7)} - \tilde{a}_{23}^{(7)} + \tilde{b}^{(7)} \right) = \\
 &= \frac{1}{15} (5m_5 + 3m_6 - 2m_9) \\
 \langle \omega^2 \omega_1^1 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^2 \int_{-\infty}^{\infty} d\omega_1 \omega_1 g_{17}(\omega, \omega_1, \mu) = \\
 &= 2 \left(3\tilde{a}_{12}^{(8)} - \tilde{a}_{14}^{(8)} - 2\tilde{a}_{15}^{(8)} + \tilde{a}_{24}^{(8)} - 3\tilde{b}_{13}^{(8)} + \tilde{b}_{14}^{(8)} + 4\tilde{b}_{15}^{(8)} + 3\tilde{b}_{35}^{(8)} - \tilde{b}_{45}^{(8)} + \tilde{c}^{(8)} \right) = \\
 &= \frac{1}{15} (5r_8 - r_9 - 3r_{10} + r_{11} + 3r_{12} + 4r_{13} + 3r_{15} - 2r_{16} + r_{17} - r_{18}) \\
 \langle \omega^1 \omega_1^2 g_{17} \rangle &\equiv \int_{-\infty}^{\bar{\Lambda}} d\omega \omega^1 \int_{-\infty}^{\infty} d\omega_1 \omega_1^2 g_{17}(\omega, \omega_1, \mu) = \\
 &= 2 \left(3\tilde{a}_{12}^{(8)} - \tilde{a}_{14}^{(8)} - 2\tilde{a}_{15}^{(8)} + \tilde{a}_{24}^{(8)} + 3\tilde{b}_{13}^{(8)} + \tilde{b}_{14}^{(8)} - 2\tilde{b}_{15}^{(8)} + 2\tilde{b}_{34}^{(8)} - \tilde{b}_{35}^{(8)} + \tilde{b}_{45}^{(8)} - \tilde{c}^{(8)} \right) = \\
 &= \frac{1}{15} (-5r_8 + r_9 + 3r_{10} + r_{11} - r_{12} - 2r_{13} + 2r_{14} + 3r_{15} - 2r_{16} + r_{17} - r_{18}) \quad (2.12)
 \end{aligned}$$

As before, there is no contribution from the structure $\not{h}\gamma^5\gamma_\alpha^\perp$. All the results for these moments are new.

3 Applications

3.1 Current numerical values of moments

The HQET parameters arising from matrix elements of HQET operators up to dimension 8 were extracted from experimental data in 2016 [22]. The authors of [22] used moments of semileptonic B decay spectra and information based on the lowest-lying state saturation approximation in [21, 23] to perform a global fit of these HQET parameters. Based on the values and standard deviations given in table 2 of [22] the non-zero moments of g_{17} are

$$\begin{aligned}
 \langle \omega^0 \omega_1^0 g_{17} \rangle &= 0.237 \pm 0.040 \text{ GeV}^2 \\
 \langle \omega^0 \omega_1^2 g_{17} \rangle &= 0.15 \pm 0.12 \text{ GeV}^4 \\
 \langle \omega^1 \omega_1^0 g_{17} \rangle &= 0.056 \pm 0.032 \text{ GeV}^3 \\
 \langle \omega^2 \omega_1^0 g_{17} \rangle &= 0.015 \pm 0.021 \text{ GeV}^4 \\
 \langle \omega^3 \omega_1^0 g_{17} \rangle &= 0.008 \pm 0.011 \text{ GeV}^5 \\
 \langle \omega^1 \omega_1^1 g_{17} \rangle &= 0.073 \pm 0.059 \text{ GeV}^4 \\
 \langle \omega^2 \omega_1^1 g_{17} \rangle &= -0.034 \pm 0.016 \text{ GeV}^5 \\
 \langle \omega^1 \omega_1^2 g_{17} \rangle &= 0.027 \pm 0.014 \text{ GeV}^5,
 \end{aligned} \tag{3.1}$$

where we have added the error bars of individual HQET parameters in quadrature. We do not include correlations as none were given in table 2 of [22].

While the relative errors are large, the moments do give useful information. For example, the two extremal models used in [5] for h_{17} , defined in (3.2), have $\langle \omega^0 \omega_1^2 g_{17} \rangle = -0.31 \text{ GeV}^4$ and $\langle \omega^0 \omega_1^2 g_{17} \rangle = 0.49 \text{ GeV}^4$. Using the value above of $0.15 \pm 0.12 \text{ GeV}^4$ this corresponds to roughly a three standard deviations range, as opposed to the one standard deviation range in (3.1). Similarly, in figure 1 we compare the models of [5] for h_{17} that used a sum of two Hermite polynomials to the sum of two Hermite polynomial model for h_{17} (defined below) with the current extremal values of $\langle \omega^0 \omega_1^2 g_{17} \rangle$.

As was alluded to in the previous section, the parameters defined in [21] and listed in [22] use the full QCD b fields, while the matrix elements we need are defined in the heavy quark limit. This implies that there are $1/m_b$ differences between, e.g., λ_2 and $\mu_G^2/3$, see [26, 27]. Since the error bars in table 2 of [22] are rather large, one might question if this issue is numerically important. To test this, we compare the value of $\mu_G^2/3$ from [22] to extractions of λ_2 (which is defined in the heavy quark limit) from B and D meson spectroscopy.

We define $\Delta m_H = m_H^* - m_H$, where m_H (m_H^*) is a pseudo-scalar (vector) heavy meson containing a heavy quark of mass m_Q . The expression for Δm_H up to order $1/m_Q^2$ can be found³ in [26]. To extract λ_2 we use isospin-averaged meson mass data from the 2019 update of the 2018 PDG review [2]. At order $1/m_Q$, $\lambda_2 = \Delta m_H m_H/2$. Thus $\lambda_2 = 0.119 \pm$

³For consistency with the rest of our paper, we do not include the scale dependance of λ_2 which is an $\mathcal{O}(\alpha_s)$ effect.

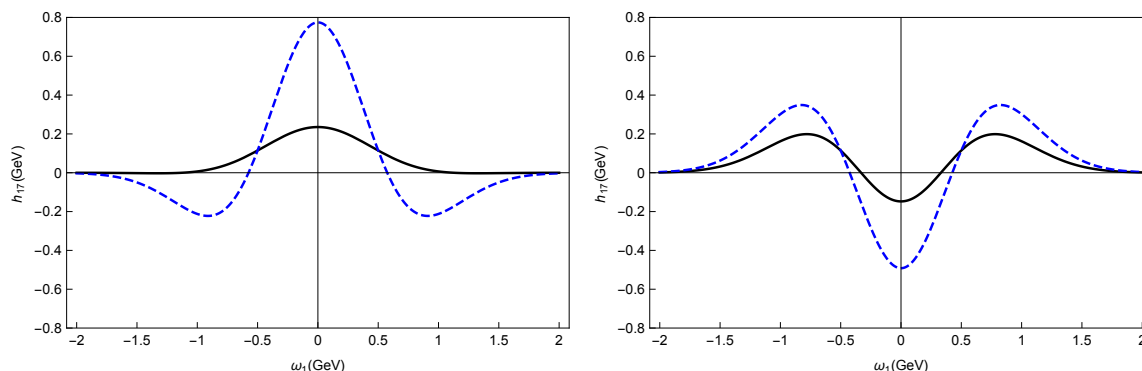


Figure 1. A comparison of the extremal models for h_{17} as a sum of two lowest even Hermite polynomials times a Gaussian of width 0.5 GeV used in [5] (dashed blue) to the same models allowed by current (2019) data (solid black). Left hand side: the model with 2010 smallest possible second moment of -0.31 GeV^4 compared to 2019 smallest possible second moment of 0.03 GeV^4 . Right hand side: the model with 2010 largest possible second moment of 0.49 GeV^4 compared to 2019 largest possible second moment of 0.27 GeV^4 .

0.001 GeV^2 from B -meson data, and $\lambda_2 = 0.13193 \pm 0.00002 \text{ GeV}^2$ from D -meson data. The errors are only from the meson masses. At order $1/m_Q^2$, $\lambda_2 = (\Delta m_B m_B^2 - \Delta m_D m_D^2)/(2m_B - 2m_D)$ [26]. Thus $\lambda_2 = 0.112 \pm 0.001 \text{ GeV}^2$. Comparing to $\mu_G^2/3 = 0.118 \pm 0.020 \text{ GeV}^2$ [22], we see that it is equal to all of these values of λ_2 within errors. Thus currently we cannot distinguish the two. To be conservative, we will use $\mu_G^2/3$ from [22]. We assume that a similar situation applies to other HQET parameters.

One would expect that in the future data from Belle II or Lattice QCD will allow to further constrain the HQET parameters and hence the moments of g_{17} .

3.2 Resolved photon contributions for $Q_1^q - Q_{7\gamma}$

The information about the moments presented above can be used to better constrain the resolved photon contribution of $Q_1^q - Q_{7\gamma}$. The observables we consider are the CP averaged rate and the CP asymmetry, both integrated over the photon energy $E_0 \leq E_\gamma \leq M_B/2$ where M_B is the B meson mass. As was discussed in [5], provided that $\Delta \equiv m_b - 2E_0$ is much larger than Λ_{QCD} , the contribution of $Q_1^q - Q_{7\gamma}$ is expressed in terms of the soft function

$$h_{17}(\omega_1, \mu) = \int \frac{dr}{2\pi} e^{-i\omega_1 r} \frac{\langle \bar{B} | (\bar{h} S_{\bar{n}})(0) \not{n} i\gamma_\alpha^\perp \bar{n}_\beta (S_{\bar{n}}^\dagger g G_s^{\alpha\beta} S_{\bar{n}})(r\bar{n}) (S_{\bar{n}}^\dagger h)(0) | \bar{B} \rangle}{2M_B}. \quad (3.2)$$

obtained from $g_{17}(\omega, \omega_1, \mu)$ by integrating over ω and omitting γ^5 [5].

For the CP averaged rate the quantity we are interested in is $\mathcal{F}_E(\Delta)$ corresponding to the relative theoretical uncertainty from the resolved photon contributions. As shown in [5] its $Q_1^u - Q_{7\gamma}$ part vanishes. Its $Q_1^c - Q_{7\gamma}$ part is

$$\mathcal{F}_E^{17} = \frac{C_1(\mu)}{C_{7\gamma}(\mu)} \frac{\Lambda_{17}(m_c^2/m_b, \mu)}{m_b}, \quad (3.3)$$

where

$$\Lambda_{17}\left(\frac{m_c^2}{m_b}, \mu\right) = e_c \operatorname{Re} \int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \left[1 - F\left(\frac{m_c^2 - i\varepsilon}{m_b \omega_1}\right) + \frac{m_b \omega_1}{12m_c^2} \right] h_{17}(\omega_1, \mu), \quad (3.4)$$

and $F(x) = 4x \arctan^2(1/\sqrt{4x-1})$. Assuming $\Delta \gg \Lambda_{\text{QCD}}$ allows to replace $g_{17}(\omega, \omega_1, \mu)$ by $h_{17}(\omega_1, \mu)$ and ignore the Δ dependence in $\mathcal{F}_E^{17}(\Delta)$. In [5] Λ_{17} was estimated to be in the range $-60 \text{ MeV} < \Lambda_{17} < 25 \text{ MeV}$. We would like to use the information about the moments to revisit this estimate.

For the CP asymmetry the quantity we are interested in is the $Q_1^q - Q_{7\gamma}$ part of the resolved photon contribution to the CP asymmetry $\mathcal{A}_{X_s\gamma}^{\text{res},17}$, defined as [13]

$$\mathcal{A}_{X_s\gamma}^{\text{res},17} = \frac{\pi}{m_b} \left\{ \operatorname{Im} \left[(1 + \epsilon_s) \frac{C_1}{C_{7\gamma}} \right] \tilde{\Lambda}_{17}^c - \operatorname{Im} \left[\epsilon_s \frac{C_1}{C_{7\gamma}} \right] \tilde{\Lambda}_{17}^u \right\}, \quad (3.5)$$

where

$$\tilde{\Lambda}_{17}^u = \frac{2}{3} h_{17}(0), \quad \tilde{\Lambda}_{17}^c = \frac{2}{3} \int_{4m_c^2/m_b}^{\infty} \frac{d\omega}{\omega} f\left(\frac{m_c^2}{m_b \omega}\right) h_{17}(\omega), \quad (3.6)$$

with

$$f(x) = 2x \ln \frac{1 + \sqrt{1-4x}}{1 - \sqrt{1-4x}}. \quad (3.7)$$

In [13] $\tilde{\Lambda}_{17}^u$ and $\tilde{\Lambda}_{17}^c$ were estimated to be in the range $-330 \text{ MeV} < \tilde{\Lambda}_{17}^u < +525 \text{ MeV}$ and $-9 \text{ MeV} < \tilde{\Lambda}_{17}^c < +11 \text{ MeV}$. We would like to use the information about the moments to revisit these estimates.

To do that, we will consider various approaches to estimate the hadronic uncertainty arising from the soft function h_{17} and its moments. We also take into account the uncertainty from the values of the charm and bottom quark masses. They appear in the functions $F(x)$ and $f(x)$ above.

As discussed in [5], the charm-quark mass enters as a running mass in charm-penguin diagrams with a soft gluon emission, which are characterized by a hard-collinear virtuality. As in [5] we use $m_c = m_c(\mu)$ defined in the $\overline{\text{MS}}$ scheme with $\mu = 1.5 \text{ GeV}$ for the CP averaged rate. As in [13] we use $m_c = m_c(\mu)$ defined in the $\overline{\text{MS}}$ scheme with $\mu = 2.0 \text{ GeV}$ for the CP asymmetry. We will comment on the choice of scales in the conclusions.

The 2019 update of the 2018 PDG listing has $m_c(m_c) = 1.27 \pm 0.02 \text{ GeV}$ [2]. This is an average of masses in other schemes converted to the $\overline{\text{MS}}$ scheme using two-loop QCD perturbation theory with $\alpha_s(\mu = m_c) = 0.38 \pm 0.03$ [2]. We use the same two-loop QCD perturbation theory [28] to find $m_c(1.5 \text{ GeV}) = 1.20 \pm 0.03 \text{ GeV}$ and $m_c(2.0 \text{ GeV}) = 1.10 \pm 0.03 \text{ GeV}$. This should be compared, for example, to the value used in [5] of 1.131 GeV derived based on a smaller value of $m_c(m_c)$ from [29] that was also used in [30, 31]. The change in the value of the charm quark mass tends to slightly change the size of Λ_{17} and $\tilde{\Lambda}_{17}^c$.

Following [5] we will use the value of the bottom quark in the shape function scheme [32]. The latest HFLAV [3] value is $m_b = 4.58 \pm 0.03 \text{ GeV}$. This should be compared to the value of 4.65 GeV used in [5].

3.3 Λ_{17} estimates based on expanded penguin function

The soft function h_{17} appears in Λ_{17} convoluted with a penguin function F that depends on the ratio m_c^2 to the anti-hard-collinear scale $m_b \omega_1$. For $x > 1/4$, $1 - F(x)$ has the expansion

$$1 - F(x) = -\frac{1}{12x} - \frac{1}{90x^2} - \frac{1}{560x^3} + \mathcal{O}\left(\frac{1}{x^4}\right) \quad (3.8)$$

Assuming that h_{17} has support for values of $\omega_1 \ll 4m_c^2/m_b \approx 1.2 - 1.3 \text{ GeV}$ we can expand the penguin function and express Λ_{17} in terms of the moments of h_{17} . From the definition of h_{17} it is clear that $\langle \omega^0 \omega_1^k g_{17} \rangle = \langle \omega_1^k h_{17} \rangle$. Thus

$$\Lambda_{17}^{\text{expanded}} = -\frac{e_c m_b^3}{560 m_c^6} \langle \omega^0 \omega_1^2 g_{17} \rangle + \dots = -6 \pm 5 \text{ MeV} + \dots, \quad (3.9)$$

where \dots denotes the contribution of higher moments in ω_1 . Odd moments in ω_1 vanish. The contribution of the zeroth moment in ω_1 is subtracted in (3.4) since it is traditionally not included in the resolved photon contributions. Its size is $-e_c m_b 2\lambda_2 / (12m_c^2) = -42 \pm 7 \text{ MeV}$. The uncertainty in (3.9) arises from $\langle \omega^0 \omega_1^2 g_{17} \rangle$, m_b , and m_c added in quadrature.

The size of the contribution of higher dimensional operators was a concern for the authors of [7–10]. They have noticed the numerical suppression arising from the expansion of the penguin function, see (3.8), but the lack of knowledge of the matrix elements prevented them from making conclusive statements. The new numerical information about the higher dimensional matrix elements allows us to address this issue for the first time.

The expansion of the penguin function generates a numerical suppression factor of ~ 50 between the first and third term⁴ in (3.8). Despite that, when combined with the second moment, the central value of the total contribution in (3.9), -6 MeV , is only suppressed by a factor of seven compared to the central value of the contribution proportional to the zeroth moment, -42 MeV . The smaller suppression is consistent with the power counting of $m_c^2 \sim m_b \Lambda_{\text{QCD}}$ which disfavors the expansion of the penguin function.

As in [5], it is instructive to look at Λ_{QCD}/m_b corrections to $\Lambda_{17}^{\text{expanded}}$. In [5] only the moment $\langle \omega \omega_1^0 g_{17} \rangle$ was considered. We are at the position now to consider other moments too. The starting point is the expression for Λ_{17} that includes the photon energy dependence beyond leading power in Λ_{QCD}/m_b [5]

$$\begin{aligned} \Lambda_{17}\left(\frac{m_c^2}{m_b}, \mu\right) &= e_c \text{Re} \int_{-\infty}^{\bar{\Lambda}} d\omega \int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \\ &\times \left\{ \left(\frac{m_b + \omega}{m_b}\right)^3 \left[1 - F\left(\frac{m_c^2 - i\varepsilon}{(m_b + \omega)\omega_1}\right) \right] + \frac{m_b \omega_1}{12m_c^2} \right\} g_{17}(\omega, \omega_1, \mu). \end{aligned} \quad (3.10)$$

We expand $F(x)$ as above and in ω/m_b and consider $1/m_b^n$ corrections to $\Lambda_{17}^{\text{expanded}}$ denoted by $\delta\Lambda_{17}^{(n)}$. By definition, $\Lambda_{17}^{\text{expanded}} = \delta\Lambda_{17}^{(0)}$. For $\delta\Lambda_{17}^{(1)}$ we have

$$\begin{aligned} \delta\Lambda_{17}^{(1)} &= -\frac{e_c}{3m_c^2} \langle \omega^1 \omega_1^0 g_{17} \rangle - \frac{e_c m_b}{18m_c^4} \langle \omega^1 \omega_1^1 g_{17} \rangle - \frac{3e_c m_b^2}{280m_c^6} \langle \omega^1 \omega_1^2 g_{17} \rangle + \dots \\ &= (-9 \pm 5 \text{ MeV}) + (-6 \pm 5 \text{ MeV}) + (-1 \pm 1 \text{ MeV}) + \dots = -16 \pm 7 \text{ MeV} + \dots \end{aligned} \quad (3.11)$$

⁴Recall that the second term combines with $\langle \omega^0 \omega_1 g_{17} \rangle$ which is zero.

We notice again a slow convergence in the series generated from the expansion of $F(x)$. Only in the third term can we see a suppression compared to the previous terms. Although nominally a Λ_{QCD}/m_b correction, $\delta\Lambda_{17}^{(1)}$ is comparable in size to $\Lambda_{17}^{\text{expanded}}$. Even if we add the contribution of $\langle\omega^0\omega_1^0g_{17}\rangle$ to $\Lambda_{17}^{\text{expanded}}$, $\delta\Lambda_{17}^{(1)}$ is only suppressed by a factor of three. The $\Lambda_{\text{QCD}}^2/m_b^2$ correction for $\Lambda_{17}^{\text{expanded}}$ is

$$\begin{aligned}\delta\Lambda_{17}^{(2)} &= -\frac{e_c}{2m_b m_c^2}\langle\omega^2\omega_1^0g_{17}\rangle - \frac{e_c}{9m_c^4}\langle\omega^2\omega_1^1g_{17}\rangle + \dots \\ &= (-0.8 \pm 1.1 \text{ MeV}) + (1.2 \pm 0.6 \text{ GeV}) + \dots = 0.4 \pm 1.3 \text{ MeV} + \dots\end{aligned}\tag{3.12}$$

Again we observe a slow convergence in the series generated from the expansion of $F(x)$. The overall magnitude in this case is consistent with a simple $\Lambda_{\text{QCD}}/m_b \sim 0.1$ expectation. Finally, the $\Lambda_{\text{QCD}}^3/m_b^3$ correction for $\Lambda_{17}^{\text{expanded}}$ is

$$\delta\Lambda_{17}^{(3)} = -\frac{e_c}{3m_b^2 m_c^2}\langle\omega^3\omega_1^0g_{17}\rangle = -0.06 \pm 0.08 \text{ MeV} + \dots\tag{3.13}$$

As for $\delta\Lambda_{17}^{(2)}$, we observe the expected order of magnitude reduction in going to the next term in Λ_{QCD}/m_b .

We see that numerically the Λ_{QCD}/m_b expansion for $\delta\Lambda_{17}$ works well with the exception of the first term. One can speculate that the vanishing of $\langle\omega^0\omega_1^1g_{17}\rangle$ makes the zeroth term in the expansion Λ_{17} of (3.10) smaller than it “should” be. Since in general for $l > 0$ the moments $\langle\omega^l\omega_1^k g_{17}\rangle$ do not vanish, there is no such suppression beyond the zeroth term. Adding the terms above linearly and their uncertainties in quadrature gives $\Lambda_{17}^{\text{expanded}} + \delta\Lambda_{17}^{(1)} + \delta\Lambda_{17}^{(2)} + \delta\Lambda_{17}^{(3)} = -22 \pm 9 \text{ MeV}$.

As was discussed in [5], the assumptions about the support of h_{17} and the resulting expansion of the penguin function are too restrictive. We turn to estimates that do not rely on this expansion.

3.4 Modeling of h_{17}

As was shown in [5], $h_{17}(\omega_1, \mu)$ is an even function. It also has a dimension of mass and in the heavy quark limit $-\infty \leq \omega_1 \leq \infty$. In modeling h_{17} it is beneficial to have a systematic expansion of h_{17} , e.g. in terms of a complete orthonormal set of basis functions. For the leading power shape function such an expansion was suggested in [33]. We will use an expansion in terms of Hermite polynomials multiplied by a Gaussian of width σ :

$$h_{17}(\omega_1, \mu) = \sum_n a_{2n} H_{2n}\left(\frac{\omega_1}{\sqrt{2}\sigma}\right) e^{-\frac{\omega_1^2}{2\sigma^2}}.\tag{3.14}$$

Since h_{17} is even, only even polynomials are needed. In the following we refer to these models by the numbers of Hermite polynomials they contain. Since the Hermite polynomials are orthogonal, the $2k$ -th moment of h_{17} only depends on the coefficients a_{2n} with $n \leq k$, for a given value of σ . In other words, the lack of knowledge of higher moments does not affect models that only use lower moments. For example, the zeroth moment of h_{17} only depends on a_0 and the second moment of h_{17} only depends on a_0 and a_2 . Conversely, we

can use the first $2k$ -th moments to determine a_{2n} with $n \leq k$. Using $\langle \omega^0 \omega_1^k g_{17} \rangle = \langle \omega_1^k h_{17} \rangle$ we have, for example, for a_0 and a_2

$$a_0 = \frac{\langle \omega_1^0 h_{17} \rangle}{\sqrt{2\pi}|\sigma|}, \quad a_2 = \frac{\langle \omega_1^2 h_{17} \rangle - \sigma^2 \langle \omega_1^0 h_{17} \rangle}{4\sqrt{2\pi}|\sigma|^3}. \quad (3.15)$$

To further constrain $h_{17}(\omega_1, \mu)$, we use the fact that it is a soft function. We limit its absolute value to 1 GeV, i.e. $|h_{17}(\omega_1, \mu)| \leq 1$ GeV and require, as in [5], that it should not have any significant structures, such as peaks or zeros, outside the range $|\omega_1| \leq 1$ GeV. This allows us to restrict the range of σ . For example, assuming a model of a sum of two Hermite polynomials, for given values of $\langle \omega_1^0 h_{17} \rangle$ and $\langle \omega_1^2 h_{17} \rangle$, the requirement on significant structures only for $|\omega_1| \leq 1$ GeV gives an upper bound on σ and the condition $|h_{17}(\omega_1, \mu)| \leq 1$ GeV gives a lower bound on σ . For example, assuming the central values for $\langle \omega_1^0 h_{17} \rangle = 0.237$ GeV² and $\langle \omega_1^2 h_{17} \rangle = 0.15$ GeV⁴ gives 0.27 GeV $< \sigma < 0.62$ GeV. For other values of $\langle \omega_1^0 h_{17} \rangle$ and $\langle \omega_1^2 h_{17} \rangle$ within their one standard deviation range, the range of σ can be larger, but we restrict σ to be less than 1 GeV. As we will see below, this does not affect our estimates in practice since the extremal values we obtain are for $\sigma < 1$ GeV anyway.

We consider models with one, and two Hermite polynomials whose coefficients are determined by the known moments as well as models with more Hermite polynomials whose coefficients depended on unknown moments.

3.4.1 One Hermite polynomial model

Since σ is not determined by the moments, a model with one Hermite polynomial can in principle be adjusted to fit both the zero and second moment of h_{17} . Notice from (3.15) and (3.1) that a_0 is never zero, so a one Hermite polynomial model must include H_0 . The one Hermite polynomial model is thus

$$h_{17}^{\text{model-1}}(\omega_1) = \frac{\langle \omega_1^0 h_{17} \rangle}{\sqrt{2\pi}|\sigma|} e^{-\frac{\omega_1^2}{2\sigma^2}}. \quad (3.16)$$

The second moment of $h_{17}^{\text{model-1}}$ implies $\sigma = \sqrt{\langle \omega_1^2 h_{17} \rangle / \langle \omega_1^0 h_{17} \rangle}$. This is also the condition for $a_2 = 0$ in (3.15).

Varying the zero and second moment within their one standard deviation ranges leads to values of σ that exceed 1 GeV. For example, this happens for $\langle \omega_1^2 h_{17} \rangle = 0.27$ GeV⁴ and for almost all values of $\langle \omega_1^0 h_{17} \rangle$ within its one standard deviation range. Based on the criterion above, we should reject such models. But even if we include them, the values of Λ_{17} , $\tilde{\Lambda}_{17}^u$, and $\tilde{\Lambda}_{17}^c$ we obtain are included in the ranges for the two Hermite polynomials model below. Thus we find $\Lambda_{17} \in [-8, -1]$ MeV, $\tilde{\Lambda}_{17}^c \in [0, 7.5]$ MeV, and $\tilde{\Lambda}_{17}^u \in [45, 220]$ MeV.

3.4.2 Sum of two Hermite polynomials model

A model that contains a sum of two Hermite polynomials for a given value of σ is determined by (3.15), i.e. by the zeroth and second moment of h_{17} . The values of a_0 and a_2 depend on σ , but for σ of the order of a few hundred MeVs they are typically of the order of 1 GeV and often smaller.

Numerically scanning over the one standard deviation range of the moments and the possible values of σ in increments of $\delta\sigma = 0.01$ GeV, and based on the restrictions above on h_{17} gives $\Lambda_{17} \in [-21, -1]$ MeV. The lower value is obtained for $\langle\omega_1^0 h_{17}\rangle = 0.197$ GeV², $\langle\omega_1^2 h_{17}\rangle = 0.27$ GeV⁴, $\sigma = 0.44$ GeV, $m_c = 1.17$ GeV, and $m_b = 4.61$ GeV. The upper value is obtained for $\langle\omega_1^0 h_{17}\rangle = 0.277$ GeV², $\langle\omega_1^2 h_{17}\rangle = 0.03$ GeV⁴, $\sigma = 0.14$ GeV, $m_c = 1.23$ GeV, and $m_b = 4.55$ GeV. Thus the extremal values are obtained for extremal values of the two moments, anti-correlated, and the extremal values of m_c and m_b , anti-correlated.

It is instructive to check the dependance on m_b and m_c . For example, consider the set $\langle\omega_1^0 h_{17}\rangle = 0.197$ GeV², $\langle\omega_1^2 h_{17}\rangle = 0.27$ GeV⁴, $\sigma = 0.44$ GeV that leads to $\Lambda_{17} = -21$ MeV. Changing $m_b = 4.61$ to $m_b = 4.55$ GeV while keeping $m_c = 1.17$ GeV changes Λ_{17} by +1 MeV. Thus the dependance on the value of m_b is rather mild. Changing $m_c = 1.17$ GeV to $m_c = 1.23$ GeV while keeping $m_b = 4.61$ GeV changes Λ_{17} by +6 MeV. Thus the dependance on the value of m_c is more pronounced. We will further comment on this in the conclusions.

Using the same method we can find the range of allowed values for $\tilde{\Lambda}_{17}^c$. We have $\tilde{\Lambda}_{17}^c \in [0, 10]$ MeV. The lower value is obtained for $\langle\omega_1^0 h_{17}\rangle = 0.277$ GeV², $\langle\omega_1^2 h_{17}\rangle = 0.03$ GeV⁴, $\sigma = 0.14$ GeV, $m_c = 1.13$ GeV, and $m_b = 4.55$ GeV. The upper value is obtained for $\langle\omega_1^0 h_{17}\rangle = 0.197$ GeV², $\langle\omega_1^2 h_{17}\rangle = 0.27$ GeV⁴, $\sigma = 0.58$ GeV, $m_c = 1.07$ GeV, and $m_b = 4.61$ GeV. Again the extremal values are obtained for extremal values of the two moments, anti-correlated, and the extremal values of m_c and m_b , anti-correlated.

Lastly, we consider $\tilde{\Lambda}_{17}^u$. Using the parameterization above we have the expression

$$\tilde{\Lambda}_{17}^u = \frac{2}{3} h_{17}(0) = \frac{3\sigma^2 \langle\omega_1^0 h_{17}\rangle - \langle\omega_1^2 h_{17}\rangle}{3\sqrt{2\pi}|\sigma|^3}. \quad (3.17)$$

Since both moments are positive within their one standard deviation range, we can easily make $h_{17}(0)$ negative by choosing a small value of σ . Thus the smallest value of $h_{17}(0)$ based on $|h_{17}(\omega_1, \mu)| \leq 1$ GeV is -1 GeV. For example, for the central values of $\langle\omega_1^0 h_{17}\rangle$ and $\langle\omega_1^2 h_{17}\rangle$, the value of $\sigma = 0.27$ GeV gives $h_{17}(0) = -1$ GeV. To make $h_{17}(0)$ reach its highest possible value, we can choose the smallest value of $\langle\omega_1^2 h_{17}\rangle$, 0.03 GeV⁴ and the largest value of $\langle\omega_1^0 h_{17}\rangle$, 0.277 GeV². The extremal value of $h_{17}(0) = 0.33$ GeV is obtained for $\sigma = \sqrt{\langle\omega_1^2 h_{17}\rangle / \langle\omega_1^0 h_{17}\rangle} = 0.33$ GeV. Based on this we find that $\tilde{\Lambda}_{17}^u \in [-660, 220]$ MeV.

3.4.3 Sum of three Hermite polynomials model

A model that contains a sum of three Hermite polynomials for a given value of σ requires the knowledge of the fourth moment of h_{17} :

$$a_4 = \frac{\langle\omega_1^4 h_{17}\rangle - 6\sigma^2 \langle\omega_1^2 h_{17}\rangle + 3\sigma^4 \langle\omega_1^0 h_{17}\rangle}{96\sqrt{2\pi}|\sigma|^5}. \quad (3.18)$$

The fourth moment is currently unknown since it is a matrix element of a dimension 9 operator. To asses the impact of such a moment if it were known, we assume the very conservative estimate of $[-0.3, 0.3]$ GeV⁶ for $\langle\omega_1^4 h_{17}\rangle$. This covers all the numerical ranges in (3.1) but with a different dimension of course. We still maintain the restrictions of the values, zeros, and extremal points of h_{17} to be below 1 GeV.

Numerically scanning over the one standard deviation range of the known zero and second moments, the range $[-0.3, 0.3] \text{ GeV}^6$ for the unknown fourth moment in increments of 0.05 GeV and the possible values of σ based on the restrictions above gives $\Lambda_{17} \in [-24, 3] \text{ MeV}$. The lower value is obtained for $\langle \omega_1^0 h_{17} \rangle = 0.277 \text{ GeV}^2$, $\langle \omega_1^2 h_{17} \rangle = 0.27 \text{ GeV}^4$, $\langle \omega_1^4 h_{17} \rangle = 0.3 \text{ GeV}^6$, $\sigma = 0.32 \text{ GeV}$, $m_c = 1.17 \text{ GeV}$, and $m_b = 4.61 \text{ GeV}$. The upper value is obtained for $\langle \omega_1^0 h_{17} \rangle = 0.237 \text{ GeV}^2$, $\langle \omega_1^2 h_{17} \rangle = 0.03 \text{ GeV}^4$, $\langle \omega_1^4 h_{17} \rangle = -0.1 \text{ GeV}^6$, $\sigma = 0.34 \text{ GeV}$, $m_c = 1.17 \text{ GeV}$, and $m_b = 4.61 \text{ GeV}$. The obtained range is only slightly different from the two Hermite polynomial model and reflects our generous range for the unknown fourth moment.

Similarly we find the range for $\tilde{\Lambda}_{17}^c$. The positive values are included in the range obtained for a sum of two Hermite polynomials. We also get negative values in the range $[-5.6, 0] \text{ MeV}$. The smallest value is obtained for $\langle \omega_1^0 h_{17} \rangle = 0.277 \text{ GeV}^2$, $\langle \omega_1^2 h_{17} \rangle = 0.03 \text{ GeV}^4$, $\langle \omega_1^4 h_{17} \rangle = -0.11 \text{ GeV}^6$, $\sigma = 0.34 \text{ GeV}$, $m_c = 1.07 \text{ GeV}$, and $m_b = 4.61 \text{ GeV}$.

Unlike the two Hermite polynomial model we can make $h_{17}(0)$ reach a value of 1 GeV . For example, taking the central values of the zeroth and second moment $\langle \omega_1^0 h_{17} \rangle = 0.237 \text{ GeV}^2$, $\langle \omega_1^2 h_{17} \rangle = 0.15 \text{ GeV}^4$ we find that for $\langle \omega_1^4 h_{17} \rangle = 0.1 \text{ GeV}^6$ and $\sigma = 0.25 \text{ GeV}$ $h_{17}(0) = 1 \text{ GeV}$. This result is not surprising. The moments are global properties of the function and it is hard to restrict using them values of the function at a single point. We conclude that for this model $\tilde{\Lambda}_{17}^u$ can be as large as 660 MeV , which is the largest value possible under the condition $|h_{17}(\omega_1, \mu)| \leq 1 \text{ GeV}$.

3.4.4 Sum of four Hermite polynomials model

To test how typical is the change from a model with two Hermite polynomials to a model with three Hermite polynomials, we consider a model with four Hermite polynomial. We assume again the very conservative estimate of $[-0.3, 0.3] \text{ GeV}^8$ for the sixth moment $\langle \omega_1^6 h_{17} \rangle$ that determines the coefficient of H_6 in (3.14). As in the three Hermite polynomials model we assume the range of $[-0.3, 0.3] \text{ GeV}^6$ for $\langle \omega_1^4 h_{17} \rangle$.

Scanning over the values of the fourth and sixth moment we find that the smallest value of Λ_{17} is -22 MeV , i.e. in the range we obtained for three Hermite polynomials. The highest value we obtain is 5 MeV for $\langle \omega_1^0 h_{17} \rangle = 0.277 \text{ GeV}^2$, $\langle \omega_1^2 h_{17} \rangle = 0.03 \text{ GeV}^4$, $\langle \omega_1^4 h_{17} \rangle = -0.1 \text{ GeV}^6$, $\langle \omega_1^6 h_{17} \rangle = -0.2 \text{ GeV}^8$, $\sigma = 0.29 \text{ GeV}$, $m_c = 1.17 \text{ GeV}$, and $m_b = 4.61 \text{ GeV}$. This should be compared to the maximum value of -1 MeV and 3 MeV for the two and three Hermite polynomial models, respectively.

For $\tilde{\Lambda}_{17}^c$ we find positive values that are already included in the ranges of the two and three Hermite polynomial models above. The smallest negative value we find for $\tilde{\Lambda}_{17}^c$ is -7 MeV for $\langle \omega_1^0 h_{17} \rangle = 0.277 \text{ GeV}^2$, $\langle \omega_1^2 h_{17} \rangle = 0.03 \text{ GeV}^4$, $\langle \omega_1^4 h_{17} \rangle = -0.1 \text{ GeV}^6$, $\langle \omega_1^6 h_{17} \rangle = -0.2 \text{ GeV}^8$, $\sigma = 0.29 \text{ GeV}$, $m_c = 1.07 \text{ GeV}$, and $m_b = 4.61 \text{ GeV}$.

Since $\tilde{\Lambda}_{17}^u$ obtains its smallest and largest possible values for the two and three Hermite polynomial models, there is no need to check the effect of the four Hermite polynomials model.

3.4.5 Sum of five and six Hermite polynomials model

One can consider continuing in this way and check models with five and even six Hermite polynomials, based on similar assumptions for the unknown moments. Namely, we assume that the unknown k -th moment is in the range $[-0.3, 0.3] \text{ GeV}^{k+2}$. Scanning over the ranges in increments of 0.1 GeV^{k+2} we find that there are no solutions that satisfy our requirements on $h_{17}(0)$. One reason is the fast growth of the value of $H_n(0)$. To maintain a value of $|h_{17}(0)| \leq 1 \text{ GeV}$ requires that the coefficient of $H_n(0)$ be increasingly smaller.

3.4.6 Summary

Using a two Hermite polynomial model we find $\Lambda_{17} \in [-21, -1] \text{ MeV}$, $\tilde{\Lambda}_{17}^c \in [0, 10] \text{ MeV}$, and $\tilde{\Lambda}_{17}^u \in [-660, 220] \text{ MeV}$. Using a three Hermite polynomial model and *assuming* the range $[-0.3, 0.3] \text{ GeV}^6$ for $\langle \omega_1^4 h_{17} \rangle$ we find $\Lambda_{17} \in [-24, 3] \text{ MeV}$. The range for $\tilde{\Lambda}_{17}^c$ can include values $\in [-5.6, 0] \text{ MeV}$. Also, $\tilde{\Lambda}_{17}^u$ can be as large 660 MeV , which is it the largest possible value based on our assumptions for h_{17} . Using the four Hermite polynomial model with similar assumptions on the fourth and sixth moments changes the highest value of Λ_{17} to 5 MeV and the lowest value of $\tilde{\Lambda}_{17}^c$ to -7 MeV .

Combining the results above and rounding to the closest integer we have $\Lambda_{17} \in [-24, 5] \text{ MeV}$, $\tilde{\Lambda}_{17}^c \in [-7, 10] \text{ MeV}$, and $\tilde{\Lambda}_{17}^u \in [-660, 660] \text{ MeV}$.

3.5 Phenomenological estimates

Based on the analysis above we can update the results of [5] and [13]. To highlight the changes in the uncertainties, we follow the same methodology in these papers to obtain the uncertainties on the total rate and the CP asymmetry.

For the total rate we use $\Lambda_{17} \in [-24, 5] \text{ MeV}$, equation (3.3), and the values $C_1(\mu) = 1.257$, $C_{7\gamma}(\mu) = -0.407$ (calculated at $\mu = 1.5 \text{ GeV}$), and $m_b = 4.58 \text{ GeV}$ to obtain

$$\mathcal{F}_E|_{17} \in [-0.3, +1.6] \% \quad (3.19)$$

This should be compared to the range $[-1.7, +4.0] \%$ in [5].

To obtain the total uncertainty we use $\mathcal{F}_E|_{88} \in [-0.3, +1.9] \%$ from [5]. For $\mathcal{F}_E|_{78}$, we can use either the VIA value from [5] $\mathcal{F}_E|_{78}^{\text{VIA}} \in [-2.8, -0.3] \%$ or the new experimental value discussed in the introduction $\mathcal{F}_E|_{78}^{\text{exp}} \in [-1.4, +2] \%$. Scanning over the various contributions gives

$$-3.4\% < \mathcal{F}_E(\Delta) < +3.2\% \quad (\text{using VIA}), \quad (3.20)$$

using the theoretical estimate for $\mathcal{F}_E|_{78}^{\text{VIA}}$. Compared to $-4.8\% < \mathcal{F}_E(\Delta) < +5.6\%$ (using VIA) in [5], the new estimate reduces the total error by about a third. Using the experimental estimate instead, the range is

$$-2.0\% < \mathcal{F}_E(\Delta) < +5.5\% \quad (\text{using } \Delta_{0-}). \quad (3.21)$$

Compared to $-6.4\% < \mathcal{F}_E(\Delta) < +11.5\%$ (using Δ_{0-}) in [5], the new estimate reduces the total error by about a half.

For the CP asymmetry we use $\tilde{\Lambda}_{17}^c \in [-7, 10]$ MeV and $\tilde{\Lambda}_{17}^u \in [-660, 660]$ MeV. The expressions in [13] can be used to find the resulting CP asymmetry. For example, the sum of the direct and resolved contributions to the CP asymmetry in the SM is [13]

$$\mathcal{A}_{X_s\gamma}^{\text{SM}} = \left(1.15 \times \frac{\tilde{\Lambda}_{17}^u - \tilde{\Lambda}_{17}^c}{300 \text{ MeV}} + 0.71 \right) \%, \quad (3.22)$$

where we use the same⁵ parameters as in [13]. Because of the increase in the allowed range for $\tilde{\Lambda}_{17}^u$, the allowed range for $\mathcal{A}_{X_s\gamma}^{\text{SM}}$ increases to $-1.9\% < \mathcal{A}_{X_s\gamma}^{\text{SM}} < 3.3\%$. This should be compared to the range $-0.6\% < \mathcal{A}_{X_s\gamma}^{\text{SM}} < 2.8\%$ in [13]. Similar updates can be applied to the resolved $Q_1^q - Q_{7\gamma}$ contributions to the CP asymmetry in extensions of the SM.

4 Conclusions and outlook

Resolved photon contributions limit the theoretical uncertainty achievable in $\bar{B} \rightarrow X_s\gamma$ decay. They give the dominant uncertainty ($\sim 5\%$) on the SM prediction of the total rate and give the dominant effect within the SM for the CP asymmetry. In this paper we have used recent progress in the knowledge of HQET parameters to reevaluate resolved photon contribution from the interference of Q_1^q ($q = u, c$) and $Q_{7\gamma}$.

This contribution depends on a soft function $g_{17}(\omega, \omega_1, \mu)$, defined in (2.1), whose moments are related to HQET parameters. In section 2 we presented such relations for general moments in ω alone, see (2.3), in ω_1 alone, see (2.8), and general mixed moments in ω and ω_1 , see (2.11). The general decomposition of HQET matrix elements presented in [19] allows to relate these matrix elements to the basis [21] of HQET parameters arising from operators of up to (and including) dimension 8 used inclusive semileptonic B decays.

In section 3 we presented several phenomenological applications for the moment relations. First, numerical values of HQET parameters arising from operators of up to (and including) dimension 8 were obtained in [22] from a global fit to semileptonic B decay spectra and information based on the lowest-lying state saturation approximation in [21, 23]. These allow to find numerical ranges for eight non-zero lowest moments of $g_{17}(\omega, \omega_1, \mu)$. Second, *assuming* a limited support for $h_{17}(\omega_1)$, defined in (3.2), we have used the values of the moments to estimate the $Q_1^c - Q_{7\gamma}$ contribution to the total rate. We have also investigated the convergence of the expansion of the penguin function in $m_b\omega_1/m_c^2$ and the resolved photon contribution in Λ_{QCD}/m_b . The former converges slowly, consistent with the power counting of $m_c^2 \sim m_b\Lambda_{\text{QCD}}$. The latter exhibits the expected power suppression with the exception of the first term which is accidentally small due to h_{17} being an even function. Third, since the assumption about the support of h_{17} is known to be too restrictive [5], we considered a systematic improvable model for h_{17} .

To construct the model we use properties of $h_{17}(\omega_1)$. It is an even function of ω_1 . While its argument formally takes values for $-\infty \leq \omega_1 \leq \infty$, it is a soft function, so one expects

⁵Using the values $C_1(\mu) = 1.204$, $C_{7\gamma}(\mu) = -0.378$, $m_c = 1.1$ GeV (calculated at $\mu = 2.0$ GeV), and $m_b = 4.58$ GeV changes 1.15 to 1.18 and 0.71 to 0.70 in (3.22). Because of rounding there is not change to the allowed range for $\mathcal{A}_{X_s\gamma}^{\text{SM}}$.

it not to have significant structures beyond $|\omega_1| \leq 1 \text{ GeV}$. Similarly we limit its values to $|h_{17}(\omega_1)| \leq 1 \text{ GeV}$. The model we use is to express $h_{17}(\omega_1)$ as sum of even Hermite polynomials multiplied by a Gaussian of width σ , see (3.14). The coefficients of the polynomials are determined by the moments of $h_{17}(\omega_1)$. A benefit of such an approach is that the lack of knowledge of higher moments does not affect models that only use lower moments. The value of σ can be constrained by the requirement on the support and values of $h_{17}(\omega_1)$.

The current numerical knowledge of the moments allows us to determine the coefficients of a sum of the first two even Hermite polynomials. Using the requirements above, we numerically scan over the values of σ to determine the extremal possible values for the parameters $\Lambda_{17}, \tilde{\Lambda}_{17}^c, \tilde{\Lambda}_{17}^u$, defined in section 3.2. We also consider models with higher number of even Hermite polynomials by assuming a conservative range for the unknown moments. Taking the envelopes of our results, we find the estimates for the parameters in section 3.4.6. For Λ_{17} and $\tilde{\Lambda}_{17}^c$ the new analysis finds a reduction in the allowed range compared to [5, 13]. For $\tilde{\Lambda}_{17}^u$ we find an increase in the allowed range compared to [13] since it depends on the value of $h_{17}(0)$ which is not well constrained by global properties like moments. In section 3.5 we give estimates for the total rate uncertainty and the SM prediction for the CP asymmetry from resolved photon effects from $Q_1^g - Q_{7\gamma}$. For the total rate the uncertainty is reduced, but for the CP asymmetry the uncertainty is increased.

We conclude with a discussion of possible future improvements. With the new moment information we get a better handle on hadronic effects. As a result, uncertainties from perturbative effects become more conspicuous. For example, following [5, 13] the scale μ for the Wilson coefficients and the charm quark mass was taken to be 1.5 GeV for the total rate and 2 GeV for the CP asymmetry. Because the resolved photon contribution are currently treated at leading order in α_s , the scale dependance is not controlled. In order to improve on that, one needs to calculate α_s corrections to the resolved $Q_1^c - Q_{7\gamma}$ contribution. Controlling the scale dependance can also help to better estimate the uncertainty from the value of the charm quark mass. See section 3.4.2 for an example of the numerical effect of the charm quark mass.

The modeling can improve considerably if the numerical value of matrix elements of dimension 9 HQET operators were known. The first step of classifying the possible spin-dependent dimension 9 operators can be easily carried out using the methods of [19]. With the expected Belle II data, it is conceivable that the work of [21] can be extended to this level and a similar analysis to [22] can be applied to the data.

Finally, we have considered quantities that are integrated over the photon energy. One can consider also the photon energy spectrum itself, or its moments. The moment information above can be used to model the $Q_1^g - Q_{7\gamma}$ resolved photon contribution in this case. This is left for a future work.

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A A useful identity

The Wilson line

$$S_{\bar{n}}(x) = \mathbf{P} \exp \left(ig \int_{-\infty}^0 du \bar{n} \cdot A_s(x + u\bar{n}) \right), \quad (\text{A.1})$$

obeys the equation $i\bar{n} \cdot D S_{\bar{n}}(x) = 0$, where $iD^\mu = i\partial^\mu + gA^\mu$, see, e.g., [34] for a derivation. Thus $i\bar{n} \cdot \partial S_{\bar{n}}(x) = -g\bar{n} \cdot A(x)S_{\bar{n}}(x)$. Taking the Hermitian conjugate of this identity gives $i\bar{n} \cdot \partial S_{\bar{n}}^\dagger(x) = S_{\bar{n}}^\dagger(x)g\bar{n} \cdot A(x)$. Consider now $i\bar{n} \cdot \partial \left(S_{\bar{n}}^\dagger(x)O(x)S_{\bar{n}}(x) \right)$, where $O(x)$ is an operator. Using the identities above we have

$$\begin{aligned} i\bar{n} \cdot \partial \left(S_{\bar{n}}^\dagger(x)O(x)S_{\bar{n}}(x) \right) &= \\ &= (i\bar{n} \cdot \partial S_{\bar{n}}^\dagger(x))O(x)S_{\bar{n}}(x) + S_{\bar{n}}^\dagger(x)(i\bar{n} \cdot \partial O(x))S_{\bar{n}}(x) + S_{\bar{n}}^\dagger(x)O(x)(i\bar{n} \cdot \partial S_{\bar{n}}(x)) \\ &= S_{\bar{n}}^\dagger(x)g\bar{n} \cdot A(x)O(x)S_{\bar{n}}(x) + S_{\bar{n}}^\dagger(x)(i\bar{n} \cdot \partial O(x))S_{\bar{n}}(x) - S_{\bar{n}}^\dagger(x)O(x)g\bar{n} \cdot A(x)S_{\bar{n}}(x) = \\ &= S_{\bar{n}}^\dagger(x)[g\bar{n} \cdot A(x), O(x)]S_{\bar{n}}(x) + S_{\bar{n}}^\dagger(x)[i\bar{n} \cdot \partial, O(x)]S_{\bar{n}}(x) = S_{\bar{n}}^\dagger(x)[i\bar{n} \cdot D, O(x)]S_{\bar{n}}(x). \end{aligned} \quad (\text{A.2})$$

In the last line we have used the identity $[i\bar{n} \cdot \partial, O(x)]f(x) = (i\bar{n} \cdot \partial O(x))f(x)$ for an arbitrary function $f(x)$. Thus we have the identity

$$i\bar{n} \cdot \partial \left(S_{\bar{n}}^\dagger(x)O(x)S_{\bar{n}}(x) \right) = S_{\bar{n}}^\dagger(x)[i\bar{n} \cdot D, O(x)]S_{\bar{n}}(x). \quad (\text{A.3})$$

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References

- [1] M. Misiak et al., *Updated NNLO QCD predictions for the weak radiative B-meson decays*, *Phys. Rev. Lett.* **114** (2015) 221801 [[arXiv:1503.01789](https://arxiv.org/abs/1503.01789)] [[INSPIRE](#)].
- [2] PARTICLE DATA GROUP collaboration, *Review of particle physics*, *Phys. Rev. D* **98** (2018) 030001 [[INSPIRE](#)].
- [3] HFLAV collaboration, *Averages of b-hadron, c-hadron and τ -lepton properties as of summer 2016*, *Eur. Phys. J. C* **77** (2017) 895 [[arXiv:1612.07233](https://arxiv.org/abs/1612.07233)] [[INSPIRE](#)].
- [4] BELLE-II collaboration, *The Belle II physics book*, [arXiv:1808.10567](https://arxiv.org/abs/1808.10567) [[INSPIRE](#)].
- [5] M. Benzke, S.J. Lee, M. Neubert and G. Paz, *Factorization at subleading power and irreducible uncertainties in $\bar{B} \rightarrow X_s \gamma$ decay*, *JHEP* **08** (2010) 099 [[arXiv:1003.5012](https://arxiv.org/abs/1003.5012)] [[INSPIRE](#)].
- [6] A. Kapustin, Z. Ligeti and H.D. Politzer, *Leading logarithms of the b quark mass in inclusive $B \rightarrow X_s \gamma$ decay*, *Phys. Lett. B* **357** (1995) 653 [[hep-ph/9507248](https://arxiv.org/abs/hep-ph/9507248)] [[INSPIRE](#)].
- [7] M.B. Voloshin, *Large $O(m_c^{-2})$ nonperturbative correction to the inclusive rate of the decay $B \rightarrow X_s \gamma$* , *Phys. Lett. B* **397** (1997) 275 [[hep-ph/9612483](https://arxiv.org/abs/hep-ph/9612483)] [[INSPIRE](#)].

- [8] Z. Ligeti, L. Randall and M.B. Wise, *Comment on nonperturbative effects in $\bar{B} \rightarrow X_s \gamma$* , *Phys. Lett. B* **402** (1997) 178 [[hep-ph/9702322](#)] [[INSPIRE](#)].
- [9] A.K. Grant, A.G. Morgan, S. Nussinov and R.D. Peccei, *Comment on nonperturbative $O(1/m_c^2)$ corrections to $\Gamma(\bar{B} \rightarrow X_s \gamma)$* , *Phys. Rev. D* **56** (1997) 3151 [[hep-ph/9702380](#)] [[INSPIRE](#)].
- [10] G. Buchalla, G. Isidori and S.J. Rey, *Corrections of order $\Lambda_{\text{QCD}}^2/m_c^2$ to inclusive rare B decays*, *Nucl. Phys. B* **511** (1998) 594 [[hep-ph/9705253](#)] [[INSPIRE](#)].
- [11] S.J. Lee, M. Neubert and G. Paz, *Enhanced non-local power corrections to the $\bar{B} \rightarrow X_s \gamma$ decay rate*, *Phys. Rev. D* **75** (2007) 114005 [[hep-ph/0609224](#)] [[INSPIRE](#)].
- [12] J.F. Donoghue and A.A. Petrov, *Is $B \rightarrow X_s \gamma$ equal to $b \rightarrow s \gamma$? Spectator contributions to rare inclusive B decays*, *Phys. Rev. D* **53** (1996) 3664 [[hep-ph/9510227](#)] [[INSPIRE](#)].
- [13] M. Benzke, S.J. Lee, M. Neubert and G. Paz, *Long-distance dominance of the CP asymmetry in $B \rightarrow X_{s,d} + \gamma$ decays*, *Phys. Rev. Lett.* **106** (2011) 141801 [[arXiv:1012.3167](#)] [[INSPIRE](#)].
- [14] BABAR collaboration, *Measurements of direct CP asymmetries in $B \rightarrow X_s \gamma$ decays using sum of exclusive decays*, *Phys. Rev. D* **90** (2014) 092001 [[arXiv:1406.0534](#)] [[INSPIRE](#)].
- [15] BELLE collaboration, *Measurements of isospin asymmetry and difference of direct CP asymmetries in inclusive $B \rightarrow X_s \gamma$ decays*, *Phys. Rev. D* **99** (2019) 032012 [[arXiv:1807.04236](#)] [[INSPIRE](#)].
- [16] BABAR collaboration, *Measurements of the $B \rightarrow X_s \gamma$ branching fraction and photon spectrum from a sum of exclusive final states*, *Phys. Rev. D* **72** (2005) 052004 [[hep-ex/0508004](#)] [[INSPIRE](#)].
- [17] BABAR collaboration, *Measurement of the $B \rightarrow X_s \gamma$ branching fraction and photon energy spectrum using the recoil method*, *Phys. Rev. D* **77** (2008) 051103 [[arXiv:0711.4889](#)] [[INSPIRE](#)].
- [18] M. Misiak, *$\bar{B} \rightarrow X_s \gamma$: current status*, *Acta Phys. Polon. B* **40** (2009) 2987 [[arXiv:0911.1651](#)] [[INSPIRE](#)].
- [19] A. Gunawardana and G. Paz, *On HQET and NRQCD operators of dimension 8 and above*, *JHEP* **07** (2017) 137 [[arXiv:1702.08904](#)] [[INSPIRE](#)].
- [20] A. Kobach and S. Pal, *Hilbert series and operator basis for NRQED and NRQCD/HQET*, *Phys. Lett. B* **772** (2017) 225 [[arXiv:1704.00008](#)] [[INSPIRE](#)].
- [21] T. Mannel, S. Turczyk and N. Uraltsev, *Higher order power corrections in inclusive B decays*, *JHEP* **11** (2010) 109 [[arXiv:1009.4622](#)] [[INSPIRE](#)].
- [22] P. Gambino, K.J. Healey and S. Turczyk, *Taming the higher power corrections in semileptonic B decays*, *Phys. Lett. B* **763** (2016) 60 [[arXiv:1606.06174](#)] [[INSPIRE](#)].
- [23] J. Heinonen and T. Mannel, *Improved estimates for the parameters of the heavy quark expansion*, *Nucl. Phys. B* **889** (2014) 46 [[arXiv:1407.4384](#)] [[INSPIRE](#)].
- [24] T. Mannel, *Higher order $1/m$ corrections at zero recoil*, *Phys. Rev. D* **50** (1994) 428 [[hep-ph/9403249](#)] [[INSPIRE](#)].
- [25] S.W. Bosch, M. Neubert and G. Paz, *Subleading shape functions in inclusive B decays*, *JHEP* **11** (2004) 073 [[hep-ph/0409115](#)] [[INSPIRE](#)].

- [26] M. Gremm and A. Kapustin, *Order $1/m_b^3$ corrections to $B \rightarrow X_c \ell \bar{\nu}$ decay and their implication for the measurement of $\bar{\Lambda}$ and λ_1* , *Phys. Rev. D* **55** (1997) 6924 [[hep-ph/9603448](#)] [[INSPIRE](#)].
- [27] M. Neubert, *Advanced predictions for moments of the $\bar{B} \rightarrow X_s \gamma$ photon spectrum*, *Phys. Rev. D* **72** (2005) 074025 [[hep-ph/0506245](#)] [[INSPIRE](#)].
- [28] A.J. Buras, *Weak Hamiltonian, CP-violation and rare decays*, in *Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session*, Les Houches, France, 28 July–5 September 1997, pg. 281 [[hep-ph/9806471](#)] [[INSPIRE](#)].
- [29] A.H. Hoang and A.V. Manohar, *Charm quark mass from inclusive semileptonic B decays*, *Phys. Lett. B* **633** (2006) 526 [[hep-ph/0509195](#)] [[INSPIRE](#)].
- [30] M. Misiak et al., *Estimate of $\mathcal{B}(\bar{B} \rightarrow X_s \gamma)$ at $O(\alpha_s^2)$* , *Phys. Rev. Lett.* **98** (2007) 022002 [[hep-ph/0609232](#)] [[INSPIRE](#)].
- [31] M. Misiak and M. Steinhauser, *NNLO QCD corrections to the $\bar{B} \rightarrow X_s \gamma$ matrix elements using interpolation in m_c* , *Nucl. Phys. B* **764** (2007) 62 [[hep-ph/0609241](#)] [[INSPIRE](#)].
- [32] S.W. Bosch, B.O. Lange, M. Neubert and G. Paz, *Factorization and shape function effects in inclusive B meson decays*, *Nucl. Phys. B* **699** (2004) 335 [[hep-ph/0402094](#)] [[INSPIRE](#)].
- [33] Z. Ligeti, I.W. Stewart and F.J. Tackmann, *Treating the b quark distribution function with reliable uncertainties*, *Phys. Rev. D* **78** (2008) 114014 [[arXiv:0807.1926](#)] [[INSPIRE](#)].
- [34] T. Becher, A. Broggio and A. Ferroglia, *Introduction to soft-collinear effective theory*, *Lect. Notes Phys.* **896** (2015) 1 [[arXiv:1410.1892](#)] [[INSPIRE](#)].