

Eleven-dimensional supergravity in 4D, $N = 1$ superspace

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ABSTRACT: We give a formulation of linearized 11D supergravity in 4D, $N = 1$ superspace keeping all eleven bosonic coordinates. The fields are fluctuations around $\mathbf{M} = \mathbf{R}^{4,4} \times Y$, where Y is a background Riemannian 7-manifold admitting a G_2 structure. We embed the 11D fields into superfield representations of the 4D, $N = 1$ superconformal algebra. These consist of the conformal graviton superfield, seven conformal gravitino superfields, a tensor hierarchy of superfields describing the 11D 3-form, and a non-abelian Kaluza-Klein vector multiplet gauging the tensor hierarchy by diffeomorphisms on Y . The quadratic action consists of the linearization of a superspace volume term and a Chern-Simons action for the gauged hierarchy coupled to the supergravity and gravitino superfields, and the full structure is fixed by superconformal and gauge invariance. When this action is projected to components, we recover the full linearized action of 11D supergravity.

KEYWORDS: Extended Supersymmetry, Supergravity Models, Superspaces, Differential and Algebraic Geometry

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1 Introduction

Superstring compactifications on Calabi-Yau three-folds give rise to effective four-dimensional supersymmetric theories which have been extensively studied ever since the seminal work of Candelas, Horowitz, Strominger and Witten [1]. As such, they have been an important tool to construct semi-realistic models of particle phenomenology. The massless spectrum is determined by topological properties of the Calabi-Yau manifolds, and if fields are massive, their masses are too large to be observed at least when the compactification scale is high.

More generally, supersymmetric string and M-theory backgrounds are obtained by compactifying on manifolds that admit parallel spinors if fluxes are ignored. Even though not directly related to semi-realistic models of particle physics, these backgrounds have also been extensively studied. Manifolds with parallel spinors admit a metric with special holonomy, and the choices of possible holonomy groups are enumerated on Berger's list (cf. e.g. [2]). The holonomy group $SU(3)$, which is relevant for compactifications on Calabi-Yau 3-folds, is only one entry on this list.

In this paper we continue to focus on compactifications of M-theory on G_2 -holonomy manifolds, building off previous work [3–6]. Our goal is to construct the complete manifestly 4D, $N = 1$ supersymmetric space-time action. By complete we mean including all fields (also massive ones) to any order in fluctuations about a background. Supersymmetry is kept explicit by working in superspace. This is a complex venture not free of problems. The phenomenological ones will not be addressed here (although we have attempted to present our results in a form useful for model building and applications). These are, of course, very important and M-theory compactifications on G_2 -holonomy manifolds might be a good framework to address vexing physics questions. So for example, it has been argued in ref. [7] that non-perturbative corrections to the space-time action could provide a natural framework to realize de Sitter space in M-(string-)theory. Or it might be interesting to understand what type of non-abelian gauge groups can be obtained once the G_2 -holonomy manifolds approach a singular limit. The list of interesting phenomenological applications can surely be further expanded and we leave these for future research. In this paper we resolve the theoretical hurdles instead.

Even though the construction of a manifestly supersymmetric space-time action is of interest in its own right, the original motivation came from ref. [8]. That paper was concerned with the fate of classical solutions of type II or M-theory of the form $\mathbf{R}^d \times M$, with M a G_2 -holonomy manifold, once perturbative corrections in α' or $1/r$ (r being the radius of M) are included. In particular, is space-time supersymmetry spoiled by these perturbative corrections, or is it possible to perturbatively modify the solution to preserve supersymmetry? Two approaches were taken. First, the supersymmetry variation of the internal components of the gravitino was analyzed, and it was shown that the internal metric could be corrected order-by-order to make the variation vanish, provided that a closed 4-form and a closed 5-form were exact at each order. No σ -model argument was found to explain why they should be exact. The second approach to the question was via the D-dimensional effective theory, either 3D, $N = 2$ for type II or 4D, $N = 1$ for M-theory, including the full Kaluza-Klein towers of 10D or 11D fields. In the language of this effective theory, supersymmetry can only be spoiled by the generation of superpotential or FI terms, but these can be ruled out by combinations of symmetry and holomorphy arguments (the latter relying on the structure of 4D, $N = 1$ off-shell superspace). In fact, a closer analysis of the F- and D-term equations reveals the existence of an exact 4-form and an exact 5-form at each order which were conjectured to correspond to those in the first part. The expectation was that the two approaches were related by two choices of regularization scheme in the σ -model.

However, the effective field theory analysis in ref. [8] was somewhat speculative because the theory had never really been constructed, even in the classical limit ($\alpha' \rightarrow 0$ or $r \rightarrow \infty$).

The arguments in ref. [8] (as is done there, we will phrase things in 4D, $N = 1$ language) involved only vector multiplets and chiral multiplets (and assumed that the 4D 2-forms C_{mni} , m, n being 4D indices, i being internal, can be dualized into scalars which sit inside the chiral multiplets), even though there is no consistent action involving these fields alone. Rather, the assumption was that in the full theory, including also Kaluza-Klein towers of spin- $\frac{3}{2}$ and spin-2 multiplets, the essential symmetry and holomorphy arguments would remain valid. There is no reason to believe that this assumption is false, but the absence of an explicit construction of the effective theory, even in the classical limit, is somewhat unsatisfying. This paper is part of a research program, the goal of which is the construction of the 4D, $N = 1$ effective theory that is the classical limit of the theory discussed in ref. [8]. Having it in hand would lift the arguments in that paper from speculative to concrete and would open the possibility of studying corrections to the theory, relations to σ -model regularization schemes, corrections to the geometry of the G_2 moduli space, and other interesting questions.

In refs. [3–5] we constructed two important terms of the space-time action in 4D, $N = 1$ superspace that, after reducing superfields to components, agreed partially with the space-time action obtained when 11D supergravity is compactified on a G_2 -structure manifold. Even though the superspace and Kaluza-Klein actions were strikingly close (for example, the complete non-linear potential for the metric scalars is reproduced exactly) there were still some differences. These appeared already at the level of field content. As it turned out, the map between the fields arising from the Kaluza-Klein reduction of 11D supergravity [9] and the components of the superfields in refs. [3–5] was not one-to-one: some of the latter did not have an M-theory interpretation. Even then, those fields with an M-theory interpretation were not correctly treated in superspace: when reduced to components some parts of the action in refs. [3–5] did not agree with the Kaluza-Klein result. To characterize these problematic terms, it is easiest to place 4D superfields into representations of the G_2 structure group. Then any kinetic term involving fields in the **7** representation of G_2 , which could be space-time scalars or gauge fields, did not agree with the Kaluza-Klein result.

In this paper we solve these two problems. Specifically, we show how to accommodate all component fields in the Kaluza-Klein reduction of 11D supergravity into 4D, $N = 1$ superfields without introducing superfluous component fields. As argued already in ref. [5], key to this analysis are the gravitino superfields and the new gauge symmetries they imply. As we explain below, this also gives the correct kinetic terms for the fields in the **7** representation of G_2 from superspace. In this paper we work to second order in fluctuations about a background given by four-dimensional Minkowski space times a G_2 -holonomy manifold. The non-linear analysis is tractable but will appear in ref. [10], since the details would distract from the main points of this paper.

In the next section, we begin with a review of the action as constructed in references [3–5]. In section 3, we linearize this action around a background of the form $\mathbf{R}^{4|4} \times Y$ where Y is a Riemannian 7-manifold with fixed G_2 structure. This is then completed by coupling to 4D, $N = 1$ conformal supergravity and the previously-missing seven conformal gravitino superfields to quadratic order. The gauge structure of this action allows for

a Wess-Zumino gauge in which only those component fields present in 11D supergravity remain. (Our result is summarized in section 3.4, to which the reader familiar with such superspace constructions can skip directly.) In section 4, we project the linearized action to these components, demonstrating explicitly the matching of the terms and their coefficients to the Kaluza-Klein result. We recapitulate the salient points of the resulting description of 11D supergravity in terms of $N = 1$ superfields in section 5 and outline extensions of this result. Some we are already exploring, such as the full action to linear order in the seven gravitino superfields. Many other applications have not yet been worked out, but we indicate a few directions for future work we find particularly promising. Finally, three appendices are included for completeness on technical and quantitative details of G_2 geometry (section A), linearized 4D, $N = 1$ supergravity (section B), and 4D, $N = 1$ conformal gravitino superfields (section C). (The analysis in appendix C was carried out for general gravitino representations and has applications to similar constructions in dimensions other than eleven.)

2 Review and overview

Our goal is to embed the components of 11D supergravity into 4D, $N = 1$ superfields and to construct the complete two-derivative 11D action in terms of them. In this section, we review those parts of the construction already worked out in refs. [3–5]. In section 2.1, we recall how to reduce the components of 11D supergravity to 4+7 dimensions. These are embedded into $N = 1$ superfields. The superfields accommodating the 11D 3-form give rise to a tensor hierarchy. In section 2.2 we recall that the structure of this tensor hierarchy uniquely fixes the superspace Chern-Simons action. Similarly, the Kähler action is strongly constrained, as we discuss in section 2.4. The resulting component action reproduces many terms of the 11D action. In section 2.5 we describe the remaining mismatch and how it will be resolved.

2.1 Decomposition of eleven-dimensional supergravity

The components of 11D supergravity consist of the frame field $e_{\mathbf{m}}^{\mathbf{a}}$, the gravitino field $\psi_{\mathbf{m}}^{\alpha}$, and the 3-form C_{mnp} . Here $\mathbf{m}, \mathbf{n} \dots = 0, \dots, 10$ are coordinate indices, $\mathbf{a}, \mathbf{b} \dots = 0, \dots, 10$ are tangent indices, and $\alpha, \beta \dots = 1, \dots, 32$ are Majorana spinor indices. The two-derivative 11D supergravity action is given by [11]

$$\begin{aligned} \kappa^2 S_{11} = & \frac{1}{2} \int d^{11}x e \left(R - \frac{1}{48} (F_{mnpq})^2 \right) - \frac{1}{12} \int C \wedge F \wedge F \\ & + \frac{1}{2} \int d^{11}x e \left(-i \bar{\psi}_{\mathbf{m}} \gamma^{mnp} \mathcal{D}_{\mathbf{n}} \psi_{\mathbf{p}} + \frac{1}{192} \bar{\psi}_{\mathbf{r}} \gamma^{\mathbf{r}} \gamma^{mnpq} \gamma^{\mathbf{s}} \psi_{\mathbf{s}} F_{mnpq} \right) + \dots \end{aligned} \tag{2.1}$$

where we suppress ψ^4 terms for simplicity. Here R and \mathcal{D} are the Ricci scalar and covariant derivative constructed from the frame and spin connection, and $F = dC$ is the 4-form field strength.

Locally we treat the 11D space-time as a direct product $X \times Y$ with X and Y describing 4- and 7-dimensional manifolds respectively. This involves decomposing coordinate indices

	metric	gravitino	3-forms	2-forms	(axial-)vectors	(pseudo-)scalars	spinors
g_{mn}	g_{mn}	—	—	—	g_{mi}	g_{ij}	—
1	1	0	0	0	7	28	0
C_{mnp}	—	—	C_{mnp}	C_{mni}	(C_{mij})	(C_{ijk})	—
1	0	0	1	7	21	35	0
ψ_m^α	—	$\psi_m^\alpha, \psi_{mi}^\alpha$	—	—	—	—	$\chi_{ij}^\alpha, \chi_{ijk}^\alpha$
1	0	1 + 7	0	0	0	0	21 + 35

Table 1. Component spectrum for the $11 \rightarrow 4 + 7$ split. The reduction of 11D supergravity fields gives rise to a mixture of forms of various degrees. After dualizing axial-vectors and 2-forms, we find a total of $7 + 21 = 28$ vectors, $28 + 7 = 35$ scalars, and 35 pseudo-scalars of $N = 8$ supergravity [13]. The 3-form is non-dynamical on-shell. Axial-vectors and pseudo-scalars are marked with parentheses.

as $\mathbf{m} \rightarrow m, i$ with m, n, \dots denoting $GL(4)$ indices and i, j, \dots denoting $GL(7)$ indices; tangent space indices as $\mathbf{a} \rightarrow a, \underline{i}$ with $a, b, \dots = 0, \dots, 3$ denoting $SO(1, 3)$ and $\underline{i}, \underline{j}, \dots = 1, \dots, 7$ denoting $SO(7)$ indices; and spinor indices as $\alpha \rightarrow (\alpha \otimes I, \dot{\alpha} \otimes I)$, with $\bar{\alpha}$ and $\dot{\alpha}$ denoting chiral and antichiral $Spin(1, 3) \cong SL(2; \mathbf{C})$ indices and I denoting a $Spin(7)$ index.

If all supersymmetries are kept on equal footing, it is natural to augment the $Spin(7)$ symmetry (which is an R-symmetry from the 4D perspective) to $SU(8)$ [12]. This matches the full R-symmetry group of 4D, $N = 8$ supergravity, both ungauged [13] and gauged [14], and permits the scalar fields of the theory to be interpreted as parameterizing an $E_{7(7)}/SU(8)$ coset space. Then the eight gravitini and 56 spin- $\frac{1}{2}$ fermions are grouped into irreducible representations of $SU(8)$. If instead the goal is to maintain manifest 4D, $N = 1$ supersymmetry, one of the eight gravitini $\psi_m^{\alpha I}$ must be separated from the rest. This can be achieved by taking

$$\psi_m^{\alpha I} = \psi_m^\alpha \eta^I + i \psi_{mi}^\alpha (\Gamma^i)^{IJ} \bar{\eta}_J, \quad (2.2)$$

where η^I is a fixed complex spinor that selects out the preferred $N = 1$ supersymmetry and $\Gamma^i = \Gamma^{\underline{j}} e_{\underline{j}}^i$ in terms of the $Spin(7)$ gamma matrix $\Gamma^{\underline{i}}$. The additional seven gravitini ψ_{mi}^α now carry a $GL(7)$ index. A similar decomposition applies to the other 56 fermions, leading to spinors χ_{ij}^α and χ_{ijk}^α , which are totally antisymmetric in their $GL(7)$ indices. Dealing with the internal components g_{ij} of the metric rather than with the internal vielbein, all fields can thus be chosen to carry $GL(7)$ indices rather than $SO(7)$ or $Spin(7)$ indices. We summarize this field content in table 1.

To count degrees of freedom it is convenient to dualize the seven space-time 2-forms C_{mni} into seven scalars. When combined with the 28 metric scalars g_{ij} , they can be arranged into a 3-form φ_{ijk} . Invertibility of the metric requires this 3-form to be non-degenerate in a suitable sense (reviewed in appendix A). When pulled back to Y , this condition defines a G_2 structure [15, 16]. Together with the 35 pseudo-scalars C_{ijk} , these gravitational scalars can be embedded into superspace as the lowest components of a chiral pseudo-scalar superfield Φ_{ijk} .

p	lowest component	constraints	prepotential	top component
0	$F_\alpha = D_\alpha F$	$\bar{D}^2 D F = 0$	$F = \frac{1}{2i}(\Phi - \bar{\Phi})$	$F_a = \sigma_a[D, \bar{D}]F$
1	$F_{\alpha\dot{a}} = (\sigma_a)_{\alpha\dot{a}}\bar{W}^{\dot{a}}$	$D\bar{W} = 0 \ \& \ \bar{D}\bar{W} = DW$	$W = -\frac{1}{4}\bar{D}^2 DV$	$F_{ab} = D\sigma_{ab}W + \text{c.c.}$
2	$F_{\alpha\dot{a}a} = (\sigma_a)_{\alpha\dot{a}}H$	$D^2 H = 0$	$H = \frac{1}{2i}(D\Sigma - \bar{D}\bar{\Sigma})$	$F_{abc} = \epsilon_{abcd}\sigma^d[D, \bar{D}]H$
3	$F_{\alpha\beta ab} = (\sigma_{ab})_{\alpha\beta}G$	$\bar{D}G = 0$	$G = -\frac{1}{4}\bar{D}^2 X$	$F_{abcd} = i\epsilon_{abcd}D^2G + \text{c.c.}$

Table 2. Conventional embedding of p -forms in closed superforms. The component p -forms are embedded into closed super $(p + 1)$ -form field strengths F as originally shown in [17]. Each field strength can be written in terms of an invariant scalar or spinor superfield. These satisfy constraints that can be solved in terms of prepotentials. Overall numerical constants have been neglected in the first and last columns.

Note, however, that we will not be dualizing the seven space-time 2-forms into scalars, because this procedure obscures the gauge structure of the tensor hierarchy. Nevertheless, we will still embed the 28 metric scalars and 35 pseudo-scalars into a chiral superfield Φ_{ijk} . Consequently there are $35 - 28 = 7$ additional scalars not accounted for in the list of fields obtained by dimensional reduction. We will later see that these additional seven scalars are pure gauge degrees of freedom. Similarly, we embed the remaining components of the 11D 3-form into an abelian tensor hierarchy of superfields [3]. This is a chain complex of superforms constructed from the de Rham complex on Y tensored with the super-de Rham complex on $\mathbf{R}^{4|4}$. The forms in this complex are charged under diffeomorphisms on Y and so couple to the non-abelian Kaluza-Klein vector field $\mathcal{A}_a^i = -e_a^i$. Any such complex of superforms, also known as a non-abelian tensor hierarchy, has a Chern-Simons-like invariant S_{CS} [4, 6]. We now turn to the details of this embedding.

2.2 3-form hierarchy and Chern-Simons action in superspace

Four-dimensional p -forms are embedded into superfields as summarized in table 2 [17, 18]. As a guiding principle it is useful to note that the gauge transformation of a p -form is formally identical to the field strength of a $(p - 1)$ -form and similarly for their Bianchi identities. In terms of super- p -forms¹ $F_{A_1\dots A_p}$, the lowest-dimension non-vanishing component is indicated in the first column. Analyzing the Bianchi identities, the higher components are found in terms of superspace derivatives of the lowest component [17, 18] (see also appendix A of ref. [6]).

This description of p -forms in 4D, $N = 1$ superspace can be extended to accommodate both the dependence on the additional seven coordinates [3] and the minimal coupling to the non-abelian gauge field of the Kaluza-Klein vector [4]. First, the non-abelian connection \mathcal{A}^i is added to the $N = 1$ superspace derivative,

$$\mathcal{D}_C = D_C - \mathcal{L}_{\mathcal{A}_C}, \tag{2.3}$$

where \mathcal{L} denotes the Lie derivative on Y . This can be separated into the de Rham differential ∂ on Y and the contraction operator ι using Cartan's formula, $\mathcal{L}_Y = \partial\iota_Y + \iota_Y\partial$.

¹Here A_1, A_2, \dots are 4D, $N = 1$ superspace indices.

The field strength $\mathcal{W}_\alpha{}^i$ of the non-abelian connection is defined by

$$[\mathcal{D}_a, \bar{\mathcal{D}}_{\dot{\alpha}}] = -(\sigma_a)_{\alpha\dot{\alpha}} \mathcal{L}\mathcal{W}^\alpha. \quad (2.4)$$

This definition implies the conditions [18–20]

$$\bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{W}_\alpha{}^i = 0 \quad \text{and} \quad \mathcal{D}^\alpha \mathcal{W}_\alpha{}^i = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}i}. \quad (2.5)$$

The invariant field strengths of the p -form hierarchy can be written in terms of prepotentials as

$$E = \partial\Phi \quad (2.6a)$$

$$F = \frac{1}{2i} (\Phi - \bar{\Phi}) - \partial V \quad (2.6b)$$

$$\mathcal{W}_\alpha = -\frac{1}{4} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha V + \partial \Sigma_\alpha + \iota_{\mathcal{W}_\alpha} \Phi \quad (2.6c)$$

$$H = \frac{1}{2i} (\mathcal{D}^\alpha \Sigma_\alpha - \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Sigma}^{\dot{\alpha}}) - \partial X - \omega(\mathcal{W}_\alpha, V) \quad (2.6d)$$

$$G = -\frac{1}{4} \bar{\mathcal{D}}^2 X + \iota_{\mathcal{W}^\alpha} \Sigma_\alpha. \quad (2.6e)$$

Here, Φ and Σ_α are chiral superfields while V and X are real unconstrained superfields. All fields are differential forms on Y . The composite superfield ω is the Chern-Simons superfield; for any p -form scalar superfield v ,

$$\omega(\mathcal{W}_\alpha, v) := \iota_{\mathcal{W}^\alpha} \mathcal{D}_\alpha v + \iota_{\bar{\mathcal{W}}_{\dot{\alpha}}} \bar{\mathcal{D}}^{\dot{\alpha}} v + \frac{1}{2} (\iota_{\mathcal{D}^\alpha \mathcal{W}_\alpha} v + \iota_{\bar{\mathcal{D}}_{\dot{\alpha}} \bar{\mathcal{W}}^{\dot{\alpha}}} v). \quad (2.7)$$

Its name derives from the fact that $-\frac{1}{4} \bar{\mathcal{D}}^2 \omega(\mathcal{W}_\alpha, v) = \iota_{\mathcal{W}^\alpha} \chi_\alpha$, where $\chi_\alpha = -\frac{1}{4} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha v$ is the field strength superfield of v , is a product of field strengths. (This is then the superspace analog of $d\omega \sim \mathcal{F} \wedge F$.)

The prepotential superfields transform covariantly under both non-abelian internal diffeomorphisms as well as the abelian tensor hierarchy gauge transformations

$$\delta\Phi = \mathcal{L}_\tau \Phi + \partial\Lambda \quad (2.8a)$$

$$\delta V = \mathcal{L}_\tau V + \frac{1}{2i} (\Lambda - \bar{\Lambda}) - \partial U \quad (2.8b)$$

$$\delta \Sigma_\alpha = \mathcal{L}_\tau \Sigma_\alpha - \frac{1}{4} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha U + \partial \Upsilon_\alpha + \iota_{\mathcal{W}_\alpha} \Lambda \quad (2.8c)$$

$$\delta X = \mathcal{L}_\tau X + \frac{1}{2i} (\mathcal{D}^\alpha \Upsilon_\alpha - \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Upsilon}^{\dot{\alpha}}) - \omega(\mathcal{W}_\alpha, U). \quad (2.8d)$$

The abelian part of the gauge transformation is parameterized by the superfields Λ_{ij} (chiral), U_i (real), and Υ_α (chiral) encoding the components of an eleven-dimensional super-2-form. The non-abelian parameter τ^i is a real superfield describing internal diffeomorphisms. The field strengths (2.6) are invariant under the abelian transformations but transform as p -forms under internal diffeomorphisms.

Having been given explicitly in terms of the prepotential superfields, the field strengths identically satisfy the Bianchi identities

$$0 = -\partial E \tag{2.9a}$$

$$\frac{1}{2i} (E - \bar{E}) = \partial F \tag{2.9b}$$

$$-\frac{1}{4} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha F = -\partial W_\alpha - \iota_{\mathcal{W}_\alpha} E \tag{2.9c}$$

$$\frac{1}{2i} (\mathcal{D}^\alpha W_\alpha - \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}) = \partial H + \omega(W, F) \tag{2.9d}$$

$$-\frac{1}{4} \bar{\mathcal{D}}^2 H = -\partial G - \iota_{\mathcal{W}^\alpha} W_\alpha \tag{2.9e}$$

$$\bar{\mathcal{D}}_{\dot{\alpha}} G = 0. \tag{2.9f}$$

Just as in 4D, $N = 1$, these identities are the superspace analogs of $dF = 0$ [6].

The Chern-Simons action $\int C \wedge F \wedge F$ can be embedded in superspace by first constructing the superspace analogue of the closed 8-form $F \wedge F$. When rewritten in 4+7 dimensions, this form generates a hierarchy of 4D p -forms with $8 - p$ internal indices for $p = 1$ through $p = 4$. These four p -forms are embedded into $N = 1$ superfields as [6]

$$\mathbb{F} = \frac{1}{2} (E + \bar{E}) F \tag{2.10a}$$

$$\mathbb{W}_\alpha = E W_\alpha - \frac{i}{4} \bar{\mathcal{D}}^2 (F \mathcal{D}_\alpha F) \tag{2.10b}$$

$$\mathbb{H} = \frac{1}{2} (E + \bar{E}) H + \omega(W, F) - i \mathcal{D}^\alpha F \iota_{\mathcal{W}_\alpha} F + i \bar{\mathcal{D}}_{\dot{\alpha}} F \iota_{\bar{\mathcal{W}}^{\dot{\alpha}}} F \tag{2.10c}$$

$$\mathbb{G} = E G + \frac{1}{2} W^\alpha W_\alpha - \frac{i}{4} \bar{\mathcal{D}}^2 (F H) \tag{2.10d}$$

with wedge products suppressed. Here, the Chern-Simons superform $\omega(W, F)$ is constructed analogously to (2.7) but with contraction replaced by wedge product:

$$\omega(W_\alpha, v) := W^\alpha \wedge \mathcal{D}_\alpha v + \bar{W}_{\dot{\alpha}} \wedge \bar{\mathcal{D}}^{\dot{\alpha}} v + \frac{1}{2} \mathcal{D}^\alpha W_\alpha \wedge v + \frac{1}{2} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \wedge v. \tag{2.11}$$

These satisfy the descent relations (2.9) expressing the fact that the superform $F \wedge F$ is closed in the extended super-de Rham complex provided F is [6]. Because of these closure relations, the action

$$\begin{aligned} S_{\text{CS}} &= \frac{1}{\kappa^2} \int d^4x \int_Y L_{\text{CS}}, \\ L_{\text{CS}} &= -\frac{i}{12} \int d^2\theta (\Phi \mathbb{G} + \Sigma^\alpha \mathbb{W}_\alpha) - \frac{1}{12} \int d^4\theta (V \mathbb{H} - X \mathbb{F}) + \text{h.c.} \end{aligned} \tag{2.12}$$

is invariant under the non-abelian tensor hierarchy transformations (2.8).

We should emphasize that this spectrum of superfields was chosen because it is the most straight-forward embedding of the physical p -forms arising from the decomposition of the eleven-dimensional 3-form. This raises the question of whether there is any freedom in this choice. This is unlikely to be the case because (assuming no duality transformations

	κ^2	$\int d^7y$	$\int d^4x$	$\int d^4\theta$	$\int d^2\theta$	\mathcal{D}_α	∂_i	$G(X)$	$H(\Sigma)$	$W_\alpha(V)$	$F(\Phi)$	$\mathcal{W}(\mathcal{V})$
Δ	0	0	-4	2	1	$\frac{1}{2}$	0	3(2)	$2(\frac{3}{2})$	$\frac{3}{2}(0)$	0(0)	$\frac{3}{2}(0)$
w	0	0	0	0	-2	-1	0	2(0)	0(1)	1(0)	0(0)	1(0)
d	-9	-7	-4	2	1	$\frac{1}{2}$	1	0(-1)	$0(-\frac{1}{2})$	$\frac{1}{2}(-1)$	0(0)	$\frac{1}{2}(-1)$

Table 3. Superconformal weights (Δ and w) and engineering dimension (d) of various objects.

have been performed on the eleven-dimensional spectrum) the physical components are those embedded here.² Although one could presumably dualize some (super)fields to other representations, doing so would alter the component spectrum. Moreover, we will show in section 4 that the choice made here recovers the linearized action of eleven-dimensional supergravity without any such dualization.

2.3 Symmetries of the superspace embedding

Eleven-dimensional supergravity is invariant under the global “trombone” scaling symmetry [21]. This means one may assign engineering dimension to each 11D component field so that the 11D Planck constant κ^2 appears as an overall factor, as we have already chosen in eq. (2.1). Of course, the 4D Chern-Simons action must retain this feature; but, moreover, it possesses a global 4D, $N = 1$ superconformal symmetry, which is due to its p -form origin. In table 3, we give the global conformal (Δ) and chiral $U(1)_R$ (w) weights as well as the engineering dimension (d) of the various fields and operators. They are useful when studying superconformal interactions beyond the linearized approximation and in constructing models with this field content.

2.4 Kähler action

The Chern-Simons action (2.12) contains at the component level the 11D Chern-Simons action as well as kinetic terms for the vector fields. But to recover the kinetic terms for the scalars and 2-forms, an appropriate Kähler term must be added. Keeping in mind the superconformal weights and engineering dimensions of table 3, the Kähler term can be specified up to an undetermined function \mathcal{H} as

$$S_K = -\frac{3}{\kappa^2} \int d^4x \int d^7y \int d^4\theta \sqrt{g(F)} (\bar{G}G)^{1/3} \mathcal{H}. \tag{2.13}$$

Here $g(F) = \det(g_{ij}(F))$ is the determinant of the Riemannian metric on Y obtained from the G_2 -structure 3-form by replacing φ_{ijk} with the superfield F_{ijk} for which it is the background value. This Hitchin-like functional [15, 16] serves here as a measure term necessary for covariance under internal diffeomorphisms. The factor of $(G\bar{G})^{1/3}$ is chosen to provide the appropriate superconformal weight. The remaining function \mathcal{H} must be

²While there are components in this description that are not physical, as we will explain in section 2.5, these are needed to compensate conformal transformations of the graviton and gravitino multiplets.

	3-forms	2-forms	vectors	scalars	spinors	auxiliaries
X	C_{mnp}	—	—	G	ζ^α	d_X
Σ_i^α	—	C_{mni}	—	H_i	ζ_i^α	—
V_{ij}	—	—	C_{mij}	—	χ_{ij}^α	d_{ij}
Φ_{ijk}	—	—	—	C_{ijk}, F_{ijk}	χ_{ijk}^α	f_{ijk}
\mathcal{Y}^i	—	—	g_{mi}	—	$\zeta^{\alpha i}$	d^i

Table 4. Component spectrum of Chern-Simons prepotentials. We present the component fields contributed by the gauged Chern-Simons hierarchy. As explained in the text, the bosons G , H_i , and seven of the F_{ijk} can all be removed by a choice of Wess-Zumino gauge. Similarly, the fermions ζ can be gauged away or vanish on-shell (by the auxiliary field ρ of the gravitino multiplet).

weight-less and transform as a scalar under internal diffeomorphisms. For a two-derivative action, it may depend only on F_{ijk} , G , and H_i and, in order to be a weight-less scalar, only in a specific combination:

$$\mathcal{H} = \mathcal{H}(x) \quad \text{with} \quad x := (\bar{G}G)^{-2/3} g^{ij} H_i H_j. \tag{2.14}$$

The uniqueness of 11D supergravity means that the function $\mathcal{H}(x)$ must be fixed by extended supersymmetry. In a forthcoming publication [10], we confirm this, give its explicit form, and explain its origin from 11D. As we will be working to quadratic order in this paper, we will need only the two lowest-order terms in the x -expansion. Reference [5] showed that matching certain kinetic terms in 11D supergravity as well as the scalar potential required that $\mathcal{H}(0) = 1$. We define the first correction

$$c := \mathcal{H}'(0). \tag{2.15}$$

Requiring invariance under linearized extended supersymmetry (section 3.3.3) will fix $c = -\frac{1}{4}$.

A few comments are in order. First, the chiral superfield G is closely related to the conformal compensator of 4D, $N = 1$ supergravity. Its conformal weight and its role in the action suggest its identification as Φ_0^3 where $\Phi_0 = e^\sigma$ is the chiral superfield compensator in old minimal supergravity. The fact that G is built from a real prepotential X rather than a complex one means that it is a constrained chiral multiplet: one of its auxiliary scalar fields is replaced by the dual of the 4-form field strength F_{mnpq} of eleven-dimensional supergravity. We review the construction of old minimal supergravity and this modification in appendix B.

2.5 Faithful embedding and compensating superfields

We have embedded the components of eleven-dimensional supergravity up through spin-1 into 4D, $N = 1$ superspace, but in doing so we have been forced to introduce additional components not present in the 11D spectrum of table 1. We list in table 4 the component fields of the various superfields of the non-abelian tensor hierarchy. Compared to the 11D spectrum, there are 16 additional scalar fields: two from the lowest component of the

	conformal vierbein	conformal gravitini	auxiliaries
H_a	e_m^a	ψ_m^α	d_m
Ψ_i^α	—	ψ_{mi}^α	$y_{mi}, t_{abi}^-, \rho_{\alpha i}$

Table 5. Component spectrum of superspin- $\frac{3}{2}$ and superspin-1 superfields.

chiral superfield G , seven from H_i , and another seven from F_{ijk} . The latter arise because, as mentioned at the end of section 2.1, F_{ijk} should encode only the 28 degrees of freedom of the internal metric. (Using the background value of F_{ijk} as the G_2 -structure 3-form, the scalars in F_{ijk} may be decomposed into G_2 representations as $\mathbf{35} = \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{1}$. The troublesome scalars are contained in the $\mathbf{7}$.) There is a similar surfeit of fermions. Aside from the expected 56 fermions χ_{ij} and χ_{ijk} , there are 15 more: ζ , ζ_i , and ζ^i . In addition to this overcounting of the spin ≤ 1 degrees of freedom, the $N = 1$ description does not yet encompass the spin- $\frac{3}{2}$ and spin-2 degrees of freedom.

It was proposed in [5] to introduce the graviton by minimally coupling to $N = 1$ conformal supergravity. This can be done either covariantly — by introducing an appropriate measure E and defining a curved space covariant derivative ∇_A to replace \mathcal{D}_A — or by explicitly coupling to the gravitational prepotential H^a [18, 20]. As we review in appendix B, this is a real, Lorentz 4-vector-valued superfield analogous to the Yang-Mills prepotential. Using the Pauli matrices $(\sigma_a)_{\alpha\dot{\alpha}}$, we can write any real vector as a Hermitian matrix $H_{\alpha\dot{\alpha}}$.³ Under linearized local superconformal transformations, this superfield transforms as

$$\delta H_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} L_\alpha - D_\alpha \bar{L}_{\dot{\alpha}}, \quad (2.16)$$

defining it as an irreducible superspin- $\frac{3}{2}$ representation. Its component field content is given in table 5 and consists of the 4D vierbein e_m^a , the $N = 1$ gravitino ψ_m^α , and the (non-propagating) $U(1)_R$ gauge field d_m . Due to the local superconformal symmetry, the trace of the graviton and the gamma-trace of the gravitino are absent and supplied instead by the compensator G via its bottom component $|G|$ and the fermion ζ . In addition, the phase of G is eaten by the $U(1)_R$ gauge field, becoming the massive vector auxiliary field of modified old minimal supergravity.

The remaining superfluous fields, corresponding to two sets of seven fermions ζ^i and ζ_i and the two sets of seven scalars in H_i and in the $\mathbf{7}$ projection of F_{ijk} may naturally be explained as compensators for various symmetries introduced by the $N = 1$ formalism but not present in the 11D dynamics. This is the interpretation advocated in [5], where it was demonstrated that the (complete non-linear) scalar potential of the component theory is reproduced under the assumption that the lowest components of H_i and G can be gauged to 0 and 1 respectively. Although it was not mentioned in [5], the correct normalizations for all spin-0 and spin-1 kinetic terms were also recovered except for those in the $\mathbf{7}$ of G_2 . Based on these facts, it was proposed that the inclusion of the superfields for the additional gravitini resolves the remaining $\mathbf{7}$ problem as well.

³In this work, we will freely switch back and forth between real 4-vectors and Hermitian matrices using $H_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^a H_a$ and $H_a = -\frac{1}{2} \bar{\sigma}_a^{\dot{\alpha}\alpha} H_{\alpha\dot{\alpha}}$. In such conversions, contractions give factors of -2 : $H_{\alpha\dot{\alpha}} H^{\alpha\dot{\alpha}} = -2H_a H^a$.

In section 3, we will finally prove these claims by explicitly constructing the linearized gravitino couplings and the associated additional gauge symmetries. We briefly sketch the mechanism here by reviewing the $N = 1$ multiplet of a single extra gravitino living purely in four dimensions; details are provided in appendix C. The 4D, $N = 1$ gravitino is described by a spinor superfield Ψ_α subject to the linearized gauge transformations [22]

$$\delta\Psi_\alpha = \Xi_\alpha + D_\alpha\Omega, \tag{2.17}$$

where Ξ_α is chiral and Ω is an unconstrained complex superfield. The physical content of this multiplet is sketched in table 5 and consists of the spin- $\frac{3}{2}$ gravitino, an auxiliary (non-propagating) complex vector field $y_m \neq \bar{y}_m$, an auxiliary anti-self-dual antisymmetric tensor t_{ab}^- , and an auxiliary fermion ρ_α . The large gauge freedom ensures that the gravitino may be taken to be gamma-traceless, while the vector field y_m is a gauge field subject to (complex) abelian gauge transformations. It is important to note that while the dimensions of the bosonic auxiliaries are such that they may appear quadratically in a two-derivative action, the auxiliary fermion must appear as a Lagrange multiplier.⁴

In eleven dimensions, we have seven such superfields $\Psi_{\alpha i}$. The fermionic Lagrange multipliers $\rho_{\alpha i}$ can kill the seven extra spinors $\zeta^{\alpha i}$, while $\zeta_{\alpha i}$ can provide the missing gamma-trace of the gravitini. Meanwhile, the complex gauge vectors y_{mi} can eat the remaining 14 extra bosonic degrees of freedom.

As we will show, this is manifested at the superfield level, where we must assign Ξ and Ω transformations to the prepotentials of section 2.2, and in doing so, it becomes apparent that some of their degrees of freedom may be removed. For example, $\Sigma_{\alpha i}$ must be chosen to transform as a Stückelberg field under Ξ as $\delta\Sigma_{\alpha i} = -\Xi_{\alpha i}$, which allows it be gauged away. These steps lead to the proper 11D spectrum. As we have mentioned, some of the kinetic terms in the 7 require correction terms. These terms (and only these terms) are expected to receive correction, since the complex vectors y_{mi} and the self-dual tensors t_{abi}^- naturally couple *only* to kinetic terms for the propagating fields; when these auxiliaries are integrated out, the kinetic terms in the 7 are modified. In appendix C, we demonstrate this explicitly for the linearized action.

3 Linearized eleven-dimensional superspace action

Because we do not know *a priori* the correct non-linear version of the gravitino transformation (2.17) or the corresponding matter field transformations, we cannot immediately couple the gravitino to the full action. One solution to this problem would be to dimensionally reduce 11D superspace to reconstruct the necessary $N = 1$ superfields. As we are interested only in the linearized action for the moment, a simpler and more expedient approach is to just bootstrap the necessary transformations via the Noether procedure.

⁴These features can be understood by considering the field content of 4D $N = 2$ conformal supergravity [23]. Then y_m corresponds to part of the $SU(2)_R$ gauge field, t_{ab}^- is the bosonic auxiliary field, and ρ corresponds to one of the fermionic auxiliaries χ_i . The additional $N = 2$ constituents fill out an $N = 1$ vector multiplet.

We begin by linearizing the prepotentials about a fixed (on-shell) $N = 1$ supersymmetric background. For simplicity, we will take the 4D space-time to be Minkowski (super)space but we will let Y have an arbitrary (but fixed) G_2 structure. Backgrounds being on-shell, the G_2 3-form $\varphi_{ijk}(y)$ must be closed and co-closed. We turn off all background flux, so that the background 3-form potential C_{ijk} vanishes. This fixes the background value of Φ to $\langle \Phi_{ijk} \rangle = i\varphi_{ijk}$. We take $\langle V_{ij} \rangle$, $\langle \Sigma_{\alpha i} \rangle$, and $\langle \mathcal{V}^i \rangle$ to vanish to eliminate any space-time flux. Because G is interpreted as a scale compensator, its background sets the Planck scale. In the normalization of (2.13), this corresponds to $\langle G \rangle = 1$, which amounts to setting $\langle X \rangle = \theta^2$.

To find the quadratic action describing fluctuations about this background, we replace

$$\Phi_{ijk} \rightarrow i\varphi_{ijk} + \tilde{\Phi}_{ijk}, \quad V_{ij} \rightarrow \tilde{V}_{ij}, \quad \Sigma_{\alpha i} \rightarrow \tilde{\Sigma}_{\alpha i}, \quad X \rightarrow \theta^2 + \tilde{X} \quad (3.1)$$

and work to second order in the tilded fields. The linearized field strengths of the tensor hierarchy are defined as

$$\tilde{F} = \frac{1}{2i} (\tilde{\Phi} - \tilde{\bar{\Phi}}) - \partial \tilde{V}, \quad (3.2a)$$

$$\tilde{W}_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha \tilde{V} + \partial \tilde{\Sigma}_\alpha, \quad (3.2b)$$

$$\tilde{H} = \frac{1}{2i} (D^\alpha \tilde{\Sigma}_\alpha - \bar{D}_{\dot{\alpha}} \tilde{\Sigma}^{\dot{\alpha}}) - \partial \tilde{X}, \quad (3.2c)$$

$$\tilde{G} = -\frac{1}{4} \bar{D}^2 \tilde{X}, \quad (3.2d)$$

and these obey the Bianchi identities

$$\partial \tilde{F} = \frac{1}{2i} \partial (\tilde{\Phi} - \tilde{\bar{\Phi}}), \quad (3.3a)$$

$$-\frac{1}{4} \bar{D}^2 D_\alpha \tilde{F} = -\partial \tilde{W}_\alpha, \quad (3.3b)$$

$$\frac{1}{2i} (D \tilde{W} - \bar{D} \tilde{\bar{W}}) = \partial \tilde{H}, \quad (3.3c)$$

$$-\frac{1}{4} \bar{D}^2 \tilde{H} = -\partial \tilde{G}, \quad (3.3d)$$

$$\bar{D}_{\dot{\alpha}} \tilde{G} = 0. \quad (3.3e)$$

For increased readability, we will drop the tildes from now on.

For the moment, we neglect the non-abelian gauge prepotential \mathcal{V}^i *except for its field strength* $\mathcal{W}_\alpha{}^i$. Then the second-order Kähler and Chern-Simons Lagrangians are

$$L_K = \sqrt{g} \int d^4\theta \left[-\frac{1}{3} G \bar{G} - \frac{1}{9} (F_1)^2 - \frac{1}{12} (F_7)^2 + \frac{1}{12} (F_{27})^2 + c H_i H_j g^{ij} - \frac{1}{18} (G + \bar{G}) \varphi^{ijk} F_{ijk} \right], \quad (3.4)$$

$$L_{CS} = \int d^2\theta \left[\frac{1}{4} G \partial \Phi \wedge \varphi - \frac{i}{8} \Phi \wedge \partial \Phi + \frac{1}{8} W^\alpha \wedge W_\alpha \wedge \varphi + \frac{i}{4} \partial \Sigma^\alpha \wedge \iota_{W_\alpha} \varphi \wedge \varphi - \frac{1}{24} \iota_{W^\alpha} \varphi \wedge \iota_{W_\alpha} \varphi \wedge \varphi \right] + \text{c.c.} \quad (3.5)$$

where c is the as-yet-undetermined constant (2.15). To derive this result, one needs the expression (A.12) for the Hitchin metric (A.11) on the space of 3-forms, which leads to the perturbative expansion

$$\begin{aligned} \frac{\sqrt{g(F)}}{\sqrt{g(F_0)}} &= 1 + \frac{1}{18} F_0^{ijk} F_{ijk} - \frac{1}{2} G_0^{ijk,mnp} F_{ijk} F_{mnp} + O(F^3) \\ &= 1 + \frac{1}{18} \varphi^{ijk} F_{\mathbf{1}ijk} - \frac{1}{2 \cdot 18} \left(-\frac{4}{3} F_{\mathbf{1}ijk}^2 - F_{\mathbf{7}ijk}^2 + F_{\mathbf{27}ijk}^2 \right) + O(F^3). \end{aligned} \quad (3.6)$$

Here the bold subscripts denote the projection of F_{ijk} onto the corresponding G_2 representations with the background $F_{0ijk} = \varphi_{ijk}$. The explicit form of the projectors is given in (A.4). These Lagrangians are incomplete: we require couplings to the explicit non-abelian gauge prepotential \mathcal{V}^i , the (conformal) supergravity prepotential $H_{\alpha\dot{\alpha}}$, and the gravitino superfield $\Psi_{\alpha i}$, which we work out sequentially in the next three subsections.

3.1 The non-abelian gauge prepotential

We pause to remind the reader of the distinction between covariant and chiral gauge transformations in $N = 1$ superspace. (For a more complete discussion, we refer to the textbooks [18–20].) In a covariant framework, gauge transformations involve unconstrained (usually real) parameters and the transformation rule of a superfield resembles that of its the bottom component. In addition, the superspace derivatives carry a connection, which transforms as a connection should, mirroring the structure of gauge theories in components. Chiral superfields are chiral with respect to the covariant derivative, which is why they may transform with an unconstrained gauge parameter. The gauge prepotential is not explicitly present; rather it is encoded in the covariant derivative and in the chiral superfields themselves.

Until this point, we have used a covariant framework for the non-abelian gauge transformations. The non-abelian gauge prepotential \mathcal{V}^i was already encoded in the Kähler (2.13) and Chern-Simons actions (2.12) through the covariant derivative (2.3) and the covariantly chiral superfields. We will need to work with the prepotential explicitly, since it will eventually be required to transform under the hidden supersymmetry. While it could be unpackaged from these objects, the easiest way to restore it in the second-order approximation is to use the Noether method. Since the only y -dependent background is $\langle \Phi_{ijk} \rangle = i\varphi_{ijk}$, the only superfields that transform under the linearized non-abelian gauge transformations are Φ_{ijk} and \mathcal{V}^i itself. Their transformations are⁵

$$\delta \mathcal{V}^i = \lambda^i + \bar{\lambda}^i, \quad \delta \Phi = 2i \mathcal{L}_\lambda \langle \Phi \rangle = -2 \mathcal{L}_\lambda \varphi, \quad \delta \bar{\Phi} = -2i \mathcal{L}_{\bar{\lambda}} \langle \bar{\Phi} \rangle = -2 \mathcal{L}_{\bar{\lambda}} \varphi, \quad (3.7)$$

where λ^i is chiral and $\bar{\lambda}^i$ is antichiral. Here and henceforth, Φ is chiral with respect to the flat superspace derivatives D_A . A straightforward calculation shows that the following terms must be added to L_K

$$L_K \ni \sqrt{g} \int d^4\theta \mathcal{V}^i \left[i \partial_i (G - \bar{G}) - \frac{1}{3} \varphi^{jkl} (\partial_{[i} \Phi_{jkl]} + \partial_{[i} \bar{\Phi}_{jkl]}) \right], \quad (3.8)$$

while L_{CS} is gauge-invariant on its own.

⁵One way to motivate these transformations is that in a convenient gauge, the covariantly chiral Φ is given by $e^{-i\mathcal{L}\mathcal{V}}\Phi$, which transforms as in (2.8) with $\tau^i = i(\lambda^i - \bar{\lambda})$.

3.2 Coupling to $N = 1$ supergravity

As we have already mentioned, the superfield G for the space-time 4-form field strength enters the action in a way that suggests it is a conformal compensator. For this reason, we ought to be able to construct a *locally* superconformal quadratic action by coupling to the (conformal) supergravity prepotential $H_{\alpha\dot{\alpha}}$. Local $N = 1$ superconformal transformations are encoded in an unconstrained spinor superfield L_α under which $H_{\alpha\dot{\alpha}}$ transforms as (2.16). G must also transform as

$$\delta G = -\frac{1}{4}\bar{D}^2 D^\alpha L_\alpha \quad \Rightarrow \quad \delta X = D^\alpha L_\alpha + \bar{D}_{\dot{\alpha}} \bar{L}^{\dot{\alpha}}, \quad (3.9)$$

which is consistent with the interpretation of G as the conformal compensator of (modified) old minimal supergravity. We review this formulation of 4D, $N = 1$ supergravity in appendix B. Covariantizing the $G\bar{G}$ term of L_K leads to the linearized (modified) old minimal supergravity Lagrangian (cf. eq. (B.5)),

$$\begin{aligned} L_K \ni \sqrt{g} \int d^4\theta \mathcal{L}_{\text{OMSG}}, \\ \mathcal{L}_{\text{OMSG}} = -\frac{1}{3}\bar{G}G + \frac{2i}{3}(G - \bar{G})\partial_a H^a - H^a \square H_a \\ + \frac{1}{8}D^2 H_a \bar{D}^2 H^a - (\partial_a H^a)^2 + \frac{1}{48}([D_\alpha, \bar{D}_{\dot{\alpha}}]H^{\alpha\dot{\alpha}})^2. \end{aligned} \quad (3.10)$$

Because we have set the background value of G to 1, we should no longer refer to the conformal or $U(1)_R$ weights of any quantities. However, since G does not carry engineering dimension, this remains a useful weight. In particular, L_α must carry engineering dimension $d = -\frac{3}{2}$. Then it is not possible to assign linearized L_α transformations to any of the other tensor hierarchy prepotentials on dimensional grounds. (Choosing $\delta\Sigma_{\alpha i} \propto \partial_i L_\alpha$ would violate chirality.)

What about the rest of L_K ? In order to covariantize the $(G + \bar{G})\varphi^{ijk}F_{ijk}$ term in (3.4), we can replace it with the combination

$$L_K \ni \sqrt{g} \int d^4\theta \left[-\frac{1}{18} \left(G + \bar{G} - \frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]H^{\alpha\dot{\alpha}} \right) \varphi^{ijk} F_{ijk} \right]. \quad (3.11)$$

Under L_α transformations, the part of F_{ijk} involving Φ_{ijk} drops out, but we are left with the piece involving $\partial_{[i}V_{jk]}$. This can be rearranged to the combination

$$\delta L_K \ni \sqrt{g} \int d^4\theta \left[-\frac{1}{2} \partial_i L^\alpha W_{\alpha jk} \varphi^{ijk} \right], \quad (3.12)$$

which is invariant under the abelian tensor hierarchy transformations. However, it cannot be countered by introducing a term involving $H_{\alpha\dot{\alpha}}$. Similarly, the term (3.8) we wish to add to restore gauge invariance under the non-abelian gauge group gives an L_α transformation that cannot be canceled. The most we can do is to covariantize it by replacing (3.8) with

$$L_K \ni \sqrt{g} \int d^4\theta \mathcal{V}^i \left[\partial_i (i(G - \bar{G}) - 2\partial_a H^a) - \frac{1}{3} \varphi^{jkl} (\partial_{[i} \Phi_{jkl]} + \partial_{[i} \bar{\Phi}_{jkl]}) \right], \quad (3.13)$$

so that the L_α transformation simplifies to

$$\delta L_K \ni \sqrt{g} \int d^4\theta \left[-i \partial_i L^\alpha \mathcal{W}_\alpha^i \right]. \quad (3.14)$$

The only other term in L_K that requires covariantization is the $(H_i)^2$ term, since the field strength H_i depends on $\partial_i X$ and X transforms according to (3.9). However, this again leads to an L_α transformation that cannot be countered by $H_{\alpha\dot{\alpha}}$ itself.

Finally, eleven-dimensional Lorentz invariance requires that we add to L_K the “mass” term for the conformal graviton [5]

$$\int d^4\theta (\partial_i H^a)^2 = - \int d^4\theta H^a \partial^i \partial_i H_a \quad (3.15)$$

normalized to combine with the $H^a \square H_a$ term in (3.10) to give the 11D d’Alembertian.⁶ The reason it must be *explicitly* included is that while mass terms are often generated by integrating auxiliary fields out of the component action, this term carries spin-2 and none of the auxiliary fields carry spin > 1 . However, this also leads to a $\partial_i L_\alpha$ term.

It does not seem possible to make L_K invariant under L_α transformations *just* by coupling to $H_{\alpha\dot{\alpha}}$. In fact, we encounter the same problem when taking the L_α transformation of the Chern-Simons Lagrangian L_{CS} , which can be written

$$\delta L_{CS} = \int d^4\theta \left[\frac{i}{2} \partial L^\alpha \wedge D_\alpha F \wedge \varphi + \text{c.c.} \right], \quad (3.16)$$

again with the same combination of $\partial_i L_\alpha$.

All of these problems have the same solution. While none of the other tensor hierarchy fields may transform under L_α , the conformal gravitino superfield can, as it carries the same superspin as L_α . In the next section, we will see how this works.

3.3 The conformal gravitino superfield

Now we will show how to incorporate the seven missing conformal gravitino superfields Ψ_i^α and construct their couplings to the “matter” superfields of the previous section. Being the least familiar multiplet in our construction, we give a self-contained presentation of the 4D, $N = 1$ “matter gravitino” in appendix C.

The conformal gravitino multiplets have a large linearized gauge transformation

$$\delta \Psi_{\alpha i} = \Xi_{\alpha i} + D_\alpha \Omega_i + 2i \partial_i L_\alpha, \quad i = 1, \dots, 7 \quad (3.17)$$

with chiral parameter Ξ and complex unconstrained Ω describing the irreducible superspin-1 multiplet: at the component level, Ψ contains only spins $\frac{3}{2}$ and 1. The inclusion of $\partial_i L_\alpha$ is necessary to counter all of the $\partial_i L_\alpha$ -dependent terms.⁷ We have already mentioned that the engineering dimension of L_α must be $d = -\frac{3}{2}$. This determines the engineering dimension of Ψ and Ξ to be $d = -\frac{1}{2}$ and that of Ω to be $d = -1$. We record these in table 6 along with the engineering dimensions of $H_{\alpha\dot{\alpha}}$ and \mathcal{V} .

⁶As we review in appendix B, the other quadratic terms for H_a in (3.10) vanish in the superspace Lorentz gauge (B.9).

⁷The factor of $2i$ in (3.17) just fixes the normalization and phase of $\Psi_{\alpha i}$.

	$H_{\alpha\dot{\alpha}}$	$\Psi_{\alpha i}$	\mathcal{V}^i	L_α	$\Xi_{\alpha i}$	Ω_i	ζ^i
d	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	-1

Table 6. Engineering dimensions. Engineering dimension (d) of the graviton, gravitini, and Kaluza-Klein prepotentials and their gauge parameters. Because $\langle G \rangle = 1$, we do not assign conformal and $U(1)_R$ weights.

3.3.1 Linearized Kähler action

First, let us see how the Lagrangian L_K may be made fully L_α -invariant. The only way to covariantize the mass term is to replace $\partial_i H^a$ with the combination [24]

$$P_{\alpha\dot{\alpha}i} := \frac{1}{2i} (\bar{D}_{\dot{\alpha}} \Psi_{\alpha i} + D_\alpha \bar{\Psi}_{\dot{\alpha}i}) - \partial_i H_{\alpha\dot{\alpha}}. \quad (3.18)$$

This is both L_α and Ξ -invariant. Thus the ΞL -covariantized mass term (3.15) becomes

$$\sqrt{g} \int d^4\theta g^{ij} P_{ai} P_j^a. \quad (3.19)$$

Similarly, to covariantize the $(H_i)^2$ term, we can define the combination

$$\mathbf{H}_i := H_i + \frac{1}{2i} (D^\alpha \Psi_{\alpha i} - \bar{D}_{\dot{\alpha}} \bar{\Psi}_i^{\dot{\alpha}}). \quad (3.20)$$

This effectively replaces $\Sigma_{\alpha i}$ in the definition of H_i with $\Sigma + \Psi$. This combination can also be made Ξ -invariant provided the 2-form gauge superfield shifts as⁸

$$\delta_\Xi \Sigma_{\alpha i} = -\Xi_{\alpha i}. \quad (3.21)$$

Countering the residual transformations (3.12) and (3.14) requires

$$L_K \ni \sqrt{g} \int d^4\theta \left[\frac{1}{2} \Psi_i^\alpha \left(\mathcal{W}_\alpha^i - \frac{i}{2} W_{\alpha jk} \varphi^{ijk} \right) - \frac{i}{4} \varphi^{ijk} \Psi_i^\alpha \partial_j \Psi_{\alpha k} + \text{c.c.} \right]. \quad (3.22)$$

The first term gives the necessary counter-terms, while the second ensures that the full combination is Ξ -invariant.

Including all of the terms, we have found

$$L_K = \sqrt{g} \int d^4\theta [\mathcal{L}_{\text{OMSG}} + P_{ai}^2 + c\mathbf{H}_i^2 + \mathcal{L}_Y + \mathcal{L}_\mathcal{V}] \quad (3.23)$$

consisting of the following parts:

- $\mathcal{L}_{\text{OMSG}}$ is the Lagrangian of (modified) old minimal supergravity in the quadratic approximation (3.10), see appendix B. This covariantizes the $G\bar{G}$ term in the quadratic action under L_α .

⁸Since these are superfields of the same type (both chiral), the Ξ parameter could be used to eliminate $\Sigma_{\alpha i}$; however, when we go to components, we will use the Ξ transformations to instead impose Wess-Zumino gauge for $\Psi_{\alpha i}$ (cf. section C.1).

- The P^2 term is the L -covariantization (3.19) of the graviton mass term (3.15) by the gravitino superfield.
- The quadratic term in the 3-form field strength H_i comes from expanding the function \mathcal{H} in (2.13). Here, H_i is covariantized to \mathbf{H}_i (3.20) when coupling to the gravitino. As discussed in section 2.4, $\mathcal{H}(x) = 1 + cx + O(x^2)$ for some real constant c (2.15). Requiring invariance under the Ω transformations that describe extended supersymmetry will determine $c = -\frac{1}{4}$.
- \mathcal{L}_Y consists of all terms arising from the expansion of the Riemannian volume density on Y ,

$$\mathcal{L}_Y = -\frac{1}{9}F_{1ijk}^2 - \frac{1}{12}F_{7ijk}^2 + \frac{1}{12}F_{27ijk}^2 - \frac{1}{18}(G + \bar{G} - \frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]H^{\dot{\alpha}\alpha})\varphi^{ijk}F_{1ijk}. \quad (3.24)$$

This contains not only the kinetic terms of the scalars but also the interaction term between the 4D trace of the metric and the volume modulus $\varphi^{ijk}F_{1ijk}$ of Y .

- Finally, \mathcal{L}_Y is the covariantization under non-abelian gauge transformations (i.e. the internal diffeomorphisms on Y)

$$\mathcal{L}_Y = \mathcal{V}^i \left[\partial_i (i(G - \bar{G}) - 2\partial_a H^a) - \frac{1}{3}\varphi^{jkl}(\partial_{[i}\Phi_{jkl]} + \partial_{[i}\bar{\Phi}_{jkl]}) \right]. \quad (3.25)$$

This gives the component coupling of the Kaluza-Klein gauge field, which appears when one extracts the connection from the covariant derivative (2.3).

3.3.2 Quadratic Chern-Simons action

Now let us make the Chern-Simons Lagrangian (3.5) invariant as well. It helps to first rewrite it as

$$L_{CS} = \int d^2\theta \left[-\frac{i}{8}\Phi \wedge \partial\Phi + \frac{1}{4}G \partial\Phi \wedge \varphi + \frac{1}{8}W^\alpha \wedge W_\alpha \wedge \varphi - \frac{1}{24}\iota_{W^\alpha}\varphi \wedge \iota_{W_\alpha}\varphi \wedge \varphi \right] + \text{c.c.} - \frac{1}{2} \int d^4\theta \partial H \wedge \iota_Y \varphi \wedge \varphi. \quad (3.26)$$

As before, we replace H_i with \mathbf{H}_i given by (3.20). This is not actually necessary for L_α invariance (since H_i appears under the Y differential), but it ensures Ξ -invariance of this term. Under an L_α transformation, we have already found (3.16), which can be canceled by adding the additional term

$$L_{CS} \ni -\frac{1}{4} \int d^4\theta [\Psi^\alpha \wedge D_\alpha F \wedge \varphi + \text{c.c.}] . \quad (3.27)$$

Remarkably, the Ξ transformation of this term precisely cancels that of the $(W_\alpha)^2$ term in (3.26). This feature is quite non-trivial and relies upon the Bianchi identity (3.3b). The

resulting Lagrangian

$$\begin{aligned}
 L_{\text{CS}} = & \int d^2\theta \left[-\frac{i}{8} \Phi \wedge \partial\Phi + \frac{1}{4} G \partial\Phi \wedge \varphi + \frac{1}{8} W^\alpha \wedge W_\alpha \wedge \varphi \right. \\
 & \left. - \frac{1}{24} \iota_{\mathcal{W}^\alpha} \varphi \wedge \iota_{\mathcal{W}_\alpha} \varphi \wedge \varphi \right] + \text{c.c.} \\
 & + \frac{1}{2} \int d^4\theta \left[\partial\mathbf{H} \wedge \iota_{\mathcal{V}} \varphi \wedge \varphi - \frac{1}{2} \Psi^\alpha \wedge D_\alpha F \wedge \varphi - \frac{1}{2} \bar{\Psi}_{\dot{\alpha}} \wedge \bar{D}^{\dot{\alpha}} F \wedge \varphi \right] \quad (3.28)
 \end{aligned}$$

is invariant under both L_α and Ξ_α . This can be rewritten with explicit indices as

$$\begin{aligned}
 L_{\text{CS}} = & \int d^2\theta \left[-\frac{i}{288} \epsilon^{ijklmnp} \Phi_{ijk} \partial_l \Phi_{mnp} + \frac{1}{24} G \tilde{\varphi}^{ijkl} \partial_i \Phi_{jkl} \right. \\
 & \left. + \frac{1}{32} \tilde{\varphi}^{ijkl} W_{ij}^\alpha W_{\alpha kl} + \frac{1}{4} \mathcal{W}^{\alpha i} \mathcal{W}_\alpha^j g_{ij} \right] + \text{c.c.} \\
 & + \int d^4\theta \left[-\partial_i \mathbf{H}_j \varphi^{ij} \mathcal{V}^k - \frac{1}{24} \Psi_i^\alpha D_\alpha F_{jkl} \tilde{\phi}^{ijkl} - \frac{1}{24} \bar{\Psi}_{\dot{\alpha} i} \bar{D}^{\dot{\alpha}} F_{jkl} \tilde{\phi}^{ijkl} \right]. \quad (3.29)
 \end{aligned}$$

The F-term contains both the G_2 superpotential $\Phi\partial\Phi$ and the kinetic terms for the vector multiplets.

3.3.3 Invariance under extended supersymmetry

We have not yet discussed the Ω part of the linearized gravitino transformation. Requiring gauge invariance of the combined Kähler-Chern-Simons action must fix the linearized Ω transformations of the other fields. These turn out to be

$$\delta\Phi_{ijk} = -\frac{i}{2} \tilde{\varphi}_{ijkl} \bar{D}^2 \bar{\Omega}^l, \quad (3.30a)$$

$$\delta V_{ij} = -\frac{i}{2} \varphi_{ijk} (\Omega^k - \bar{\Omega}^k), \quad (3.30b)$$

$$\delta \mathcal{V}^i = -\frac{1}{2} (\Omega^i + \bar{\Omega}^i), \quad (3.30c)$$

with $\Sigma_{\alpha i}$, X , and $H_{\alpha\dot{\alpha}}$ invariant. While they can be determined directly by requiring invariance of the action, the structure of the transformations (up to normalization) can be determined purely on the grounds of symmetry and a few observations. These transformations also fix the constant in (2.15) to $c = \mathcal{H}'(0) = -\frac{1}{4}$.

Let's briefly motivate why the structure of the Ω transformations must be of this form. Since they are linearized, they may contain only the background metric and φ_{ijk} in addition to Ω and its derivatives. The engineering dimension forbids any derivatives from appearing in δV_{ij} and $\delta \mathcal{V}^i$ while $\delta\Phi$'s dimension and chirality permit only \bar{D}^2 . To determine that \bar{D}^2 acts on $\bar{\Omega}$ rather than Ω (or both), one must recall that Ω appears in its defining transformation (3.17) under a D_α , so it can be shifted by an anti-chiral superfield. This is a superfield version of a gauge-for-gauge symmetry, and it is necessary so that the physical content of Ω (and Ξ) are precisely enough to adopt a proper Wess-Zumino gauge condition for Ψ . (We will discuss the physics of this in the next section.) This gauge-for-gauge symmetry is manifestly maintained in $\delta\Phi$ only for $\bar{D}^2\bar{\Omega}$, while for δV_{ij} and

$\delta\mathcal{V}^i$ the shift in $\bar{\Omega}$ can be countered by a certain non-abelian λ^i transformation combined with an abelian Λ_{ij} transformation. Finally, the phase in $\delta\Phi_{ijk}$ and the requirement that the imaginary part of Ω be used for δV_{ij} and the real part for $\delta\mathcal{V}^i$ can be determined by requiring invariance under space-time parity.

3.4 Assimilation and summary

We now collect all the terms we have worked out for the linearized eleven-dimensional supergravity action. This section summarizes our main result.

The complete action for eleven-dimensional supergravity (to quadratic order in fields) is given by the sum of the Kähler (3.23) and Chern-Simons actions (3.29). As we have emphasized, this action is invariant under a large set of superspace gauge transformations:⁹

1. The abelian tensor hierarchy transformations [3] (cf. (2.8))

$$\delta_{ATH}\Phi_{ijk} = 3\partial_{[i}\Lambda_{jk]} \quad (3.31a)$$

$$\delta_{ATH}V_{ij} = \frac{1}{2i}(\Lambda_{ij} - \bar{\Lambda}_{ij}) - 2\partial_{[i}U_{j]} \quad (3.31b)$$

$$\delta_{ATH}\Sigma_{\alpha i} = -\frac{1}{4}\bar{D}^2 D_{\alpha}U_i + \partial_i\Upsilon_{\alpha} \quad (3.31c)$$

$$\delta_{ATH}X = \frac{1}{2i}(D^{\alpha}\Upsilon_{\alpha} - \bar{D}_{\dot{\alpha}}\bar{\Upsilon}^{\dot{\alpha}}) . \quad (3.31d)$$

2. The non-abelian gauge transformations (i.e. the internal diffeomorphisms) with chiral parameter λ^i [3] (cf. (3.7))

$$\delta_{\lambda}\mathcal{V}^i = \lambda^i + \bar{\lambda}^i , \quad (3.32a)$$

$$\delta_{\lambda}\Phi_{ijk} = -6\partial_{[i}(\varphi_{jk]l}\lambda^l) . \quad (3.32b)$$

The transformation of Φ_{ijk} can be interpreted as a certain abelian Λ_{ij} transformation. This means one can choose to define a covariantized non-abelian transformation that is often easier to work with:

$$\delta'_{\lambda}\mathcal{V}^i = \lambda^i + \bar{\lambda}^i , \quad (3.33a)$$

$$\delta'_{\lambda}V_{ij} = -i\varphi_{ijk}(\lambda^k - \bar{\lambda}^k) . \quad (3.33b)$$

3. The superconformal gravitino transformations with chiral parameter $\Xi_{\alpha i}$ and complex parameter Ω^i

$$\delta_{\Xi\Omega}\Psi_{\alpha i} = \Xi_{\alpha i} + D_{\alpha}\Omega_i \quad (3.34a)$$

$$\delta_{\Xi\Omega}\Phi_{ijk} = \frac{1}{2i}\tilde{\varphi}_{ijkl}\bar{D}^2\bar{\Omega}^l \quad (3.34b)$$

$$\delta_{\Xi\Omega}V_{ij} = \frac{1}{2i}\varphi_{ijk}(\Omega^k - \bar{\Omega}^k) \quad (3.34c)$$

$$\delta_{\Xi\Omega}\Sigma_{\alpha i} = -\Xi_{\alpha i} \quad (3.34d)$$

$$\delta_{\Xi\Omega}\mathcal{V}^i = -\frac{1}{2}(\Omega^i + \bar{\Omega}^i) . \quad (3.34e)$$

⁹All fields transform under the full supergravity gauge group, but we are presenting only the non-vanishing linearized transformations here. The non-linear corrections are important for the full action, but we defer this to future work.

4. The local superconformal transformations with parameter L_α under which

$$\delta_L H_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} L_\alpha - D_\alpha \bar{L}_{\dot{\alpha}} \quad (3.35a)$$

$$\delta_L X = D^\alpha L_\alpha + \bar{D}_{\dot{\alpha}} \bar{L}^{\dot{\alpha}} \quad (3.35b)$$

$$\delta_L \Psi_{\alpha i} = 2i \partial_i L_\alpha. \quad (3.35c)$$

We present the complete linearized action invariant under these transformations in terms of D- and F-term integrals

$$S = \frac{1}{\kappa^2} \int d^4x d^7y \int d^4\theta L_D + \frac{1}{\kappa^2} \int d^4x d^7y \left[\int d^2\theta L_F + \text{h.c.} \right] \quad (3.36)$$

in 4D, $N = 1$ superspace extended to Y . Putting together the pieces of the previous sections, we find the quadratic Lagrangians to be

$$L_D = -H^a \square H_a + \frac{1}{8} D^2 H_a \bar{D}^2 H^a - (\partial_a H^a)^2 + \frac{1}{48} ([D_\alpha, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}})^2 - \frac{1}{3} \bar{G} G + \frac{2i}{3} (G - \bar{G}) \partial_a H^a \quad (3.37a)$$

$$- \frac{1}{18} (G + \bar{G} - \frac{1}{2} [D_\alpha, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}}) \varphi^{ijk} F_{ijk} - \frac{1}{9} F_{1ijk}^2 - \frac{1}{12} F_{7ijk}^2 + \frac{1}{12} F_{27ijk}^2 \quad (3.37b)$$

$$- \frac{1}{2} \left[\partial_i H_{\alpha\dot{\alpha}} - \frac{1}{2i} (\bar{D}_{\dot{\alpha}} \Psi_{\alpha i} + D_\alpha \bar{\Psi}_{\dot{\alpha} i}) \right]^2 - \frac{1}{4} \left[H_i + \frac{1}{2i} (D^\alpha \Psi_{\alpha i} - \bar{D}_{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha} i}) \right]^2 \quad (3.37c)$$

$$+ \frac{1}{2} \left\{ \Psi_i^\alpha \left[\mathcal{W}_\alpha^i - \frac{i}{2} \varphi^{ijk} (\partial_j \Psi_{\alpha k} + W_{\alpha jk}) - \frac{1}{12} \tilde{\varphi}^{ijkl} D_\alpha F_{jkl} \right] + \text{h.c.} \right\} \quad (3.37d)$$

$$+ \mathcal{V}^i \left[i \partial_i (G - \bar{G}) - 2 \partial_i \partial_a H^a - \frac{1}{3} \varphi^{jkl} (\partial_{[i} \Phi_{jkl]} + \partial_{[i} \bar{\Phi}_{jkl]}) - \varphi_i^{jk} \partial_j \left(H_k + \frac{1}{2i} (D^\alpha \Psi_{\alpha k} - \bar{D}_{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha} k}) \right) \right] \quad (3.37e)$$

$$L_F = -\frac{i}{288} \epsilon^{ijklmnp} \Phi_{ijk} \partial_l \Phi_{mnp} + \frac{1}{24} \tilde{\varphi}^{ijkl} G \partial_i \Phi_{jkl} + \frac{1}{32} \tilde{\varphi}^{ijkl} W_{ij}^\alpha W_{\alpha kl} + \frac{1}{4} g_{ij} \mathcal{W}^{\alpha i} \mathcal{W}_\alpha^j. \quad (3.37f)$$

We have organized the terms as follows:

(3.37a) This is the action of linearized old minimal supergravity (3.10).

(3.37b) These are the terms (3.24). They describe the “radion coupling” between the 4D and 7D volume terms and F_{ijk} kinetic terms.

(3.37c) Quadratic gravitino terms and “mass” terms for the prepotentials H^a and X of modified old minimal supergravity have the sum-of-squares form as follows from L -invariance.

(3.37d) This includes the gravitino current, describing the linear couplings of Ψ to the Kaluza-Klein gauge fields, the Y components of the 3-form, and the “mass” of the gravitino.

(3.37e) Kaluza-Klein gauge field couplings are needed to covariantize the linearized diffeomorphisms of Y . In a more covariant description, these couplings are hidden in the covariant derivative \mathcal{D} (2.3).

(3.37f) The F-term contains the G_2 superpotential $\Phi\partial\Phi$ [8] and gauge kinetic terms. The gauge symmetry (3.34b) (or more precisely (C.10)) explains the consistency of this superpotential for the first time.

This action and its gauge transformations are the main result of this paper. We have endeavored to present it in a way that motivates the roles of the myriad parts and how they relate to one another under. Pragmatically, the presentation of the foregoing sections can be skipped, and the claim that the action (3.36)–(3.37) is invariant under the transformations (3.31)–(3.35) can be checked directly.

In the next section, we project our action to components to demonstrate explicitly that this is indeed the superspace representation of the linearized action of eleven-dimensional supergravity.

4 Components

We now want to confirm that the Lagrangian (3.37) produces the correct component action of eleven-dimensional supergravity on $M = \mathbf{R}^4 \times Y$ and elucidate the required auxiliary field mechanisms. The part of the Lagrangian with all derivatives and polarizations along \mathbf{R}^4 corresponds to 4D, $N = 1$ supergravity, while the part with all derivatives and polarizations along Y (corresponding to the scalar potential in 4D) was demonstrated at the fully non-linear level in [5]. Thus, we will mainly be interested in the mixed part at the linearized level.

To compare to the quadratic approximation of the eleven-dimensional component action of reference [9], we rewrite that result in terms of G_2 representations. Using various G_2 identities which we collect in appendix A leads to the Lagrangian

$$\kappa^2 L^{(2)} = -\frac{1}{8}(\partial_c h_{ab})^2 + \frac{1}{4}(\partial^b h_{ab})^2 + \frac{1}{8}(\partial_a h)^2 + \frac{1}{4}h\partial^a\partial^b h_{ab} \quad (4.1a)$$

$$-\frac{1}{8}(\partial_i h_{ab})^2 + \frac{1}{8}(\partial_i h)^2 \quad (4.1b)$$

$$+\frac{1}{36}\left(\partial^a\partial^b h_{ab} - \square h\right)\left(\varphi^{ijk}F_{1ijk}\right) \quad (4.1c)$$

$$-3h\partial^i(\tau_1)_i \quad (4.1d)$$

$$-\frac{1}{8}(\mathcal{F}_{ab}^i)^2 + \frac{1}{2}\partial_a h^{ab}\partial_i \mathcal{A}_b^i - \frac{1}{2}\partial^a h\partial_i \mathcal{A}_a^i \quad (4.1e)$$

$$-\frac{1}{96}F_{abcd}^2 - \frac{1}{24}F_{abci}^2 - \frac{1}{16}\left(F_{7abij}^2 + F_{14abij}^2\right) \quad (4.1f)$$

$$+\frac{1}{18}(\partial_a\phi_{1ijk})^2 - \frac{1}{24}(\partial_a\phi_{27ijk})^2 - \frac{1}{24}\left(F_{1aijk}^2 + F_{7aijk}^2 + F_{27aijk}^2\right) \quad (4.1g)$$

$$+\frac{21}{16}\tau_0^2 + 15\tau_1^2 - \frac{1}{8}\tau_2^2 - \frac{1}{24}\tau_3^2 \quad (4.1h)$$

$$-\frac{7}{4}\sigma_0^2 - 9\sigma_1^2 - \frac{1}{24}\sigma_3^2 \quad (4.1i)$$

The notation and structure of this action are as follows:

- (4.1a) is the linearized four-dimensional Einstein-Hilbert action (B.21).
- (4.1b) extends the derivatives on the linearized metric to Y . From the point of view of four-dimensional compactifications, these look like mass terms for the graviton.
- (4.1c) gives the “radion coupling”, that is, the coupling between the graviton and the volume modulus of Y .
- (4.1d) gives the analogous coupling with Y derivatives. The torsion class τ_1 is the $\mathbf{7}$ -projection (A.13a) of $\partial_{[i}F_{jkl]}$, the differential of the fluctuation around the G_2 -structure 3-form.
- (4.1e) are the kinetic terms and spin-2 mixing terms of the Kaluza-Klein gauge field (mixed components of the frame) $\mathcal{A}_a^i = -e_a^i$. At quadratic order, this is the entire contribution to the action from this field. (There is no $\varphi_{ijk}\mathcal{F}_{ab}^i F_7^{abjk}$ term; the combination that appears instead is the cubic term $C_{ijk}\mathcal{F}_{ab}^i F_7^{abjk}$, which we ignore in the quadratic approximation.)
- (4.1f) are the covariantized kinetic terms for the 3-form C_{abc} , seven 2-form C_{abi} , and twenty-one 1-form C_{aij} components of the eleven-dimensional 3-form C_{abc} .
- (4.1g) gives the kinetic terms for the scalars. The first two terms are those for the 28 metric scalars g_{ij} written in terms of the differential of the fluctuations around the G_2 -structure 3-form. (These have no $\mathbf{7}$ part.) The remaining terms are the projections of the covariantized kinetic term of the 35 scalars C_{ijk} .
- (4.1h) is the Einstein-Hilbert term $\frac{1}{2}\sqrt{g}R(g)$ on Y written in terms of torsion classes (A.13a) using a result due to Bryant [25]. From the four-dimensional point of view, these resemble potential terms.
- (4.1i) gives the Maxwellian contribution $-\frac{1}{4!}\sqrt{g}F_{ijkl}^2$ on Y in a form analogous to (4.1h). These also look like potential terms from the four-dimensional perspective.

We now confirm that our Lagrangian (3.37) reproduces the bosonic components (4.1), beginning with the parts that have already been verified. The first line (4.1a) is the linearized gravity action. It comes from the component projection (B.21) of linearized old minimal supergravity which we review in appendix B. The last two lines (4.1h) and (4.1i) have already been confirmed at the non-linear level in reference [5]. It was also mentioned there that the first term in (4.1f) was used to fix the G -dependence of the volume functional (2.13). The coefficient of the mass term for the spin-2 field (the first one in line (4.1b)) must be the same as the first term in the pure gravity sector (4.1a). This becomes clear when working in transverse-traceless gauge wherein the statement amounts to one of 11D Lorentz invariance. (Lorentz invariance guarantees that the trace mass works as well, but we will discuss it in more detail after we have understood the terms in the $\mathbf{7}$ representation.)

The kinetic terms for fields in the **14** and **27** are particularly simple to work out as these are not corrected by integrating out any auxiliary fields. (The only vector auxiliaries are in the **1** and **7** representations of G_2 .) As an example, we check the **27**. Consider the components that would result from the terms ((3.37b) = (3.24)) in the superfield Lagrangian. (These arose from the expansion (3.6) of the Hitchin functional and the superspace volume measure.) To obtain the coefficient of $F_{\mathbf{27}aijk}^2$, we need only take into account a factor of $-\frac{1}{2}$ that results from the component projection¹⁰ to get $-\frac{1}{2} \times \frac{1}{12} = -\frac{1}{24}$. Since the $(\partial_a \phi_{\mathbf{27}ijk})^2$ term is the partner of this, it gets the same factor.

Next, consider the terms $\phi_{\mathbf{1}}$ and $F_{\mathbf{1}a}$ in (4.1f) and (4.1g). The first of these comes out correctly from the expansion (3.24) with the same factor: $-\frac{1}{2} \times (-\frac{1}{9}) = \frac{1}{18}$. Since $F_{\mathbf{1}aijk}$ is the pseudoscalar partner of $\partial_a \phi_{\mathbf{1}}$, it too will come with a factor of $\frac{1}{18}$. However, this field strength couples to the conformal supergravity auxiliary field d^a (B.21) giving a correction

$$\frac{4}{3}d_a^2 + \frac{1}{18}d^a \varphi^{ijk} F_{aijk} = \frac{4}{3} \left(d_a + \frac{1}{48} \varphi^{ijk} F_{aijk} \right)^2 - \frac{4}{3} \cdot \frac{42}{48^2} F_{\mathbf{1}aijk}^2 \rightarrow -\frac{7}{72} F_{\mathbf{1}aijk}^2, \quad (4.2)$$

where we used $(\varphi^{ijk} F_{aijk})^2 = (\varphi^{ijk} F_{\mathbf{1}aijk})^2 = 42 F_{\mathbf{1}aijk}^2$ (A.3). This changes $\frac{1}{18} \rightarrow \frac{1}{18} - \frac{7}{72} = -\frac{1}{24}$, which is the correct coefficient. We thus reproduce the terms in (4.1f) and (4.1g).

Next, we will look at the radion couplings (4.1c) which come from the first term in (3.37b). In the Wess-Zumino gauge adopted in section B.1,

$$\frac{1}{36} \int d^4\theta [D_\alpha, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}} \varphi^{ijk} F_{ijk} = \frac{1}{36} \partial^a \partial^b h_{ab}^{\text{spin-2}} \varphi^{ijk} F_{ijk} + \dots \quad (4.3)$$

gives one of these couplings directly for the traceless part of the metric (B.11), and

$$-\frac{1}{18} \int d^4\theta (G + \bar{G}) \varphi^{ijk} F_{ijk} = -\frac{1}{48} \square h \varphi^{ijk} F_{ijk} - \frac{1}{8} \varphi^{ijk} d_{ij} \partial_k h + \dots \quad (4.4)$$

gives couplings involving the trace of the metric, $h = \frac{8}{3} \text{Re } G$ (B.20a). Together, these give the correct radion coupling (4.1c) in the basis in which the spin-0 part of the metric is separated out: $\frac{1}{36} (\partial^a \partial^b h_{ab} - \square h) = \frac{1}{36} \partial^a \partial^b h_{ab}^{\text{spin-2}} - \frac{1}{48} \square h$. The elided term in (4.3) involves d_a , which we have already accounted for, while those in (4.4) involve the auxiliary fields of G , V_{ij} and Φ_{ijk} already included in the analysis of [5], where they were shown to generate e.g. the correct normalization of the 3-form kinetic term. This leaves the term that involves the **7**-projection $d^k := \frac{1}{6} \varphi^{ijk} d_{ij}$ (A.5a) of the auxiliary field of V_{ij} . Additional terms involving this projection come from the F^2 terms in (3.37b) and the $\tilde{\varphi} WW$ the term in (3.37f) which were important in [5] in obtaining the correct τ_1 contribution to the scalar potential. Here they will contribute to the trace mass in (4.1b) and the mixed term (4.1d). Explicitly, we find

$$\begin{aligned} & -\frac{1}{8} \varphi^{ijk} d_{ij} \partial_k h - 2 \varphi^{ijk} d_{ij} (\tau_1)_k + \frac{1}{4} \tilde{\varphi}^{ijkl} d_{ij} d_{kl} \\ & = -\frac{3}{4} d^k \partial_k h - 12 d^k (\tau_1)_k - 6 d_k^2 + \mathbf{14}\text{-term} \end{aligned} \quad (4.5)$$

¹⁰The fermionic integral of the square of one of the field strengths F gives $\int d^4\theta F^2 = -\frac{1}{2} (\partial_a F)^2 - \frac{1}{2} F_a^2 + \dots$ where the ellipses stand for fermionic terms and auxiliary fields. On the right-hand side F stands for the $\theta \rightarrow 0$ component of the superfield F and F_a is the $\theta \rightarrow 0$ projection of $-\frac{1}{4} (\sigma_a)^{\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] F$.

where we used (A.7). Integrating out d_k gives

$$\frac{3}{2} \cdot \left(\frac{1}{8}\right)^2 (\partial_i h)^2 - \frac{3}{4} h \partial^i (\tau_1)_i. \quad (4.6)$$

There are two other sources contributing to such terms. The first is due to the **7** projections f^i (A.5b) of the Φ auxiliaries f_{ijk} . The square of these terms comes from the F^2 part of the Kähler action with the linear terms coming from the Chern-Simons terms making up the first line of (3.37f)

$$-\frac{1}{24} |f_{ijk}|^2 + \frac{1}{24} \cdot \frac{3}{8} \tilde{\varphi}^{ijkl} \text{Re} f_{ijk} \partial_l h = -\frac{1}{2} (\text{Re} f_k)^2 + \frac{3}{16} \text{Re} f^k \partial_k h + \dots \quad (4.7)$$

where the ellipses stand for terms irrelevant to this calculation. Integrating out $\text{Re} f^i$ gives

$$\left(\frac{3}{8}\right)^2 (\partial_i h)^2 - \frac{9}{4} h \partial^i (\tau_1)_i. \quad (4.8)$$

The second contribution comes directly from the 2-form field strength H_i in (3.37c). This term gives only a trace mass correction $-\frac{1}{4} \int d^4 \theta (\partial_i X)^2 = -\frac{1}{2} \cdot \left(\frac{3}{8}\right)^2 (\partial_i h)^2$. Adding this to (4.6) and (4.8), we find

$$\left(\frac{3}{8}\right)^2 \left[\frac{1}{6} + 1 - \frac{1}{2} \right] (\partial_i h)^2 + \left[-\frac{3}{4} - \frac{9}{4} \right] h \partial^i (\tau_1)_i = \frac{3}{32} (\partial_i h)^2 - 3 h \partial^i (\tau_1)_i \quad (4.9)$$

giving the correct mixed term (4.1d). Recombining with the traceless part of the metric $-\frac{1}{8} (\partial_i h_{ab})^2 + \frac{1}{8} (\partial_i h)^2 = -\frac{1}{8} (\partial_i h_{ab}^{\text{spin-2}})^2 + (-\frac{1}{32} + \frac{4}{32}) (\partial_i h)^2$ shows that this is also the correct trace mass (4.1b).

4.1 Component fields in the **7** representation of G_2

Now we finally come to the analysis of the recalcitrant terms in the **7** representation of G_2 . For this, we apply the general analysis of the gravitino and its compensators worked out in detail in appendix C. To understand what this general analysis implies for eleven-dimensional supergravity, we compare the Lagrangians (C.18) and (3.37). This gives the coefficients

$$a = -\frac{1}{4}, \quad b = \frac{1}{2}, \quad \text{and} \quad c = -\frac{1}{4} \quad (4.10)$$

for linearized eleven-dimensional supergravity. Substituting into the component result (C.27), gives (using (A.6))

$$\begin{aligned} L_{(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2})} &= \frac{1}{16} \tilde{H}_a^i \tilde{H}_i^a - \frac{1}{4} F_i^a F_a^i - \frac{3}{8} F_i^{ab} F_{ab}^i - \frac{1}{8} \mathcal{F}_i^{ab} \mathcal{F}_{ab}^i \\ &= -\frac{1}{24} (F_{abci})^2 - \frac{1}{24} (F_{\mathbf{7}aijk})^2 - \frac{1}{16} (F_{\mathbf{7}abij})^2 - \frac{1}{8} (\mathcal{F}_{ab}^i)^2 \end{aligned} \quad (4.11)$$

These are the correct coefficients of the **7**-projection of the component theory as found in lines (4.1f), (4.1g), and (4.1e). Note that the cancellation of $(\partial_a F_i)^2 \propto (\partial_a F_{\mathbf{7}ijk})^2$ is

important to recover the eleven-dimensional theory, since the $\mathbf{7}$ projection of F_{ijk} does not correspond to any physical field.

At this point we have verified all the components in (4.1) except for the $\partial h \partial \mathcal{A}$ terms in (4.1e). These terms are not subtle, coming directly from (3.37e): the third $\mathcal{V} \partial \Phi$ term contributes only to the potential [5] and the first two integrate to

$$\begin{aligned} \int d^4\theta \mathcal{V}^i \partial_i [i(G - \bar{G}) - 2\partial_a H^a] &= \partial_i \mathcal{A}_a^i \left[\frac{1}{2} \partial_b h^{\text{spin-2 } ab} - \partial^a \text{Re} G \right] \\ &= \frac{1}{2} \partial_i \mathcal{A}_a^i \left[\partial_b h^{ab} - \partial^a h \right]. \end{aligned} \tag{4.12}$$

This completes the verification that the superspace action (3.36) reproduces the bosonic action (4.1) of eleven-dimensional supergravity in the quadratic approximation.

5 Conclusions and outlook

We have extended the construction of the embedding of eleven-dimensional supergravity into 4D, $N = 1$ superspace of ref. [5] to quadratic order in the gravitino superfield $\Psi_{\alpha i}$. This extension is needed to prove that the spectrum is represented faithfully in terms of 4D, $N = 1$ superfields, and to show that the dynamics of the eleven-dimensional components is that required by eleven-dimensional Poincaré invariance. The result is that the action is given by (3.36) in terms of the Lagrangian (3.37). In addition to being manifestly invariant under local 4D, $N = 1$ supergravity transformations (3.35), the complete action is invariant under the tensor hierarchy gauge transformations (3.31) and (3.32) and the extended supersymmetry transformations (3.34). This level of approximation (i.e. to quadratic order in the gravitini) suffices to demonstrate the consistency of this superspace description of eleven-dimensional supergravity. (In particular, it realizes the linearized gauge symmetry and associated compensator mechanism advocated in [5].) Additionally, we expect it to be adequate for most applications. For example, already at this level, the conformal graviton propagator and all other superspace Feynman rules needed for perturbative calculations can be deduced as was done for five-dimensional supergravity in ref. [26].

There are three directions in which we are currently extending this analysis. The first is that we would like to complete the quadratic gravitino action to all orders in the remaining fields. (The analogous $O(H_a, \Psi)$ but all orders in remaining fields has already been worked out and will be presented in a separate article [10].) Secondly, we would like to construct the terms cubic and higher in the gravitino multiplet. Although this may initially appear a daunting task, current results suggest that it is possible to construct fields strengths invariant under the Ξ transformations. As the Kähler part of the action is non-polynomial in the field strengths F , H , G , and \mathcal{W} , this fact goes some way toward generating all of the higher-order terms. Finally, the tight structure of this formulation of eleven-dimensional supergravity seems well-suited to the study of higher-derivative corrections to the action, although this may be easier to demonstrate once we have presented the non-linear couplings [10]. (The coupling of the gravitino multiplets to conformal supergravity of [27] might be useful in this context.)

Besides the immediate extensions just mentioned, and which are needed to really complete the embedding of M-theory into 4D, $N = 1$ superspace, there are some applications of this result and lower-dimensional analogues which could be worked out. First, closely related to the present story should be the superspace description of type IIA string theory compactified on G_2 -structure manifolds. Our choice to focus on M-theory on G_2 -structure backgrounds was motivated by minimality: eleven-dimensional supergravity has the most economic field content of all higher-dimensional supergravity theories, and 4D, $N = 1$ is the most familiar superspace. Describing type IIA string theory in superspace is potentially messier given the larger number of fields but can, in principle, be obtained from our formalism by dimensional reduction.

Potentially less straightforward is the description of type IIB string theory on G_2 -structure backgrounds. It would be interesting to work this out to elucidate how mirror symmetry is realized in superspace. In fact, quite recently, a proposal for mirror symmetry for G_2 -manifolds applicable to the twisted connected sum construction of ref. [28] was made in ref. [29]. It should then be possible to find a map between the (super)space-time actions for type IIA and type IIB string theory compactified on mirror G_2 -manifolds, resembling the c-map in ref. [30]. In addition to mirror symmetry, other dualities can be considered. In the context of duality between M-theory, heterotic, and F-theory we could try to make contact with the recent paper [31], at least in the case of smooth manifolds.

Another scenario that would be worth exploring in our formalism is the case when the internal manifold has a resolved orbifold singularity, as discussed in [32]. For a local model, we would consider internal manifolds of the form $M \times Q$, where M is a resolved ADE singularity and Q is a three-manifold. Away from the singular point, the massless fields give only a $U(1)^r$ gauge group corresponding to the harmonic 2-forms on M , but as we approach the singular limit, various massive fields become light and the gauge group enhances to something non-abelian. In terms of the space-time effective action, there is a contribution to the superpotential in the form of a complex Chern-Simons invariant on Q , as explained in [32]. Since we are keeping all KK-modes, we might be able to usefully study this limit, see more direct evidence for the enhancement, and compute the relevant superpotential terms.

In flux compactifications there is typically a warp factor multiplying the space-time part of the metric, and this has complicated the analysis of the effective theory, particularly for the purposes of constructing an $N = 1$ superspace action [33]. Our approach is applicable to those scenarios and has the potential to simplify the analysis substantially.

It would also be interesting to make contact with ref. [34] where the space-time action for massless fields obtained from a compactification of eleven-dimensional supergravity on twisted connected sum G_2 -manifolds was presented. In our analysis all fields (not only the massless ones) are taken into account so we could, for example, compute the gravitino mass matrix and analyze how it behaves in various limits (for example, as a function of the gluing modulus). It would, of course, be interesting to develop these ideas further and analyze the (super)space-time action for compactification of the extra twisted connected sum type of ref. [35], particularly since this might help elucidate the physical significance of the new homotopy invariant introduced by Crowley and Nordström in ref. [36]. Also quite

recently, a new construction of G_2 -holonomy manifolds was found in ref. [37], and it would be interesting to consider the corresponding (super)space-time action. In short, there has been a proliferation of new results in the mathematics literature concerning G_2 -holonomy manifolds and it will be fascinating to work out the physical implications.

Finally, it would be desirable to understand the truncations of this formulation of eleven-dimensional supergravity to other dimensions and/or extensions to superspaces with more supersymmetries manifest. In this work, we chose the superspace and superfield content to realize four supersymmetries linearly. This is natural because compactification to 4D on a G_2 holonomy manifold gives this structure. Moreover, because the number of internal dimensions ($11 - 4$) equals the number of non-manifest 4D supersymmetries ($8 - 1$), it is possible for all superfields, including the gravitino multiplets, to be put into $GL(7)$ representations. This would not have been the case, for example, had we sought to manifest 4D $N = 2$ supersymmetry in describing M-theory. Instead, 4D $N = 2$ superspace should more naturally describe 10D type II supergravities, as compactifications of superstrings on Calabi-Yau 3-folds naturally give $N = 2$ theories. Similarly, compactifications of M-theory on Calabi-Yau 3-folds naturally lead to 5D $N = 1$ theories, and so describing M-theory with eight supercharges manifest suggests the use of 5D $N = 1$ superspace. Although such constructions would be natural, it is currently not clear whether this is even possible. (An immediate obstacle to finding such an extension is our poor understanding of the super-de Rham complexes in such superspaces [38–40].)¹¹ Consistent truncations of such constructions would make contact with (and potentially simplify) the phenomenological literature on five-dimensional supergravity [24, 26, 43–46] and extensions to six-dimensions [47–49].

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A G_2 in a nutshell

In this appendix, we collect some useful definitions and formulæ of G_2 -structure manifolds [2, 15, 16, 25, 50]. Let φ be a 3-form on Y and define the symmetric bilinear form $g_{ij}(\varphi)$ through the non-linear equation

$$\sqrt{g}g_{ij} := -\frac{1}{144}\epsilon^{abcdefg}\varphi_{iab}\varphi_{cde}\varphi_{jfg}, \tag{A.1}$$

where $g = \det(g_{ij})$. The 3-form φ is *stable* if this determinant is non-zero everywhere and *positive* if, in addition, g_{ij} is a Riemannian metric. These are open conditions, so if

¹¹Presumably, one should be able to start with (on-shell) 11D superspace [41, 42] and reduce directly to 5D $N = 1$ to resolve this question. In our forthcoming publication [10], we will show that reducing 11D superspace to 4D $N = 1$ indeed leads to the formulation we have found.

we start at a three-form for which they hold, then they will also hold for nearby three-forms. Throughout this paper, we will assume that these conditions hold without further qualification.

Using the metric, we define the Hodge dual

$$\tilde{\varphi} := *\varphi. \quad (\text{A.2})$$

This equation is highly non-linear in φ since the $*$ operation is non-linear in $g(\varphi)$ which is, itself, non-linear in φ . The tensors φ , $g(\varphi)$, and $\tilde{\varphi}(\varphi)$ satisfy the algebraic identities

$$\begin{aligned} \tilde{\varphi}^{ijkl} \tilde{\varphi}_{i'j'k'l} &= 6\delta_{[i'}^i \delta_{j'}^j \delta_{k'}^k] - \varphi_{i'j'k'} \varphi^{ijk} - 9\delta_{[i'}^i \tilde{\varphi}_{j'k']}^{jk}], \\ \varphi^{ijk} \varphi_{i'j'k'} &= 2\delta_{[j'}^j \delta_{k'}^k] - \tilde{\varphi}_{j'k'}^{jk}, & \tilde{\varphi}^{ijkl} \tilde{\varphi}_{ijk'l'} &= 8\delta_{[k'}^k \delta_{l'}^l] - 2\tilde{\varphi}_{k'l'}^{kl}, \\ \varphi^{ijk} \varphi_{ijk'} &= 6\delta_{k'}^k, & \tilde{\varphi}^{ijkl} \tilde{\varphi}_{ijkl'} &= 24\delta_{l'}^l, & \varphi_i{}^{lm} \tilde{\varphi}_{jklm} &= -4\varphi_{ijk}, \end{aligned} \quad (\text{A.3})$$

where indices are raised and lowered with the metric (A.1).

A stable 3-form on the tangent space of Y reduces the structure group $\text{GL}(7) \rightarrow G_2$ so that Y is a G_2 -structure manifold. Under this reduction, the **21**-dimensional space of 2-forms on Y decomposes into G_2 representations as $\mathbf{21} = \mathbf{7} \oplus \mathbf{14}$. Similarly, the **35**-dimensional space of 3-forms on Y decomposes as $\mathbf{35} = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27}$. For any p -form ω , let $\omega_{\mathbf{i}} := \pi_{\mathbf{i}}\omega$ denote the projection to the \mathbf{i} -dimensional representation. Explicitly, for any 2-form η and 3-form ω ,

$$\pi_{\mathbf{7}}\eta_{ij} = \left(\frac{1}{3}\delta_i^k \delta_j^l - \frac{1}{6}\tilde{\varphi}_{ij}{}^{kl} \right) \eta_{kl}, \quad (\text{A.4a})$$

$$\pi_{\mathbf{14}}\eta_{ij} = \left(\frac{2}{3}\delta_i^k \delta_j^l + \frac{1}{6}\tilde{\varphi}_{ij}{}^{kl} \right) \eta_{kl}, \quad (\text{A.4b})$$

$$\pi_{\mathbf{1}}\omega_{ijk} = \frac{1}{42}\varphi_{ijk}\varphi^{i'j'k'}\omega_{i'j'k'}, \quad (\text{A.4c})$$

$$\pi_{\mathbf{7}}\omega_{ijk} = \left(\frac{1}{4}\delta_i^{i'} \delta_j^{j'} \delta_k^{k'} - \frac{3}{8}\tilde{\varphi}_{[ij}{}^{i'j'} \delta_{k]}^{k'} - \frac{1}{24}\varphi_{ijk}\varphi^{i'j'k'} \right) \omega_{i'j'k'}, \quad (\text{A.4d})$$

$$\pi_{\mathbf{27}}\omega_{ijk} = \left(\frac{3}{4}\delta_i^{i'} \delta_j^{j'} \delta_k^{k'} + \frac{3}{8}\tilde{\varphi}_{[ij}{}^{i'j'} \delta_{k]}^{k'} + \frac{1}{56}\varphi_{ijk}\varphi^{i'j'k'} \right) \omega_{i'j'k'}. \quad (\text{A.4e})$$

The $\mathbf{7}$ -projections of 2- and 3-forms play an important role in the gravitino analysis. We define for such projections the vectors fields¹²

$$\eta^i := \frac{1}{6}\varphi^{ijk}\pi_{\mathbf{7}}\eta_{jk} = \frac{1}{6}\varphi^{ijk}\eta_{jk} \Leftrightarrow \pi_{\mathbf{7}}\eta_{ij} = \varphi_{ijk}\eta^k \quad (\text{A.5a})$$

$$\omega^i := \frac{1}{12}\tilde{\varphi}^{ijkl}\pi_{\mathbf{7}}\omega_{jkl} = \frac{1}{12}\tilde{\varphi}^{ijkl}\omega_{jkl} \Leftrightarrow \pi_{\mathbf{7}}\omega_{ijk} = -\frac{1}{2}\tilde{\varphi}_{ijkl}\omega^l. \quad (\text{A.5b})$$

Note that this implies that there are conversion factors in squares

$$(\eta^i)^2 = \frac{1}{6}(\pi_{\mathbf{7}}\eta_{ij})^2 \quad \text{and} \quad (\omega^i)^2 = \frac{1}{6}(\pi_{\mathbf{7}}\omega_{ijk})^2. \quad (\text{A.6})$$

¹²That is, for any $\eta \in \Lambda^2(Y)$ and $\omega \in \Lambda^3(Y)$, we are defining the vectors $\vec{\eta}$ and $\vec{\omega}$ on Y such that $\iota_{\vec{\eta}}\varphi = \pi_{\mathbf{7}}\eta$ and $\iota_{\vec{\omega}}\tilde{\varphi} = 2\pi_{\mathbf{7}}\omega$.

These factors appear when we confirm the coefficients of the kinetic terms of all gauge fields in the **7** in (4.11). The dual 4-form $\tilde{\varphi}$ acts on 2-forms as $\tilde{\varphi}_{ij}{}^{kl}\eta_{\tau kl} = -4\eta_{\tau ij}$ and $\tilde{\varphi}_{ij}{}^{kl}\eta_{\mathbf{14}kl} = 2\eta_{\mathbf{14}ij}$ or

$$\tilde{\varphi}^{ijkl}\eta_{ij}\eta_{kl} = -4\eta_{\mathbf{7}ij}^2 + 2\eta_{\mathbf{14}ij}^2 = -24(\eta^i)^2 + 2\eta_{\mathbf{14}ij}^2. \quad (\text{A.7})$$

Momentarily, we will use similar equations on the space of 3-forms,

$$\begin{aligned} \omega^{ijk}\omega_{ijk} &= \omega_{\mathbf{1}}^2 + \omega_{\mathbf{7}}^2 + \omega_{\mathbf{27}}^2 \\ g^{ii'}\tilde{\varphi}^{jj'k'}\omega_{ijk}\omega_{i'j'k'} &= -4\omega_{\mathbf{1}}^2 - 2\omega_{\mathbf{7}}^2 + \frac{2}{3}\omega_{\mathbf{27}}^2 \\ (\varphi^{ijk}\omega_{ijk})^2 &= 42\omega_{\mathbf{1}}^2. \end{aligned} \quad (\text{A.8})$$

The Hitchin functional is defined as the Riemannian volume

$$S_H = \int d^7y \sqrt{g(\varphi)}. \quad (\text{A.9})$$

Since this will be the main ingredient in our Kähler term, it will prove useful to derive the first few functional derivatives. The first derivative is the dual of φ

$$3!\epsilon_{ijklmnp}\frac{\delta K}{\delta F_{mnp}} = \tilde{\varphi}_{ijkl}, \quad (\text{A.10})$$

and the second derivative is (proportional to) the Hitchin metric on the moduli space of G_2 structures [15]

$$G^{ijk,mnp} := -\frac{\partial^2 \sqrt{g(\varphi)}}{\partial \varphi_{ijk} \partial \varphi_{mnp}} = \frac{1}{3! \cdot 3!} \sqrt{g} \left(g^{[i|m} g^{|j|n} g^{k]p} + \frac{1}{18} \varphi^{ijk} \varphi^{mnp} + \frac{3}{2} g^{[m|[i} \tilde{\varphi}^{jk]|np]} \right). \quad (\text{A.11})$$

The contractions (A.8) can be used to compute the signature

$$18\omega_{ijk}G^{ijk,lmn}\omega_{lmn} = -\frac{4}{3}\omega_{\mathbf{1}ijk}^2 - \omega_{\mathbf{7}ijk}^2 + \omega_{\mathbf{27}ijk}^2 \quad (\text{A.12})$$

in terms of G_2 projections for any 3-form ω .

It will also be useful to introduce the intrinsic torsion forms τ_μ for $\mu = 0, 1, 2, 3$ and analogous quantities σ_μ for $\mu = 0, 1, 3$ defined by [25]

$$d\varphi = \tau_0\tilde{\varphi} + 3\tau_1\varphi + *\tau_3, \quad d\tilde{\varphi} = 4\tau_1\tilde{\varphi} + \tau_2\varphi, \quad (\text{A.13a})$$

$$dC = \sigma_0\tilde{\varphi} + 3\sigma_1\varphi + *\sigma_3, \quad (\text{A.13b})$$

where the subscripts indicate the degree as a form, and where we impose that τ_2 transforms in the **14** (so $\pi_{\mathbf{14}}\tau_2 = \tau_2$) and that τ_3 and σ_3 transform in the **27**. (We could make the analogous definition for the components of $d * C$ but the action depends only on C and dC ; the C -field analogue of the torsion class τ_2 is not gauge invariant.)

B Old minimal supergravity

In this section we review the elements of old minimal supergravity (see e.g. [18–20]) used in this work. The component fields of spin 2 and $\frac{3}{2}$ are described in terms of the superspace analog of the conformal Weyl tensor C_{abcd} (the trace-free part of the Riemann tensor R_{abcd}). Converting to spinor indices, $C_{\alpha\beta\gamma\delta} = C_{(\alpha\beta\gamma\delta)}$ is totally symmetric. (Any anti-symmetric part can be isolated with $\varepsilon_{\alpha\beta}$ and corresponds to a trace.) Its spin- $\frac{3}{2}$ analog is the gravitino curl $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$. Together, they are contained within a superfield $W_{\alpha\beta\gamma}$ (for which the gravitino curl is the bottom component) subject to the conditions

$$\bar{D}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 \quad \text{and} \quad \partial_{\dot{\alpha}}^{\beta} D^{\gamma} W_{\gamma\beta\alpha} = -\partial_{\dot{\alpha}}^{\beta} \bar{D}^{\dot{\gamma}} \bar{W}_{\dot{\gamma}\beta\dot{\alpha}}. \quad (\text{B.1})$$

Together these imply that $W_{\alpha\beta\gamma}$ contains, in addition to the gravitino field strength, the component Weyl tensor as $C_{\alpha\beta\gamma\delta} = D_{(\delta} W_{\alpha\beta\gamma)}$ and a $U(1)_R$ field strength as $D^{\gamma} W_{\gamma\beta\alpha}$. For old minimal supergravity, the $U(1)_R$ connection is auxiliary and pure gauge, so that

$$D^{\gamma} W_{\alpha\beta\gamma} \stackrel{om}{=} 0, \quad (\text{B.2})$$

where $\stackrel{om}{=} 0$ indicates that this combination vanishes only on-shell.

The full set of Bianchi identities is an off-shell version of these constraints. They may be solved in terms of the conformal supergravity prepotential H^a . (We will need only the linearized expressions.) Converting its 4-vector index into a bi-spinor index using the Pauli matrices, the conformal supertensor is given in terms of it as

$$W_{\alpha\beta\gamma} = \frac{i}{8} \bar{D}^2 D_{(\alpha} \partial_{\beta}^{\dot{\gamma}} H_{\gamma)\dot{\gamma}} \quad \Rightarrow \quad \delta H_{\gamma\dot{\gamma}} = \bar{D}_{\dot{\gamma}} L_{\gamma} - D_{\gamma} \bar{L}_{\dot{\gamma}}. \quad (\text{B.3})$$

This combination of D 's projects onto the desired irreducible superspin- $\frac{3}{2}$ representation consisting of component spins $(\frac{3}{2}, 2)$ [51] (see also [18, 52]). Equivalently, it is invariant under the huge gauge transformation involving the unconstrained superfield parameter L_{α} , which allows one to gauge away all but the spin-2 conformal graviton and spin- $\frac{3}{2}$ conformal gravitino.

For Poincaré supergravity, this gauge transformation is too large: we are required to reinstate the spin-0 part of the graviton (trace) and the spin- $\frac{1}{2}$ part of the gravitino (gamma-trace). A closely-related statement is that it is not possible to construct a two-derivative action from this representation alone.¹³ Following [53, 54], this is done by coupling conformally to a superfield with a scalar component that has a non-vanishing background value. This scalar field is the conformal compensator. Different off-shell supergravity theories correspond to different choices for this scale compensator.¹⁴ Old minimal supergravity involves a chiral scalar superfield $\Phi_0 = e^{\sigma}$, often written in an exponential form as its background value is taken to be 1. Its linearized gauge transformation is

$$\delta\sigma = -\frac{1}{12} \bar{D}^2 D^{\alpha} L_{\alpha}. \quad (\text{B.4})$$

¹³The conformal supergravity action $\int d^2\theta W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}$ is the supersymmetrization of the four-derivative Weyl² action.

¹⁴For a particularly enlightening classification of irreducible quadratic Poincaré supergravity actions and treatment of scale compensators, see [55].

The quadratic action of old minimal supergravity is [18, 20]

$$S_{\text{OMSG}} = \frac{1}{\kappa^2} \int d^4x \int d^4\theta L_{\text{OMSG}}, \quad (\text{B.5})$$

$$L_{\text{OMSG}} = \frac{1}{8} H_a D^\beta \bar{D}^2 D_\beta H^a - (\partial_a H^a)^2 + \frac{1}{48} ([D_\alpha, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}})^2 - 3\bar{\sigma}\sigma + 2i(\sigma - \bar{\sigma})\partial_a H^a.$$

There is a modification [56, 57] of this action in which the conformal chiral compensator Φ_0 is replaced with a slightly different representation in terms of a real prepotential. Generally, the chirality constraint on a generic chiral field can be solved in terms of a complex scalar superfield $X_{\mathbf{C}}$,

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad \Rightarrow \quad \Phi = -\frac{1}{4}\bar{D}^2 X_{\mathbf{C}}. \quad (\text{B.6})$$

But a closer inspection of the components reveals that the same *physical* components result from the restriction $\bar{X}_{\mathbf{C}} = X_{\mathbf{C}} =: X$. In this representation, the scalar that was the imaginary part of the F-component of Φ is replaced by the divergence of a vector,

$$iD^2\Phi - i\bar{D}^2\bar{\Phi} = -\frac{i}{4}[D^2, \bar{D}^2]X = -\partial^{\alpha\dot{\alpha}}([D_\alpha, \bar{D}_{\dot{\alpha}}]X), \quad (\text{B.7})$$

or, equivalently, the dual of a four-form field strength. This is just the superspace representation of the gauge 3-form C_{abc} [17], and one recognizes the superfield G and its gauge 3-form prepotential X . It was already observed in [5] that this field strength plays the role of the conformal compensator (at least in the gauge where $H_i \rightarrow 0$). This suggests that the 4D, $N = 1$ supergravity formulation best suited to the description of eleven-dimensional supergravity is given by this modification of old minimal supergravity [56, 57] with the replacements¹⁵

$$e^{3\sigma} \rightarrow G = -\frac{1}{4}\bar{D}^2 X \quad \text{with} \quad \delta_L X = D^\alpha L_\alpha + \bar{D}_{\dot{\alpha}} \bar{L}^{\dot{\alpha}}. \quad (\text{B.8})$$

The L_α gauge invariance can be exploited in several different ways. One choice is to fix $\sigma \rightarrow 0$. Another choice is to impose the Lorentz gauge

$$D^\alpha H_{\alpha\dot{\alpha}} \rightarrow 0 \quad \Rightarrow \quad L_{\text{OMSG}} \rightarrow -H_a \square H^a - 3\bar{\sigma}\sigma. \quad (\text{B.9})$$

In this gauge, the component action reduces to that of linearized supergravity in transverse-traceless gauge. Both of these are manifestly supersymmetric gauge choices, but leave some unphysical component fields unfixed.

B.1 Wess-Zumino gauge and components

It is usually more convenient to impose a Wess-Zumino gauge choice that eliminates all but the physical components. Using a vertical bar $|$ to denote projection to $\theta = 0$, fixing

¹⁵The relative normalizations of σ and G arise because G has conformal weight 3 while Φ_0 is normalized to have weight 1.

certain components of H_a to zero restricts the superfield L_α :

$$H_{\alpha\dot{\alpha}}| = 0 \quad \Rightarrow \quad D_\alpha \bar{L}_{\dot{\alpha}}| = \bar{D}_{\dot{\alpha}} L_\alpha|, \quad (\text{B.10a})$$

$$D^\beta H_{\beta\dot{\alpha}}| = 0 \quad \Rightarrow \quad D^2 L_{\dot{\alpha}}| = -\bar{D}_{\dot{\alpha}} D^\alpha L_\alpha|, \quad (\text{B.10b})$$

$$D_{(\beta} H_{\alpha)\dot{\alpha}}| = 0 \quad \Rightarrow \quad D_{(\beta} \bar{D}_{\dot{\alpha}} L_\alpha| = \bar{D}_{\dot{\alpha}} D_{(\beta} L_\alpha) = 0, \quad (\text{B.10c})$$

$$D^2 H_{\alpha\dot{\alpha}}| = 0 \quad \Rightarrow \quad D^2 \bar{D}_{\dot{\alpha}} L_\alpha = 0. \quad (\text{B.10d})$$

In addition, the ability to shift L_α by a chiral spinor implies we can always take $L_\alpha| = D_\beta L_\alpha| = D^2 L_\alpha| = 0$. At the $\theta\bar{\theta}$ level, we identify the spin-2 part of the graviton as

$$h_{\dot{\beta}\dot{\alpha}\dot{\gamma}\dot{\delta}}^{\text{spin-2}} := -[D_{\dot{\beta}}, \bar{D}_{\dot{\gamma}}] H_{\alpha\dot{\alpha}}| \quad (\text{B.11})$$

and impose another WZ condition to ensure that the right-hand side is symmetric in $\beta\alpha$ and $\dot{\beta}\dot{\alpha}$ so that $h_{\dot{\beta}\dot{\alpha}}^{\text{spin-2}}$ is symmetric and traceless. This fixes

$$D_{(\beta} \bar{D}^2 L_\alpha| = 0, \quad \text{Re } D^\alpha \bar{D}^2 L_\alpha| = \partial_a \xi^a, \quad (\text{B.12})$$

where ξ^m is the linearized diffeomorphism

$$\xi_{\alpha\dot{\alpha}} := -i(\bar{D}_{\dot{\alpha}} L_\alpha + D_\alpha \bar{L}_{\dot{\alpha}})|, \quad \delta h_{ab}^{\text{spin-2}} = 2\partial_{(a} \xi_{b)} - \frac{1}{2} \eta_{ab} \partial_c \xi^c. \quad (\text{B.13})$$

The $N = 1$ supersymmetry parameter is

$$\xi_\alpha = -\frac{1}{4} \bar{D}^2 L_\alpha| = \frac{1}{4} D_\alpha \bar{D}_{\dot{\alpha}} L_{\dot{\alpha}}|, \quad (\text{B.14})$$

and the spin- $\frac{3}{2}$ part of the $N = 1$ gravitino is

$$\psi_{\dot{\beta}\dot{\alpha}}^{\text{spin-}\frac{3}{2}} = -\frac{i}{4} \bar{D}^2 D_{(\beta} H_{\alpha)\dot{\beta}}|, \quad \delta \psi_{\dot{\beta}\dot{\alpha}} = 2\partial_{\dot{\beta}} \xi_{\dot{\alpha}}. \quad (\text{B.15})$$

The final WZ condition guarantees that no other fermions appear at the $\theta\bar{\theta}^2$ level,

$$\bar{D}^2 D^\alpha H_{\alpha\dot{\alpha}}| = 0 \quad \Rightarrow \quad \bar{D}^2 D^2 \bar{L}_{\dot{\alpha}}| = 2i \partial_{\alpha\dot{\alpha}} \bar{D}^2 L^\alpha|. \quad (\text{B.16})$$

The top component of $H_{\alpha\dot{\alpha}}$ corresponds to the $U(1)_R$ gauge field,

$$d_{\alpha\dot{\alpha}} = -\frac{1}{16} D^\beta \bar{D}^2 D_\beta H_{\alpha\dot{\alpha}}|, \quad \delta d_a = \partial_a \omega, \quad \omega = -\frac{1}{8} \text{Im } D^\alpha \bar{D}^2 L_\alpha|. \quad (\text{B.17})$$

While $H_{\alpha\dot{\alpha}}$ contains the component field content of $N = 1$ conformal supergravity, the compensator G contains a propagating complex scalar and Weyl fermion. These transform as

$$\delta G| = -\frac{1}{4} \bar{D}^2 D^\alpha L_\alpha| = \frac{3}{4} \partial_a \xi^a + 2i\omega, \quad (\text{B.18})$$

$$\delta D_\alpha G| = -\frac{1}{4} D_\alpha \bar{D}^2 D^\beta L_\beta| = 3i \partial_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}. \quad (\text{B.19})$$

We identify the spin-0 part of the metric as the real part of $G|$, and the spin- $\frac{1}{2}$ part of the gravitino as the fermion $D_\alpha G|$, so that

$$h_{ba} := \frac{1}{2} \sigma_b^{\dot{\beta}\beta} [D_\beta, \bar{D}_{\dot{\beta}}] H_a| + \frac{1}{3} \eta_{ba} (G + \bar{G})|, \quad (\text{B.20a})$$

$$\psi_{\beta\dot{\beta}\alpha} := -\frac{i}{4} \bar{D}^2 D_{(\beta} H_{\alpha)\dot{\beta}}| - \frac{i}{3} \epsilon_{\beta\alpha} \bar{D}_{\dot{\beta}} \bar{G}|. \quad (\text{B.20b})$$

The $U(1)_R$ gauge transformation associated with ω may be used to set the imaginary part of $G|$ to zero; equivalently, it is eaten by the auxiliary gauge field d_a .¹⁶ The remaining degrees of freedom are the θ^2 components of G . These are the real auxiliary d_X and the 4-form field strength F_{abcd} , given by

$$F_{abcd} := \frac{i}{8} \epsilon_{abcd} (D^2 G - \bar{D}^2 \bar{G})| \quad (\text{B.20c})$$

$$d_X := \frac{1}{32} \{D^2, \bar{D}^2\} X|. \quad (\text{B.20d})$$

Then the component Lagrangian for modified old minimal supergravity can be written

$$\begin{aligned} \kappa^2 L_{\text{OMSG}} = & -\frac{1}{8} (\partial_c h_{ab})^2 + \frac{1}{4} (\partial^b h_{ab})^2 + \frac{1}{8} (\partial_a h)^2 + \frac{1}{4} h \partial^a \partial^b h_{ab} \\ & + \frac{4}{3} d_a^2 - \frac{1}{3} (d_X^2 + F_{abcd}^2). \end{aligned} \quad (\text{B.21})$$

We will use this when comparing to the linearized eleven-dimensional action in section 4.

C Gravitino superfields

In this section, we will work out in detail the quadratic superspace actions for a free spin- $\frac{3}{2}$ field and its spin-1 superpartner. In manifestly supersymmetric language, we are considering the free massless superspin-1 multiplet which is often referred to as a (matter) gravitino multiplet. Investigations into the precise form of this action date back to the work of Ogievetsky and Sokatchev [58]. A second formulation was discovered by de Wit and van Holten [59] and Fradkin and Vasiliev [60]. The relation between these theories was elucidated at the level of superfield representation theory in [22], at the level of supergeometry in [61], and from 4D, $N = 2$ superspace in [62]. In [24], the 5D gravitino superfield was discovered to be neither of these multiplets.

In modern terms, the basic matter gravitino model can be defined by the conformal gravitino field strength $W_{\alpha\beta i}$ describing a set of irreducible super-spin-1 superfields, with i denoting the additional gravitini. On-shell such a representation consists of a spin- $\frac{3}{2}$ gravitino and a spin-1 “graviphoton”. This implies that the superfield satisfies

$$\bar{D}_{\dot{\alpha}} W_{\alpha\beta i} = 0 \quad \text{and} \quad D^\beta W_{\alpha\beta i} \stackrel{eom}{=} 0, \quad (\text{C.1})$$

¹⁶An alternative WZ gauge-fixing involves setting $G| = 0$ and $D_\alpha G| = 0$. Then the spin-0 part of the metric and spin- $\frac{1}{2}$ part of the gravitino are also contained within H_a . This alters the definitions (B.20a), (B.20b) but leaves the Lagrangian (B.21) unchanged.

where the second equation is required to hold only on-shell.¹⁷ We suppress the full set of off-shell Bianchi identities which can be solved in terms of an unconstrained gravitino prepotential superfield as

$$W_{\alpha\beta i} = -\frac{1}{4}\bar{D}^2 D_{(\alpha}\Psi_{\beta)i}. \quad (\text{C.2})$$

This expression has a large (pre-)gauge symmetry

$$\delta\Psi_{\alpha i} = \Xi_{\alpha i} + D_{\alpha}\Omega_i \quad \text{with} \quad \bar{D}_{\dot{\alpha}}\Xi_{\alpha i} = 0 \quad (\text{C.3})$$

and Ω_i complex and unconstrained. We will be interested in the case where $\Psi_{\alpha i}$ transforms also under the $N = 1$ conformal supergravity L_{α} transformation as in (3.17). Before discussing the actions and compensating mechanisms in detail, it is useful to discuss the off-shell components of $\Psi_{\alpha i}$.

C.1 Wess-Zumino analysis

The gravitino superfield $\Psi_{\alpha i}$, like the $N = 1$ supergravity prepotential H_a , is subject to a large set of gauge transformations, here encoded in the parameters $\Xi_{\alpha i}$ and Ω . The gauge parameter Ω appears under D_{α} , which means that it is defined only up to a shift by an antichiral superfield. This means we may assume (without loss of generality) $\Omega_i| = \bar{D}_{\dot{\alpha}}\Omega_i| = \bar{D}^2\Omega_i| = 0$. Turning to $\Psi_{\alpha i}$ itself, a number of WZ conditions can be imposed. These in turn constrain the residual gauge symmetries within Ξ and Ω :

$$\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad \Xi_{\alpha i}| = -D_{\alpha}\Omega_i|, \quad (\text{C.4a})$$

$$D^{\alpha}\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad D^{\alpha}\Xi_{\alpha i}| = -D^2\Omega_i|, \quad (\text{C.4b})$$

$$D_{(\beta}\Psi_{\alpha)i}| = 0 \quad \Rightarrow \quad D_{(\beta}\Xi_{\alpha)i}| = 0, \quad (\text{C.4c})$$

$$\bar{D}_{\dot{\alpha}}\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad \bar{D}_{\dot{\alpha}}D_{\alpha}\Omega_i| = \partial_i\xi_{\alpha\dot{\alpha}}, \quad (\text{C.4d})$$

$$D^2\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad D^2\Xi_{\alpha i}| = 0, \quad (\text{C.4e})$$

$$D^{\alpha}\bar{D}_{\dot{\alpha}}\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad D^{\alpha}\bar{D}_{\dot{\alpha}}D_{\alpha}\Omega_i| = 8i\partial_i\bar{\xi}_{\dot{\alpha}}, \quad (\text{C.4f})$$

$$\bar{D}^2\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad \bar{D}^2D_{\alpha}\Omega_i| = 8i\partial_i\xi_{\alpha}, \quad (\text{C.4g})$$

$$\bar{D}^2D^{\alpha}\Psi_{\alpha i}| = 0 \quad \Rightarrow \quad \bar{D}^2D^2\Omega_i| = 6i\partial_i\partial_a\xi^a. \quad (\text{C.4h})$$

Note that certain components of Ω_i are related to the internal derivatives of the diffeomorphism and $N = 1$ supersymmetry parameters ξ_a and ξ_{α} . This is a consequence of the $\partial_i L_{\alpha}$ term in the gravitino superfield transformation (3.17).

The spin- $\frac{3}{2}$ part of the extended gravitino is defined as the remaining $\theta\bar{\theta}$ component:

$$\psi_{(\beta\dot{\beta}\alpha)i} := -iD_{(\alpha}\bar{D}_{\dot{\beta}}\Psi_{\beta)i}|, \quad \delta\psi_{(\beta\dot{\beta}\alpha)i} = 2\partial_{\dot{\beta}(\beta}\xi_{\alpha)i}|, \quad \xi_{\alpha i} := \Xi_{\alpha i}|. \quad (\text{C.5})$$

¹⁷There is a dual formulation of the superspin-1 multiplet with prepotential $\Psi_{\alpha\beta\dot{\beta}}$ and field strength $W_{\alpha\beta} \sim \bar{D}^{\dot{\gamma}}D_{(\alpha}[2D^{\gamma}\Psi_{\beta)\gamma\dot{\gamma}} - \bar{D}^{\dot{\beta}}\bar{\Psi}_{\beta)\dot{\beta}\dot{\gamma}}]$ [63] (see also section 6.9 of [20]). We thank Sergei Kuzenko for emphasizing this point to us.

The remaining components of $\Psi_{\alpha i}$ are auxiliary fields that can be defined as

$$y_{\alpha\dot{\alpha}i} := -\frac{1}{4}D^2\bar{D}_{\dot{\alpha}}\Psi_{\alpha i}|, \quad (\text{C.6a})$$

$$t_{\alpha\beta i} := -\frac{1}{4}\bar{D}^2D_{(\alpha}\Psi_{\beta)i}|, \quad (\text{C.6b})$$

$$\rho_{\alpha i} := \frac{1}{3}D^{\beta}\bar{D}^2D_{(\alpha}\Psi_{\beta)i}|. \quad (\text{C.6c})$$

The component $t_{\alpha\beta i} = W_{\alpha\beta i}|$ describes an anti-self-dual rank-two tensor, and $\rho_{\alpha i}$ must play the role of a Lagrange multiplier due to its high dimension. Being Ω and Ξ invariant, these only transform under the L_{α} transformations. The auxiliary vector y_{ai} is in contrast subject to complex gauge transformations

$$\delta y_{ai} = \frac{i}{2}\partial_a D^2\Omega_i|. \quad (\text{C.7})$$

In Wess-Zumino gauge, the only residual gauge symmetries are the extended supersymmetry (C.5) and the bosonic symmetry associated with $D^2\Omega_i|$.

The residual gauge transformation of the gravitino multiplet associated with the 7 complex parameters $D^2\Omega_i|$ will allow 14 of the residual bosonic fields to be eliminated, or equivalently, eaten by the auxiliary field y_{ai} . We have already mentioned that the bottom component of H_i must be unphysical. Its gauge transformation turns out to be

$$\delta H_i| = \frac{i}{2}D^{\alpha}\Xi_{\alpha i}| - \frac{i}{2}\bar{D}_{\dot{\alpha}}\Xi^{\dot{\alpha}}_i| = \text{Im}D^2\Omega_i|, \quad (\text{C.8})$$

and so one can exploit half of the residual gauge symmetry of the gravitino multiplet to eliminate it. This fixes $D^2\Omega^i| = \bar{D}^2\bar{\Omega}^i|$. The other scalar fields are contained within $\Phi_{ijk}|$. These transform as (3.34b), which in the gauge $H_i| = 0$ implies

$$\delta\bar{\Phi}_{ijk}| = -\frac{i}{2}\tilde{\varphi}_{ijkl} \text{Re}D^2\Omega^l|. \quad (\text{C.9})$$

This ensures that only the imaginary part of $\Phi_{ijk}|$ transforms,

$$\delta F_{ijk} = -\frac{1}{2}\tilde{\varphi}_{ijkl} \text{Re}D^2\Omega^l| \quad (\text{C.10})$$

while the 3-form C_{ijk} is invariant. The residual gauge symmetry associated with $\text{Re}D^2\Omega^l|$ ensures that we can eliminate the **7**-component of F_{ijk} .

Finally, we find that the KK vector field transforms under Ω_i as

$$\delta\mathcal{A}^i_{\alpha\dot{\alpha}} = \frac{1}{2}[D_{\alpha}, \bar{D}_{\dot{\alpha}}]\delta\mathcal{V}^i| = -\frac{1}{4}[D_{\alpha}, \bar{D}_{\dot{\alpha}}](\Omega^i + \bar{\Omega}^i)| = \partial^i\xi_{\alpha\dot{\alpha}}, \quad (\text{C.11})$$

consistent with its interpretation as the component g_{mi} of the 11D metric. Note this result requires a precise interplay between the Ω_i and L_{α} transformations of the gravitino superfield in WZ gauge. In contrast, the Ω_i transformation leaves the vector fields A_{ija} inert, as expected at the linearized level.

C.2 Actions and compensators

Similarly to conformal supergravity, it is not possible to write a 1-derivative Rarita-Schwinger action for the conformal gravitino alone. (The only conformal invariant is the 2-derivative action $\int d^4x d^2\theta W^{\alpha\beta i} W_{\alpha\beta i}$.) To write an action, we require the analog of a scale compensator. Recalling the $N = 1$ old minimal supergravity action, we would expect such a compensator to provide (in WZ gauge) the missing spin- $\frac{1}{2}$ component of the gravitino as well as the longitudinal mode of the auxiliary vector y_{ai} (C.6a).

In this section we will include all possible compensator couplings from the outset. This includes superspins $\frac{1}{2}^+ \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^- \oplus 0$ corresponding to real and imaginary vector multiplets, a tensor multiplet, and a scalar multiplet [22]. For simplicity, we will ignore any internal y derivatives here. The most general quadratic action is of the form

$$L = \int d^4\theta L_D + \int d^2\theta L_F + \int d^2\bar{\theta} \bar{L}_F \quad (\text{C.12})$$

with (suppressing the index i now)

$$L_D = a_0 \bar{E}^{\alpha\dot{\alpha}} E_{\alpha\dot{\alpha}} + a_1 E^{\alpha\dot{\alpha}} E_{\alpha\dot{\alpha}} + \bar{a}_1 \bar{E}^{\alpha\dot{\alpha}} \bar{E}_{\alpha\dot{\alpha}} + a_2 \bar{B}B + a_3 B^2 + \bar{a}_3 \bar{B}^2 \\ + [\Psi^\alpha (a_4 \mathcal{W}_\alpha + a_5 W_\alpha + a_6 D_\alpha F + a_7 D_\alpha H) + \text{h.c.}] + a_8 F^2 + a_9 H^2 \quad (\text{C.13a})$$

$$L_F = a_{10} \mathcal{W}^\alpha \mathcal{W}_\alpha + a_{11} W^\alpha W_\alpha. \quad (\text{C.13b})$$

Here we have defined the complex potentials

$$E_{\alpha\dot{\alpha}} := \bar{D}_{\dot{\alpha}} \Psi_\alpha \quad \text{and} \quad B := D^\alpha \Psi_\alpha. \quad (\text{C.14})$$

This action is required to be invariant under the gauge transformations (C.3) and

$$\delta \Sigma_\alpha = -\bar{\Xi}_\alpha, \quad \delta V = \text{Im } \Omega, \quad \delta \Phi = -i \bar{D}^2 \bar{\Omega}, \quad \delta \mathcal{V} = -\text{Re } \Omega \quad (\text{C.15})$$

or, for the compensator “field strengths”

$$\delta H = \frac{1}{2i} (\bar{D} \bar{\Xi} - D \Xi) \quad (\text{C.16a})$$

$$\delta W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha \text{Im } \Omega \quad (\text{C.16b})$$

$$\delta F = -\frac{1}{2} (D^2 \Omega + \bar{D}^2 \bar{\Omega}) \quad (\text{C.16c})$$

$$\delta \mathcal{W}_\alpha = \frac{1}{4} \bar{D}^2 D_\alpha \text{Re } \Omega. \quad (\text{C.16d})$$

The Ξ transformation imposes the strong condition that Ψ and Σ appear only in the combination $\mathbf{\Psi} := \Psi + \Sigma$. The form of the field strength H then implies $(a_2, a_3, a_7, a_9) = (\frac{c}{2}, -\frac{c}{4}, ic, c)$ with $c \in \mathbf{R}$, so that these terms must all appear in the combination

$$\mathbf{H} = \frac{1}{2i} (D \mathbf{\Psi} - \bar{D} \bar{\mathbf{\Psi}}) = H + \frac{1}{2i} (B - \bar{B}) \quad \Rightarrow \quad \delta \mathbf{H} = \frac{1}{2i} (D^2 \Omega - \bar{D}^2 \bar{\Omega}) \quad (\text{C.17})$$

as a square. Imposing Ω invariance, we find that $(a_0, a_1) = (a, b) \in \mathbf{R} \times \mathbf{C}$ remain undetermined with $(a_4, a_5, a_6, a_8, a_{10}, a_{11}) = (2(a+b), 2i(a-b), -(a+c), a+c, a+b, a-b)$. Then, the action can be written as

$$\begin{aligned} L_D &= a\bar{E}^{\alpha\dot{\alpha}}E_{\alpha\dot{\alpha}} + \frac{b}{4}E^{\alpha\dot{\alpha}}E_{\alpha\dot{\alpha}} + \frac{\bar{b}}{4}\bar{E}^{\alpha\dot{\alpha}}\bar{E}_{\alpha\dot{\alpha}} + c\mathbf{H}^2 + (a+c)F^2 + \Psi^\alpha J_\alpha + \bar{\Psi}_{\dot{\alpha}}\bar{J}^{\dot{\alpha}} \\ L_F &= (a+b)\mathcal{W}^\alpha\mathcal{W}_\alpha + (a-b)W^\alpha W_\alpha \end{aligned} \quad (\text{C.18})$$

in terms of the matter current coupling

$$J_\alpha = 2(a+b)\mathcal{W}_\alpha + 2i(a-b)W_\alpha - (a+c)D_\alpha F. \quad (\text{C.19})$$

We now project this action to components. The bosonic Lagrangian is

$$\begin{aligned} L &= \frac{c}{2}(\partial_a H)^2 - \frac{c}{2}\tilde{H}_a^2 - \frac{a+c}{2}F_a^2 - \frac{a+c}{2}(\partial_a F)^2 - (a-b)F_{ab}^2 - (a+b)\mathcal{F}_{ab}^2 \\ &\quad - \frac{a}{2}(y_a + \bar{y}_a)^2 + \frac{a+c}{2}(y_a - \bar{y}_a)^2 + \frac{b}{2}t^{\alpha\beta}t_{\alpha\beta} + \frac{\bar{b}}{2}\bar{t}_{\dot{\alpha}\dot{\beta}}\bar{t}^{\dot{\alpha}\dot{\beta}} \\ &\quad + (y^a + \bar{y}^a)\hat{j}_a + i(y^a - \bar{y}^a)\check{j}_a + \frac{1}{2}t^{\alpha\beta}j_{\alpha\beta} + \frac{1}{2}\bar{t}_{\dot{\alpha}\dot{\beta}}\bar{j}^{\dot{\alpha}\dot{\beta}}. \end{aligned} \quad (\text{C.20})$$

Here we are defining the component currents

$$\begin{aligned} \hat{j}_a &= (a+c)F_a + c\partial_a H, & \check{j}_a &= (a+c)\partial_a F - c\tilde{H}_a \\ j_{\alpha\beta} &= 2(a+b)\mathcal{F}_{\alpha\beta} + 2i(a-b)F_{\alpha\beta} \end{aligned} \quad (\text{C.21})$$

where

$$F_{\alpha\beta} := D_{(\alpha}W_{\beta)}, \quad F_{ab} = -(\sigma_{ab})^{\alpha\beta}F_{\alpha\beta} - (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}F_{\dot{\alpha}\dot{\beta}}, \quad (\text{C.22})$$

$$F_{\alpha\dot{\alpha}} := \frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]F, \quad \tilde{H}_{\alpha\dot{\alpha}} := \frac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]H \quad (\text{C.23})$$

The auxiliary fields are contained in the second and third lines of (C.20), which we denote L_{aux} . Note that if $c = -a$, $\text{Im } y_a$ becomes a Lagrange multiplier.¹⁸ Integrating out auxiliary fields (assuming none of a , b , or $c+a$ vanish),

$$\begin{aligned} L_{aux} &\rightarrow -\frac{c}{2}(\partial_a H)^2 + \frac{c^2}{2(a+c)}\tilde{H}^a\tilde{H}_a + \frac{(a+c)^2}{2a}F^a F_a + \frac{a+c}{2}(\partial_a F)^2 \\ &\quad + \frac{m_{11}}{2}F_{ab}^2 + m_{12}F^{ab}\mathcal{F}_{ab} + \frac{m_{22}}{2}\mathcal{F}_{ab}^2, \end{aligned} \quad (\text{C.24})$$

where we have dropped total derivative terms like $F \wedge F$. The coefficients of the gauge field kinetic terms are complicated and given by

$$\begin{aligned} m_{11} &= -\frac{1}{b}(a-b)^2 - \frac{1}{\bar{b}}(a-\bar{b})^2, & m_{12} &= -\frac{i}{b}(a^2-b^2) + \frac{i}{\bar{b}}(a^2-\bar{b}^2), \\ m_{22} &= \frac{1}{b}(a+b)^2 + \frac{1}{\bar{b}}(a+\bar{b})^2. \end{aligned} \quad (\text{C.25})$$

¹⁸This is the case in the five-dimensional model [24], where $\text{Im } y_a$ trivializes the dynamics of the would-be 2-form which is not in the spectrum of five-dimensional fields. This interpretation is confirmed by an analysis of the field strengths of the theory [64].

This simplifies when b^2 is a real number. The case of most interest to us (and extended supergravity in general) is when b is a nonzero real number,

$$m_{11} = -\frac{2}{b}(a-b)^2, \quad m_{12} = 0, \quad m_{22} = \frac{2}{b}(a+b)^2, \quad \text{for } b \in \mathbf{R}^\times. \quad (\text{C.26})$$

Including the kinetic terms in the first line of (C.20) gives

$$L_{(a,b,c)} = -\frac{ac}{2(a+c)}\tilde{H}_a^2 + \frac{c(a+c)}{2a}F_a^2 - \frac{a}{b}(a-b)F_{ab}^2 + \frac{a}{b}(a+b)\mathcal{F}_{ab}^2. \quad (\text{C.27})$$

The scalar fields F and H have dropped out of the action for any value of the parameters a , b , and c . This is a straightforward consequence of gauge invariance, as in Wess-Zumino gauge, the scalars F and H transform as

$$\delta F = -\frac{1}{2}(D^2\Omega + \bar{D}^2\bar{\Omega})|, \quad \delta H = \frac{1}{2i}(D^2\Omega - \bar{D}^2\bar{\Omega})|. \quad (\text{C.28})$$

They can both be set to zero by a $D^2\Omega|$ gauge transformation. Equivalently, they are always eaten by the auxiliary gauge field y_a (C.6a), which is then integrated out.

C.3 Comments

The (matter) gravitino multiplet is encoded in a spinor superfield, which is reducible as a representation of the 4D, $N = 1$ super-Poincaré algebra containing superspins $1 \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^- \oplus \frac{1}{2}^- \oplus \frac{1}{2}^- \oplus 0$ [22].¹⁹ (Strictly speaking, retaining all superspins does not give a gauge multiplet.) At special points in the space of quadratic gravitino Lagrangians, this superspin content is reduced. When $b = a$, one of the vector multiplets decouples and we recover the results of Butter and Kuzenko. Further setting $c = -a$ recovers the model of Ogievetsky and Sokatchev, whereas setting $c = 0$ gives that of de Wit and van Holten and Fradkin and Vasiliev. Note that flipping the sign of b simply interchanges the role of the two vector multiplets. The gravitino sector of the 5D, $N = 1$ supergravity model of [24, 26, 64] has $(a, b, c) = (-1, 2, 1)$. Although this latter model was presented with its compensators gauge-fixed, we see from this analysis that these correspond to two vector multiplets and a tensor multiplet with the scalar decoupling since $a + c = 0$. Finally, we found in section 3.4 that eleven-dimensional supergravity corresponds to the point $(a, b, c) = \frac{1}{4} \times (-1, 2, -1)$. Note that this is a quite generic gravitino multiplet, missing only one superspin $\frac{1}{2}^-$ representation. These special values are collected in table 7.

Returning to the case appropriate to 11D, an important fact is that the ratio $\frac{a}{b} = -\frac{1}{2}$ is the same for the eleven-dimensional theory as it is for the five-dimensional one. The significance of this is that the Lagrangian depends only on $E_{\underline{a}}^i - \bar{E}_{\underline{a}}^i$ and not the other linear combination. We already saw the analogous statement for B : the action depends

¹⁹This language is that of irreducible representations of the four-dimensional super-Poincaré algebra. Superspin 1 contains ordinary spins $(\frac{3}{2}, 1)$, $\frac{1}{2}^+$ contains $(1, \frac{1}{2})$ corresponding to a vector multiplet. Superspins $\frac{1}{2}^-$ and 0 both correspond to spins $(\frac{1}{2}, 0)$, but the first is a 2-form gauge supermultiplet and the second is a scalar multiplet.

theory	references	b	c	supergravity	superspin [22]
OS	[58]	a	$-a$	4D, $N = 2$	$1 \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^-$
dWvHFV	[59, 60]	a	0	4D, $N = 2$	$1 \oplus \frac{1}{2}^+ \oplus 0$
LLP	[24, 64]	$-2a$	$-a$	5D, $N = 1$	$1 \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^+$
BBGRL	[5]	$-2a$	a	11D	$1 \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^- \oplus 0$

Table 7. Gravitino multiplets. The generic spinor superfield contains superspins $1 \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^+ \oplus \frac{1}{2}^- \oplus \frac{1}{2}^- \oplus 0$. At special points in the parameter space of gravitino Lagrangians $L_{(a,b,c)}$ (C.18) this superspin content is reduced. The two 4D, $N = 2$ entries correspond to distinct off-shell 4D, $N = 1$ embeddings [62].

only on the imaginary combination $B - \bar{B}$ and not the real one so

$$L_{(a,-2a,c)} = \int d^4\theta \left[-\frac{a}{2} \mathbf{E}_i^a \mathbf{E}_a^i + c \mathbf{H}^i \mathbf{H}_i + (a+c) F_i F^i + \Psi_i^\alpha J_\alpha^i + \bar{\Psi}_{\dot{\alpha}i} \bar{J}^{\dot{\alpha}i} \right] + 2a \int d^2\theta [3W_i^\alpha W_\alpha^i - \mathcal{W}_i^\alpha \mathcal{W}_\alpha^i] \tag{C.29}$$

where $\mathbf{E}_{ai} := E_{ai} - \bar{E}_{ai} = \bar{D}_{\dot{\alpha}} \Psi_{\alpha i} + D_\alpha \bar{\Psi}_{\dot{\alpha}i}$. Both of these statements are important when considering Y -dependence of the supergravity gauge parameters L_α , because they imply that it is possible to covariantize the conformal supergravity “mass” terms $(\partial_i H_a)^2 \rightarrow (\partial_i H_a + \mathbf{E}_{ai})^2$ by defining the gravitino transformation $\delta \Psi_{\alpha i} \sim 2i \partial_i L_\alpha$ [24].

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References

- [1] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum Configurations for Superstrings*, *Nucl. Phys.* **B 258** (1985) 46 [INSPIRE].
- [2] D.D. Joyce. *Compact Manifolds with Special Holonomy*, Oxford mathematical monographs. Oxford University Press, (2000).
- [3] K. Becker, M. Becker, W.D. Linch III and D. Robbins, *Abelian tensor hierarchy in 4D, $N = 1$ superspace*, *JHEP* **03** (2016) 052 [arXiv:1601.03066] [INSPIRE].
- [4] K. Becker, M. Becker, W.D. Linch III and D. Robbins, *Chern-Simons actions and their gaugings in 4D, $N = 1$ superspace*, *JHEP* **06** (2016) 097 [arXiv:1603.07362] [INSPIRE].
- [5] K. Becker, M. Becker, S. Guha, W.D. Linch III and D. Robbins, *M-theory potential from the G_2 Hitchin functional in superspace*, *JHEP* **12** (2016) 085 [arXiv:1611.03098] [INSPIRE].
- [6] K. Becker, M. Becker, W.D. Linch III, S. Randall and D. Robbins, *All Chern-Simons Invariants of 4D, $N = 1$ Gauged Superform Hierarchies*, *JHEP* **04** (2017) 103 [arXiv:1702.00799] [INSPIRE].

- [7] J. Halverson and D.R. Morrison, *The landscape of M-theory compactifications on seven-manifolds with G_2 holonomy*, *JHEP* **04** (2015) 047 [[arXiv:1412.4123](#)] [[INSPIRE](#)].
- [8] K. Becker, D. Robbins and E. Witten, *The α' Expansion On A Compact Manifold Of Exceptional Holonomy*, *JHEP* **06** (2014) 051 [[arXiv:1404.2460](#)] [[INSPIRE](#)].
- [9] K. Becker, M. Becker and D. Robbins, *Kaluza-Klein Theories Without Truncation*, *JHEP* **02** (2015) 140 [[arXiv:1412.8198](#)] [[INSPIRE](#)].
- [10] K. Becker, M. Becker, D. Butter, S. Guha, W.D. Linch III and D. Robbins, in preparation.
- [11] E. Cremmer, B. Julia and J. Scherk, *Supergravity Theory in Eleven-Dimensions*, *Phys. Lett.* **B 76** (1978) 409 [[INSPIRE](#)].
- [12] B. de Wit and H. Nicolai, *$d = 11$ Supergravity With Local $SU(8)$ Invariance*, *Nucl. Phys.* **B 274** (1986) 363 [[INSPIRE](#)].
- [13] E. Cremmer and B. Julia, *The $SO(8)$ Supergravity*, *Nucl. Phys.* **B 159** (1979) 141 [[INSPIRE](#)].
- [14] B. de Wit, H. Samtleben and M. Trigiante, *The maximal $D = 4$ supergravities*, *JHEP* **06** (2007) 049 [[arXiv:0705.2101](#)] [[INSPIRE](#)].
- [15] N.J. Hitchin, *The Geometry of Three-Forms in Six Dimensions*, *J. Diff. Geom.* **55** (2000) 547 [[math/0010054](#)] [[INSPIRE](#)].
- [16] N.J. Hitchin, *Stable forms and special metrics*, [math/0107101](#) [[INSPIRE](#)].
- [17] S.J. Gates Jr., *Super p -Form Gauge Superfields*, *Nucl. Phys.* **B 184** (1981) 381 [[INSPIRE](#)].
- [18] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, *Front. Phys.* **58** (1983) 1 [[hep-th/0108200](#)] [[INSPIRE](#)].
- [19] J. Wess and J. Bagger. *Supersymmetry and supergravity*, Princeton University Press, Princeton, U.S.A. (1992), [[INSPIRE](#)].
- [20] I.L. Buchbinder and S.M. Kuzenko. *Ideas and methods of supersymmetry and supergravity: Or a walk through superspace*, IOP, Bristol, U.K. (1998), [[INSPIRE](#)].
- [21] E. Cremmer, H. Lü, C.N. Pope and K.S. Stelle, *Spectrum generating symmetries for BPS solitons*, *Nucl. Phys.* **B 520** (1998) 132 [[hep-th/9707207](#)] [[INSPIRE](#)].
- [22] S.J. Gates Jr. and W. Siegel, *$(3/2, 1)$ Superfield of $O(2)$ Supergravity*, *Nucl. Phys.* **B 164** (1980) 484 [[INSPIRE](#)].
- [23] B. de Wit, J.W. van Holten and A. Van Proeyen, *Structure of $N = 2$ Supergravity*, *Nucl. Phys.* **B 184** (1981) 77 [*Erratum* *ibid.* **B 222** (1983) 516] [[INSPIRE](#)].
- [24] W.D. Linch III, M.A. Luty and J. Phillips, *Five-dimensional supergravity in $N = 1$ superspace*, *Phys. Rev.* **D 68** (2003) 025008 [[hep-th/0209060](#)] [[INSPIRE](#)].
- [25] R.L. Bryant, *Some remarks on G_2 -structures*, [math/0305124](#) [[INSPIRE](#)].
- [26] I.L. Buchbinder et al., *Supergravity loop contributions to brane world supersymmetry breaking*, *Phys. Rev.* **D 70** (2004) 025008 [[hep-th/0305169](#)] [[INSPIRE](#)].
- [27] S.M. Kuzenko, R. Manvelyan and S. Theisen, *Off-shell superconformal higher spin multiplets in four dimensions*, *JHEP* **07** (2017) 034 [[arXiv:1701.00682](#)] [[INSPIRE](#)].
- [28] A. Corti, M. Haskins, J. Nordström and T. Pacini, *G_2 -manifolds and associative submanifolds via semi-Fano 3-folds*, *Duke Math. J.* **164** (2015) 1971 [[arXiv:1207.4470](#)] [[INSPIRE](#)].

- [29] A.P. Braun and M. Del Zotto, *Mirror Symmetry for G_2 -Manifolds: Twisted Connected Sums and Dual Tops*, *JHEP* **05** (2017) 080 [[arXiv:1701.05202](#)] [[INSPIRE](#)].
- [30] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, *Int. J. Mod. Phys. A* **4** (1989) 2475 [[INSPIRE](#)].
- [31] A.P. Braun and S. Schäfer-Nameki, *Compact, Singular G_2 -Holonomy Manifolds and M /Heterotic/ F -Theory Duality*, [arXiv:1708.07215](#) [[INSPIRE](#)].
- [32] B.S. Acharya, *A moduli fixing mechanism in M -theory*, [hep-th/0212294](#) [[INSPIRE](#)].
- [33] G. Shiu, G. Torroba, B. Underwood and M.R. Douglas, *Dynamics of Warped Flux Compactifications*, *JHEP* **06** (2008) 024 [[arXiv:0803.3068](#)] [[INSPIRE](#)].
- [34] T.C. d.C. Guio, H. Jockers, A. Klemm and H.-Y. Yeh, *Effective action from M -theory on twisted connected sum G_2 -manifolds*, [arXiv:1702.05435](#) [[INSPIRE](#)].
- [35] D. Crowley, S. Goette and J. Nordström, *An analytic invariant of G_2 manifolds*, (2015). [arXiv:1505.02734](#).
- [36] D. Crowley and J. Nordström, *New invariants of G_2 -structures*, [arXiv:1211.0269](#).
- [37] D. Joyce and S. Karigiannis, *A new construction of compact G_2 -manifolds by gluing families of Eguchi-Hanson spaces*, [arXiv:1707.09325](#).
- [38] C. Arias, W.D. Linch III and A.K. Ridgway, *Superforms in six-dimensional superspace*, *JHEP* **05** (2016) 016 [[arXiv:1402.4823](#)] [[INSPIRE](#)].
- [39] S.J. Gates Jr., W.D. Linch and S. Randall, *Superforms in Five-Dimensional, $N = 1$ Superspace*, *JHEP* **05** (2015) 049 [[arXiv:1412.4086](#)] [[INSPIRE](#)].
- [40] W.D. Linch III and S. Randall, *Superspace de Rham Complex and Relative Cohomology*, *JHEP* **09** (2015) 190 [[arXiv:1412.4686](#)] [[INSPIRE](#)].
- [41] E. Cremmer and S. Ferrara, *Formulation of Eleven-Dimensional Supergravity in Superspace*, *Phys. Lett. B* **91** (1980) 61 [[INSPIRE](#)].
- [42] L. Brink and P.S. Howe, *Eleven-Dimensional Supergravity on the Mass-Shell in Superspace*, *Phys. Lett. B* **91** (1980) 384 [[INSPIRE](#)].
- [43] F. Paccetti Correia, M.G. Schmidt and Z. Tavartkiladze, *Superfield approach to 5D conformal SUGRA and the radion*, *Nucl. Phys. B* **709** (2005) 141 [[hep-th/0408138](#)] [[INSPIRE](#)].
- [44] H. Abe and Y. Sakamura, *Dynamical radion superfield in 5D action*, *Phys. Rev. D* **71** (2005) 105010 [[hep-th/0501183](#)] [[INSPIRE](#)].
- [45] Y. Sakamura, *Superfield description of gravitational couplings in generic 5D supergravity*, *JHEP* **07** (2012) 183 [[arXiv:1204.6603](#)] [[INSPIRE](#)].
- [46] Y. Sakamura, *One-loop Kähler potential in 5D gauged supergravity with generic prepotential*, *Nucl. Phys. B* **873** (2013) 165 [*Erratum ibid.* **B 873** (2013) 728] [[arXiv:1302.7244](#)] [[INSPIRE](#)].
- [47] H. Abe, Y. Sakamura and Y. Yamada, *$N = 1$ superfield description of vector-tensor couplings in six dimensions*, *JHEP* **04** (2015) 035 [[arXiv:1501.07642](#)] [[INSPIRE](#)].
- [48] H. Abe, Y. Sakamura and Y. Yamada, *$\mathcal{N} = 1$ superfield description of six-dimensional supergravity*, *JHEP* **10** (2015) 181 [[arXiv:1507.08435](#)] [[INSPIRE](#)].
- [49] H. Abe, S. Aoki and Y. Sakamura, *Full diffeomorphism and Lorentz invariance in 4D $\mathcal{N} = 1$ superfield description of 6D SUGRA*, [arXiv:1708.09106](#) [[INSPIRE](#)].

- [50] S. Karigiannis, *Deformations of G_2 and Spin(7) Structures on Manifolds*, *Can. J. Math.* **57** (2005) 1012 [[math/0301218](#)].
- [51] E. Sokatchev, *Projection Operators and Supplementary Conditions for Superfields with an Arbitrary Spin*, *Nucl. Phys. B* **99** (1975) 96 [[INSPIRE](#)].
- [52] W. Siegel and S.J. Gates Jr., *Superprojectors*, *Nucl. Phys. B* **189** (1981) 295 [[INSPIRE](#)].
- [53] W. Siegel, *Solution to Constraints in Wess-Zumino Supergravity Formalism*, *Nucl. Phys. B* **142** (1978) 301 [[INSPIRE](#)].
- [54] W. Siegel and S.J. Gates Jr., *Superfield Supergravity*, *Nucl. Phys. B* **147** (1979) 77 [[INSPIRE](#)].
- [55] S.J. Gates Jr., S.M. Kuzenko and J. Phillips, *The off-shell (3/2, 2) supermultiplets revisited*, *Phys. Lett. B* **576** (2003) 97 [[hep-th/0306288](#)] [[INSPIRE](#)].
- [56] M.T. Grisaru and W. Siegel, *Supergraphity. Part 1. Background Field Formalism*, *Nucl. Phys. B* **187** (1981) 149 [[INSPIRE](#)].
- [57] S.J. Gates Jr. and W. Siegel, *Variant Superfield Representations*, *Nucl. Phys. B* **187** (1981) 389 [[INSPIRE](#)].
- [58] V.I. Ogievetsky and E. Sokatchev, *On Gauge Spinor Superfield*, *JETP Lett.* **23** (1976) 58 [[INSPIRE](#)].
- [59] B. de Wit and J.W. van Holten, *Multiplets of Linearized SO(2) Supergravity*, *Nucl. Phys. B* **155** (1979) 530 [[INSPIRE](#)].
- [60] E.S. Fradkin and M.A. Vasiliev, *Minimal Set Of Auxiliary Fields And S-matrix For Extended Supergravity*, *Lett. Nuovo Cim.* **25** (1979) 79 [[INSPIRE](#)].
- [61] S.J. Gates Jr. and V.A. Kostelecky, *Supersymmetric Matter Gravitino Multiplets*, *Nucl. Phys. B* **248** (1984) 570 [[INSPIRE](#)].
- [62] D. Butter and S.M. Kuzenko, *$N = 2$ supergravity and supercurrents*, *JHEP* **12** (2010) 080 [[arXiv:1011.0339](#)] [[INSPIRE](#)].
- [63] S.M. Kuzenko and A.G. Sibiryakov, *Massless gauge superfields of higher integer superspins*, *JETP Lett.* **57** (1993) 539 [[INSPIRE](#)].
- [64] S.J. Gates Jr., W.D. Linch III and J. Phillips, *Field strengths of linearized 5D, $N = 1$ superfield supergravity on a three-brane*, *JHEP* **02** (2005) 036 [[hep-th/0311153](#)] [[INSPIRE](#)].