

A proposal for (0,2) mirrors of toric varieties

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ABSTRACT: In this paper we propose (0,2) mirrors for general Fano toric varieties with special tangent bundle deformations, corresponding to subsets of toric deformations. Our mirrors are of the form of (B/2-twisted) (0,2) Landau-Ginzburg models, matching Hori-Vafa mirrors on the (2,2) locus. We compare our predictions to (0,2) mirrors obtained by Chen et al. for certain examples of toric varieties, and find that they match. We also briefly outline conjectures for analogous results for hypersurfaces in Fano toric varieties. Our methods utilize results from supersymmetric localization, which allows us to incidentally gain occasional further insights into GLSM-based (2,2) mirror constructions. For example, we explicitly verify that closed string correlation functions of the original A-twisted GLSM match those of the mirror B-twisted Landau-Ginzburg model, as well as (0,2) deformations thereof.

KEYWORDS: Sigma Models, Supersymmetric Gauge Theory, Topological Field Theories, Supersymmetry and Duality

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1 Introduction

Ordinary mirror symmetry has had a long history in string theory. This paper concerns a heterotic generalization of ordinary mirror symmetry, sometimes known as (0,2) mirror symmetry. Whereas ordinary mirror symmetry relates, in simple cases, pairs of Calabi-Yau spaces X_1, X_2 , (0,2) mirror symmetry relates pairs $(X_1, \mathcal{E}_1), (X_2, \mathcal{E}_2)$, where $\mathcal{E}_i \rightarrow X_i$ is a holomorphic vector bundle such that $\text{ch}_2(\mathcal{E}_i) = \text{ch}_2(TX_i)$.

Ordinary mirror symmetry is now well-understood, but (0,2) mirror symmetry is still under development, and has been for a number of years (see e.g. [1–7]). Many basics have been worked out: there is a (0,2) version [2] of the Greene-Plesser orbifold construction [8], there has been an attempt [3] to duplicate GLSM-based dualities [9, 10], and for the case that \mathcal{E} is a deformation of the tangent bundle of a ‘reflexively plain’ Calabi-Yau hypersurface, there is a (0,2) analogue [4] of Batyrev’s construction [11–13]. Furthermore, there is now a (0,2) analogue of quantum cohomology, known as quantum sheaf cohomology, which has been developed in e.g. [14–29]. The present state-of-the-art is that quantum sheaf cohomology has been computed for toric varieties and Grassmannians with deformations of the tangent bundle. (At present, however, a heterotic analogue of Gromov-Witten invariants [30] is not yet known.)

Ideally, one would like to understand mirrors to basic cases such as the quintic with a tangent bundle deformation. At present, not even such basic examples are understood. One strategy to construct such mirrors would be to use abelian duality [9, 10] to construct mirrors to toric ambient spaces, and then standard tricks to extrapolate to conjectures for mirrors to compact Calabi-Yau hypersurfaces.

Such a strategy was attempted in [3], who discovered that the methods previously applied in [10] seem to crucially require (2,2) supersymmetry — or at least a (0,2) extension will require new ideas. As a result, the (0,2) version of abelian duality is not presently understood. Worse, unlike the case when [10] was written, until recently there were no known examples of (0,2) Landau-Ginzburg mirrors to Fano spaces, not even for simple cases such as $\mathbb{P}^1 \times \mathbb{P}^1$, which complicates efforts to extend abelian duality to (0,2) cases.

As part of a program of better understanding (0,2) mirror symmetry, one of the authors has been engaged with various collaborators in a program of constructing such Landau-Ginzburg mirrors for Fano spaces [6, 7], to help cut through the difficulties above. In those works, mirrors were constructed for (0,2) GLSMs for products of projective spaces, toric del Pezzo surfaces, and Hirzebruch surfaces,¹ with (Euler-type) tangent bundle deformations. In each case, mirrors were constructed in a laborious non-systematic piecemeal fashion by guessing ansatzes and comparing chiral rings and correlation functions to determine coefficients — no systematic formulas applicable to all cases were produced.

In this paper we propose formulas for (0,2) B/2-twisted mirrors to A/2 models on toric Fano spaces (and closely related toric varieties), and present corresponding conjectures for hypersurfaces, for a special class of Euler-type tangent bundle deformations corresponding

¹Most Hirzebruch surfaces are not Fano, but as discussed in [7], one expects them to flow to isolated vacua in the IR, so one expects to be able to use the same techniques to build a mirror to the GLSM, which is more properly interpreted as the mirror to a different geometric phase (the UV phase) of the GLSM.

to a subset of ‘toric’ deformations. (To be clear, we are proposing a formula for Landau-Ginzburg mirrors, but we are not claiming to have a worldsheet dualization procedure along the lines of [10].)

We will check that our systematic construction successfully duplicates results (for this special class of deformations) for the examples of toric Fano surfaces described in [6, 7]. The methods we present here will only apply to a subset of the deformations considered in [6, 7], but will produce mirrors systematically and quickly, unlike the methods used in [6, 7] to arrive at the results presented there.

Our methods will use ideas and results from supersymmetric localization [31], first applied to two-dimensional GLSMs in [32, 33].

We begin in section 2 by quickly reviewing existing results on GLSM-based mirror constructions in theories with (2,2) supersymmetry. In section 3 we describe our proposal for (0,2) mirrors to toric Fano varieties. In section 4 we describe formal arguments for why correlation functions match between the original A/2-twisted GLSM and the mirror B/2-twisted Landau-Ginzburg model. In section 5 we describe several examples, checking that the predictions of our proposal match existing results worked out in [6, 7]. In section 6 we describe how to formally extend these results to hypersurfaces, following the same pattern that has been followed for (2,2) mirror symmetry.

Other recent work on two-dimensional (0,2) theories from different directions includes e.g. [34–44].

2 Review of (2,2) Fano mirrors

Let us quickly review the mirror ansatz for abelian (2,2) GLSMs for Fano toric varieties in [10].

2.1 General aspects

2.1.1 Basics

First, we consider a GLSM with gauge group $U(1)^k$ and N chiral superfields, with charges encoded in charge matrix (Q_i^a) .

Following [10], the mirror is a theory with k superfields Σ_a , as many as $U(1)$ s in the original GLSM, and N twisted chiral fields Y_i , as many as chiral multiplets in the original GLSM, of periodicity $2\pi i$, with superpotential

$$W = \sum_{a=1}^k \Sigma_a \left(\sum_{i=1}^N Q_i^a Y_i - t_a \right) + \mu \sum_{i=1}^N \exp(-Y_i), \tag{2.1}$$

where μ is a scale factor.

In the expression above, the Σ_a act effectively as Lagrange multipliers, generating constraints

$$\sum_{i=1}^N Q_i^a Y_i = t_a \tag{2.2}$$

originating with the D terms of the original theory. We can solve these constraints formally² by writing

$$Y_i = \sum_{A=1}^{N-k} V_i^A \theta_A + \tilde{t}_i \tag{2.3}$$

where θ_A are the surviving physical degrees of freedom, \tilde{t}_i are solutions of

$$\sum_{i=1}^N Q_i^a \tilde{t}_i = t_a, \tag{2.4}$$

and V_i^A is a rank- $(N - k)$ matrix solving

$$\sum_{i=1}^N Q_i^a V_i^A = 0. \tag{2.5}$$

(The rank requirement goes hand-in-hand with the statement that there are $N - k$ independent θ_A 's.) The periodicity of the Y_i 's will lead to interpretations of the space of θ_A 's in terms of LG orbifolds and character-valued fields, as we shall review later. Note that for t_i, V_i^A satisfying the equation above,

$$\sum_{i=1}^N Q_i^a Y_i = \sum_i Q_i^a \left(\sum_A V_i^A \theta_A + \tilde{t}_i \right) = t_a,$$

and so the V_i^A encode a solution of the D-term constraints.

After integrating out the Lagrange multipliers, the superpotential can be rewritten as

$$W = \mu \sum_{i=1}^N \left(e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A) \right). \tag{2.6}$$

In this language, the (2,2) mirror map between A- and B-model operators is (partially) defined by

$$\sum_{a=1}^k Q_i^a \sigma_a \leftrightarrow \mu \exp(-Y_i) = \mu e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A), \tag{2.7}$$

which can be derived by differentiating (2.1) with respect to Y_i . (See for example [10, Section 3.2], where this is derived as the equations of motion of the mirror theory. In the next section, we will also see that this map is consistent with axial R symmetries.) In fact, this overdetermines the map — only a subset of the Y_i 's will be independent variables solving the constraints (2.2). As we will see explicitly later, the redundant equations are equivalent to chiral ring relations (as must follow since they all arise as the same equations of motion in the mirror), and are also specified by the equations of motion derived from the superpotential W above.

²The expressions given here are entirely formal, and there can be subtleties. For example, if the entries in V_i^A are fractional, then as is well-known, the mirror may have orbifolds.

In appendix A we will briefly outline a variation on the usual GLSM-based mirror derivation. Regardless of how the B-model mirror superpotential is obtained, it can be checked by comparing closed-string A model correlation functions between the mirror and the original A-twisted GLSM using supersymmetric localization. For (2,2) theories, this can be done at arbitrary genus using the methods of [45, 46], whereas for (0,2) theories, we can only apply analogous tests at genus zero. We will perform such correlation function checks later in this paper.

2.1.2 R charges

Let us take a moment to consider R charges. In the A-twisted theory, the axial R-charge is in general broken by nonperturbative effects, so that under an axial symmetry transformation, anomalies induce a shift in the theta angle³ by

$$\theta^a \mapsto \theta^a + 2\alpha \sum_i Q_i^a, \quad t_a \mapsto t_a + 2i\alpha \sum_i Q_i^a,$$

for α parametrizing axial R symmetry rotations. The shift above can formally be described as

$$\tilde{t}_i \mapsto \tilde{t}_i + 2i\alpha,$$

(using the relation between \tilde{t}_i and t^a in (2.4)). In the same vein, under the same axial R symmetry, the mirror field Y_i transforms as

$$Y_i \mapsto Y_i + 2i\alpha,$$

so that $\exp(-Y_i)$ has axial R-charge 2. If we take Σ_a to also have axial R-charge 2, then it is easy to verify that the entire mirror superpotential (2.1) has axial R-charge 2, as desired, taking the t 's to have nonzero R-charge as described. In addition, the operator mirror map (2.7) is also consistent with axial R-charges in that case.

2.1.3 Twisted masses

One can also consider adding twisted masses. Recall that a twisted mass can be thought of as the vev of a vector multiplet, gauging some flavor symmetry. Taking the vev removes the gauge field, gauginos, and auxiliary field, and replaces them with a single mass parameter \tilde{m} , corresponding to the vev of the σ field. In the notation of [47, Eq. (2.19)], this means, for a single U(1) flavor symmetry that acts on a field ϕ_i with charge $Q_{F,i}$, we add terms to the action of the form

$$-2|\tilde{m}|^2 \sum_i Q_{F,i}^2 |\phi_i|^2 - \sqrt{2} \sum_i Q_{F,i} (\tilde{m} \bar{\psi}_{+,i} \psi_{-,i} + \tilde{m} \bar{\psi}_{-,i} \psi_{+,i}).$$

In the present case, for a toric variety with no superpotential, there are at least as many flavor symmetries as chiral superfields modulo gauged U(1)s, i.e. at least $N - k$ U(1) flavor symmetries. (There can also be nonabelian components.) For simplicity, we will

³This should not be confused with the fundamental field θ_A defined earlier.

simply allow for a twisted mass \tilde{m}_i associated to each chiral superfield, and will not try to distinguish between those related by gauge U(1)s.

Including twisted masses \tilde{m}_i , the full mirror superpotential (before integrating out Σ 's) takes the form

$$W = \sum_{i=1}^N \left(\sum_{a=1}^k \Sigma_a Q_i^a + \tilde{m}_i \right) (Y_i - \tilde{t}_i) + \mu \sum_{i=1}^N \exp(-Y_i). \tag{2.8}$$

This expression manifestly has consistent axial R-charge 2 (using the ‘modified’ R-charge that acts on \tilde{t}_i). It differs from the more traditional expression [10, Eq. (3.86)]

$$W = \sum_{a=1}^k \Sigma_a \left(\sum_i Q_i^a Y_i - t_a \right) + \sum_{i=1}^N \tilde{m}_i Y_i + \mu \sum_{i=1}^N \exp(-Y_i), \tag{2.9}$$

by a constant term (proportional to $\sum_{i=1}^N \tilde{m}_i \tilde{t}_i$), and so defines the same physics.

After including twisted masses, the operator mirror map becomes

$$\sum_{a=1}^k Q_i^a \sigma_a + \tilde{m}_i \leftrightarrow \mu \exp(-Y_i).$$

Note that both sides of this expression are consistent with the (modified) R-charge assignments described above.

Generically in this paper we will absorb μ into a redefinition of the Y_i 's, and so not write it explicitly, but we mention it here for completeness.

Finally, we should remind the reader that in addition to the superpotential above, one may also need to take an orbifold to define the theory, as is well-known. This will happen if, for example, some of the entries in (V_i^A) are fractions, in order to reflect ambiguities in taking the roots implicit in resulting expressions such as $\exp(-V_i^A \theta_A)$.

2.2 Example with twisted masses

To give another perspective, in this section we will review the (2,2) mirror to the GLSM for $\text{Tot}(\mathcal{O}(-n) \rightarrow \mathbb{P}^2)$, for $n \leq 3$ (and no superpotential), and to make this interesting, we will include twisted masses \tilde{m}_i , corresponding to phase rotations of each field.

The charge matrix for this GLSM is

$$Q = (1, 1, 1, -n),$$

and following the usual procedure, the D terms constrain the dual (twisted) chiral superfields as

$$Y_1 + Y_2 + Y_3 - nY_p = t.$$

The standard procedure at this point is to eliminate Y_p , and write the dual potential in terms of Y_{1-3} , taking a \mathbb{Z}_n orbifold to account for the fractional coefficients of the Y_i and its periodicity. In other words,

$$Y_p = \frac{1}{n} (Y_1 + Y_2 + Y_3 - t),$$

hence the (2,2) superpotential is given by

$$\begin{aligned} W &= \sum_i \tilde{m}_i Y_i + \exp(-Y_1) + \exp(-Y_2) + \exp(-Y_3) + \exp(-Y_p), \\ &= \sum_i \tilde{m}_i Y_i + (\exp(-Y_1/n))^n + (\exp(-Y_2/n))^n + (\exp(-Y_3/n))^n \\ &\quad + \exp(-t/n) \exp(-Y_1/n) \exp(-Y_2/n) \exp(-Y_3/n). \end{aligned}$$

Phrased more simply, if we define $Z_i = \exp(-Y_i/n)$, then the (2,2) mirror theory is, as expected, a \mathbb{Z}_n orbifold with superpotential

$$W = - \sum_i \tilde{m}_i n \ln Z_i + Z_1^n + Z_2^n + Z_3^n + \exp(-t/n) Z_1 Z_2 Z_3,$$

with the understanding that the fundamental fields are Y_i s not Z_i s. (For hypersurfaces, the fundamental fields will change.)

Later, we will use the matrices (V_i^A) extensively, so in that language, the change of variables above is encoded in

$$(V_i^A) = \begin{bmatrix} 1 & 0 & 0 & 1/n \\ 0 & 1 & 0 & 1/n \\ 0 & 0 & 1 & 1/n \end{bmatrix}.$$

Then, we write $Y_i = V_i^A \theta_A$, and so

$$Y_1 = \theta_1, \quad Y_2 = \theta_2, \quad Y_3 = \theta_3, \quad Y_p = (1/n)(\theta_1 + \theta_2 + \theta_3 - t).$$

Let us next discuss the operator mirror map. This is given by

$$\begin{aligned} \exp(-Y_1) &= Z_1^n \leftrightarrow \sigma, \\ \exp(-Y_2) &= Z_2^n \leftrightarrow \sigma, \\ \exp(-Y_3) &= Z_3^n \leftrightarrow \sigma, \\ \exp(-Y_p) &= Z_1 Z_2 Z_3 \exp(-t/n) \leftrightarrow -n\sigma. \end{aligned}$$

2.3 (2,2) in (0,2) language

Now, let us describe (2,2) mirrors in (0,2) language, as preparation for describing more general (0,2) mirrors. Let (Σ_a, Υ_a) be the (0,2) chiral and Fermi components of Σ_a , and (Y_i, F_i) the (0,2) chiral and Fermi components of Y_i . Then, the (2,2) superpotential (2.8) is given in (0,2) superspace by

$$W = \sum_{a=1}^k \left[\Upsilon_a \left(\sum_{i=1}^N Q_i^a Y_i - t_a \right) + \sum_{i=1}^N \Sigma_a Q_i^a F_i \right] - \mu \sum_{i=1}^N F_i \exp(-Y_i) + \sum_{i=1}^N \tilde{m}_i F_i. \quad (2.10)$$

We integrate out Σ_a, Υ_a to get the constraints

$$\sum_{i=1}^N Q_i^a Y_i = t_a, \quad \sum_{i=1}^N Q_i^a F_i = 0,$$

which we solve with the V_i^A by writing

$$Y_i = \sum_{A=1}^{N-k} V_i^A \theta_A + \tilde{t}_i, \quad F_i = \sum_{A=1}^{N-k} V_i^A G_A,$$

where (θ_A, G_A) are the chiral and Fermi components of the (2,2) chiral superfields θ_A . After integrating out the constraints, the (0,2) superpotential becomes

$$W = \sum_{i=1}^N \sum_{A=1}^{N-k} G_A V_i^A (\tilde{m}_i - \mu \exp(-Y_i)) = \sum_{i=1}^N \sum_{A=1}^{N-k} G_A V_i^A \left(\tilde{m}_i - \mu e^{\tilde{t}_i} \prod_{B=1}^{N-k} \exp(-V_i^B \theta_B) \right). \quad (2.11)$$

As is standard, we remind that reader that depending upon the entries in (V_i^A) , the mirror may be a LG orbifold, which are required to leave W invariant.

In this language, the (2,2) mirror map between A- and B-model operators is (partially) defined by

$$\sum_{a=1}^k Q_i^a \sigma_a + \tilde{m}_i \leftrightarrow \mu \exp(-Y_i) = \mu e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A), \quad (2.12)$$

which can be derived by differentiating (2.10) with respect to F_i .

In most of the rest of this paper, we will absorb μ into a field redefinition of the Y_i s for simplicity, but we include it here for completeness.

3 Proposal for (0,2) Fano mirrors

We restrict to (0,2) theories obtained by (some) toric deformations of abelian (2,2) GLSMs for Fano spaces, by which we mean physically that we choose E 's such that $E_i \propto \phi_i$, where on the (2,2) locus ϕ_i is the chiral superfield paired with the Fermi superfield whose superderivative is E_i .

In addition, to define a mirror, we also make another choice, namely we pick an invertible⁴ $k \times k$ submatrix, of the charge matrix (Q_i^a) , which we will denote S . The choice of S will further constrain the allowed toric deformations — for a given S , we only consider some toric deformations. Our mirror will depend upon the choice of S , and since different S 's will yield different allowed bundle deformations, there need not be a simple coordinate transformation relating results for different choices of S in general. Furthermore, S is only relevant for bundle deformations — it does not enter (2,2) locus computations, and so it has no analogue within [10].

For a given choice of S , in the A/2 model, write

$$E_i = \sum_{a=1}^k \sum_{j=1}^N (\delta_{ij} + B_{ij}) Q_j^a \sigma_a \phi_i,$$

⁴We assume that the charge matrix does indeed have an invertible $k \times k$ submatrix. If not, then the theory has at least one free decoupled U(1), and after performing a change of basis to explicitly decouple those U(1)'s, our analysis can proceed on the remainder.

where in the expression above, we do not sum over i 's. The (0,2) deformations we will consider are encoded in the matrices B_{ij} , where $B_{ij} = 0$ if i defines a column of the matrix S . Note that, at least on its face, this does not describe all possible Euler-sequence-type (0,2) deformations, but only a special subset. We will give a mirror construction for that special subset.

Then, the mirror can be described by a collection of \mathbb{C}^\times -valued fields Y_i (just as on the (2,2) locus, dual to the chiral superfields of the original theory), satisfying the same D-term constraints as on the (2,2) locus, and with (0,2) superpotential

$$W = \sum_{a=1}^k \left[\Upsilon_a \left(\sum_{i=1}^N Q_i^a Y_i - t_a \right) + \sum_{i=1}^N \Sigma_a Q_i^a F_i \right] - \mu \sum_i F_i \exp(-Y_i) + \mu \sum_i F_i \left(\sum_{i_S, j, a} B_{ij} Q_j^a [(S^{-1})^T]_{ai_S} \exp(-Y_{i_S}) \right), \quad (3.1)$$

where i_S denotes an index running through the columns of S , and where the second term was chosen so that the resulting equations of motion duplicate the chiral ring. (For the moment, we have assumed no twisted masses are present; we will return to twisted masses at the end of this section.)

Now, to do meaningful computations, we must apply the D-term constraints to both Y_i 's and F_i 's. Applying the D-term constraints to the F_i 's to write them in terms of G_A 's (i.e. integrating out Σ_a 's), and for simplicity suppressing the Υ_a constraints and setting the mass scale μ to unity, we have the expression

$$W = - \sum_{A=1}^{N-k} G_A \left(\sum_i V_i^A \exp(-Y_i) + \sum_{i_S} D_{i_S}^A \exp(-Y_{i_S}) \right), \quad (3.2)$$

where

$$D_{i_S}^A = - \sum_{i, j} \sum_a V_i^A B_{ij} Q_j^a [(S^{-1})^T]_{ai_S} \quad (3.3)$$

Note when $B = 0$, $D = 0$, and the expression for W above immediately reduces to its (2,2) locus form. We will derive this expression for D below.

In this language, the mirror map between A/2- and B/2-model observables is defined by

$$\sum_{a=1}^k \sum_{j=1}^N (\delta_{ij} + B_{ij}) Q_j^a \sigma_a \leftrightarrow \exp(-Y_i) = e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A). \quad (3.4)$$

(Strictly speaking, we will see in examples that these equations define not only the operator mirror map plus some of the chiral ring relations.)

We can derive the operator mirror map above from the superpotential (3.1) by taking a derivative with respect to F_i , as before. Doing so, one finds

$$Q_i^a \sigma_a - \exp(-Y_i) + \sum_{i_S, j, a} B_{ij} Q_j^a [(S^{-1})^T]_{ai_S} \exp(-Y_{i_S}) = 0.$$

For i corresponding to columns of S , $B_{ij} = 0$, and the expression above simplifies to

$$S_{i_S}^a \sigma_a = \exp(-Y_{i_S}).$$

Plugging this back in, we find

$$Q_i^a \sigma_a - \exp(-Y_i) + \sum_{j,a} B_{ij} Q_j^a \sigma_a = 0,$$

which is easily seen to be the operator mirror map (3.4).

We can apply the operator mirror map as follows. Recall that the constraints imply

$$\sum_i Q_i^a Y_i = t_a$$

hence

$$\prod_i \exp(-Q_i^a Y_i) = \exp(-t_a) = q_a,$$

hence plugging in the proposed map (3.4) above, we have

$$\prod_i \left(\sum_{a=1}^k \sum_{j=1}^N (\delta_{ij} + B_{ij}) Q_j^a \sigma_a \right)^{Q_i^a} = q_a,$$

which is the chiral ring relation in the A/2-twisted GLSM.

In passing, to make the method above work, it is important that the determinants appearing in quantum sheaf cohomology relations in e.g. [22–25] all factorize. In other words, recall that for a general tangent bundle deformation, the quantum sheaf cohomology ring relations take the form

$$\prod_{\alpha} (\det M_{\alpha})^{Q_{\alpha}^a} = q_a,$$

where α denotes a block of chiral fields with the same charges, and M_{α} encodes the E 's, which will mix chiral superfields of the same charges. In order for the operator mirror map construction we have outlined above to work, it is necessary that each $\det M_{\alpha}$ factorize into a product of factors, one for each matter chiral multiplet. This is ultimately the reason why in this paper we have chosen to focus on ‘toric’ deformations, in which each E 's do not mix different matter chiral multiplets.

Now, in terms of the operator mirror map, let us derive the form of D above in equation (3.3). The equations of motion from the superpotential (3.2) are given by

$$\frac{\partial W}{\partial G_A} = \sum_i V_i^A \exp(-Y_i) + \sum_{i_S} D_{i_S}^A \exp(-Y_{i_S}) = 0.$$

Now, we plug in the operator mirror map (3.4) above to get

$$\sum_i V_i^A \left(\sum_a \sum_j (\delta_{ij} + B_{ij}) Q_j^a \sigma_a \right) + \sum_{i_S} D_{i_S}^A \left(\sum_a \sum_j (\delta_{i_S j} + B_{i_S j}) Q_j^a \sigma_a \right) = 0.$$

Using the constraint

$$\sum_i V_i^A Q_i^a = 0,$$

the first δ_{ij} term vanishes, and furthermore, since the matrix B is defined to vanish for indices from columns of S , we see that in the second term, $B_{i_S j} = 0$, hence the equation above reduces to

$$\sum_{i,j} \sum_a V_i^A B_{ij} Q_j^a \sigma_a + \sum_{i_S} \sum_a D_{i_S}^A S_{i_S}^a \sigma_a = 0.$$

Since this should hold for all σ_a , we have that

$$\sum_{i,j} V_i^A B_{ij} Q_j^a + \sum_{i_S} D_{i_S}^A S_{i_S}^a = 0,$$

which can be solved to give expression (3.3) for D above.

Thus, the expression for the superpotential (3.2) together with the operator mirror map (3.4) has equations of motion that duplicate the chiral ring.

In passing, one could also formally try to consider more general cases in which a submatrix $S \subset Q$ is not specified. One might then try to take the expression for the mirror superpotential to be of the form

$$W = - \sum_{A=1}^{N-k} G_A \left(\sum_i V_i^A \exp(-Y_i) + \sum_i D_i^A \exp(-Y_i) \right),$$

where now the i index on D is allowed to run over all chiral superfields, not just a subset. Following the methods above, one cannot uniquely solve for D — one gets families of possible D 's with undetermined coefficients, and we do not know how to argue that the correlation functions match for all such coefficients without restricting to subsets defined by choices $S \subset Q$.

Now, in principle, for (0,2) theories defined by deformations of the (2,2) locus, there is an analogue of twisted masses that one can add to the theory. In the (2,2) case, twisted masses corresponded to replacing a vector multiplet by its vevs, so that only a residue of σ survived. In (0,2), by contrast, the vector multiplet does not contain σ , only the gauge field, gauginos, and auxiliary fields D , so we can no longer interpret the twisted mass in terms of replacing a vector multiplet with its vevs.

Instead, we can understand the analogue of a twisted mass in a (0,2) theory corresponding to a deformation of the (2,2) locus in terms of additions to $E_i = \overline{D}_+ \Lambda^i$, for Fermi superfields Λ^i . In particular, the (2,2) vector multiplet's σ field enters GLSMs written in (0,2) superfields as a factor in such E 's, so twisted masses enter similarly, as terms of the form

$$E_i = \tilde{m}_i \phi_i$$

(where as usual we are admitting the possibility of several toric symmetries, and simply giving each chiral superfield the possibility of its own twisted mass). Such terms are only possible if the (0,2) superpotential has compatible J 's, meaning that in order for

supersymmetry to hold, one requires $E \cdot J = 0$, as usual. This is a residue of the requirement in the (2,2) theory that twisted masses arise from flavor symmetries.

We have already seen, in section 2.3, how (2,2) twisted masses can be represented in the mirror, described in (0,2) superspace. To describe their combination with E deformations is straightforward. Briefly, the (0,2) mirror superpotential takes the form

$$W = - \sum_{A=1}^{N-k} G_A \left(\sum_i V_i^A \exp(-Y_i) + \sum_{i_S} D_{i_S}^A \exp(-Y_{i_S}) \right) + \sum_{i=1}^N \sum_{A=1}^{N-k} G_A V_i^A \tilde{m}_i, \quad (3.5)$$

with $(D_{i_S}^A)$ defined as in (3.3), and the operator mirror map has the form

$$\sum_{a=1}^k \sum_{j=1}^N (\delta_{ij} + B_{ij}) Q_j^a \sigma_a + \tilde{m}_i \leftrightarrow \exp(-Y_i) = e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A). \quad (3.6)$$

4 Correlation functions

In this section, we will argue formally that correlation functions in our proposed (0,2) mirrors match those of the original theory. More precisely, we will compare closed-string correlation functions of A- or A/2-twisted GLSM σ 's to corresponding correlation functions in B- or B/2-twisted Landau-Ginzburg models. (Often, the Landau-Ginzburg mirror will be an orbifold; we will only compare against untwisted sector correlation functions in such orbifolds.) Our computations will focus on genus zero computations, but in (2,2) cases, in principle can be generalized to any genus. (See also [48] for related work.)

Before doing so, let us first outline in what sense correlation functions match. There are two possibilities:

- First, for special matrices (V_i^A) , we will argue that correlation functions match on the nose. In order for this to happen, we will need to require that the determinant of an invertible $k \times k$ submatrix of the charge matrix Q , match (up to sign) the determinant of a complementary⁵ $(N - k) \times (N - k)$ submatrix of (V_i^A) .
- Alternatively, we can always formally rescale some of the Y_i s (without introducing or removing orbifolds) to arrange for the determinants above to match, up to sign. In this case, the correlation functions of one theory are isomorphic to those of the other theory, but the numerical factors will not match on the nose. (Instead, the relations between numerical factors will be determined by the rescaling of the Y_i s.)

In either event, correlation functions will match.

⁵‘Complementary’ in this case means that if the $k \times k$ matrix is defined by i 's corresponding to certain chiral superfields, then those same chiral superfields cannot appear corresponding to any i 's in the $(N - k) \times (N - k)$ submatrix of (V_i^A) .

4.1 (2,2) supersymmetric cases

We will first check that on the (2,2) locus, the ansatz described above (i.e. the ansatz of [10]) generates matching correlation functions between the A-twisted GLSM and its B-twisted Landau-Ginzburg model mirror. (See also [49] for an analogous comparison of partition functions.)

First, let us consider correlation functions in an A-twisted GLSM. An exact expression is given for fully massive cases in e.g. [50, Eq. (4.77)]:

$$\langle \mathcal{O} \rangle = \frac{(-)^{N_c}}{|W|} \frac{1}{(-2\pi i)^{\text{rk } G}} \sum_{\sigma_P} \mathcal{O} \frac{Z_{1\text{-loop}}}{H}$$

where G is the GLSM gauge group, W its Weyl group, N_c its rank,

$$Z_{1\text{-loop}} = \prod_{i=1}^N \left(\sum_{a=1}^k Q_i^a \sigma_a \right)^{R(\Phi_i)-1},$$

($R(\Phi_i)$ the R-charge, which for simplicity we will assume to vanish,) σ_P the vacua, and H is the Hessian of the twisted one-loop effective action, meaning

$$H = \det \left(\sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right), \tag{4.1}$$

using (up to factors) the twisted one-loop effective action in e.g. [51, Eq. (3.36)].

Now, up to irrelevant overall factors, there is an essentially identical expression for Landau-Ginzburg correlation functions [52], involving the Hessian of the superpotential rather than $H/Z_{1\text{-loop}}$ above. Therefore, to show that correlation functions match, we will argue that the $H/Z_{1\text{-loop}}$ above, computed for the A-twisted GLSM, matches the Hessian of superpotential derivatives for the mirror Landau-Ginzburg model.

First, since we are only interested in the determinant, we can rotate the charge matrix (Q_i^a) by an element $U \in \text{SL}(k, \mathbb{C})$ without changing the determinant:

$$\det \left(\sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right) \mapsto (\det U)^2 \det \left(\sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right) = \det \left(\sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right).$$

Thus, it will be convenient to rotate the charge matrix to the form⁶

$$Q_i^a = \begin{pmatrix} a_1 & & Q_{k+1}^1 & \cdots & Q_N^1 \\ & \ddots & \vdots & \ddots & \vdots \\ & & a_k & Q_{k+1}^k & \cdots & Q_N^k \end{pmatrix}. \tag{4.2}$$

Note that for the charge matrix in this form,

$$Z_{1\text{-loop}} = \left(\prod_{i=1}^k a_i \sigma_i \right)^{-1} \left(\prod_{i=k+1}^N \left(\sum_{a=1}^k Q_i^a \sigma_a \right) \right)^{-1}.$$

⁶As we are not conjugating the charge matrix, but rather multiplying on one side only, it should be possible to arrange for a $k \times k$ submatrix to be diagonal, not just in Jordan normal form.

To put the charge matrix in this form, write

$$Q = [S|*] = (\det S)^{1/k}[S'|*],$$

where S is $k \times k$. Then, multiply in $(S')^{-1}$, to get

$$(S')^{-1}Q = (\det S)^{1/k}[I|*],$$

which is now diagonal.

It is straightforward to compute

$$\begin{aligned} H &= \det \left(\sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right) \\ &= \det \begin{pmatrix} \frac{a_1}{\sigma_1} + \frac{(Q_{k+1}^1)^2}{Q_{k+1}^c \sigma_c} + \dots + \frac{(Q_N^1)^2}{Q_N^c \sigma_c} & \frac{Q_{k+1}^1 Q_{k+1}^2}{Q_{k+1}^c \sigma_c} + \dots + \frac{Q_N^1 Q_N^2}{Q_N^c \sigma_c} & \dots \\ \frac{Q_{k+1}^2 Q_{k+1}^1}{Q_{k+1}^c \sigma_c} + \dots + \frac{Q_N^2 Q_N^1}{Q_N^c \sigma_c} & \frac{a_2}{\sigma_2} + \frac{(Q_{k+1}^2)^2}{Q_{k+1}^c \sigma_c} + \dots + \frac{(Q_N^2)^2}{Q_N^c \sigma_c} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \quad (4.3)$$

We define⁷ $t_i = a_i/\sigma_i$ and $E_i^a = Q_i^a/\sqrt{\sum_c Q_i^c \sigma_c}$, then the matrix in the above determinant becomes

$$\begin{pmatrix} t_1 + (E_{k+1}^1)^2 + \dots + (E_N^1)^2 & E_{k+1}^1 E_{k+1}^2 + \dots + E_N^1 E_N^2 & \dots \\ E_{k+1}^2 E_{k+1}^1 + \dots + E_N^2 E_N^1 & t_2 + (E_{k+1}^2)^2 + \dots + (E_N^2)^2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (4.4)$$

When all the t_i vanish, one can straightforwardly see that the matrix above is the product $(E^T)^T E^T$, for matrix E

$$E = \begin{pmatrix} E_{k+1}^1 & E_{k+2}^1 & \dots & E_N^1 \\ E_{k+1}^2 & E_{k+2}^2 & \dots & E_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ E_{k+1}^k & \dots & \dots & E_N^k \end{pmatrix}. \quad (4.5)$$

Using standard results from linear algebra, the generalized characteristic polynomial of matrix (4.4), in terms of the variables t_i , is given by

$$\sum_{m=0}^k \left(\sum_{a_1 < \dots < a_m} t_{a_1} \dots t_{a_m} \det(M_{a_1 \dots a_m}) \right), \quad (4.6)$$

where the matrix $M_{a_1 \dots a_m}$ denotes the submatrix of $M = (E^T)^T E^T$ by omitting rows $a_1 \dots a_m$ and columns $a'_1 \dots a'_m$ (i.e. a principal minor of M of size $k - m$). (In our conventions, the determinant vanishes if M has no entries.) Notice that $M = (E^T)^T E^T$, so the determinant can be written more simply as

$$\det M + t_1 \dots t_k + \sum_{m=1}^{k-1} \left(\sum_{a_1 < \dots < a_m} t_{a_1} \dots t_{a_m} \left(\sum_{i_1 < \dots < i_{N+m-2k}} (\det E_{a_1 \dots a_m, i_1 \dots i_{N+m-2k}})^2 \right) \right), \quad (4.7)$$

⁷The reader should note that the t_i in this section, defined above, is not related to t 's used earlier to describe FI parameters.

where $\det E_{a_1 \dots a_m, i_1 \dots i_{N+m-2k}}$ denotes the determinant of the submatrix of E formed by omitting rows $a_1 \dots a_m$ and columns $i_1 \dots i_{N+m-2k}$. (We formally require it to be zero when $N + m - 2k < 0$.)

Finally, we divide by $Z_{1\text{-loop}}$ to get an expression for $H/Z_{1\text{-loop}}$ where H is the Hessian (4.1):

$$\frac{\det M + t_1 \dots t_k}{Z_{1\text{-loop}}} + \sum_{m=1}^{k-1} \left(\sum_{a_1 < \dots < a_m} (a_{a_1})^2 \dots (a_{a_m})^2 \left(\prod_{i \notin \{a_1, \dots, a_m\}} \left(\sum_{a=1}^k Q_i^a \sigma_a \right) \right) B_{a_1 \dots a_m} \right), \quad (4.8)$$

for

$$B_{a_1 \dots a_m} = \sum_{i_1 < \dots < i_{N+m-2k}} (\det E_{a_1 \dots a_m, i_1 \dots i_{N+m-2k}})^2,$$

where $\det M$ vanishes for $N < 2k$. For later use, note that for $N \leq 2k$ we can expand

$$\frac{\det M}{Z_{1\text{-loop}}} = \left(\prod_{i=1}^k a_i \sigma_i \right) \left(\prod_{i \notin \{i_1, \dots, i_k\}} \left(\sum_{c=1}^k Q_{k+i}^c \sigma_c \right) (A_{i_1, \dots, i_k})^2 \right), \quad (4.9)$$

and the terms for $1 \leq m \leq k-1$ are given by

$$a_{a_1}^2 \dots a_{a_m}^2 \left(\prod_{b \notin \{a_1, \dots, a_m\}} a_b \sigma_b \right) \left(\prod_{i \notin \{i_{m+1}, \dots, i_k\}} \left(\sum_{c=1}^k Q_{k+i}^c \sigma_c \right) (A_{i_{m+1}, \dots, i_k})^2 \right), \quad (4.10)$$

where A_{i_{m+1}, \dots, i_k} denotes the sum of determinants of all $(k-m) \times (k-m)$ submatrices of the charge matrix (Q_i^a) for values of $i > k$.

Next, we need to compare the ratio $H/Z_{1\text{-loop}}$ above to the analogous Hessian arising in the mirror B-twisted Landau-Ginzburg model. Here, there is a nearly identical computation in which the Hessian we just computed is replaced with the determinant of second derivatives of the mirror superpotential (2.6):

$$\begin{aligned} \frac{\partial^2 W}{\partial \theta_A \partial \theta_B} &= - \sum_{i=1}^N \left(e^{\tilde{t}_i} \left(\prod_{C=1}^{N-k} \exp(-V_i^C \theta_C) \right) V_i^A V_i^B \right), \\ &= - \sum_{i=1}^N \left(\left(\sum_{a=1}^k Q_i^a \sigma_a \right) V_i^A V_i^B \right), \end{aligned}$$

using the operator mirror map (2.7).

Thus, we need to compute

$$\det \left(\sum_i V_i^A V_i^B \sum_c Q_i^c \sigma_c \right),$$

and compare to the ratio $H/Z_{1\text{-loop}}$ from the A model that we computed previously. In principle, the argument here is very similar to the argument just given for the determinant

defined by charge matrices. First, using the fact that V has rank $N - k$, inside the determinant we can rotate V to the more convenient form

$$V_i^A = \begin{pmatrix} V_1^1 & \cdots & V_k^1 & \lambda^1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ V_1^k & \cdots & V_k^k & & & \lambda^{(N-k)} \end{pmatrix}. \quad (4.11)$$

In fact, we can say more. Given that the V matrix was originally defined to satisfy

$$\sum_i Q_i^a V_i^A = 0,$$

after the rotation above inside the determinant, the V matrix should in fact have the form

$$V_i^A = \begin{pmatrix} -\frac{\lambda^1 Q_{k+1}^1}{a_1} & \cdots & -\frac{\lambda^1 Q_{k+1}^k}{a_k} & \lambda^1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ -\frac{\lambda^{(N-k)} Q_N^1}{a_1} & \cdots & -\frac{\lambda^{(N-k)} Q_N^k}{a_k} & & & \lambda^{(N-k)} \end{pmatrix}. \quad (4.12)$$

Then, using the more convenient form of V above, we find that we can write the matrix

$$\begin{aligned} & \left(\sum_i V_i^A V_i^B \sum_c Q_i^c \sigma_c \right) \\ &= \begin{pmatrix} (\lambda^1)^2 \left[\frac{(Q_{k+1}^1)^2 \sigma_1}{a_1} + \cdots + \frac{(Q_{k+1}^k)^2 \sigma_k}{a_k} + Q_{k+1}^c \sigma_c \right] & \lambda^1 \lambda^2 \left[\frac{Q_{k+1}^1 Q_{k+2}^1 \sigma_1}{a_1} + \cdots + \frac{Q_{k+1}^k Q_{k+2}^k \sigma_k}{a_k} \right] & \cdots \\ \lambda^2 \lambda^1 \left[\frac{Q_{k+1}^1 Q_{k+2}^1 \sigma_1}{a_1} + \cdots + \frac{Q_{k+1}^k Q_{k+2}^k \sigma_k}{a_k} \right] & (\lambda^2)^2 \left[\frac{(Q_{k+2}^1)^2 \sigma_1}{a_1} + \cdots + \frac{(Q_{k+2}^k)^2 \sigma_k}{a_k} + Q_{k+2}^c \sigma_c \right] & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \quad (4.13)$$

Similarly, we define $s_i = (\lambda^i)^2 Q_{k+i}^c \sigma_c$ and $F_i^a = \lambda^i Q_{k+i}^a \sqrt{\sigma_a / a_a}$ (without summing over the index a). Then, the matrix above can be written as

$$\begin{pmatrix} s_1 + (F_1^1)^2 + \cdots + (F_1^k)^2 & F_1^1 F_2^1 + \cdots + F_1^k F_2^k & \cdots \\ F_2^1 F_1^1 + \cdots + F_2^k F_1^k & s_2 + (F_2^1)^2 + \cdots + (F_2^k)^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (4.14)$$

When all s_i vanish, one can observe that the matrix is the product $F^T F$, for

$$F = \begin{pmatrix} F_1^1 & F_2^1 & \cdots & F_{N-k}^1 \\ F_1^2 & F_2^2 & \cdots & F_{N-k}^2 \\ \vdots & \vdots & \ddots & \vdots \\ F_1^k & F_2^k & \cdots & F_{N-k}^k \end{pmatrix}. \quad (4.15)$$

By using the same technique we can show that the determinant of (4.13) vanishes for $N > 2k$, and for $N \leq 2k$ is

$$\begin{aligned} & \det(F^T F) + s_1 \cdots s_{N-k} \\ &+ \sum_{n=1}^{N-k-1} \left(\sum_{i_1 < \cdots < i_n} (s_{i_1} s_{i_2} \cdots s_{i_n}) \left(\sum_{a_1 < \cdots < a_{2k-N+n}} (\det F_{i_1 \cdots i_n, a_1 \cdots a_{2k-N+n}})^2 \right) \right) \end{aligned} \quad (4.16)$$

For later use, note that

$$\begin{aligned} \det F^T F &= \sum_{a_1 < \dots < a_{2k-N}} (\det F_{a_1 \dots a_{2k-N}})^2, \\ &= \left(\prod_{A=1}^{N-k} (\lambda^A)^2 \right) \left(\prod_{b=1}^k \frac{\sigma_b}{a_b} \right) \left(\sum_{i_1 < \dots < i_k} \left(\sum_{a_1, \dots, a_k} Q_{k+i_1}^{a_1} \dots Q_{k+i_k}^{a_k} \epsilon_{a_1 \dots a_k} \right)^2 \right), \end{aligned}$$

where $F_{a_1 \dots a_{2k-N}}$ denotes the submatrix of F_i^a formed by deleting columns a_1 through a_{2k-N} .

Next, we plug

$$s_{i_j} = (\lambda^{i_j})^2 Q_{k+i_j}^c \sigma_c$$

into equation (4.16), and compare equation (4.8). First, note that we can expand

$$\frac{t_1 \dots t_k}{Z_{1\text{-loop}}} = \left(\prod_{i=1}^k a_i^2 \right) \left(\prod_{j=1}^N \left(\sum_{a=1}^k Q_j^a \sigma_a \right) \right),$$

which matches

$$s_1 \dots s_{N-k} = \prod_{A=1}^{N-k} (\lambda^A)^2 \left(\sum_{a=1}^k Q_{k+A}^a \sigma_a \right)$$

so long as

$$\prod_{A=1}^{N-k} \lambda^A = \pm \prod_{i=1}^k a_i. \tag{4.17}$$

Analogous results hold for other terms, as we now verify. First we consider the case $N \geq 2k$. The term $\det M/Z_{1\text{-loop}}$ in the previous determinant corresponds to the term $n = N - 2k$ in the expansion (4.16), which is given by

$$\left(\prod_{A=1}^{N-k} \lambda^A \right)^2 \prod_{a=1}^k \frac{\sigma_a}{a_a} \prod_{i \notin \{i_1, \dots, i_k\}} \left(\sum_{c=1}^k Q_{k+i}^c \sigma_c \right) (A_{i_1 \dots i_k})^2,$$

for $A_{i_1 \dots i_k}$ defined previously. It is easy to verify that this matches equation (4.9) for $\det M/Z_{1\text{-loop}}$ so long as condition (4.17) is satisfied, just as before. The remaining terms in expansion (4.16) for any given n correspond to terms in (4.8) with m related by $n = N - 2k + m$, and have the form

$$\left(\prod_{A=1}^{N-k} \lambda^A \right)^2 \prod_{b \notin \{a_1, \dots, a_m\}} \frac{\sigma_b}{a_b} \left(\prod_{i \notin \{i_{m+1}, \dots, i_k\}} \left(\sum_{c=1}^k Q_{k+i}^c \sigma_c \right) (A_{i_{m+1} \dots i_k})^2 \right),$$

and it is easy to verify that this matches equation (4.10) so long as condition (4.17) is satisfied, just as before. The reader can now straightforwardly verify that analogous statements hold for the cases $k < N < 2k$, which exhausts all nontrivial possibilities.

Thus, we see that correlation functions will match so long as equation (4.17) holds. Furthermore, we can always arrange for equation (4.17) to hold. If it does not do so initially,

then as discussed at the start of this section, we can perform field redefinitions and rescale Y_i s to arrange for it to hold, at the cost of making the isomorphism between the correlation functions of either theory a shade more complicated. For example, the coefficient of

$$\left(\sum_c Q_{k+1}^c \sigma_c\right) \cdots \left(\sum_d Q_N^d \sigma_d\right)$$

in equation (4.8) is $(a_1 a_2 \cdots a_k)^2$, and the coefficient of the term of the same order in equation (4.16) is $(\lambda^1 \lambda^2 \cdots \lambda^{N-k})^2$. We see that equation (4.17) is required for equality hold, and the choice of sign in equation (4.17) should not have any physical significance.

So far, we have worked at genus zero, but the same argument also implies that the same closed-string correlation functions match at arbitrary genus. At genus g , A-twisted GLSM correlation functions are computed in the same fashion albeit with a factor of $(H/Z_{1\text{-loop}})^{g-1}$ (see e.g. [45, Section 4], [46, Section 5.1]), whereas B-twisted Landau-Ginzburg model correlation functions (in the untwisted sector) are computed with a factor of $(H')^{g-1}$ [52], for H' the determinant of second derivatives of the mirror superpotential. Demonstrating that $H/Z_{1\text{-loop}} = H'$ therefore not only demonstrates that genus zero correlation functions match, but also higher-genus correlation functions. (For (0,2) theories, by contrast, higher genus correlation functions are not yet understood, so there we will only be able to compare genus zero correlation functions.)

Essentially the same argument applies if one adds twisted masses to the theory. One simply makes the substitution

$$\sum_a Q_i^a \sigma_a \rightarrow \sum_a Q_i^a \sigma_a + \tilde{m}_i, \tag{4.18}$$

where \tilde{m}_i is the twisted mass. The details of the proof above are essentially unchanged. Also note that we are free to redefine σ_a to $\sigma_a + c_a$, and we can use this to set the first k twisted masses to zero. This leaves $N - k$ twisted masses, consistent with a global flavor symmetry $U(1)^{N-k}$.

The arguments above hold so long as one can integrate out all of the matter Higgs fields, to obtain a pure Coulomb branch. In the (2,2) theory one expects that one should be able to do this if one adds sufficient twisted masses (see e.g. [45, Section 2.3]). (In particular, adding twisted masses can act as a substitute for going far out along the Coulomb branch, which also makes the matter fields massive.)

4.2 (0,2) supersymmetric cases

We will now generalize the previous argument to (0,2) cases.

Our argument here will be very similar to that given for (2,2) cases. We will compare results for correlation functions in A/2-twisted GLSMs computed with supersymmetric localization to results for correlation functions computed in B/2-twisted (0,2) Landau-Ginzburg models.

First, as before, applying supersymmetric localization to an A/2-twisted GLSM, there is an exact formula for (genus zero) (0,2) correlation functions [25], which has more or

less the same form as in the (2,2) case, now involving a Hessian of derivatives of a twisted one-loop (0,2) effective action [22], which takes the form

$$H = \det \left(\sum_i \frac{\sum_j Q_i^a A_{ij} Q_j^b}{\sum_m A_{im} Q_m^c \sigma_c} \right), \quad (4.19)$$

where $A_{ij} = \delta_{ij} + B_{ij}$.

We assume without loss of generality that the invertible S submatrix of the charge matrix corresponds to the first k columns of Q . Then, one can show that the determinant (4.19) above is equal to

$$\det \begin{pmatrix} \frac{a_1}{\sigma_1} + \frac{Q_{k+1}^1(Q_{k+1}^1 + \varepsilon_{k+1}^1)}{(Q_{k+1}^a + \varepsilon_{k+1}^a)\sigma_a} + \dots + \frac{Q_N^1(Q_N^1 + \varepsilon_N^1)}{(Q_N^a + \varepsilon_N^a)\sigma_a} & \frac{Q_{k+1}^1(Q_{k+1}^2 + \varepsilon_{k+1}^2)}{(Q_{k+1}^a + \varepsilon_{k+1}^a)\sigma_a} + \dots + \frac{Q_N^1(Q_N^2 + \varepsilon_N^2)}{(Q_N^a + \varepsilon_N^a)\sigma_a} & \dots \\ \frac{Q_{k+1}^2(Q_{k+1}^1 + \varepsilon_{k+1}^1)}{(Q_{k+1}^a + \varepsilon_{k+1}^a)\sigma_a} + \dots + \frac{Q_N^2(Q_N^1 + \varepsilon_N^1)}{(Q_N^a + \varepsilon_N^a)\sigma_a} & \frac{a_2}{\sigma_2} + \frac{Q_{k+1}^2(Q_{k+1}^2 + \varepsilon_{k+1}^2)}{(Q_{k+1}^a + \varepsilon_{k+1}^a)\sigma_a} + \dots + \frac{Q_N^2(Q_N^2 + \varepsilon_N^2)}{(Q_N^a + \varepsilon_N^a)\sigma_a} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.20)$$

where $\varepsilon_i^a = \sum_j B_{ij} Q_j^a$.

Now, in the B/2-twisted (0,2) Landau-Ginzburg model, there is an analogous expression for correlation functions [17], involving the Hessian

$$\det \frac{\partial^2 W}{\partial G_A \partial \theta_B}.$$

One can similarly show that the Hessian above is given by (using the (0,2) operator mirror map (3.4))

$$\det \begin{pmatrix} (\lambda^1)^2 \left[\sum_{b=1}^k \frac{Q_{k+1}^b(Q_{k+1}^b + \varepsilon_{k+1}^b)\sigma_b}{a_b} + S_{k+1} \right] & \lambda^1 \lambda^2 \left[\sum_{b=1}^k \frac{(Q_{k+1}^b + \varepsilon_{k+1}^b) Q_{k+2}^b \sigma_b}{a_b} \right] & \dots \\ \lambda^2 \lambda^1 \left[\sum_{b=1}^k \frac{Q_{k+1}^b(Q_{k+2}^b + \varepsilon_{k+2}^b)\sigma_b}{a_b} \right] & (\lambda^2)^2 \left[\sum_{b=1}^k \frac{Q_{k+2}^b(Q_{k+2}^b + \varepsilon_{k+2}^b)\sigma_b}{a_b} + S_{k+2} \right] & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.21)$$

where $S_{k+i} = \sum_a (Q_{k+i}^a + \varepsilon_{k+i}^a) \sigma_a$.

Finally, following the same steps as for (2,2), one can show that the ratio $H/Z_{1\text{-loop}}$ appearing in the A/2-twisted GLSM matches the Hessian appearing in the B/2-twisted Landau-Ginzburg model,

$$\det \left(\sum_i \frac{\sum_j Q_i^a A_{ij} Q_j^b}{\sum_m A_{im} Q_m^c \sigma_c} \right) \left(\prod_{i=1}^k a_i \sigma_i \right) \left(\prod_{j=k+1}^N (Q_j^a + \varepsilon_j^a) \sigma_a \right) = \det \frac{\partial^2 W}{\partial G_A \partial \theta_B},$$

so long as

$$\prod_{i=1}^{N-k} \lambda^i = \pm \prod_{i=1}^k a_i.$$

(As before, if this does not hold, we can always perform field redefinitions to rescale some of the Y_i s and corresponding Fermi fields F_i , at the cost of making the isomorphism between

correlation functions of either theory slightly more complicated.) Thus, the A/2-twisted GLSM Hessian matches that arising in B/2-twisted Landau-Ginzburg model correlation functions [17]. Since correlation functions in the A/2-twisted GLSM and the B/2-twisted (0,2) Landau-Ginzburg model have essentially the same form, albeit with potential different Hessians, and we have now demonstrated that the Hessians match, it follows that correlation functions match.

5 Examples

So far we have presented formal arguments for a (0,2) mirror defined by a (0,2) Landau-Ginzburg theory with the same chiral ring and correlation functions⁸ as the original A/2 theory. In this section we will verify that this proposal reproduces known results in specific examples.

To be specific, we will compare predictions to mirror proposals previously made in [6, 7]. Those papers were originally written by guessing ansatzes for possible mirrors, constrained to match known results on the (2,2) locus and to have the correct correlation functions and chiral ring relations. Here, we will see that the proposal we have presented correctly and systematically reproduces the results obtained by much more laborious methods in [6, 7]. This will implicitly also provide tests that correlation functions and chiral rings do indeed match, as argued formally in the last section.

That said, the systematic proposal of this paper will only apply to special, ‘toric’ deformations, not all tangent bundle deformations, not even all tangent bundle deformations realizable by Euler-type sequences. Curiously, the terms in the mirrors described in [6, 7] that are not realized are nonlinear in the fields, suggesting that toric deformations are mirror to linear terms. We will not pursue this direction further in this paper, but mention it here for completeness.

5.1 $\mathbb{P}^1 \times \mathbb{P}^1$

We begin by reviewing the now-nearly-canonical example of $\mathbb{P}^1 \times \mathbb{P}^1$. The charge matrix of the GLSM for the chiral fields is of the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and deformations of the tangent bundle are described mathematically as the cokernel of the short exact sequence

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0,$$

where the map $*$ is given by

$$* = \begin{bmatrix} \tilde{A}x & \tilde{B}x \\ \tilde{C}y & \tilde{D}y \end{bmatrix},$$

⁸Technically, untwisted sector correlation functions, if the mirror involves an orbifold.

where \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} are constant 2×2 matrices, and x , y are two-component vectors of homogeneous coordinates on either \mathbb{P}^1 factor.

We have the same constraints on fields from D terms as on the (2,2) locus, namely:

$$Y_1 + Y_2 = t_1, \quad Y_3 + Y_4 = t_2,$$

where $Y_{1,2}$ are dual to the variables for one \mathbb{P}^1 , and $Y_{3,4}$ are dual to the variables for the other.

Let us solve the constraints above by taking

$$\tilde{t}_1 = 0, \quad \tilde{t}_2 = t_1, \quad \tilde{t}_3 = 0, \quad \tilde{t}_4 = t_2,$$

and

$$V_i^A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

so that

$$\begin{aligned} Y_1 &= \theta, & Y_2 &= t_1 - \theta, & G^1 &= F_1 = -F_2, \\ Y_3 &= \tilde{\theta}, & Y_4 &= t_2 - \tilde{\theta}, & G^2 &= F_3 = -F_4. \end{aligned}$$

5.1.1 First choice of S

We will first consider the case that the submatrix S is given by the first and third columns of the charge matrix Q above, so that S is the identity matrix. Then, the allowed deformations are defined by

$$A_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 1 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

(Rows correspond to fixed i index, and reflect the fact that for i a row of S , values are fixed to those of the (2,2) locus.) We can find the corresponding tangent bundle deformations by comparing the E_i 's. In terms of the matrix above,

$$\begin{aligned} E_1 &= \sum_{j,a} A_{1j} Q_j^a \sigma_a \phi_1, \\ &= \sigma_1 \phi_1, \\ E_2 &= ((A_{21} + A_{22})\sigma_1 + (A_{23} + A_{24})\sigma_2) \phi_2, \\ E_3 &= \sigma_2 \phi_3, \\ E_4 &= (A_{41} + A_{42})\sigma_1 + (A_{43} + A_{44})\sigma_2 \phi_4, \end{aligned}$$

whereas in terms of \tilde{A} , \tilde{B} , \tilde{C} , \tilde{D} , we have

$$\begin{aligned} E_1 &= (\tilde{A}_{11}\phi_1 + \tilde{A}_{12}\phi_2)\sigma_1 + (\tilde{B}_{11}\phi_1 + \tilde{B}_{12}\phi_2)\sigma_2, \\ E_2 &= (\tilde{A}_{21}\phi_1 + \tilde{A}_{22}\phi_2)\sigma_1 + (\tilde{B}_{21}\phi_1 + \tilde{B}_{22}\phi_2)\sigma_2, \\ E_3 &= (\tilde{C}_{11}\phi_3 + \tilde{C}_{12}\phi_4)\sigma_1 + (\tilde{D}_{11}\phi_3 + \tilde{D}_{12}\phi_4)\sigma_2, \\ E_4 &= (\tilde{C}_{21}\phi_3 + \tilde{C}_{22}\phi_4)\sigma_1 + (\tilde{D}_{21}\phi_3 + \tilde{D}_{22}\phi_4)\sigma_2, \end{aligned}$$

for $\phi_{1,2}$ corresponding to homogeneous coordinates on the first \mathbb{P}^1 factor and $\phi_{3,4}$ corresponding to homogeneous coordinates on the second. Comparing these two expressions, we find that in terms of the original 2×2 matrices, our ansatz is equivalent to the special case

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} 1 & 0 \\ 0 & A_{21} + A_{22} \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} 0 & 0 \\ 0 & A_{23} + A_{24} \end{bmatrix}, \\ \tilde{C} &= \begin{bmatrix} 0 & 0 \\ 0 & A_{41} + A_{42} \end{bmatrix}, & \tilde{D} &= \begin{bmatrix} 1 & 0 \\ 0 & A_{43} + A_{44} \end{bmatrix}.\end{aligned}$$

From formula (3.3), we have that

$$(D_{i_S}^A) = \begin{bmatrix} A_{21} + A_{22} - 1 & A_{23} + A_{24} \\ A_{41} + A_{42} & A_{43} + A_{44} - 1 \end{bmatrix},$$

where the A index counts across rows, and the i_S index counts across columns.

The (0,2) superpotential of the proposed mirror is then given by (3.2)

$$\begin{aligned}W &= -G_1 (e^{-Y_1} - e^{-Y_2} + (A_{21} + A_{22} - 1)e^{-Y_1} + (A_{23} + A_{24})e^{-Y_3}) \\ &\quad - G_2 (e^{-Y_3} - e^{-Y_4} + (A_{41} + A_{42})e^{-Y_1} + (A_{43} + A_{44} - 1)e^{-Y_3}),\end{aligned}$$

If we define $X_i = \exp(-Y_i)$, then the (0,2) superpotential above can be written

$$\begin{aligned}W &= -G_1 \left((A_{21} + A_{22})X_1 - \frac{q_1}{X_1} + (A_{23} + A_{24})X_3 \right) \\ &\quad - G_2 \left((A_{43} + A_{44})X_3 - \frac{q_2}{X_3} + (A_{41} + A_{42})X_1 \right).\end{aligned}$$

The operator mirror map (3.4) in this case implies

$$\begin{aligned}X_1 &\leftrightarrow \sigma_1, \\ X_3 &\leftrightarrow \sigma_2.\end{aligned}$$

In fact, the operator mirror map statement earlier also implies

$$X_2 = \frac{q_1}{X_1} \leftrightarrow (A_{21} + A_{22})\sigma_1 + (A_{23} + A_{24})\sigma_2, \quad (5.1)$$

$$X_4 = \frac{q_2}{X_3} \leftrightarrow (A_{41} + A_{42})\sigma_1 + (A_{43} + A_{44})\sigma_2. \quad (5.2)$$

These two statements are also redundant consequences of the equations of motion $\partial W / \partial G_{1,2} = 0$, which are themselves consequences of the quantum sheaf cohomology relations, as we shall see momentarily.

Now, let us compare results from [6]. There, it was argued that the (0,2) mirror could be represented as

$$\begin{aligned}W &= -G_1 \left(aX'_1 + b \frac{(X'_2)^2}{X'_1} + \mu X'_2 - \frac{q_1}{X'_1} \right) \\ &\quad - G_2 \left(dX'_2 + c \frac{(X'_1)^2}{X'_2} + \nu X'_1 - \frac{q_2}{X'_2} \right),\end{aligned}$$

where

$$\begin{aligned}
 a &= \det \tilde{A}, & b &= \det \tilde{B}, & c &= \det \tilde{C}, & d &= \det \tilde{D}, \\
 \mu &= \det(\tilde{A} + \tilde{B}) - \det \tilde{A} - \det \tilde{B}, \\
 \nu &= \det(\tilde{C} + \tilde{D}) - \det \tilde{C} - \det \tilde{D},
 \end{aligned}$$

and with operator mirror map

$$X'_1 \leftrightarrow \sigma_1, \quad X'_2 \leftrightarrow \sigma_2.$$

In the present case,

$$a = A_{21} + A_{22}, \quad b = 0, \quad c = 0, \quad d = A_{43} + A_{44}, \quad \mu = A_{23} + A_{24}, \quad \nu = A_{41} + A_{42},$$

and it is easy to see that the mirror superpotential proposed here matches the specialization of that in [6], if we identify $X_1 = X'_1$, $X_3 = X'_2$.

In addition, the quantum sheaf cohomology ring in this model is [22–25]

$$a\sigma_1^2 + b\sigma_2 + \mu\sigma_1\sigma_2 = q_1, \quad c\sigma_1^2 + d\sigma_2^2 + \nu\sigma_1\sigma_2 = q_2,$$

which in the present case matches the remaining mirror map equations (5.1), (5.2) above.

Altogether, we see that the mirror proposed here matches results in [6], giving us a consistency check not only of proposed mirror superpotentials but also implicitly correlation functions and chiral rings.

5.1.2 Second choice of S

Next, we consider the case that the submatrix S is given by the first and fourth columns of the charge matrix Q , so that S is the identity matrix, and the allowed deformations are

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Rows correspond to fixed i index, and reflect the fact that for i a row of S , values are fixed to those of the (2,2) locus.) In other words, in terms of the original 2×2 matrices, we restrict to the special case

$$\begin{aligned}
 \tilde{A} &= \begin{bmatrix} 1 & 0 \\ 0 & A_{21} + A_{22} \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} 0 & 0 \\ 0 & A_{23} + A_{24} \end{bmatrix}, \\
 \tilde{C} &= \begin{bmatrix} A_{31} + A_{32} & 0 \\ 0 & 0 \end{bmatrix}, & \tilde{D} &= \begin{bmatrix} A_{33} + A_{34} & 0 \\ 0 & 1 \end{bmatrix}.
 \end{aligned}$$

From formula (3.3), we have that

$$(D_{i_S}^A) = \begin{bmatrix} A_{21} + A_{22} - 1 & A_{23} + A_{24} \\ -(A_{31} + A_{32}) & -(A_{33} + A_{34} - 1) \end{bmatrix},$$

where the A index counts across rows, and the i_S index counts across columns.

The (0,2) superpotential of the proposed mirror (3.2) is then given by

$$\begin{aligned}
 W &= -G_1 (e^{-Y_1} - e^{-Y_2} + (A_{21} + A_{22} - 1)e^{-Y_1} + (A_{23} + A_{24})e^{-Y_4}) \\
 &\quad -G_2 (e^{-Y_3} - e^{-Y_4} - (A_{31} + A_{32})e^{-Y_1} - (A_{33} + A_{34} - 1)e^{-Y_4}), \\
 &= -G_1 \left((A_{21} + A_{22})X_1 - \frac{q_1}{X_1} + (A_{23} + A_{24})X_4 \right) \\
 &\quad -G_2 \left(\frac{q_2}{X_4} - (A_{31} + A_{32})X_1 - (A_{33} + A_{34})X_4 \right),
 \end{aligned}$$

where $X_i = \exp(-Y_i)$ and with hindsight we have chosen to write the superpotential in terms of X_1 and X_4 , to illuminate the relation to other mirrors.

Furthermore, the operator mirror map (3.4) implies

$$X_1 \leftrightarrow \sigma_1, \tag{5.3}$$

$$X_4 \leftrightarrow \sigma_2 \tag{5.4}$$

and also implies

$$X_2 = \frac{q_1}{X_1} \leftrightarrow (A_{21} + A_{22})\sigma_1 + (A_{23} + A_{24})\sigma_2, \tag{5.5}$$

$$X_3 = \frac{q_2}{X_4} \leftrightarrow (A_{31} + A_{32})\sigma_1 + (A_{33} + A_{34})\sigma_2 \tag{5.6}$$

The latter two relations are a redundant restatement of the chiral ring of the theory, which can be seen by comparing the equations of motion of the superpotential above, defined by $\partial W / \partial G_A = 0$.

Now, in this case the mirror given in [6] is defined by

$$a = A_{21} + A_{22}, \quad b = c = 0, \quad d = A_{33} + A_{34}, \quad \mu = A_{23} + A_{24}, \quad \nu = A_{31} + A_{32},$$

and so has the form

$$\begin{aligned}
 W &= -G_1 \left((A_{21} + A_{22})X'_1 + (A_{23} + A_{24})X'_2 - \frac{q_1}{X'_1} \right) \\
 &\quad -G_2 \left((A_{33} + A_{34})X'_2 + (A_{31} + A_{32})X'_1 - \frac{q_2}{X'_2} \right),
 \end{aligned}$$

with operator mirror map

$$X'_1 \leftrightarrow \sigma_1, \quad X'_2 \leftrightarrow \sigma_2.$$

With the dictionary $X_1 = X'_1$, $X_4 = X'_2$ and a sign change on G_2 , we see that the superpotential and operator mirror map predicted here match that in [6]. In passing, note that the fact that we reproduce the mirror of [6] implicitly gives an independent check of matching of correlation functions.

5.1.3 Comment on Adams-Basu-Sethi result

Let us very briefly compare to the analysis of [3], which also examined (0,2) mirrors to $\mathbb{P}^1 \times \mathbb{P}^1$. They primarily considered tangent bundle deformations of the form [3, Eq. (255)]

$$\tilde{A} = I = \tilde{D}, \quad \tilde{C} = 0, \quad \tilde{B} = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix}.$$

In principle, a deformation of this form is compatible with making a non-invertible choice of S , specifically the last two columns of the charge matrix, so that

$$S = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since S is not invertible, it is not possible to uniquely solve for $(D_{iS}^A)_i$ in our approach, and we find it intriguing that in their analysis, they also were not able to uniquely solve for the mirror superpotential without doing further worldsheet instanton computations to solve for the values of some otherwise-undetermined parameters.

5.2 \mathbb{F}_n

Next we consider Hirzebruch surfaces \mathbb{F}_n . Now, for $n > 1$, these are not Fano, but nevertheless one can write down a mirror for the GLSM (which for the non-Fano cases is more properly interpreted as a mirror to a different geometric phase, the UV phase, of the GLSM), as discussed in [7]. The charge matrix of the GLSM is

$$\begin{bmatrix} 1 & 1 & n & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and deformations of the tangent bundle are described mathematically as the cokernel \mathcal{E} of the short exact sequence

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{*} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(n,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$* = \begin{bmatrix} \tilde{A}x & \tilde{B}x \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t & \alpha_2 t \end{bmatrix}.$$

In the expression above, x is a two-component vector of homogeneous coordinates of charge $(1,0)$, s is a homogeneous coordinate of charge $(n,1)$, and t is a homogeneous coordinate of charge $(0,1)$, A, B are constant 2×2 matrices, and $\gamma_{1,2}, \alpha_{1,2}$ are constants. (In principle, nonlinear deformations are also possible, but as observed previously in e.g. [22–25], do not contribute to quantum sheaf cohomology or A/2-model correlation functions, so we omit nonlinear deformations.) The (2,2) locus is given by the special case

$$A = I, \quad B = 0, \quad \gamma_1 = n, \quad \gamma_2 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 1.$$

We have the same constraints on fields from D terms as on the (2,2) locus, namely

$$Y_1 + Y_2 + nY_s = t_1, \quad Y_s + Y_t = t_2,$$

where $Y_{1,2}$ are dual to the x 's, Y_3 is dual to s , and Y_4 is dual to t . We can solve them by taking

$$\begin{aligned} \tilde{t}_1 = 0, \quad \tilde{t}_2 = t_1, \quad \tilde{t}_s = 0, \quad \tilde{t}_t = t_2, \\ (V_i^A) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -n & 1 & -1 \end{bmatrix}, \end{aligned}$$

so that

$$\begin{aligned} Y_1 = \theta, & & Y_2 = t_1 - \theta - n\tilde{\theta}, & & G_1 = F_1 = -F_2 - nG_2, \\ Y_3 = \tilde{\theta}, & & Y_4 = t_2 - \tilde{\theta}, & & G_2 = F_3 = -F_4. \end{aligned}$$

5.2.1 First choice of S

We take the submatrix $S \subset Q$ to correspond to the first and third columns of the charge matrix Q , i.e.

$$S = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

The allowed deformations are

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 1 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

To find the corresponding elements of \tilde{A} , \tilde{B} , $\gamma_{1,2}$, $\alpha_{1,2}$, we compare the E 's. For the deformations defined by A_{ij} ,

$$\begin{aligned} E_1 &= \sum_a Q_a^a \sigma_a \phi_1 = \sigma_1 \phi_1, \\ E_2 &= \sum_{j,a} A_{2j} Q_j^a \sigma_a \phi_2, \\ &= (A_{21}\sigma_1 + A_{22}\sigma_1 + A_{23}(n\sigma_1 + \sigma_2) + A_{24}\sigma_2) \phi_2, \\ E_s &= (n\sigma_1 + \sigma_2)s, \\ E_t &= (A_{41}\sigma_1 + A_{42}\sigma_1 + A_{43}(n\sigma_1 + \sigma_2) + A_{44}\sigma_2) t, \end{aligned}$$

whereas for the bundle deformation parameters,

$$\begin{aligned} E_1 &= (\tilde{A}_{11}\phi_1 + \tilde{A}_{12}\phi_2) \sigma_1 + (\tilde{B}_{11}\phi_1 + \tilde{B}_{12}\phi_2) \sigma_2, \\ E_2 &= (\tilde{A}_{21}\phi_1 + \tilde{A}_{22}\phi_2) \sigma_1 + (\tilde{B}_{21}\phi_1 + \tilde{B}_{22}\phi_2) \sigma_2, \\ E_s &= \gamma_1 s \sigma_1 + \gamma_2 s \sigma_2, \\ E_t &= \alpha_1 t \sigma_1 + \alpha_2 t \sigma_2, \end{aligned}$$

from which we read off

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & A_{21} + A_{22} + nA_{23} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & A_{23} + A_{24} \end{bmatrix},$$

$$\begin{aligned} a = \det \tilde{A} &= A_{21} + A_{22} + nA_{23}, & b = \det \tilde{B} &= 0, & \mu &= A_{23} + A_{24}, \\ \gamma_1 &= n, & \gamma_2 &= 1, & \alpha_1 &= A_{41} + A_{42} + nA_{43}, & \alpha_2 &= A_{43} + A_{44}. \end{aligned}$$

Next, let us construct the mirror. From formula (3.3),

$$(D_{i_S}^A) = \begin{bmatrix} A_{21} + A_{22} - nA_{24} - 1 & A_{23} + A_{24} \\ n(A_{21} + A_{22} - nA_{24}) + (A_{41} + A_{42} - nA_{44}) & n(A_{23} + A_{24}) + (A_{43} + A_{44}) - 1 \end{bmatrix},$$

From equation (3.2), the proposed mirror superpotential is then

$$\begin{aligned} W &= -G_1 (e^{-Y_1} - e^{-Y_2} + (A_{21} + A_{22} - nA_{24} - 1)e^{-Y_1} + (A_{23} + A_{24})e^{-Y_3}) \\ &\quad -G_2 \left(-ne^{-Y_2} + e^{-Y_3} - e^{-Y_4} \right. \\ &\quad \quad \left. + (n(A_{21} + A_{22} - nA_{24}) + (A_{41} + A_{42} - nA_{44}))e^{-Y_1} \right. \\ &\quad \quad \left. + (n(A_{23} + A_{24}) + (A_{43} + A_{44}) - 1)e^{-Y_3} \right), \\ &= -G_1 \left((A_{21} + A_{22} - nA_{24})X_1 - \frac{q_1}{X_1 X_3^n} + (A_{23} + A_{24})X_3 \right) \\ &\quad -G_2 \left(-n\frac{q_1}{X_1 X_3^n} + (n(A_{23} + A_{24}) + (A_{43} + A_{44}))X_3 - \frac{q_2}{X_3} \right. \\ &\quad \quad \left. + (n(A_{21} + A_{22} - nA_{24}) + (A_{41} + A_{42} - nA_{44}))X_1 \right), \end{aligned}$$

where $X_i = \exp(-Y_i)$, with operator mirror map (3.4)

$$\begin{aligned} X_1 &\leftrightarrow \sigma_1, \\ X_2 &= \frac{q_1}{X_1 X_3^n} \leftrightarrow (A_{21} + A_{22})\sigma_1 + A_{23}(n\sigma_1 + \sigma_2) + A_{24}\sigma_2, \\ X_3 &\leftrightarrow n\sigma_1 + \sigma_2, \\ X_4 &= \frac{q_2}{X_3} \leftrightarrow (A_{41} + A_{42})\sigma_1 + A_{43}(n\sigma_1 + \sigma_2) + A_{44}\sigma_2. \end{aligned}$$

Note that the operator mirror map relations for X_2, X_4 are consequences of the equations of motion $\partial W / \partial G_A = 0$.

For these deformations, the quantum sheaf cohomology ring is given by [22–25]

$$Q_{(k)}Q_{(s)}^n = q_1, \quad Q_{(s)}Q_{(t)} = q_2,$$

where

$$\begin{aligned} Q_{(k)} &= (A_{21} + A_{22} + nA_{23})\sigma_1^2 + (A_{23} + A_{24})\sigma_1\sigma_2, \\ Q_{(s)} &= n\sigma_1 + \sigma_2, \quad Q_{(t)} = (A_{41} + A_{42} + nA_{43})\sigma_1 + (A_{43} + A_{44})\sigma_2. \end{aligned}$$

It is straightforward to check that these relations are implied by the mirror map equations above.

A proposal was made in [7] for the Toda dual to a (GLSM for a) Hirzebruch surface. Briefly, the mirror superpotential had the form

$$W = -G_1 J_1 - G_2 J_2$$

for [7, Section 4.2]

$$J_1 = aX_1 + \mu_{AB}(X_3 - nX_1) + b \frac{(X_3 - nX_1)^2}{X_1} - q_1 X_1^{-1} (\gamma_1 X_1 + \gamma_2 (X_3 - nX_1))^{-n}, \quad (5.7)$$

$$J_2 = n \left(aX_1 + \mu_{AB}(X_3 - nX_1) + b \frac{(X_3 - nX_1)^2}{X_1} \right) - \frac{nq_1}{X_1 (\gamma_1 X_1 + \gamma_2 (X_3 - nX_1))^n} - \frac{q_2}{X_3} + \frac{(\gamma_1 X_1 + \gamma_2 (X_3 - nX_1)) (\alpha_1 X_1 + \alpha_2 (X_3 - nX_1))}{X_3}, \quad (5.8)$$

with operator mirror map

$$X_1 \leftrightarrow \sigma_1, \quad X_3 \leftrightarrow n\sigma_1 + \sigma_2.$$

It is straightforward to check that the proposal of [7], reviewed above, specializes to our proposal here.

5.2.2 Second choice of S

Next, consider the case that the submatrix $S \subset Q$ is taken to correspond to the first and fourth columns of Q , i.e.

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The allowed deformations are

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proceeding as before, the corresponding \tilde{A} , \tilde{B} , $\gamma_{1,2}$, $\alpha_{1,2}$ are given by

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & A_{21} + A_{22} + nA_{23} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & A_{23} + A_{24} \end{bmatrix},$$

$$\begin{aligned} a &= \det \tilde{A} = A_{21} + A_{22} + nA_{23}, & b &= \det \tilde{B} = 0, & \mu &= A_{23} + A_{24}, \\ \gamma_1 &= A_{31} + A_{32} + nA_{33}, & \gamma_2 &= A_{33} + A_{34}, \\ \alpha_1 &= 0, & \alpha_2 &= 1. \end{aligned}$$

Next, let us construct the mirror. From formula (3.3),

$$(D_{iS}^A) = \begin{bmatrix} A_{21} + A_{22} + nA_{23} - 1 & A_{23} + A_{24} \\ n(A_{21} + A_{22} + nA_{23}) - (A_{31} + A_{32} + nA_{33}) & n(A_{23} + A_{24}) - (A_{33} + A_{34} - 1) \end{bmatrix}.$$

From equation (3.2), the proposed mirror superpotential is then

$$\begin{aligned}
W &= -G_1 (e^{-Y_1} - e^{-Y_2} + (A_{21} + A_{22} + nA_{23} - 1)e^{-Y_1} + (A_{23} + A_{24})e^{-Y_4}) \\
&\quad -G_2 \left(-ne^{-Y_2} + e^{-Y_3} - e^{-Y_4} \right. \\
&\quad \quad \left. + (n(A_{21} + A_{22} + nA_{23}) - (A_{31} + A_{32} + nA_{33}))e^{-Y_1} \right. \\
&\quad \quad \left. + (n(A_{23} + A_{24}) - (A_{33} + A_{34} - 1))e^{-Y_4} \right), \\
&= -G_1 \left((A_{21} + A_{22} + nA_{23})X_1 - \frac{q_1 X_4^n}{q_2^n X_1} + (A_{23} + A_{24})X_4 \right) \\
&\quad -G_2 \left(-n \frac{q_1 X_4^n}{q_2^n X_1} + \frac{q_2}{X_4} + (n(A_{23} + A_{24}) - (A_{33} + A_{34}))X_4 \right. \\
&\quad \quad \left. + (n(A_{21} + A_{22} + nA_{23}) - (A_{31} + A_{32} + nA_{33}))X_1 \right),
\end{aligned}$$

where $X_i = \exp(-Y_i)$, with operator mirror map (3.4)

$$\begin{aligned}
X_1 &\leftrightarrow \sigma_1, \\
X_2 &= \frac{q_1 X_4^n}{q_2^n X_1} \leftrightarrow (A_{21} + A_{22})\sigma_1 + A_{23}(n\sigma_1 + \sigma_2) + A_{24}\sigma_2, \\
X_3 &= \frac{q_2}{X_4} \leftrightarrow (A_{31} + A_{32})\sigma_1 + A_{33}(n\sigma_1 + \sigma_2) + A_{34}\sigma_2, \\
X_4 &\leftrightarrow \sigma_2.
\end{aligned}$$

The operator mirror map relation for X_2 is a consequence of $\partial W/\partial G_1 = 0$, and the operator mirror map relation for X_3 is a consequence of that plus $\partial W/\partial G_2 = 0$.

A second proposal was made in [7] for the Toda dual to a (GLSM for a) Hirzebruch surface, in which the mirror superpotential had the form

$$W = -G_1 J_1 - G_2 J_2,$$

for [7, Section 4.2]

$$J_1 = \left(aX_1 + \mu_{AB}X_4 + b \frac{X_4^2}{X_1} \right) - \frac{q_1 (\alpha_1 X_1 + \alpha_2 X_4)^n}{q_2^n X_1}, \tag{5.9}$$

$$\begin{aligned}
J_2 &= -n \left(aX_1 + \mu_{AB}X_4 + b \frac{X_4^2}{X_1} - \frac{q_1 (\alpha_1 X_1 + \alpha_2 X_4)^n}{q_2^n X_1} \right) \\
&\quad + \left(\alpha_2 \gamma_2 X_4 + \gamma_1 \alpha_1 \frac{X_1^2}{X_4} + (\gamma_1 \alpha_2 + \gamma_2 \alpha_1)X_1 \right) - \frac{q_2}{X_4}.
\end{aligned} \tag{5.10}$$

with operator mirror map

$$X_1 \leftrightarrow \sigma_1, \quad X_4 \leftrightarrow \sigma_2.$$

It is straightforward to check that this proposal of [7] specializes to our proposal.

5.3 dP_2

Let us now consider the del Pezzo surface dP_2 , corresponding to \mathbb{P}^2 blown up at two points, which was also considered in [7]. The charge matrix of the GLSM for the chiral fields is of the form

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and deformations of the tangent bundle are described mathematically as the cokernel \mathcal{E} of the short exact sequence

$$0 \longrightarrow \mathcal{O}^3 \xrightarrow{*} \mathcal{O}(1, 0, 1) \oplus \mathcal{O}(1, 0, 0) \oplus \mathcal{O}(1, 1, 0) \oplus \mathcal{O}(0, 1, 0) \oplus \mathcal{O}(0, 0, 1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$* = \begin{bmatrix} \alpha_1\phi_1 & \alpha_2\phi_2 & \alpha_3\phi_3 \\ \beta_1\phi_1 & \beta_2\phi_2 & \beta_3\phi_3 \\ \gamma_1\phi_1 & \gamma_2\phi_2 & \gamma_3\phi_3 \\ \delta_1\phi_1 & \delta_2\phi_2 & \delta_3\phi_3 \\ \epsilon_1\phi_1 & \epsilon_2\phi_2 & \epsilon_3\phi_3 \end{bmatrix}.$$

The (2,2) locus is given by the special case

$$\begin{aligned} \alpha_1 &= 1, & \alpha_2 &= 0, & \alpha_3 &= 1, \\ \beta_1 &= 1, & \beta_2 &= \beta_3 = 0, & & \\ \gamma_1 &= \gamma_2 = 1, & \gamma_3 &= 0, & & \\ \delta_1 &= 0, & \delta_2 &= 1, & \delta_3 &= 0, \\ \epsilon_1 &= \epsilon_2 = 0, & \epsilon_3 &= 1. & & \end{aligned}$$

We will take

$$(V_i^A) = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{bmatrix}.$$

5.3.1 First choice of S

For our first choice of S , we will take the first, third, and fifth columns of the charge matrix, so that

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

With this choice of S , the allowed deformations are

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ 0 & 0 & 1 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We compute the corresponding bundle deformation parameters by comparing E 's:

$$\begin{aligned}
 E_1 &= (\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)\phi_1 = (\sigma_1 + \sigma_3)\phi_1, \\
 E_2 &= (\beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3)\phi_2, \\
 &= (A_{21}(\sigma_1 + \sigma_3) + A_{22}\sigma_1 + A_{23}(\sigma_1 + \sigma_2) + A_{24}\sigma_2 + A_{25}\sigma_3)\phi_2, \\
 E_3 &= (\gamma_1\sigma_1 + \gamma_2\sigma_2 + \gamma_3\sigma_3)\phi_3 = (\sigma_1 + \sigma_2)\phi_3, \\
 E_4 &= (\delta_1\sigma_1 + \delta_2\sigma_2 + \delta_3\sigma_3)\phi_4, \\
 &= (A_{41}(\sigma_1 + \sigma_3) + A_{42}\sigma_1 + A_{43}(\sigma_1 + \sigma_2) + A_{44}\sigma_2 + A_{45}\sigma_3)\phi_4, \\
 E_5 &= (\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3)\phi_5 = \sigma_3\phi_5,
 \end{aligned}$$

from which we find

$$\begin{aligned}
 \vec{\alpha} &= (1, 0, 1), \quad \vec{\gamma} = (1, 1, 0), \quad \vec{\epsilon} = (0, 0, 1), \\
 \vec{\beta} &= (A_{21} + A_{22} + A_{23}, A_{23} + A_{24}, A_{21} + A_{25}), \\
 \vec{\delta} &= (A_{41} + A_{42} + A_{43}, A_{43} + A_{44}, A_{41} + A_{45}).
 \end{aligned}$$

From formula (3.3), we have

$$(D_{i_S}^A) = \begin{bmatrix} A_{21} + A_{22} - 1 - A_{24} & A_{23} + A_{24} & -A_{22} + A_{24} + A_{25} + 1 \\ A_{21} + A_{22} - A_{24} + & A_{23} + A_{24} + A_{43} + A_{44} - 1 & -A_{22} + A_{24} + A_{25} \\ A_{41} + A_{42} - A_{44} & & -A_{42} + A_{44} + A_{45} \end{bmatrix}.$$

The proposed mirror superpotential (3.2) is then

$$\begin{aligned}
 W &= -G_1 \left[(A_{21} + A_{22} - A_{24})X_1 - \frac{q_1}{X_1 X_3} + (A_{23} + A_{24})X_3 + (-A_{22} + A_{24} + A_{25})X_5 \right] \\
 &\quad - G_2 \left[(A_{21} + A_{22} - A_{24} + A_{41} + A_{42} - A_{44})X_1 \right. \\
 &\quad \quad - \frac{q_1}{X_1 X_3} + (A_{23} + A_{24} + A_{43} + A_{44})X_3 \\
 &\quad \quad \left. - \frac{q_2}{X_3} + (-A_{22} + A_{24} + A_{25} - A_{42} + A_{44} + A_{45})X_5 \right],
 \end{aligned}$$

with operator mirror map (3.4)

$$\begin{aligned}
 X_1 &\leftrightarrow \sigma_1 + \sigma_3, \\
 X_2 &= \frac{q_1}{X_1 X_3} \leftrightarrow (A_{21} + A_{22} + A_{23})\sigma_1 + (A_{23} + A_{24})\sigma_2 + (A_{21} + A_{25})\sigma_3, \\
 X_3 &\leftrightarrow \sigma_1 + \sigma_2, \\
 X_4 &= \frac{q_2}{X_3} \leftrightarrow (A_{41} + A_{42} + A_{43})\sigma_1 + (A_{43} + A_{44})\sigma_2 + (A_{41} + A_{45})\sigma_3, \\
 X_5 &= \frac{q_3}{X_1} \leftrightarrow \sigma_3.
 \end{aligned}$$

(It is straightforward to check that the nontrivial operator mirror map relations above are equivalent to the equations of motion derived from $\partial W / \partial G_A = 0$.)

Now, let us compare to the first (0,2) mirror proposal for dP_2 in [7, Section 3.2.2]. In their notation

$$\begin{aligned}\alpha \cdot X &= X_1, \quad \gamma \cdot X = X_3, \quad \epsilon \cdot X = X_5, \\ \beta \cdot X &= (A_{21} + A_{22} - A_{24})X_1 + (A_{23} + A_{24})X_3 + (A_{24} + A_{25} - A_{22})X_5, \\ \delta \cdot X &= (A_{41} + A_{42} - A_{44})X_1 + (A_{43} + A_{44})X_3 + (A_{44} + A_{45} - A_{22})X_5,\end{aligned}$$

and

$$\begin{aligned}J_1 &= -\frac{q_1}{X_1 X_3} + Z \frac{q_3}{X_1 X_5} + X_5 + \beta \cdot X, \\ J_3 &= -\frac{q_2}{X_3} - \frac{q_1}{X_1 X_3} + \beta \cdot X + \delta \cdot X, \\ J_5 &= X_5 + Z \frac{q_3}{X_1 X_5}, \\ J_Z &= 1 - \frac{q_3}{X_1 X_5}.\end{aligned}$$

Solving $J_Z = 0$, we get $X_5 = q_3/X_1$, and solving $J_5 = 0$, we get $Z = -X_5$. Plugging in, we are left with two independent J 's:

$$\begin{aligned}J_1 &= -\frac{q_1}{X_1 X_3} + (A_{21} + A_{22} - A_{24})X_1 + (A_{23} + A_{24})X_3 + (A_{24} + A_{25} - A_{22})\frac{q_3}{X_1}, \\ J_3 &= -\frac{q_2}{X_3} - \frac{q_1}{X_1 X_3} + (A_{21} + A_{22} - A_{24} + A_{41} + A_{42} - A_{44})X_1 \\ &\quad + (A_{23} + A_{24} + A_{43} + A_{44})X_3 + (A_{24} + A_{25} - A_{22} + A_{44} + A_{45} - A_{42})\frac{q_3}{X_1},\end{aligned}$$

which precisely matches the prediction of our proposal.

Finally, let us compare the quantum sheaf cohomology ring relations. There are three quantum sheaf cohomology ring relations, but only two relations appearing above in the operator mirror map and equations of motion. Specifically, in this case, the quantum sheaf cohomology ring relations [22–24] take the form

$$(\sigma_1 + \sigma_3)(\sigma_1 + \sigma_2)Q_{(2)} = q_1, \tag{5.11}$$

$$(\sigma_1 + \sigma_2)Q_{(4)} = q_2, \tag{5.12}$$

$$(\sigma_1 + \sigma_3)\sigma_3 = q_3, \tag{5.13}$$

where

$$\begin{aligned}Q_{(2)} &= (A_{21} + A_{22} + A_{23})\sigma_1 + (A_{23} + A_{24})\sigma_2 + (A_{21} + A_{25})\sigma_3, \\ Q_{(4)} &= (A_{41} + A_{42} + A_{43})\sigma_1 + (A_{43} + A_{44})\sigma_2 + (A_{41} + A_{45})\sigma_3.\end{aligned}$$

The first two quantum sheaf cohomology ring relations correspond to two of the operator mirror map statements. The third is realized on the mirror as the relation

$$X_1 X_5 = q_3,$$

which is a consequence of the D terms, and so automatic.

5.3.2 Second choice of S

For our second choice of S , we will take the second, fourth, and fifth columns of the charge matrix, so that S is the identity. With that choice of S , the allowed deformations are

$$(A_{ij}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ 0 & 1 & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

To find the corresponding bundle deformation parameters, we compare the E 's:

$$\begin{aligned} E_1 &= (\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)\phi_1, \\ &= (A_{11}(\sigma_1 + \sigma_3) + A_{12}\sigma_1 + A_{13}(\sigma_1 + \sigma_2) + A_{14}\sigma_2 + A_{15}\sigma_3)\phi_1, \\ E_2 &= (\beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3)\phi_2 = \sigma_1\phi_2, \\ E_3 &= (\gamma_1\sigma_1 + \gamma_2\sigma_2 + \gamma_3\sigma_3)\phi_3, \\ &= (A_{31}(\sigma_1 + \sigma_3) + A_{32}\sigma_1 + A_{33}(\sigma_1 + \sigma_2) + A_{34}\sigma_2 + A_{35}\sigma_3)\phi_3, \\ E_4 &= (\delta_1\sigma_1 + \delta_2\sigma_2 + \delta_3\sigma_3)\phi_4 = \sigma_2\phi_4, \\ E_5 &= (\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3)\phi_5 = \sigma_3\phi_5, \end{aligned}$$

from which we conclude

$$\begin{aligned} \vec{\alpha} &= (A_{11} + A_{12} + A_{13}, A_{13} + A_{14}, A_{11} + A_{15}), \\ \vec{\beta} &= (1, 0, 0), \quad \vec{\delta} = (0, 1, 0), \quad \vec{\epsilon} = (0, 0, 1), \\ \vec{\gamma} &= (A_{31} + A_{32} + A_{33}, A_{33} + A_{34}, A_{31} + A_{35}). \end{aligned}$$

Next, we construct the mirror. From formula (3.3), we have

$$(D_{iS}^A) = - \begin{bmatrix} A_{11} + A_{12} + A_{13} - 1 & A_{13} + A_{14} & A_{11} + A_{15} - 1 \\ A_{31} + A_{32} + A_{33} - 1 & A_{33} + A_{34} - 1 & A_{31} + A_{35} \end{bmatrix},$$

then the proposed mirror superpotential (3.2) is

$$\begin{aligned} W &= -G_1 \left(\frac{q_1 X_4}{q_2 X_2} - (A_{11} + A_{12} + A_{13})X_2 - (A_{13} + A_{14})X_4 - (A_{11} + A_{15})X_5 \right) \\ &\quad - G_2 \left(-(A_{31} + A_{32} + A_{33})X_2 + \frac{q_2}{X_4} - (A_{33} + A_{34})X_4 - (A_{31} + A_{35})X_5 \right), \end{aligned}$$

where $X_i = \exp(-Y_i)$.

The operator mirror map (3.4) is then given by

$$\begin{aligned} X_1 &= \frac{q_1 X_4}{q_2 X_2} \leftrightarrow A_{11}(\sigma_1 + \sigma_3) + A_{12}\sigma_1 + A_{13}(\sigma_1 + \sigma_2) + A_{14}\sigma_2 + A_{15}\sigma_3, \\ X_2 &\leftrightarrow \sigma_1, \\ X_3 &= \frac{q_2}{X_4} \leftrightarrow A_{31}(\sigma_1 + \sigma_3) + A_{32}\sigma_1 + A_{33}(\sigma_1 + \sigma_2) + A_{34}\sigma_2 + A_{35}\sigma_3, \\ X_4 &\leftrightarrow \sigma_2, \\ X_5 &\leftrightarrow \sigma_3. \end{aligned}$$

From [22–24], the quantum sheaf cohomology relations in this model are given by

$$\sigma_1 Q_{(1)} Q_{(3)} = q_1, \tag{5.14}$$

$$\sigma_2 Q_{(3)} = q_2, \tag{5.15}$$

$$\sigma_3 Q_{(1)} = q_3, \tag{5.16}$$

where

$$Q_{(1)} = (A_{11} + A_{12} + A_{13})\sigma_1 + (A_{13} + A_{14})\sigma_2 + (A_{11} + A_{15})\sigma_3,$$

$$Q_{(3)} = (A_{31} + A_{32} + A_{33})\sigma_1 + (A_{33} + A_{34})\sigma_2 + (A_{31} + A_{35})\sigma_3.$$

This first two quantum sheaf cohomology relations can be seen to correspond to both the operator mirror map relations and the equations of motion. The third relation (5.16) is effectively redundant on the mirror. To see this, note that it corresponds to the statement

$$X_5 Q_{(1)} = q_3.$$

However, $X_5 = q_3/X_1$ from the D-term relations, so this is equivalent to

$$X_1 = Q_{(1)},$$

which is the first mirror-map relation.

6 Hypersurfaces in toric varieties

6.1 General aspects

An extension of ordinary mirror symmetry from toric varieties to hypersurfaces therein was discussed in [10] and further justified in [53]. Let us begin our discussion here by very briefly reviewing this for the special case of the (2,2) quintic in \mathbb{P}^4 . One begins with the Toda dual of Tot ($\mathcal{O}(-5) \rightarrow \mathbb{P}^4$), the ambient GLSM for the quintic albeit with vanishing superpotential. The Toda dual is defined by a (twisted) superpotential of the form

$$W = \sum_{i=1}^5 \exp(-Y_i) + \exp(-Y_p),$$

where the D term constraint requires

$$-5Y_p + \sum_{i=1}^n Y_i = t.$$

The effect of dualizing the GLSM with a nonzero superpotential, according to [10, 53] is to change the fundamental fields from Y_i to $X_i \equiv \exp(-Y_i/5)$, which also introduces \mathbb{Z}_5 orbifolds. As a result, after eliminating Y_p with the D term constraint and the change of variables, the mirror superpotential becomes (\mathbb{Z}_5 orbifolds of)

$$W = \sum_i X_i^5 + q \prod_i X_i.$$

The fact that the B-twisted mirror sits at a Landau-Ginzburg point reflects the fact that the B model is independent of Kähler moduli, and so topological field theory computations can be computed at any point on the Kähler moduli space — the Landau-Ginzburg orbifold point in the moduli space of a hypersurface being a convenient example.

Before considering (0,2) analogues, let us quickly outline a formal justification for the change of variables above, in TFT correlation functions. We claim that at the level of untwisted-sector correlation functions, it is formally equivalent to insertions needed to restrict to the hypersurface. The argument below omits questions of counting and degeneracy of vacua, as well as how the kinetic terms of the fields change, but is sufficiently tantalizing that we mention it here.

Begin with a (2,2) supersymmetric GLSM for Tot ($\mathcal{O}(-5) \rightarrow \mathbb{P}^4$), labelled the V^+ model in [51]. To compute a correlation function matching one on the hypersurface, in principle one should insert $(-)(-5\sigma)^2$ (see e.g. [51, Eq. (5.8)]). Correlation functions in the untwisted sector of the mirror B-twisted Landau-Ginzburg model, corresponding to certain computations on the hypersurface, should then have the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\text{quintic}} = \langle \mathcal{O}_1 \cdots \mathcal{O}_n (-)(-5\sigma)^2 \rangle_{V^+} \propto \sum_{\text{vacua}} \frac{\mathcal{O}_1 \cdots \mathcal{O}_n \exp(-2Y_p)}{\det(\partial_{Y_i} \partial_{Y_j} W)}$$

where we have used the mirror map

$$\exp(-Y_p) = e^{-t/5} \prod_i \exp(-Y_i/5) \leftrightarrow -5\sigma.$$

Now, we change variables from Y_i to $X_i \equiv \exp(-Y_i/5)$. Formally, in the denominator,

$$\begin{aligned} \frac{\partial}{\partial Y_i} W &= \frac{\partial X_j}{\partial Y_i} \frac{\partial W}{\partial X_j} = -\frac{1}{5} X_i \frac{\partial W}{\partial X_i}, \\ \frac{\partial^2 W}{\partial Y_i \partial Y_j} &= \left(-\frac{1}{5}\right)^2 X_i X_j \frac{\partial^2 W}{\partial X_i \partial X_j} \text{ (no sum on } i, j), \end{aligned}$$

along the critical locus. Now, strictly speaking, after the change of variables, the critical locus becomes degenerate, the second derivative of the superpotential vanishes at the critical locus. One could attempt to solve this by turning on twisted masses, possible for certain special superpotentials, but as our aim here is a formal observation, we shall move on. The point we wish to make is that formally, if one plugs into the expression for correlation functions, then glossing over questions of number and degeneracy of vacua,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\text{quintic}} = \sum_{\text{vacua}} \frac{\mathcal{O}_1 \cdots \mathcal{O}_n \exp(-2Y_p)}{(-1/5)^{10} (\prod_i X_i)^2 \det(\partial_{X_i} \partial_{X_j} W)},$$

then along the vacua, the factor of $\exp(-2Y_p) = e^{-2t/n} (\prod_i X_i)^2$ largely cancels out the $(\prod_i X_i)^2$ in the denominator, leaving an expression which (setting aside questions of counting and degeneracy of vacua, as well as how the change of variables would operate on kinetic terms) formally duplicates the usual expression in the orbifold:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\text{quintic}} \propto \sum_{\text{vacua}} \frac{\mathcal{O}_1 \cdots \mathcal{O}_n}{\det(\partial_{X_i} \partial_{X_j} W)}.$$

Now, setting aside formal justifications, it is, of course, tempting to conjecture a (0,2) analogue of the story above. The justification for the (2,2) case in [53] was based on the claim in [54] that the A model on a hypersurface is equivalent to the A model on a supermanifold over the ambient space. Unfortunately, at present no (0,2) analogue of [54] exists in the literature. Mathematically, the idea of inserting Chern classes to restrict to hypersurfaces is more properly understood in terms of insertions of Mathai-Quillen forms. A (0,2) analogue of Mathai-Quillen forms has been sketched out (see [55] for definitions and conjectures), but one open question in that work is whether they have all the pertinent properties of ordinary Mathai-Quillen forms.

Another question concerns the appropriateness of getting a mirror at a Landau-Ginzburg orbifold point. In (2,2) theories, for hypersurfaces, since the B model is independent of Kähler structures, we could evaluate a B-twisted mirror at any convenient point on the Kähler moduli space, and the Landau-Ginzburg orbifold point is a convenient such point. By contrast, (0,2) theories are somewhat more complicated. It was argued in [22–25] that A/2-twisted GLSMs are independent of complex structure moduli and some bundle moduli (J s), and B/2-twisted GLSMs are independent of Kähler moduli and other bundle moduli (E s), but this seems to be an accident of the GLSM’s presentation of moduli, and we do not understand how to formulate analogous statements in IR nonlinear sigma models — or indeed if it is even possible to formulate such statements in IR theories. In any event, for our purposes, this will suffice to justify describing B/2-twisted mirrors at Landau-Ginzburg orbifold points.

With all that in mind, we will now describe some formal computations for (0,2) theories, generated by proceeding along the same lines as (2,2) theories, albeit with less justification.

Note that in (0,2) theories, in addition to performing a variable change on (0,2) chiral superfields, we must also perform a variable change on (0,2) Fermi superfields. For example, suppose that in the mirror to the ambient GLSM, the fundamental fields are (θ_A, G_A) , where θ_{AS} are (0,2) chiral superfields and G_{AS} are (0,2) Fermi superfields, which on the (2,2) locus together form a (2,2) chiral multiplet. If we define

$$X_A = \exp(-\theta_A/n),$$

and take that to be a fundamental chiral superfield then to get a Landau-Ginzburg model that matches the (2,2) mirror on the (2,2) locus, we must also define new fundamental Fermi superfields Λ_A , as

$$\Lambda_A = \frac{\partial X_A}{\partial \theta_B} G_B.$$

In the example above,

$$\frac{\partial X_A}{\partial \theta_B} = -\frac{1}{n} X_A \delta_A^B,$$

hence

$$\Lambda_A = -\frac{1}{n} X_A G_A \text{ (no sum on } A),$$

and we should take Λ_A to be the fundamental field in the mirror to the hypersurface, replacing G_A .

6.2 Example

For an example, we will study the mirror to a (0,2) GLSM describing a degree $(n, 0)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$, with a tangent bundle deformation. There are two dueling constraints that make finding interesting examples, somewhat nontrivial:

- Our E 's, J 's must satisfy $E \cdot J = 0$ in order to preserve supersymmetry,
- Simultaneously, choices of $S \subset Q$ constrain the allowed deformations.

We will deal with these issues by considering a hypersurface in the first \mathbb{P}^2 factor, paired with a bundle deformation over the second \mathbb{P}^1 factor, so that the two constraints are fundamentally uncoupled from one another. We will ‘follow our nose’ and work out what naive expectations would predict for the (0,2) mirror given the proposal of this paper; however, we have not checked any correlation functions or performed other independent tests to ensure that we have in fact produced the correct (0,2) mirror.

Our GLSM will have six chiral superfields, which we label $\phi_{1,\dots,5}$ and p , with charge matrix

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -n \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

We interpret ϕ_{1-3} as related to homogeneous coordinates on first factor (\mathbb{P}^2), and ϕ_{4-5} as related to homogeneous coordinates on the second factor (\mathbb{P}^1).

Given the charge matrix Q above, we will take

$$(V_i^A) = \begin{bmatrix} n & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1/n \\ 0 & 0 & 1 & 0 & 0 & 1/n \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}.$$

Now, since we are going to apply our mirror construction, we pick an invertible submatrix $S \subset Q$. To be specific, we will take S to correspond to the fourth and sixth rows of Q , so that

$$S = \begin{bmatrix} 0 & -n \\ 1 & 0 \end{bmatrix},$$

and we will consider deformations

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, the choice of S above is compatible with a more complicated deformation matrix (A_{ij}) — for example, the first, second, and third rows could have entries different from those of the identity matrix. However, to avoid running into difficulties with the constraint

$E \cdot J = 0$, for simplicity in this example we pick the E 's to be trivial along directions with nonzero J 's, and so we take the matrix to be of the more specialized form above.

With these choices, it is straightforward to compute

$$(D_{iS}^A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ (A_{54} + A_{55} - 1)/n & -(A_{51} + A_{52} + A_{53} - nA_{56}) \end{bmatrix}.$$

Thus, the (0,2) mirror to the GLSM *without superpotential* is given by a Landau-Ginzburg model with superpotential

$$\begin{aligned} W = & -G_1(n \exp(-Y_1) + \exp(-Y_p)) - G_2(\exp(-Y_2) + (1/n) \exp(-Y_p)) \\ & -G_3(\exp(-Y_3) + (1/n) \exp(-Y_p)) \\ & -G_4\left(-\exp(-Y_4) + \exp(-Y_5) + \frac{1}{n}(A_{54} + A_{55} - 1) \exp(-Y_4) \right. \\ & \left. -(A_{51} + A_{52} + A_{53} - nA_{56}) \exp(-Y_p)\right), \end{aligned}$$

subject to the usual D-term constraints, which allow us to rewrite Y 's in terms of θ 's. If we pick

$$(\tilde{t}_i) = (0, 0, 0, t_2, 0, -t_1/n),$$

then

$$\theta_1 = Y_1/n, \quad \theta_2 = Y_2, \quad \theta_3 = Y_3, \quad \theta_4 = Y_5,$$

and

$$Y_p = \theta_1 + \theta_2/n + \theta_3/n - t_1/n, \quad Y_4 = -\theta_4 + t_2,$$

so if we define

$$Z_1 = \exp(-\theta_1), \quad Z_{2,3} = \exp(-\theta_{2,3}/n), \quad Z_4 = \exp(-\theta_4),$$

the mirror superpotential above becomes

$$\begin{aligned} W = & -G_1\left(nZ_1^n + Z_1Z_2Z_3e^{-t_1/n}\right) - G_2\left(Z_2^n + \frac{1}{n}Z_1Z_2Z_3e^{-t_1/n}\right) \\ & -G_3\left(Z_3^n + \frac{1}{n}Z_1Z_2Z_3e^{-t_1/n}\right) \\ & -G_4\left(Z_4 - \frac{q_2}{Z_4} + \frac{1}{n}(A_{54} + A_{55} - 1) \frac{q_2}{Z_4} - (A_{51} + A_{52} + A_{53} - nA_{56}) Z_1Z_2Z_3e^{-t_1/n}\right), \end{aligned}$$

and the theory has a $(\mathbb{Z}_n)^2$ orbifold, acting on $Z_{2,3}$, in which the group action preserves the superpotential above.

For completeness, the operator mirror map takes the following form:

$$\begin{aligned}
 \exp(-Y_1) &= Z_1^n \leftrightarrow \sigma_1, \\
 \exp(-Y_2) &= Z_2^n \leftrightarrow \sigma_1, \\
 \exp(-Y_3) &= Z_3^n \leftrightarrow \sigma_1, \\
 \exp(-Y_4) &= \frac{q_2}{Z_4} \leftrightarrow \sigma_2, \\
 \exp(-Y_5) &= Z_4 \leftrightarrow (A_{51} + A_{52} + A_{53} - nA_{56})\sigma_1 + (A_{54} + A_{55})\sigma_2, \\
 \exp(-Y_p) &= Z_1 Z_2 Z_3 e^{-t_1/n} \leftrightarrow -n\sigma_1.
 \end{aligned}$$

We could also add a twisted mass along the p direction, for example. Doing so would add the following terms to the mirror Landau-Ginzburg model superpotential

$$G_A V_p^A \tilde{m}_p = \tilde{m}_p \left(G_1 + \frac{1}{n} G_2 + \frac{1}{n} G_3 \right),$$

and also alter the operator mirror map, by modifying the map for $\exp(-Y_p)$ as

$$\exp(-Y_p) \leftrightarrow -n\sigma_1 + \tilde{m}_p.$$

Now, let us consider restricting to the hypersurface. This will involve changing the fundamental fields, from θ 's to X 's and G 's to Λ 's.

First, let us consider changing variables amongst chiral superfields. We will take Z_i s to be the fundamental variables. (Physically, this means changing e.g. kinetic terms, and so changing the physical theory, but here we will primarily focus on the superpotential.)

In addition, we must also change the Fermi fields, as mentioned previously. Following the pattern discussed previously, we define the new fundamental Fermi superfields

$$\Lambda_1 = -Z_1 G_1, \quad \Lambda_{2,3} = -\frac{1}{n} Z_{2,3} G_{2,3}, \quad \Lambda_4 = -Z_4 G_4.$$

Rewriting the mirror superpotential above in terms of the new fundamental fields Z_i , Λ_i , we find it takes the form

$$\begin{aligned}
 W &= \Lambda_1 \left(n Z_1^{n-1} + Z_2 Z_3 e^{-t_1/n} \right) + \Lambda_2 \left(n Z_2^{n-1} + Z_1 Z_3 e^{-t_1/n} \right) \\
 &\quad + \Lambda_3 \left(n Z_3^{n-1} + Z_1 Z_2 e^{-t_1/n} \right) \\
 &\quad + \Lambda_4 \left(1 - \frac{q_2}{Z_4^2} + \frac{1}{n} (A_{54} + A_{55} - 1) \frac{q_2}{Z_4^2} - (A_{51} + A_{52} + A_{53} - nA_{56}) \frac{Z_1 Z_2 Z_3}{Z_4} e^{-t_1/n} \right),
 \end{aligned}$$

again with a $(\mathbb{Z}_n)^2$ orbifold group action.

The reader should note that the first three terms in the superpotential above appear identical to those one would expect in a (0,2) expansion of a (2,2) superpotential of the form

$$Z_1^n + Z_2^n + Z_3^n + Z_1 Z_2 Z_3 e^{-t_1/n},$$

exactly as one would expect for the mirror to a hypersurface of degree n in \mathbb{P}^2 . Similarly, the first two terms in the last line appear to be a (0,2) expansion of the (2,2) Toda dual to \mathbb{P}^1 , defined by the superpotential

$$Z_4 + \frac{q_2}{Z_4},$$

again exactly as one would expect. The remaining terms encode the bundle deformation.

6.3 Brief comment on the quintic

Next, let us consider an example of a different character. Consider a GLSM for a quintic in \mathbb{P}^4 . As is well-known (see e.g. [56, Section 15.6.3]), one can deform it to a (0,2) theory. To see this, first rewrite the (2,2)superpotential in (0,2) language as

$$W = \sum_i \Lambda^i J_i + \Lambda^p J_p,$$

for

$$J_i = p \frac{\partial G}{\partial \phi_i}, \quad J_p = G,$$

for G a quintic polynomial in the chiral superfields ϕ_i . Then, one typically deforms off the (2,2) locus by deforming

$$J_i = p \frac{\partial G}{\partial \phi_i} \mapsto p \frac{\partial G}{\partial \phi_i} + G_i,$$

where the functions G_i are constrained to obey

$$\sum_i \phi_i G_i = 0,$$

Now, let us consider this theory in the context of the mirror proposals of this paper. The A/2-twist only depends upon E 's, not J 's, so the deformation above is invisible in the twisted theory. The dualization procedure we have described does not seem to involve the G_i 's above — we take a tangent bundle deformation on the ambient space and restrict to a hypersurface, but the G_i 's cannot be understood as a tangent bundle deformation of the ambient space. As a result, using the methods here, the G_i 's are effectively invisible. On the other hand, since the A/2 theory is independent of the G_i 's, it is nevertheless a sensible mirror for the A/2 theory.

7 Conclusions

In this paper we have given a proposal for (0,2) mirrors to toric Fano varieties with special tangent bundle deformations, subsets of toric deformations, and also described restrictions to hypersurfaces. Our methods do not apply to all tangent bundle deformations, only a subset of the toric deformations. We have given formal arguments that the resulting correlation functions always match, and also checked in examples that our methods reproduce previous results for (0,2) mirrors produced by laboriously guessing and tuning ansatzes.

It would be interesting to generalize the results presented here to all tangent bundle deformations. On the one hand, for more general tangent bundle deformations, previous methods have generated nonlinear superpotential terms in the mirror, which we do not see. On the other hand, naive moduli counting arguments suggest that in many cases it may be possible to use field redefinitions to rewrite all the GLSM-realizable tangent bundle deformations in the form we use in this paper.

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A Brief notes on (2,2) mirror ansatz

In this appendix we will briefly outline how symmetries and the operator mirror map partially determine the exponential terms in the (2,2) GLSM mirror superpotential. Suppose we have not derived the instanton-generated terms, and only have an ansatz for the mirror superpotential of the form

$$W = \sum_{a=1}^k \Sigma_a \left(\sum_{i=1}^N Q_i^a Y_i - t_a \right) + g(Y_i), \quad (\text{A.1})$$

for some unknown function $g(Y_i)$. (Requiring R-charges match only fixes terms $\exp(-Y_i)$ up to an R-invariant function.) Instead of deriving g from a direct instanton computation in the A-twisted theory, we outline here how the same result could be obtained using other properties of the theory.

Now, previously we derived the operator mirror map (2.7) from the form of the mirror superpotential, but one can outline an independent justification, and then use it to demonstrate the form of g . To see this, use the relation [10, Eq. (3.17)]

$$Y + \bar{Y} = 2\bar{\Phi}e^{2QV}\Phi,$$

which implies component relations [10, Eqs. (3.20), (3.21)]

$$\chi_+ = 2\bar{\psi}_+\phi, \quad \bar{\chi}_- = -2\phi^\dagger\psi_-,$$

for χ the superpartners of Y and ψ the superpartners of ϕ . From [47, Eq. (2.19)], the equations of motion of $\bar{\sigma}_a$ (in the limit $e^2 \rightarrow \infty$, so that the kinetic terms drop out) are

$$\left(\sum_{i=1}^N Q_i^a |\phi_i|^2 \right) \sigma_a \propto \sum_{i=1}^N \bar{\psi}_{+i} \psi_{-i}.$$

If we add twisted masses so that only one ϕ field is light, then this becomes

$$Q_i^a \sigma_a + \tilde{m}_i \propto \frac{\bar{\psi}_{+i} \psi_{-i}}{|\phi_i|^2} \propto \chi_{+i} \bar{\chi}_{-i}.$$

Now, the $\chi_+ \bar{\chi}_-$ could come from a Y^2 , but that has the wrong R charge to make the expression sensible. However, $\exp(-Y_i)$ has the correct R charge and contains Y^2 , so up to overall factors, which can be reabsorbed into field redefinitions, this suggests

$$Q_i^a \sigma_a + \tilde{m}_i = \exp(-Y_i),$$

which is the operator mirror map (2.7). (Granted, we are again using axial R-charges, but here since we know some components, there is less ambiguity.)

Returning to the ansatz (A.1), we can now determine the function g . The equations of motion from the superpotential above imply

$$\begin{aligned}\frac{\partial W}{\partial Y_i} &= Q_i^a \sigma_a + \frac{\partial g}{\partial Y_i}, \\ &= 0,\end{aligned}$$

and the operator mirror map implies

$$Q_i^a \sigma_a + \tilde{m}_a = \exp(-Y_i),$$

hence

$$\frac{\partial g}{\partial Y_i} = -Q_i^a \sigma_a = \tilde{m}_i - \exp(-Y_i),$$

hence

$$g(Y_i) = \tilde{m}_i Y_i + \sum_i \exp(-Y_i),$$

up to an irrelevant additive constant.

It is tempting to apply the same methods to (0,2) theories. Unfortunately, the decreased symmetry leads to multiple possible potential (0,2) mirror superpotentials, derived from applying the operator mirror map in different ways, which must be independently tested against chiral rings and correlation functions.

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