

# Mutual information of excited states and relative entropy of two disjoint subsystems in CFT

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**ABSTRACT:** In this paper, we first study mutual information of excited states in the small subsystem size limit in generic conformal field theory. We then discuss relative entropy of two disjoint subsystems in the same limit.

**KEYWORDS:** Conformal Field Theory, Field Theories in Higher Dimensions, AdS-CFT Correspondence

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**1 Introduction**

Quantum information theoretic approaches have been uncovering several profound properties of quantum field theory. In these approaches, nice features of mutual information and relative entropy, such as positivity or monotonicity play a crucial role in deriving these results [1–10]. Furthermore, these properties can be used to constrain the dual gravitational dynamics and spacetime structure through holography. For example, positivity of relative entropy implies bulk linearized equations of motion [11–13] and constraints bulk matter theory [14–18].

Mutual information  $I(A, B)$  in quantum field theory measures the correlations between two regions A and B. More precisely, this quantity gives an upper bound on correlations between A and B [19],

$$I(A, B) \geq \frac{(\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle)^2}{\|O_A\|^2 \|O_B\|^2}, \tag{1.1}$$

where  $O_A$  and  $O_B$  denote bounded operators on the region A and B respectively. In [20, 21, 25] the mutual information of vacuum in two dimensional conformal field theory was computed in the limit where the distance between A and B is large.<sup>1</sup> This was done by expanding the four point function of the twist operators using their OPE, and gathering the leading order contributions in this limit. This calculation is generalized to CFT in higher dimensions [22, 23], by introducing the notion of a higher dimensional twist operator [22]. A nice feature of this formalism is that it makes the relation between 2d calculation and its higher dimensional counterpart transparent.

In the first part of this paper, we calculate the mutual information of an arbitrary excited state  $|V\rangle$  in generic conformal field theory in the small subsystem size limit. We derive this by computing a correlation function involving twist operators and operators representing the excited state. The result is given by,

$$I(A, B) = (l_A)^{2\Delta} (l_B)^{2\Delta} \frac{\Gamma(\frac{3}{2})\Gamma(2\Delta + 1)}{2\Gamma(2\Delta + \frac{3}{2})} \left[ \langle V | \mathcal{O}_\alpha \mathcal{O}_\beta | V \rangle - \langle V | \mathcal{O}_\alpha | V \rangle \langle V | \mathcal{O}_\beta | V \rangle \right]^2, \quad (1.2)$$

where  $\mathcal{O}$  is the lightest non vacuum operator in the CFT,<sup>2</sup> and  $\Delta$  is the conformal dimension of it.  $(l_A, \alpha)$  and  $(l_B, \beta)$  denote (radius, location) of the subsystem A and B respectively. Note that we can uniquely specify the locations  $\alpha, \beta$  since we are considering the small subsystem size limit. The result is essentially given by square of the connected two point function of the operator  $\mathcal{O}$  evaluated on the excited state  $|V\rangle$ .

Relative entropy  $S(\rho||\sigma)$  measures the distance between two density matrices  $\rho$  and  $\sigma$ , and is defined by

$$S(\rho||\sigma) = \text{tr} \rho \log \rho - \text{tr} \rho \log \sigma. \quad (1.3)$$

When the subsystem A is a single connected region, the relative entropy  $S_A(\rho_V||\rho_W)$  between two reduced density matrices  $\rho_V, \rho_W$  of excited states  $|V\rangle, |W\rangle$ ,  $\rho_V = \text{tr}_{A^c} |V\rangle\langle V|$ ,  $\rho_W = \text{tr}_{A^c} |W\rangle\langle W|$  was computed in the small subsystem limit, both in 2d CFT [27] and CFT in higher dimensions [28] by using the replica trick introduced in [29].

In the second part of this paper, we derive a general formula for the relative entropy  $S_{A \cup B}(\rho_V||\rho_W)$  in the small interval limit  $l_A, l_B \rightarrow 0$ . We do this by reading off the form of the modular Hamiltonian of the excited state  $K_{A \cup B}^W$  from the mutual information result of the first part.

The organization of this paper is as follows: in section 2, and section 3, we discuss the mutual information of an excited state in the small subsystem size limit in 2d CFT. In section 4 we generalize this result to higher dimensions. In section 5, we derive an expression of the relative entropy  $S(\rho_V||\rho_W)$  of the disjoint subsystems in the same limit by using the results of the previous two sections.

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<sup>1</sup>This is equivalent to the small subsystem size limit  $|A|, |B| \rightarrow 0$  with the distance between them kept fixed.

<sup>2</sup>In this paper, we assume that this operator  $\mathcal{O}$  is a scalar. It should be straightforward to generalize this to other cases where  $\mathcal{O}$  has spin.

## 2 Mutual information in 2d CFT

In this section, we focus the on the mutual information,

$$I(A : B) = S_A + S_B - S_{A \cup B}, \quad (2.1)$$

of an excited state in 2d CFT. We consider a CFT defined on a cylinder  $S^1 \times \mathbb{R}$ , and take the subsystems  $A = [l_1, l_2]$ ,  $B = [l_3, l_4]$  on a time slice  $S^1$  of the cylinder.

We compute the mutual information (2.1) by using the replica trick,

$$I(A : B) = \lim_{n \rightarrow 1} \frac{1}{n-1} I_n(A : B), \quad (2.2)$$

with

$$I_n(A : B) = \log \text{tr} \rho_{A \cup B}^n - \log \text{tr} \rho_A^n - \log \text{tr} \rho_B^n. \quad (2.3)$$

Each term in (2.3) can be expressed by a path integral on an  $n$ -sheeted cylinder with the cut on the subsystem. By the exponential map  $w = e^{iz}$ , each cylinder sheet is mapped to the plane, and employing the orbifold description of the theory on the  $n$  sheeted plane  $\Sigma_n$ , one can express them as the correlation functions involving the twist operators  $\sigma_n, \tilde{\sigma}_{-n}$  [26],

$$\text{tr} \rho_{A \cup B}^n = \langle V_n(\infty) \sigma_n(z_1) \tilde{\sigma}_{-n}(z_2) \sigma_n(z_3) \tilde{\sigma}_{-n}(z_4) V_n(0) \rangle, \quad V_n \equiv V^{\otimes n}, \quad (2.4)$$

$$\text{tr} \rho_A^n = \langle V_n(\infty) \sigma_n(z_1) \tilde{\sigma}_{-n}(z_2) V_n(0) \rangle, \quad \text{tr} \rho_B^n = \langle V_n(\infty) \sigma_n(z_3) \tilde{\sigma}_{-n}(z_4) V_n(0) \rangle, \quad (2.5)$$

where

$$z_1 = e^{il_1}, z_2 = e^{il_2}, \quad z_3 = e^{il_3}, \quad z_4 = e^{il_4}. \quad (2.6)$$

By using these, we obtain,

$$I_n(A : B) = \log \left[ \frac{\langle V_n(\infty) \sigma_n(z_1) \tilde{\sigma}_{-n}(z_2) \sigma_n(z_3) \tilde{\sigma}_{-n}(z_4) V_n(0) \rangle}{\langle V_n(\infty) \sigma_n(z_1) \tilde{\sigma}_{-n}(z_2) V_n(0) \rangle \langle V_n(\infty) \sigma_n(z_3) \tilde{\sigma}_{-n}(z_4) V_n(0) \rangle} \right]. \quad (2.7)$$

## 3 Small interval expansion

### 3.1 Substituting the OPEs

The objective of this section is to find the leading behavior of the mutual information (2.1) in the small subsystem limit,  $l_{12} \equiv l_1 - l_2 \rightarrow 0$ ,  $l_{34} \equiv l_3 - l_4 \rightarrow 0$  with  $l_3$  kept fixed. To see the leading behavior, we use the OPE of the twist operators [20, 21, 25],

$$\sigma_n(z_1) \tilde{\sigma}_{-n}(z_2) = \langle \sigma_n(z_1) \tilde{\sigma}_{-n}(z_2) \rangle \left\{ 1 + \sum_{j=1}^{n/2} C_{\sigma_n \tilde{\sigma}_{-n}}^{T^j} T^j(z_2) (l_{12})^{\Delta_T} + \dots \right\}, \quad (3.1)$$

with

$$\begin{aligned} T^j(z_2) &= \left[ \mathcal{O}(z_2) \otimes I^{\otimes(j-1)} \mathcal{O}(z_2) \otimes I^{\otimes(n-j+1)} \right]_{\text{sym}} \\ &\equiv \frac{1}{n} \sum_{p_1} B_{z_2}(p_1, p_1 + j), \end{aligned} \quad (3.2)$$

where  $\mathcal{O}$  is the lightest non vacuum state of the seed theory with the conformal dimension  $\Delta$ , therefore  $\Delta_T = 2\Delta$ , and we have defined

$$B_{z_2}(p_1, p_1 + j) \equiv I^{\otimes(p_1-1)} \otimes \mathcal{O}(z_2) \otimes I^{\otimes(j-1)} \otimes \mathcal{O}(z_2) \otimes I^{\otimes(n-p_1-j)}. \quad (3.3)$$

There is a similar OPE for  $\sigma_n(z_3)\tilde{\sigma}_{-n}(z_4)$ .

Substituting these OPEs into (2.7), we get,

$$I_n(A : B) = (l_{12})^{2\Delta}(l_{34})^{2\Delta} \sum_{a,b}^{n/2} C_{\sigma_n\tilde{\sigma}_{-n}}^{T_a} C_{\sigma_n\tilde{\sigma}_{-n}}^{T_b} \langle V_n(\infty) T^a(z_2) T^b(z_4) V_n(0) \rangle, \quad (3.4)$$

in the small subsystem size limit,  $l_{12}, l_{34} \rightarrow 0$ . Notice that since

$$\sum_j^n C_{\sigma_n\tilde{\sigma}_{-n}}^{T_j} T_j = \sum_{(m,k)=1, m \neq k}^n C_{\sigma_n\tilde{\sigma}_{-n}}^{B(m,k)} B(m, k), \quad (3.5)$$

it can be written in terms of  $B(j, k)$ ,

$$\begin{aligned} I_n(A : B) & \\ &= \frac{1}{4} (l_{12} l_{34})^{2\Delta} \sum_{(j,p_1)=1, j \neq p_1}^n \sum_{(k,p_2)=1, k \neq p_2}^n C_{\sigma_n\tilde{\sigma}_{-n}}^{B(j,p_1)} C_{\sigma_n\tilde{\sigma}_{-n}}^{B(k,p_2)} \langle V_n(\infty) B_{z_2}(p_1, j) B_{z_4}(p_2, k) V_n(0) \rangle. \end{aligned} \quad (3.6)$$

### 3.2 Computation of $I_n(A : B)$

The strategy to compute (3.6) is, first computing the sum with respect to  $j$  and  $k$

$$\begin{aligned} I_{p_1 p_2} &= \sum_{k=1, k \neq p_1}^n \sum_{j=1, j \neq p_2}^n \langle V_n(\infty) B_{z_2}(p_1, j) B_{z_4}(p_2, k) V_n(0) \rangle C_{\sigma_n\tilde{\sigma}_{-n}}^{B(j,p_1)} C_{\sigma_n\tilde{\sigma}_{-n}}^{B(k,p_2)} \\ &\equiv \sum_{k=1, k \neq p_1}^n \sum_{j=1, j \neq p_2}^n I_{p_1 p_2}^{k,j}, \end{aligned} \quad (3.7)$$

while keeping  $p_1, p_2$  fixed, then perform the sum with respect to  $p_1, p_2$ . The precise form of the summand  $I_{p_1, p_2}^{k,j}$  depends on the value of the indices, for example  $k = j_2$  or  $k \neq j_2$ . We classify them and perform the sum carefully in appendix A.

After these calculations, we get following expression of  $I_n(A : B)$ ,

$$\begin{aligned} I_n(A : B) &= \frac{1}{4} (l_{12})^{2\Delta} (l_{34})^{2\Delta} \left( \sum_{p_1 \neq p_2} I_{p_1, p_2} + \sum_{p_1} I_{p_1, p_1} \right) \\ &= \frac{1}{2} (l_{12})^{2\Delta} (l_{34})^{2\Delta} [\langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle]^2 \sum_{k=1, k \neq p_1}^n C(p_1 - k)^2, \end{aligned} \quad (3.8)$$

here we introduced the simplified notations,

$$C(p_1 - p_2) \equiv C_{\sigma_n\tilde{\sigma}_{-n}}^{B(p_1, p_2)} \quad \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \equiv \langle V(\infty) \mathcal{O}(z_2) \mathcal{O}(z_4) V(0) \rangle, \quad \langle \mathcal{O}_\alpha \rangle \equiv \langle V(\infty) \mathcal{O}_\alpha V(0) \rangle. \quad (3.9)$$

and we have omitted terms which are not surviving the final  $n \rightarrow 1$  limit in (2.2).

The  $n \rightarrow 1$  limit of (3.8) can be easily taken by using the formula [20, 21],

$$f(\Delta, n) = \sum_{k=1, k \neq p_1}^n C(p_1 - k)^2 = \sum_{k=1}^{n-1} \frac{1}{(2n \sin \frac{\pi k}{n})^{4\Delta}} \rightarrow (n-1) \frac{\Gamma(3/2)\Gamma(2\Delta+1)}{2^{4\Delta}\Gamma(2\Delta+3/2)}, \quad (3.10)$$

therefore we get,

$$I(A, B) = (l_A)^{2\Delta} (l_B)^{2\Delta} \frac{\Gamma(\frac{3}{2})\Gamma(2\Delta+1)}{2\Gamma(2\Delta+\frac{3}{2})} \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right]^2. \quad (3.11)$$

Here we introduced radii  $l_A, l_B$  of the subsystems  $2l_A \equiv l_{12}, 2l_B \equiv l_{34}$ .

## 4 Mutual information in higher dimensions

In this section we discuss the higher dimensional generalization of the above 2d calculation, by using higher dimensional twist operators  $\Sigma_A$  developed recently in [22–24].

We will see that the result (3.11) remains to be true in this case, once we interpret  $l_A, l_B$  as radii of the regions  $A, B$ .

### 4.1 Setups

We start from a d-dimensional conformal field theory on a cylinder  $S^{d-1} \times \mathbb{R}$ . We define the standard metric on this manifold,

$$ds^2 = dt^2 + (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2), \quad (4.1)$$

where  $d\Omega_{d-2}^2$  denotes the metric on the d-2 dimensional sphere  $S^{d-2}$ .

We choose the subsystem to be the union of two disjoint cap like regions  $A \cup B$  on the spatial manifold  $S^{d-1}$  at  $t = 0$ . We again assume that two regions  $A$  and  $B$  are small, therefore we can define radii  $l_A, l_B$  of the regions.

As in two dimensions, the trace of the reduced density matrix  $\text{tr } \rho_{AUB}^n$  of an excited state  $|V\rangle$  is given by the path integral on the n-sheet cover of  $S^{d-1} \times \mathbb{R}$  with the cut on the subsystem  $A \cup B$ .

It is again useful to map the replica manifold to the n-sheet plane  $C_{AUB}^{(n)}(\mathbb{R}^d)$  with the radial coordinate  $r$ , by the conformal map  $t = \log r$ . On the n-sheet plane, two regions  $A$  and  $B$  are mapped to regions in  $r = 1$  sphere, the excited states are located at  $r = 0, \infty$ . By using state operator correspondence, we obtain,

$$\text{tr } \rho_{AUB}^n = Z_{AUB}^{(n)} \left\langle \prod_{j=1}^n V(\infty_j) V(0_j) \right\rangle_{C_{AUB}^{(n)}(\mathbb{R}^d)}, \quad (4.2)$$

where  $Z_{AUB}^{(n)}$  denotes the partition function on the n-sheet plane  $C_{AUB}^{(n)}(\mathbb{R}^d)$ .

## 4.2 Higher dimensional twist operators

In this section we briefly review the concept of a higher dimensional twist operator [22]. A higher dimensional twist operator  $\Sigma_A$  of a region  $A$  is a non local operator defined in the  $n$  copies of the original CFT (below this theory is denoted by  $\text{CFT}^{\otimes n}$ ), and satisfies

$$\langle X \rangle_{C_A^{(n)}} = \frac{(Z_A^{(1)})^n}{Z_A^{(n)}} \langle X \Sigma_A \rangle_{\text{CFT}^{\otimes n}}, \quad (4.3)$$

where  $X$  is a product of local operators of the CFT on  $C_A^{(n)}$ . Note that we can naturally interpret  $X$  to be an operator of  $\text{CFT}^{\otimes n}$ .

In the small subsystem size limit  $|A| \rightarrow 0$ , one can expand the twist operator by a set of local operators of  $\text{CFT}^{\otimes n}$  on the region  $A$  [22],

$$\Sigma_A = \frac{Z_A^{(n)}}{(Z_A^{(1)})^n} \sum_{\{k_j\}} C_{\{k_j\}}^A \otimes_{j=1}^n \mathcal{O}_{k_j}(r_A), \quad (4.4)$$

Furthermore, the coefficient  $C_{\{k_j\}}^A$  is given by the correlation function on  $C_A^{(n)}$ ,

$$C_{\{k_j\}}^A = \lim_{r \rightarrow \infty} \langle \prod_{j=1}^n \mathcal{O}_{k_j}(r) \rangle_{C_A^{(n)}}. \quad (4.5)$$

When the original theory is defined on a conformally flat space, these coefficients are related to correlation functions on  $S^1 \times H^{d-1}$ , where  $H^{d-1}$  is  $d-1$  dimensional hyperbolic space [23]. If we pick up the first few terms in the small subsystem size expansion (4.4), we get

$$\Sigma_A = \frac{Z_A^{(n)}}{(Z_A^{(1)})^n} \left( 1 + \frac{1}{2} \sum_{(p_1, j)=1, j \neq p_1}^n C_{(p_1, j)}^A B_{r_A}(p_1, j) + \dots \right), \quad (4.6)$$

where  $B_{r_A}(p_1, j)$  is defined by

$$B_{r_A}(p_1, j) = I^{\otimes(p_1-1)} \otimes \mathcal{O}(r_A) \otimes I^{\otimes(j-p_1-1)} \otimes \mathcal{O}(r_A) \otimes I^{\otimes n-j}, \quad (4.7)$$

and  $\mathcal{O}$  is the lightest non vacuum operator with the conformal dimension  $\Delta$ . From the conformal map of interest, we can specify the subsystem size dependence of the coefficient [23],

$$C_{p_1, j}^A = (2l_A)^\Delta \tilde{C}_{p_1, j}, \quad (4.8)$$

and  $\tilde{C}_{p_1, j}$  is independent of  $l_A$ .

## 4.3 The expression of $I_n(A, B)$

By using the twist operators  $\Sigma_A, \Sigma_B$ , we can write  $\text{tr } \rho_V^n(A \cup B)$  in term of the correlation function in  $\text{CFT}^{\otimes n}$

$$\text{tr } \rho_V^n(A \cup B) = \langle V^{\otimes n}(\infty) \Sigma_A \Sigma_B V^{\otimes n}(0) \rangle_{\text{CFT}^{\otimes n}}. \quad (4.9)$$

There are similar expressions of  $\text{tr}\rho_V^n(A), \text{tr}\rho_V^n(B)$ . By substituting them into the definition (2.3) of  $I_n(A, B)$ , and by using the expansion (4.6), we get

$$\begin{aligned}
 I_n(A, B) & \tag{4.10} \\
 &= \frac{(4l_A l_B)^{2\Delta}}{4} \sum_{(p_1, j)=1, p_1 \neq j}^n \sum_{(p_2, k)=1, p_2 \neq k}^n \tilde{C}_{(p_1, j)} \tilde{C}_{(p_2, k)} \langle V^{\otimes n} B_{r_A}(p_1, j)(\infty) B_{r_B}(p_2, k) V^{\otimes n}(0) \rangle.
 \end{aligned}$$

Note that this expression of  $I_n(A, B)$  is essentially same as that of the 2d counterpart (3.6). This in particular means that (3.8) still holds in the higher dimensional case, once we do the replacement  $C(p_1 - j) \rightarrow \tilde{C}_{(p_1, j)}$ . In [23] it was shown that

$$\sum_{k=1, k \neq p_1}^n \tilde{C}_{(p_1, j)}^2 \rightarrow (n-1) \frac{\Gamma(3/2)\Gamma(2\Delta+1)}{2^{4\Delta}\Gamma(2\Delta+3/2)}, \quad n \rightarrow 1, \tag{4.11}$$

therefore the result (3.11) remains to be true in higher dimensions.

## 5 Relative entropy of two disjoint intervals

In this section we discuss the relative entropy  $S_{A \cup B}(\rho_V || \rho_W)$  of two disjoint subsystems  $A \cup B$  in the small subsystem size limit. We take  $\rho_V, \rho_W$  to be the reduced density matrices of two excited states  $|V\rangle, |W\rangle$  on  $A \cup B$ . We calculate the relative entropy by finding the form of the modular Hamiltonian  $K_{A \cup B}^W$  from entanglement first law,  $\delta S = \langle K_{A \cup B}^W \delta \rho \rangle$ . Similar trick was used in [28] to derive the first asymmetric part of the relative entropy  $S_A(\rho_V || \rho_W)$  of a connected region A.

### 5.1 Summary of entanglement entropy

Here we summarize the result of the entanglement entropy  $S_{A \cup B}$  of an excited state  $|W\rangle$  on  $A \cup B$  in the small interval limit. By using mutual information  $I(A, B)$  we can write this,

$$S_{A \cup B} = S_A + S_B - I(A, B). \tag{5.1}$$

$S_A$  is the entanglement entropy of the region A, and in the small subsystem size limit  $l_A \rightarrow 0$ , it is given by (see for example [28]),

$$S_A(\rho_W) = \langle W | K_A^0 | W \rangle - c_A l_A^{2\Delta} \langle W | \mathcal{O}_A | W \rangle^2 + \dots \quad c_A \equiv \frac{\Gamma(\frac{3}{2})\Gamma(\Delta+1)}{2\Gamma(\Delta+\frac{3}{2})}. \tag{5.2}$$

where  $\dots$  denotes the terms of  $O(l_A^{3\Delta})$ .  $K_A^0$  is the modular Hamiltonian of the vacuum,  $\mathcal{O}$  is the lightest non vacuum operator, and  $\Delta$  is the conformal dimension of the operator  $\mathcal{O}$ .

There is a similar expression for  $S_B(\rho_W)$ . When the original state is not pure but a mixed state

$$\sigma = \sum_w p_w |w\rangle \langle w|, \tag{5.3}$$

then the entanglement entropy of  $\sigma$  in this limit is given by

$$S_A(\sigma) = \text{tr} [K_A^0 \sigma] - c_A l_A^{2\Delta} \text{tr} [\sigma \mathcal{O}_A]^2 + \dots \tag{5.4}$$



The last term in (5.1) is the mutual information, in the small subsystem size limit  $l_A, l_B \rightarrow 0$ , it is given by (1.2) which we reproduce here,

$$I(A, B) = (l_A)^{2\Delta} (l_B)^{2\Delta} \frac{\Gamma(\frac{3}{2})\Gamma(2\Delta + 1)}{2\Gamma(2\Delta + \frac{3}{2})} \left[ \langle W|\mathcal{O}_A\mathcal{O}_B|W\rangle - \langle W|\mathcal{O}_A|W\rangle\langle W|\mathcal{O}_B|W\rangle \right]^2 + \dots \quad (5.5)$$

By defining

$$c_{AB} = (l_A)^{2\Delta} (l_B)^{2\Delta} \frac{\Gamma(\frac{3}{2})\Gamma(2\Delta + 1)}{2\Gamma(2\Delta + \frac{3}{2})}$$

$$M_{AB}^W = \langle W|\mathcal{O}_A\mathcal{O}_B|W\rangle - \langle W|\mathcal{O}_A|W\rangle\langle W|\mathcal{O}_B|W\rangle, \quad (5.6)$$

we can write it as

$$I^W(A, B) = c_{AB}(l_A, l_B)(M_{AB}^W)^2 + \dots \quad (5.7)$$

## 5.2 Modular Hamiltonian

If we slightly deform the reduced density matrix of  $|W\rangle$ ,  $\rho_W \rightarrow \rho_W + \delta\rho$ , then the entanglement entropy changes as

$$\delta S_{AB} = \langle K_{AUB}^W \delta\rho\rangle, \quad K_{AUB}^W = -\log \rho_W, \quad (5.8)$$

$K_{AUB}^W$  is the modular Hamiltonian of the state  $|W\rangle$ . It is convenient to divide it into three parts

$$K_{AUB}^W = K_A^W + K_B^W + K_{AB}^W, \quad (5.9)$$

and we define each part of the right hand side of (5.9) by

$$\delta S_A = \langle K_A \delta\rho\rangle \quad \delta S_B = \langle K_B \delta\rho\rangle, \quad (5.10)$$

$$-\delta I^W(A, B) = \langle K_{AB}^W \delta\rho\rangle. \quad (5.11)$$

By taking the variation of (5.4) with respect to the deformation of  $\rho_W$  and combine it with (5.10), we get,

$$\delta S_A = \langle K_A^0 \delta\rho\rangle - 2c_A l_A^{2\Delta} \langle \mathcal{O}_A \rangle_W \langle \mathcal{O}_A \delta\rho\rangle = \langle K_A^W \delta\rho\rangle, \quad \langle \mathcal{O}_A \rangle_W \equiv \langle W|\mathcal{O}_A|W\rangle. \quad (5.12)$$

Since the above equation holds for any deformation of the reduced density matrix  $\delta\rho$ , we can read off the expression of the modular Hamiltonian  $K_A^W$ ,<sup>3</sup>

$$K_A^W = K_A^0 - 2c_A l_A^{2\Delta} \langle \mathcal{O}_A \rangle_W \mathcal{O}_A + \dots \quad (5.13)$$

in the small interval limit,  $l_A \rightarrow 0$ .

By using a similar argument for (5.5), we obtain

$$K_{AB}^W = -2M_{AB}^W c_{AB} \left[ \mathcal{O}_A \mathcal{O}_B - \langle \mathcal{O}_B \rangle_W \mathcal{O}_A - \langle \mathcal{O}_A \rangle_W \mathcal{O}_B \right] + \dots \quad (5.14)$$

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<sup>3</sup>We thank S.Leichenauer for discussion on this.

### 5.3 Relative entropy of two disjoint interval

By putting everything together, the relative entropy  $S_{AUB}(\rho_V||\rho_W)$  is now given by

$$\begin{aligned}
 S_{AUB}(\rho_V||\rho_W) &= \Delta\langle K_{AUB}^W \rangle - \Delta S_{AUB} \\
 &= \left( \Delta\langle K_A^W \rangle - \Delta S_A \right) + \left( \Delta\langle K_B^W \rangle - \Delta S_B \right) + \left( \langle K_{AB}^W \rangle + \Delta I(A, B) \right) \\
 &= S_A(\rho_V||\rho_W) + S_B(\rho_V||\rho_W) + c_{AB} \left[ (M_{AB}^V - M_{AB}^W)^2 - 2M_{AB}^W (\langle \mathcal{O} \rangle_V - \langle \mathcal{O} \rangle_W)^2 \right]. \quad (5.15)
 \end{aligned}$$

$S_A(\rho_V||\rho_W)$  denotes the relative entropy of the region A [27, 28], which is given by

$$S_A(\rho_V||\rho_W) = c_A l_A^{2\Delta} (\langle V|\mathcal{O}_A|V \rangle - \langle W|\mathcal{O}_A|W \rangle)^2 + \dots, \quad (5.16)$$

in the small subsystem size limit. We have a similar expression for  $S_B(\rho_V||\rho_W)$ .

The last two terms of (5.15) involve both region A and B. Note that the last term is asymmetric under the exchange of two states  $|V\rangle \leftrightarrow |W\rangle$ .

## 6 Conclusions

In the first part of this paper, we derived a general formula for the mutual information  $I(A, B)$  of an arbitrary excited state in the small subsystem size limit. In the context of holography, this result should agree with the mutual information  $I(\hat{A}, \hat{B})$  of the bulk quantum field theory of the regions  $\hat{A}, \hat{B}$  which are enclosed by the corresponding boundary subsystems and the bulk RT surfaces [31, 32], according to the FLM conjecture [30]. It would be interesting to check this statement concretely. When the CFT state is vacuum, this was explicitly confirmed in [23] by calculating the mutual information on the dual geometry, ie, anti de Sitter space. To reproduce our result (1.2) by a bulk calculation, we need to generalize their work to the asymptotically AdS space with the back reaction of the scalar condensate dual to  $\langle \mathcal{O} \rangle$ .

In the second part, we derived a formula for the relative entropy  $S_{AUB}(\rho_V||\rho_W)$  of disjoint subsystems in the small subsystem size limit. We did this by reading off the form of the modular Hamiltonian from the entanglement first law. It will be an interesting future work to study the relative entropy numerically.

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## A Some details of the calculation of $I_n(A, B)$

In this appendix we explain the detail of the calculation of  $I_n(A, B)$  defined by

$$\begin{aligned}
 I_n(A : B) & \tag{A.1} \\
 &= \frac{1}{4}(l_{12}l_{34})^{2\Delta} \sum_{(j,p_1)=1, j \neq p_1}^n \sum_{(k,p_2)=1, k \neq p_2}^n C_{\sigma_n \bar{\sigma}_{-n}}^{B(j,p_1)} C_{\sigma_n \bar{\sigma}_{-n}}^{B(k,p_2)} \langle V_n(\infty) B_{z_2}(p_1, j) B_{z_4}(p_2, k) V_n(0) \rangle.
 \end{aligned}$$

To do this we first compute the sum,

$$\begin{aligned}
 I_{p_1 p_2} &= \sum_{j=1, k \neq p_1}^n \sum_{k=1, k \neq p_2}^n \langle V_n(\infty) B_{z_2}(p_1, j) B_{z_4}(p_2, k) V_n(0) \rangle C_{\sigma_n \bar{\sigma}_{-n}}^{B(j,p_1)} C_{\sigma_n \bar{\sigma}_{-n}}^{B(k,p_2)} \\
 &\equiv \sum_{j=1, k \neq p_1}^n \sum_{k=1, k \neq p_2}^n I_{p_1 p_2}^{k,j}, \tag{A.2}
 \end{aligned}$$

for fixed  $p_1, p_2$ , by classifying the possible forms of the summand  $I_{p_1, p_2}^{j,k}$ , then performing the sum with respect to  $p_1, p_2$ .

### A.1 $I_{p_1, p_2} : p_1 \neq p_2$ case

This case is further classified into four possibilities,  $\{(j=p_2), (k=p_1)\}$ ,  $\{(j \neq p_2), (k=p_1)\}$ ,  $\{(j=p_2), (k \neq p_1)\}$ ,  $\{(j \neq p_2), (k \neq p_1)\}$ .

$\{(j=p_2), (k=p_1)\}$ . When  $\{(j=p_2), (k=p_1)\}$  the summand in (3.4) is given by

$$I_{p_1 p_2}^{p_2, p_1} = C(p_1 - p_2)^2 \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle^2, \tag{A.3}$$

here we introduced the simplified notations,

$$C(p_1 - p_2) \equiv C_{\sigma_n \bar{\sigma}_{-n}}^{T(p_1 - p_2)} = C_{\sigma_n \bar{\sigma}_{-n}}^{B(p_1, p_2)}, \quad \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \equiv \langle V(\infty) \mathcal{O}(z_2) \mathcal{O}(z_4) V(0) \rangle. \tag{A.4}$$

$\{(j=p_2), (k \neq p_1)\}$ . When  $\{(j=p_2), (k \neq p_1)\}$ , The summand is given by

$$I_{p_1, p_2}^{p_2, k} = C(p_2 - p_1) C(k - p_2) \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle, \tag{A.5}$$

with

$$\langle \mathcal{O}_\alpha \rangle = \langle V(\infty) \mathcal{O}(z_2) V(0) \rangle, \quad \langle \mathcal{O}_\beta \rangle = \langle V(\infty) \mathcal{O}(z_4) V(0) \rangle. \tag{A.6}$$

The sum with respect to  $k$  is given by,

$$\begin{aligned}
 \sum_{k=1, k \neq p_1, p_2}^n I_{p_1 p_2}^{p_2, k} &= \sum_{k=1, k \neq p_1, p_2}^n C(p_1 - p_2) C(k - p_2) \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \\
 &= C(p_1 - p_2) \left[ \sum_{k=1, k \neq p_2}^n C(k - p_2) - C(p_1 - p_2) \right] \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle. \tag{A.7}
 \end{aligned}$$

$\{(j \neq p_2), (k = p_1)\}$ . Similarly, when  $\{(j \neq p_2), (k = p_1)\}$

$$\sum_{j=1, j \neq p_1, p_2}^n I_{p_1, p_2}^{j, p_1} = C(p_1 - p_2) \left[ \sum_{j=1, j \neq p_1}^n C(j - p_1) - C(p_1 - p_2) \right] \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle. \quad (\text{A.8})$$

$\{(j \neq p_2), (k \neq p_1)\}$ . Let us finally consider the  $\{(j \neq p_2), (k \neq p_1)\}$  case. It is useful to further separate this case into  $j = k$ , and  $j \neq k$ . The  $j = k$  result is given by

$$\sum_{k=1, k \neq p_1, p_2} I_{p_1, p_2}^{k, k} = \sum_{k=1, k \neq p_1, p_2} C(k - p_1) C(k - p_2) \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle. \quad (\text{A.9})$$

The  $j \neq k$  result is

$$\begin{aligned} & \sum_{j=1, j \neq p_1}^n \sum_{k=1, k \neq p_1, j}^n I_{p_1, p_2}^{j, k} \\ &= \sum_{j=1, j \neq p_1}^n \sum_{k=1, k \neq p_1, j}^n C(j - p_1) C(k - p_2) \langle \mathcal{O}_\alpha \rangle^2 \langle \mathcal{O}_\beta \rangle^2 \\ &= \left[ \left( \sum_{j=1, j \neq p_1}^n C(j - p_1) \right) \left( \sum_{k=1, k \neq p_2}^n C(k - p_2) \right) - \sum_{k=1, k \neq p_1, p_2}^n C(k - p_1) C(k - p_2) \right. \\ & \quad \left. - C(p_1 - p_2) \left( \sum_{j=1, j \neq p_1}^n C(j - p_1) + \sum_{k=1, k \neq p_2}^n C(k - p_1) - C(p_1 - p_2) \right) \right] \langle \mathcal{O}_\alpha \rangle^2 \langle \mathcal{O}_\beta \rangle^2. \end{aligned} \quad (\text{A.10})$$

**Net result for  $I_{p_1 p_2}$ .** By putting (A.7), (A.8), (A.9), (A.10) together, we get,

$$\begin{aligned} I_{p_1, p_2} &= \sum_{j=1, k \neq p_1}^n \sum_{k=1, k \neq p_2}^n I_{p_1, p_2}^{j, k} = C(p_1 - p_2)^2 \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right]^2 \\ & \quad + \left[ \left( \sum_{k=1, k \neq p_1, p_2}^n C(k - p_1) C(k - p_2) \right) + C(p_1 - p_2) \left( \sum_{k=1, k \neq p_2}^n C(k - p_2) + \sum_{j=1, j \neq p_1}^n C(j - p_1) \right) \right] \\ & \quad \times \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right] \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \\ & \rightarrow C(p_1 - p_2)^2 \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right]^2 \\ & \quad + \sum_{k=1, k \neq p_1, p_2}^n C(k - p_1) C(k - p_2) \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right] \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle, \quad n \rightarrow 1. \end{aligned} \quad (\text{A.11})$$

In the last line we only picked up terms which are proportional to  $(n-1)$  after the final analytic continuation. Terms which are ignored in the last line are proportional to  $(n-1)^2$  in  $I_n(A : B)$ . For example

$$\sum_{p_1=1, p_1 \neq p_2}^n \sum_{k=1, k \neq p_2}^n C(p_1 - p_2) C(k - p_2) = f(n, \Delta/2)^2 \propto (n-1)^2, \quad n \rightarrow 1. \quad (\text{A.12})$$

where  $f(n, \Delta)$  is defined in (3.10),

### A.2 $I_{p_1, p_2} : p_1 = p_2$ case

When  $j = k$ ,

$$\sum_{k=1, k \neq p_1}^n I_{p_1, p_1}^{k, k} = \sum_{k=1, k \neq p_1}^n C(p_1 - k)^2 \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle^2. \quad (\text{A.13})$$

$j \neq k$  result is

$$\begin{aligned} \sum_{j=1, j \neq p_1}^n \sum_{k=1, k \neq j, p_1}^n I_{p_1, p_1}^{k, j} &= \sum_{j=1, j \neq p_1}^n \sum_{k=1, k \neq j, p_1}^n C(j - p_1) C(k - p_1) \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \\ &= \left[ \left( \sum_{k=1, k \neq p_1}^n C(k - p_1) \right)^2 - \sum_{k=1, k \neq p_1}^n C(k - p_1)^2 \right] \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle. \end{aligned} \quad (\text{A.14})$$

Therefore

$$I_{p_1, p_1} \rightarrow \sum_{k=1, k \neq p_1}^n C(p_1 - k)^2 \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right] \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle. \quad (\text{A.15})$$

Again we only picked up terms surviving  $n \rightarrow 1$  limit.

### A.3 The final result

By combining (A.15) and (A.11), and using the identity

$$\sum_{p_1 \neq p_2} \sum_{k \neq p_1, p_2} C(k - p_1) C(k - p_2) = n \left( \sum_{p_1=1, p_1 \neq k} C(k - p_1) \right)^2 - \sum_k \sum_{p_1 \neq k} C(p_1 - k)^2, \quad (\text{A.16})$$

we get the expression of  $I_n(A : B)$ ,

$$\begin{aligned} I_n(A : B) &= \frac{1}{4} (l_{12})^\Delta (l_{34})^\Delta \left( \sum_{p_1 \neq p_2} I_{p_1, p_2} + \sum_{p_1} I_{p_1, p_1} \right) \\ &= \frac{1}{2} (l_{12})^\Delta (l_{34})^\Delta \left[ \langle \mathcal{O}_\alpha \mathcal{O}_\beta \rangle - \langle \mathcal{O}_\alpha \rangle \langle \mathcal{O}_\beta \rangle \right]^2 \sum_{k=1, k \neq p_1}^n C(p_1 - k)^2. \end{aligned} \quad (\text{A.17})$$

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