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A superfield realization of the integrated vertex operator in an $AdS_5 \times S^5$ background

Osvaldo Chandia[a] **and Brenno Carlini Vallilo**[b]

[a] *Departamento de Ciencias, Facultad de Artes Liberales & UAI Physics Center, Universidad Adolfo Ibáñez, Santiago, Chile*

[b] *Departamento de Ciencias Físicas, Universidad Andres Bello, Sazie 2212, Santiago, Chile*

E-mail: ochandiaq@gmail.com, vallilo@gmail.com

Abstract: The integrated massless vertex operator in an $AdS_5 \times S^5$ background in the pure spinor formalism is constructed in terms of superfields.

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Contents

1 Introduction

A covariantly quantizable action for the type IIB superstring was proposed some time ago using the pure spinor formalism [1, 2]. In this formalism, the physical states are defined as cohomology elements of a nilpotent BRST-like charge. The light-cone spectrum obtained [3] coincides with the answer one for a superstring in flat ten dimensional space. This was done using non-covariant methods. The covariant description of states and vertex operators is much less known. Only the first massive state of the open superstring was studied covariantly [4]. An attempt to describe a specific massive state in the $AdS_5 \times S^5$ background was made in [5].

The condition of having the BRST-like symmetry in generic background fields constrains them to be on-shell. This was done for supregravity [6]. For the type IIB superstring in an $AdS_5 \times S^5$ background, it was shown that this is exact for all orders in perturbation theory [7]. Fluctuations around a given background are the vertex operators of the theory. They are also necessary to compute amplitudes. The problem of finding explicit descriptions of specific physical states of the superstring in curved backgrounds is very difficult both in RNS of GS-like descriptions. Apart from BMN limits of AdS backgrounds [8, 9], as far as we know, the only construction of a vertex operator for some specific state in curved backgrounds has been done very recently in [10] for the ambitwistor RNS superstring [11, 12] in a plane wave background.

The structure of the unintegrated vertex operator for $AdS_5 \times S^5$ was first described in [2]. A formal construction for the integrated vertex operator in this case was done in [13] using the string Lax pair [14–16]. The explicit form of the integrated vertex operator was not considered in this reference. The purpose of this paper is to construct the explicit superfields of the integrated vertex operator and their constraints. Due to the lack of a fundamental b ghost, the relation between the unintegrated vertex operator U and the integrated vertex operator V are better described by a descent procedure

$$QU = 0, \quad \partial U = QW, \quad \bar{\partial} U = Q\bar{W}, \quad QV = \partial \bar{W} - \bar{\partial} W, \tag{1.1}$$

where Q is the pure spinor BRST-like charge and $(W, \bar{W})$ are operators defined by these equations.

This paper is organized as follows. Section 2 contains a short review of the pure spinor formalism and a detailed description of its massless vertex operators. In section 3 we begin with the description of the pure spinor version of type IIB superstring in an $AdS_5 \times S^5$ background. After it we discuss the unintegrated vertex operator, pointing out some similarities with the flat space case. In section 4 we find the chain of operators (1.1) for $AdS_5 \times S^5$ background with all superfields containing the physical fluctuations. We conclude the work in section 5, discussing future problems and possible applications.

2 Massless vertex operators in flat space for type II superstring

We will begin with a short review of the pure spinor type II superstring in flat space. The closed string vertex operator was studied in detail in [17], here we will review some aspects which will be relevant later. The fundamental variables are those of $N = 2$ ten dimensional superspace $(X, \theta, \bar{\theta})$ plus the conjugate momenta of the odd variables $(p, \bar{p})$ and a set of ghosts. The action is

$$S_0 = \int d^2z \left(\frac{1}{2} \partial X_m \bar{\partial} X^m + p_\alpha \bar{\partial} \theta^\alpha + \bar{p}_{\bar{\alpha}} \partial \bar{\theta}^{\bar{\alpha}} + \omega_\alpha \bar{\partial} \lambda^\alpha + \bar{\omega}_{\bar{\alpha}} \partial \bar{\lambda}^{\bar{\alpha}} \right), \tag{2.1}$$

and the BRST symmetry is generated by

$$Q = \oint \left(\lambda^\alpha d_\alpha + \bar{\lambda}^{\bar{\alpha}} \bar{d}_{\bar{\alpha}} \right), \tag{2.2}$$

where

$$d_\alpha = p_\alpha + \frac{1}{2} (\gamma^m \theta)_\alpha \left(\partial X_m - \frac{1}{4} (\theta \gamma_m \partial \theta) \right), \quad \bar{d}_{\bar{\alpha}} = \bar{p}_{\bar{\alpha}} + \frac{1}{2} (\gamma^m \bar{\theta})_{\bar{\alpha}} \left(\bar{\partial} X_m - \frac{1}{4} (\bar{\theta} \gamma_m \bar{\partial} \bar{\theta}) \right), \tag{2.3}$$

where the γ^m's are the symmetric 16×16 gamma matrices in ten dimensions. The BRST charge is nilpotent when the ghosts $(\lambda, \bar{\lambda})$ satisfy

$$\lambda \gamma^m \lambda = 0 = \bar{\lambda} \gamma^m \bar{\lambda}. \tag{2.4}$$

These conditions also imply that the anti-ghosts are defined up to

$$\delta \omega_\alpha = a_m (\gamma^m \lambda)_\alpha, \quad \delta \bar{\omega}_{\bar{\alpha}} = b_m (\gamma^m \bar{\lambda})_{\bar{\alpha}}, \tag{2.5}$$

for any local parameters (a_m, b_m).

JHEP10(2017)178

It is useful to work with supersymmetric combinations of the world-sheet variables. They are the $d, \bar{d}$ defined above, world-sheet derivatives of $\theta, \bar{\theta}$ and

$$\Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma_m\partial\theta) + \frac{1}{2}(\bar{\theta}\gamma^m\partial\bar{\theta}), \quad \bar{\Pi}^m = \bar{\partial} X^m + \frac{1}{2}(\theta\gamma_m\bar{\partial}\theta) + \frac{1}{2}(\bar{\theta}\gamma^m\bar{\partial}\bar{\theta}). \tag{2.6}$$

The BRST transformations of these supersymmetric invariants are given by

$$\begin{aligned} Q\Pi^m &= (\lambda\gamma^m\partial\theta) + (\bar{\lambda}\gamma^m\partial\bar{\theta}), & Qd_\alpha &= -(\lambda\gamma_m)_\alpha\Pi^m, & & \\ Q\partial\theta^\alpha &= \partial\lambda^\alpha, & Q\omega_\alpha &= d_\alpha, & Q\lambda^\alpha &= 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} Q\bar{\Pi}^m &= (\lambda\gamma^m\bar{\partial}\theta) + (\bar{\lambda}\gamma^m\bar{\partial}\bar{\theta}), & Q\bar{d}_{\bar{\alpha}} &= -(\bar{\lambda}\gamma_m)_{\bar{\alpha}}\bar{\Pi}^m, & & \\ Q\bar{\partial}\bar{\theta}^{\bar{\alpha}} &= \bar{\partial}\bar{\lambda}^{\bar{\alpha}}, & Q\bar{\omega}_{\bar{\alpha}} &= \bar{d}_{\bar{\alpha}}, & Q\bar{\lambda}^{\bar{\alpha}} &= 0. \end{aligned} \tag{2.8}$$

Note that on-shell λ only appears in the holomorphic sector and $\bar{\lambda}$ only appears in the anti-holomorphic sector. This is due to the fact that Q is only part of a much larger symmetry generated by the holomorphic and anti-holomorphic currents $j = \lambda^\alpha d_\alpha$ and $\bar{j} = \bar{\lambda}^{\bar{\alpha}}\bar{d}_{\bar{\alpha}}$.

The BRST transformation of any superfield is $Q\Psi(X,\theta,\bar{\theta}) = \lambda^\alpha D_\alpha\Psi + \bar{\lambda}^{\bar{\alpha}} D_{\bar{\alpha}}\Psi$, where $D_\alpha = \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$ and $D_{\bar{\alpha}} = \partial_{\bar{\alpha}} + \frac{1}{2}(\gamma^m\bar{\theta})_{\bar{\alpha}}\partial_m$. The algebra of theses superspace covariant derivatives is given by

$$\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m, \quad \{D_{\bar{\alpha}}, D_{\bar{\beta}}\} = \gamma^m_{\bar{\alpha}\bar{\beta}}\partial_m, \quad \{D_\alpha, D_{\bar{\beta}}\} = 0. \tag{2.9}$$

Physical states are defined to be in the cohomology of Q. The massless states are described by the unintegrated vertex operator with vanishing classical dimension[1]

$$U = \lambda^\alpha\bar{\lambda}^{\bar{\beta}} A_{\alpha\bar{\beta}}(X,\theta,\bar{\theta}). \tag{2.10}$$

Since U is in the cohomology of Q, its gauge transformation is given by

$$\delta U = Q\Lambda = Q(\lambda^\alpha\Lambda_\alpha + \bar{\lambda}^{\bar{\alpha}}\bar{\Lambda}_{\bar{\alpha}}) = \lambda^\alpha\bar{\lambda}^{\bar{\beta}}(D_\alpha\Lambda_{\bar{\beta}} + D_{\bar{\beta}}\Lambda_\alpha) + \lambda^\alpha\lambda^\beta D_\alpha\Lambda_\beta + \bar{\lambda}^{\bar{\alpha}}\bar{\lambda}^{\bar{\beta}} D_{\bar{\alpha}}\Lambda_{\bar{\beta}}. \tag{2.11}$$

In order to preserve the original form of U the parameters $(\Lambda_\alpha, \bar{\Lambda}_{\bar{\alpha}})$ are constrained to satisfy

$$D_{(\alpha}\Lambda_{\beta)} = \gamma^m_{\alpha\beta}\Lambda_m, \quad D_{(\bar{\alpha}}\bar{\Lambda}_{\bar{\beta})} = \gamma^m_{\bar{\alpha}\bar{\beta}}\bar{\Lambda}_m. \tag{2.12}$$

These conditions resembles the equations that define two on-shell vector multiplet in ten dimensions. The main difference is that here $(\Lambda_\alpha, \bar{\Lambda}_{\bar{\alpha}})$ are functions of $N = 2$ superspace variables. From them we can also obtain

$$D_\alpha\Lambda_m - \partial_m\Lambda_\alpha = (\gamma_m)_{\alpha\beta}\Lambda^\beta, \quad D_\alpha\Lambda^\beta = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\partial_{[m}\Lambda_{n]}, \tag{2.13}$$

$$D_{\bar{\alpha}}\bar{\Lambda}_m - \partial_m\bar{\Lambda}_{\bar{\alpha}} = (\gamma_m)_{\bar{\alpha}\bar{\beta}}\bar{\Lambda}^{\bar{\beta}}, \quad D_{\bar{\alpha}}\bar{\Lambda}^{\bar{\beta}} = \frac{1}{4}(\gamma^{mn})_{\bar{\alpha}}{}^{\bar{\beta}}\partial_{[m}\bar{\Lambda}_{n]}. \tag{2.14}$$

[1] A more general form of the vertex operator is given by

$$U = \lambda^\alpha\bar{\lambda}^{\bar{\beta}} A_{\alpha\bar{\beta}}(X,\theta,\bar{\theta}) + \tfrac{1}{2}\lambda^\alpha\lambda^\beta A_{\alpha\beta}(X,\theta,\bar{\theta}) + \tfrac{1}{2}\bar{\lambda}^{\bar{\alpha}}\bar{\lambda}^{\bar{\beta}} A_{\bar{\alpha}\bar{\beta}}(X,\theta,\bar{\theta}).$$

Although this for is useful for some applications [18, 19], all physical states can be described by the gauge fixed version (2.10).

The first components of $(\Lambda_m, \bar\Lambda_m)$ are related to the diffeomorphism parameters and gauge transformation of the Kalb-Ramond field and $(\Lambda^\alpha, \bar\Lambda^{\bar\alpha})$ are the local supersymmetry parameters. We will see this explicitly later.

The condition that U is BRST closed implies

$$\begin{aligned} QU &= (\lambda^\alpha D_\alpha + \bar\lambda^{\bar\alpha} D_{\bar\alpha})\lambda^\beta \bar\lambda^{\bar\beta} A_{\beta\bar\beta}(X,\theta,\bar\theta) \\ &= \lambda^\beta \bar\lambda^{\bar\beta} \lambda^\alpha D_\alpha A_{\beta\bar\beta}(X,\theta,\bar\theta) + \lambda^\beta \bar\lambda^{\bar\beta} \bar\lambda^{\bar\alpha} D_{\bar\alpha} A_{\beta\bar\beta}(X,\theta,\bar\theta) = 0 \end{aligned} \tag{2.15}$$

which is solved by

$$D_{(\alpha} A_{\beta)\bar\gamma} = \gamma^m_{\alpha\beta} A_{m\bar\gamma}, \tag{2.16}$$

$$D_{(\bar\alpha} A_{\gamma\bar\beta)} = \gamma^m_{\bar\alpha\bar\beta} A_{\gamma m}. \tag{2.17}$$

Using the covariant derivatives algebra and gamma matrix identities these equations imply a chain of equations that define the supergravity fields as higher components of $A_{\beta\bar\beta}(X,\theta,\bar\theta)$ and put them on-shell. The first few are given by

$$D_\alpha A_{m\bar\gamma} - \partial_m A_{\alpha\bar\gamma} = (\gamma_m)_{\alpha\beta} W^\beta{}_{\bar\gamma}, \tag{2.18}$$

$$D_\alpha W^\beta{}_{\bar\gamma} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn\bar\gamma}, \tag{2.19}$$

$$D_{\bar\alpha} A_{\gamma m} - \partial_m A_{\gamma\bar\alpha} = (\gamma_m)_{\bar\alpha\bar\beta} W_\gamma{}^{\bar\beta}, \tag{2.20}$$

$$D_{\bar\alpha} W_\gamma{}^{\bar\beta} = \frac{1}{4}(\gamma^{mn})_{\bar\alpha}{}^{\bar\beta} F_{\gamma mn}, \tag{2.21}$$

where $F_{mn\bar\gamma} = \partial_{[m} A_{n]\bar\gamma}$ and $F_{\gamma mn} = \partial_{[m} A_{\gamma n]}$.

The integrated vertex operator can be seen as a deformation of the flat space action

$$S_{\rm def} = S_0 + \mu \int d^2 z\, V, \tag{2.22}$$

This deformation induces a change of order μ in the flat space BRST transformations (2.7) and (2.8). We will call the generator of these new transformations Q_1. Invariance of $S_{\rm def}$ under BRST transformations up to order μ implies that

$$(Q + Q_1)\left(S_0 + \mu \int d^2 z V\right) = Q_1 S_0 + \mu \int d^2 z Q V = 0. \tag{2.23}$$

The integrated vertex operator is not uniquely defined; we can make a non-linear field redefinition of the fundamental fields at order μ that will change V by terms proportional to the world-sheet equations of motion. Since $Q_1 S_0$ is proportional to the equations of motion, we can first solve (2.23) on-shell and require that

$$QV = \partial \bar W - \bar\partial W \tag{2.24}$$

up to terms proportional to the flat space equations of motion. The off-shell solution of (2.23) can then be constructed.

JHEP10(2017)178

Since (2.7) and (2.8) are on-shell nilpotent on all matter fields and combinations of the ghost variables that are invariant under (2.5), (2.24) implies that

$$QW = \partial U, \quad Q\bar{W} = \bar{\partial} U, \tag{2.25}$$

and finally we have that $QU = 0$, where U is the unintegrated vertex operator defined before.

The idea is to find V from U up to world-sheet equations of motion. We first start with (2.25) to find W and $\bar{W}$. Consider the equation for W first. Its form can be guessed knowing that we can use $\partial\bar{\lambda}^{\bar{\alpha}} = 0$ and that it should have classical dimension $(1,0)$. W turns out to be

$$W = \bar{\lambda}^{\bar{\beta}}\left(\partial\theta^{\alpha}A_{\alpha\bar{\beta}} + \Pi^m A_{m\bar{\beta}} + d_{\alpha}W^{\alpha}{}_{\bar{\beta}} + \frac{1}{2}N^{mn}F_{mn\bar{\beta}}\right). \tag{2.26}$$

After using the BRST transformations (2.7) and the equations

$$\bar{\lambda}^{\bar{\alpha}}\bar{\lambda}^{\bar{\beta}}D_{\bar{\alpha}}A_{m\bar{\beta}} = \bar{\lambda}^{\bar{\alpha}}\bar{\lambda}^{\bar{\beta}}D_{\bar{\alpha}}W^{\gamma}{}_{\bar{\beta}} = \bar{\lambda}^{\bar{\beta}}N^{mn}(\lambda^{a}D_{\alpha} + \bar{\lambda}^{\bar{\alpha}}D_{\bar{\alpha}})F_{mn\bar{\beta}} = 0 \tag{2.27}$$

which are consequences of (2.16), (2.18) and (2.19), we obtain that W satisfies $QW = \partial U$. We now prove that (2.26) transforms in the right way under the residual gauge symmetry from $\delta U = Q\Lambda$. The superfield $A_{\alpha\bar{\beta}}$ transforms as

$$\delta A_{\alpha\bar{\beta}} = D_{\alpha}\bar{\Lambda}_{\bar{\beta}} + D_{\bar{\beta}}\Lambda_{\alpha}, \tag{2.28}$$

where the gauge parameters Λ and $\bar{\Lambda}$ satisfy (2.12)–(2.14). The gauge transformations for the fields in (2.26) come from their definition in (2.16)–(2.19). They are

$$\delta A_{m\bar{\beta}} = \partial_m\bar{\Lambda}_{\bar{\beta}} - D_{\bar{\beta}}\Lambda_m, \quad \delta W^{\alpha}{}_{\bar{\beta}} = D_{\bar{\beta}}\Lambda^{\alpha}, \quad \delta F_{mn\bar{\beta}} = -D_{\bar{\beta}}\partial_{[m}\Lambda_{n]}. \tag{2.29}$$

Using this, W transforms as

$$\delta W = \partial(\lambda^{\alpha}\Lambda_{\alpha} + \bar{\lambda}^{\bar{\alpha}}\bar{\Lambda}_{\bar{\alpha}}) - Q\left(\partial\theta^{\alpha}\Lambda_{\alpha} + \Pi^m\Lambda_m + d_{\alpha}\Lambda^{\alpha} + \frac{1}{2}N^{mn}\partial_{[m}\Lambda_{n]}\right), \tag{2.30}$$

as required by $QW = \partial U$. Note that the second term closely resembles the the integrated vertex operator for SYM multiplet in the open superstring.

Similarly, $\bar{W}$ can be found using,

$$\lambda^{\alpha}\lambda^{\beta}D_{\alpha}A_{\beta m} = \lambda^{\alpha}\lambda^{\beta}D_{\alpha}W_{\beta}{}^{\bar{\gamma}} = \lambda^{\beta}\bar{N}^{mn}(\lambda^{\alpha}D_{\alpha} + \bar{\lambda}^{\bar{\alpha}})F_{\beta mn} = 0 \tag{2.31}$$

which are consequences of (2.16)–(2.21). We obtain that

$$\bar{W} = \lambda^{\alpha}\left(\bar{\partial}\bar{\theta}^{\bar{\beta}}A_{\alpha\bar{\beta}} + \bar{\Pi}^m A_{\alpha m} + \bar{d}_{\bar{\beta}}W_{\alpha}{}^{\bar{\beta}} + \frac{1}{2}\bar{N}^{mn}F_{\alpha mn}\right), \tag{2.32}$$

satisfies $Q\bar{W} = \bar{\partial}U$. As W above, $\bar{W}$ transforms adequately under the residual gauge transformation (2.28).

The last step is to find V such that $QV = \partial\bar{W} - \bar{\partial}W$. Knowing that V has to have classical dimension $(1,1)$, vanishing ghost number and we can ignore term proportional to the world-sheet equations of motion, we can guess the following form

$$\begin{aligned} V = {} & \partial\theta^\alpha\bar{\partial}\bar{\theta}^{\bar{\beta}}A_{\alpha\bar{\beta}} + \partial\theta^\alpha\bar{\Pi}^m A_{\alpha m} - \Pi^m\bar{\partial}\bar{\theta}^{\bar{\alpha}}A_{m\bar{\alpha}} + d_\alpha\bar{\partial}\bar{\theta}^{\bar{\beta}}W^\alpha{}_{\bar{\beta}} - \bar{d}_{\bar{\alpha}}\partial\theta^\beta W_\beta{}^{\bar{\alpha}} \\ & + \frac{1}{2}\partial\theta^\alpha\bar{N}^{mn}F_{\alpha mn} - \frac{1}{2}\bar{\partial}\bar{\theta}^{\bar{\alpha}}N^{mn}F_{mn\bar{\alpha}} + \Pi^m\bar{\Pi}^n A_{mn} + d_\alpha\bar{d}_{\bar{\beta}}P^{\alpha\bar{\beta}} \\ & + \frac{1}{4}N^{mn}\bar{N}^{pq}S_{mnpq} + \Pi^m\bar{d}_{\bar{\alpha}}E_m{}^{\bar{\alpha}} + d_\alpha\bar{\Pi}^m E_m{}^\alpha + \frac{1}{2}\Pi^m\bar{N}^{np}\Omega_{mnp} \\ & + \frac{1}{2}N^{np}\bar{\Pi}^m\bar{\Omega}_{mnp} + \frac{1}{2}d_\alpha\bar{N}^{mn}C_{mn}{}^\alpha + \frac{1}{2}N^{mn}\bar{d}_{\bar{\alpha}}C_{mn}{}^{\bar{\alpha}}. \end{aligned} \tag{2.33}$$

Using the BRST transformations and $QV = \partial\bar{W} - \bar{\partial}W$ is possible to show that A_{mn} is defined by

$$D_{(\alpha}A_{\beta)m} = \gamma^n_{\alpha\beta}A_{nm}, \quad D_{(\bar{\alpha}}A_{m\bar{\beta})} = -\gamma^n_{\bar{\alpha}\bar{\beta}}A_{mn}, \tag{2.34}$$

$E_m{}^{\bar{\gamma}}$ and $E_m{}^\gamma$ are defined by

$$D_{(\alpha}W_{\beta)}{}^{\bar{\gamma}} = -\gamma^m_{\alpha\beta}E_m{}^{\bar{\gamma}}, \quad D_{(\bar{\alpha}}W^\gamma{}_{\bar{\beta})} = \gamma^m_{\bar{\alpha}\bar{\beta}}E_m{}^\gamma, \tag{2.35}$$

Ω and $\bar{\Omega}$ are defined by

$$D_{(\alpha}F_{\beta)mn} = \gamma^p_{\alpha\beta}\Omega_{pmn}, \quad D_{(\bar{\alpha}}F_{mn\bar{\beta})} = -\gamma^p_{\bar{\alpha}\bar{\beta}}\bar{\Omega}_{pmn}, \tag{2.36}$$

P is defined by

$$D_\alpha E_m{}^{\bar{\beta}} + \partial_m W_\alpha{}^{\bar{\beta}} = -(\gamma_m)_{\alpha\gamma}P^{\gamma\bar{\beta}}, \quad D_{\bar{\alpha}}E_m{}^\beta - \partial_m W^\beta{}_{\bar{\alpha}} = (\gamma_m)_{\bar{\alpha}\bar{\gamma}}P^{\beta\bar{\gamma}}, \tag{2.37}$$

$C_{mn}{}^\beta$ and $C_{mn}{}^{\bar{\beta}}$ are defined by

$$D_\alpha\Omega_{pmn} - \partial_p F_{\alpha mn} = (\gamma_p)_{\alpha\beta}C_{mn}{}^\beta, \quad D_{\bar{\alpha}}\bar{\Omega}_{pmn} + \partial_p F_{mn\bar{\alpha}} = (\gamma_p)_{\bar{\alpha}\bar{\beta}}C_{mn}{}^{\bar{\beta}}, \tag{2.38}$$

and finally S is defined by

$$D_\alpha C_{mn}{}^\beta = \frac{1}{4}(\gamma^{pq})_\alpha{}^\beta S_{pqmn}, \quad D_{\bar{\alpha}}C_{mn}{}^{\bar{\beta}} = \frac{1}{4}(\gamma^{pq})_{\bar{\alpha}}{}^{\bar{\beta}}S_{mnpq}. \tag{2.39}$$

It can be shown that these definitions are equivalent among themselves after writing all the superfields in terms of $A_{\alpha\bar{\beta}}$ and superspace derivatives. Note that the gauge transformation of V is given by

$$\begin{aligned} \delta V = {} & \bar{\partial}\left(\partial\theta^\alpha\Lambda_\alpha + \Pi^m\Lambda_m + d_\alpha\Lambda^\alpha + \frac{1}{2}N^{mn}\partial_{[m}\Lambda_{n]}\right) \\ & - \partial\left(\bar{\partial}\bar{\theta}^{\bar{\alpha}}\bar{\Lambda}_{\bar{\alpha}} + \bar{\Pi}^m\bar{\Lambda}_m + \bar{d}_{\bar{\alpha}}\bar{\Lambda}^{\bar{\alpha}} + \frac{1}{2}\bar{N}^{mn}\partial_{[m}\bar{\Lambda}_{n]}\right), \end{aligned} \tag{2.40}$$

and vanishes after integration. There are no BRST exact terms above because the only way to have well defined covariant operators with ghost number -1 invariant under (2.5) is using the non-minimal pure spinor formalism [20]. In particular, we can see from (2.34) and (2.35) that the gauge transformations of A_{mn}, $E_m{}^{\alpha}$ and $E_m{}^{\bar\alpha}$ are given by

$$\delta A_{mn} = \partial_n \Lambda_m - \partial_m \bar\Lambda_n, \quad \delta E_m{}^{\alpha} = \partial_m \Lambda^{\alpha}, \quad \delta E_m{}^{\bar\alpha} = -\partial_m \bar\Lambda^{\bar\alpha}, \tag{2.41}$$

which are the expected gauge transformations for the graviton plus the B-field and local supersymmetries for the gravitini. Note that the γ-trace of $E_m{}^{\alpha}$ and $E_m{}^{\bar\alpha}$ are invariant by a consequence of (2.13) and (2.14) and are identified as the dilatini.

3 Unintegrated vertex operator in type IIB superstring in $AdS_5 \times S^5$

We will now focus on the Type IIB superstring in an $AdS_5 \times S^5$ background using the pure spinor description. It is well known that this background is described by the coset $\mathrm{PSU}(2,2|4)/\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)$. We will denote the generators of the $\mathfrak{psu}(2,2|4)$ algebra by $T_A = (M_{ab}, M_{a'b'}, P_{\underline{a}}, Q_\alpha, Q_{\bar\alpha})$. Their non-zero (anti-)commutators are

$$\begin{aligned}
[M_{\underline{ab}}, M_{\underline{cd}}] &= -\,\eta_{\underline{a}[\underline{c}} M_{\underline{d}]\underline{b}} + \eta_{\underline{b}[\underline{c}} M_{\underline{d}]\underline{a}}, & [P_{\underline{a}}, M_{\underline{bc}}] &= -\eta_{\underline{a}[\underline{b}} P_{\underline{c}]}, \\
[P_a, P_b] &= M_{ab}, & [P_{a'}, P_{b'}] &= -M_{a'b'}, \\
[M_{\underline{ab}}, Q_\alpha] &= \frac{1}{2}(\gamma_{\underline{ab}})_\alpha{}^\beta Q_\beta, & [M_{\underline{ab}}, Q_{\bar\alpha}] &= \frac{1}{2}(\gamma_{\underline{ab}})_{\bar\alpha}{}^{\bar\beta} Q_{\bar\beta}, \\
[P_{\underline{a}}, Q_\alpha] &= \frac{1}{2}(\gamma_{\underline{a}}\eta)_\alpha{}^{\bar\beta} Q_{\bar\beta}, & [P_{\underline{a}}, Q_{\bar\alpha}] &= -\frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar\alpha}{}^{\beta} Q_\beta, \quad \{Q_\alpha, Q_\beta\} = \gamma^{\underline{a}}_{\alpha\beta} P_{\underline{a}}, \\
\{Q_{\bar\alpha}, Q_{\bar\beta}\} &= \gamma^{\underline{a}}_{\bar\alpha\bar\beta} P_{\underline{a}}, & \{Q_\alpha, Q_{\bar\beta}\} &= -\frac{1}{2}\left((\gamma^{ab}\eta)_{\alpha\bar\beta} M_{ab} - (\gamma^{a'b'}\eta)_{\alpha\bar\beta} M_{a'b'}\right).
\end{aligned} \tag{3.1}$$

The non-vanishing structure constants $f_{BC}{}^A$ can read off from these (anti-)commutators. The geometric quantities are defined using the Maurer-Cartan currents. They are defined in terms of the coset element g as

$$(g^{-1}\partial_M g)^A T_A = J_M{}^A T_A = E_M{}^{\underline{a}} P_{\underline{a}} + E_M{}^{\alpha} Q_\alpha + E_M{}^{\bar\alpha} Q_{\bar\alpha} + \frac{1}{2}\Omega_M{}^{ab} M_{ab} + \frac{1}{2}\Omega_M{}^{a'b'} M_{a'b'}, \tag{3.2}$$

where ∂_M are derivatives with respect to local coordinates Z^M.

Additionally, we need the background values of the RR field-strength and the NSNS two-form, which are not defined by the geometry,

$$P^{\alpha\bar\beta} = -\frac{1}{2}\eta^{\alpha\bar\beta}, \quad B_{\alpha\bar\beta} = \eta_{\alpha\bar\beta}, \tag{3.3}$$

where $\eta_{\alpha\bar\beta} = \gamma^{01234}_{\alpha\bar\beta}$, $\eta^{\alpha\bar\beta} = \gamma_{01234}^{\alpha\bar\beta}$. We can also calculate the non-zero values of the torsion and the curvature

$$\begin{aligned}
&T_{\alpha\beta}{}^{\underline{a}} = -\gamma^{\underline{a}}_{\alpha\beta}, && T_{\bar\alpha\bar\beta}{}^{\underline{a}} = -\gamma^{\underline{a}}_{\bar\alpha\bar\beta}, && T_{\underline{a}\alpha}{}^{\bar\beta} = -\frac{1}{2}(\gamma_{\underline{a}}\eta)_\alpha{}^{\bar\beta}, && T_{\underline{a}\bar\alpha}{}^{\beta} = \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar\alpha}{}^{\beta} \\
&R_{\alpha\bar\beta\, ab} = (\gamma_{ab}\eta)_{\alpha\bar\beta}, && R_{\alpha\bar\beta\, a'b'} = -(\gamma_{a'b'}\eta)_{\alpha\bar\beta}, && R_{ab\, cd} = -\eta_{a[c}\eta_{d]b}, && R_{a'b'\, c'd'} = \eta_{a'[c'}\eta_{d']b'}.
\end{aligned} \tag{3.4}$$

Here $\underline{a} = (a, a')$ with $a(a')$ refers to the tangent vectorial index of $AdS_5(S^5)$. With all these ingredients, the world-sheet action in the pure spinor formalism for this background is

$$S = \int d^2z \Big[\frac{1}{2}J^{\underline{a}}\bar{J}^{\underline{b}}\eta_{\underline{ab}} + \frac{1}{2}(J^{\alpha}\bar{J}^{\bar{\beta}} - 3J^{\bar{\beta}}\bar{J}^{\alpha})\eta_{\alpha\bar{\beta}} \\ + \omega_{\alpha}\bar{\nabla}\lambda^{\alpha} + \bar{\omega}_{\bar{\alpha}}\nabla\bar{\lambda}^{\bar{\alpha}} - \frac{1}{2}(N^{ab}\bar{N}_{ab} - N^{a'b'}\bar{N}_{a'b'})\Big], \tag{3.5}$$

where we are using

$$J^A = \partial Z^M J_M{}^A, \quad \bar{J}^A = \bar{\partial} Z^M J_M{}^A, \quad N^{\underline{ab}} = \frac{1}{2}(\lambda\gamma^{\underline{ab}}\omega), \quad \bar{N}^{\underline{ab}} = \frac{1}{2}(\bar{\lambda}\gamma^{\underline{ab}}\bar{\omega}). \tag{3.6}$$

It is important to note that the action does not depend on all components of ghost currents, however, fluctuations of these background can and will depend on them. We have integrated out the d_α and $\bar{d}_{\bar{\alpha}}$ variables of the pure spinor formalism. After this, the BRST charge (2.2) becomes

$$Q = \oint \left(-2\lambda^{\alpha}\eta_{\alpha\bar{\beta}}J^{\bar{\beta}} + 2\bar{\lambda}^{\bar{\alpha}}\eta_{\beta\bar{\alpha}}\bar{J}^{\beta}\right). \tag{3.7}$$

The BRST transformation of the coset element is $Qg = g(\lambda + \bar{\lambda})$, where we used the shorthand $\lambda = \lambda^{\alpha}Q_{\alpha}$ and $\bar{\lambda} = \bar{\lambda}^{\bar{\alpha}}Q_{\bar{\alpha}}$. The Maurer-Cartan currents $J = g^{-1}\partial g$ transform as

$$QJ^{ab} = -(\lambda\gamma^{ab}\eta)_{\bar{\alpha}}J^{\bar{\alpha}} + (\bar{\lambda}\gamma^{ab}\eta)_{\alpha}J^{\alpha}, \qquad QJ^{a'b'} = (\lambda\gamma^{a'b'}\eta)_{\bar{\alpha}}J^{\bar{\alpha}} - (\bar{\lambda}\gamma^{a'b'}\eta)_{\alpha}J^{\alpha}, \tag{3.8}$$

$$QJ^{\underline{a}} = (\lambda\gamma^{\underline{a}})_{\alpha}J^{\alpha} + (\bar{\lambda}\gamma^{\underline{a}})_{\bar{\alpha}}J^{\bar{\alpha}},$$

$$QJ^{\alpha} = \nabla\lambda^{\alpha} - \frac{1}{2}(\bar{\lambda}\gamma_{\underline{a}}\eta)^{\alpha}J^{\underline{a}}, \qquad QJ^{\bar{\alpha}} = \nabla\bar{\lambda}^{\bar{\alpha}} + \frac{1}{2}(\lambda\gamma_{\underline{a}}\eta)^{\bar{\alpha}}J^{\underline{a}}. \tag{3.9}$$

and similarly for $\bar{J} = g^{-1}\bar{\partial}g$. The pure spinor ghosts $\lambda, \bar{\lambda}$ are BRST invariant and their conjugate momenta transform as

$$Q\omega_{\alpha} = -2\eta_{\alpha\bar{\alpha}}J^{\bar{\alpha}}, \quad Q\bar{\omega}_{\bar{\alpha}} = 2\eta_{\alpha\bar{\alpha}}\bar{J}^{\alpha}. \tag{3.10}$$

The structure of the unintegrated vertex operator will be the same as in the flat space case

$$U = \lambda^{\alpha}\bar{\lambda}^{\bar{\alpha}}A_{\alpha\bar{\alpha}}(g), \tag{3.11}$$

where the superfield $A_{\alpha\bar{\alpha}}(g)$ is a function of the coset element g. The equations coming from $QU = 0$ are [2]

$$\nabla_{(\alpha}A_{\beta)\bar{\gamma}} = \gamma^{\underline{a}}_{\alpha\beta}A_{\underline{a}\bar{\gamma}}, \quad \nabla_{(\bar{\alpha}}A_{\gamma\bar{\beta})} = \gamma^{\underline{a}}_{\bar{\alpha}\bar{\beta}}A_{\gamma\underline{a}}, \tag{3.12}$$

where $\nabla_A = E_A{}^M(\partial_M + \frac{1}{2}\Omega_M{}^{ab}M_{ab} + \frac{1}{2}\Omega_M{}^{a'b'}M_{a'b'})$ for $A = \{\underline{a}, \alpha, \bar{\alpha}\}$ are the covariant derivatives. The residual gauge symmetry from $\delta U = Q\Lambda$ implies that

$$\delta A_{\alpha\bar{\beta}} = \nabla_{\alpha}\bar{\Lambda}_{\bar{\beta}} + \nabla_{\bar{\beta}}\Lambda_{\alpha}, \tag{3.13}$$

where the gauge parameters Λ and $\bar{\Lambda}$ satisfy

$$\nabla_{(\alpha}\Lambda_{\beta)} = \gamma^{\underline{a}}_{\alpha\beta}\Lambda_{\underline{a}}, \quad \nabla_{(\bar{\alpha}}\bar{\Lambda}_{\bar{\beta})} = \gamma^{\underline{a}}_{\bar{\alpha}\bar{\beta}}\bar{\Lambda}_{\underline{a}}, \tag{3.14}$$

which are the generalization of (2.12) and (2.28) for the $AdS_5 \times S^5$ case. As in the flat space case, there are consequences of these constraint equations. They are

$$\nabla_{\alpha}\Lambda_{\underline{a}} - \nabla_{\underline{a}}\Lambda_{\alpha} = (\gamma_{\underline{a}})_{\alpha\beta}\Lambda^{\beta}, \quad \nabla_{\alpha}\Lambda^{\beta} - \frac{1}{2}\eta^{\beta\bar{\gamma}}\nabla_{\bar{\gamma}}\Lambda_{\alpha} = \frac{1}{4}(\gamma^{\underline{ab}})_{\alpha}{}^{\beta}\nabla_{[\underline{a}}\Lambda_{\underline{b}]}, \tag{3.15}$$

$$\nabla_{\bar{\alpha}}\bar{\Lambda}_{\underline{a}} - \nabla_{\underline{a}}\bar{\Lambda}_{\bar{\beta}} = (\gamma_{\underline{a}})_{\bar{\alpha}\bar{\beta}}\bar{\Lambda}^{\bar{\beta}}, \quad \nabla_{\bar{\alpha}}\bar{\Lambda}^{\bar{\beta}} + \frac{1}{2}\eta^{\gamma\bar{\beta}}\nabla_{\gamma}\bar{\Lambda}_{\bar{\alpha}} = \frac{1}{4}(\gamma^{\underline{ab}})_{\bar{\alpha}}{}^{\bar{\beta}}\nabla_{[\underline{a}}\bar{\Lambda}_{\underline{b}]}. \tag{3.16}$$

These equations are similar to the equation satisfied by the super-Maxell fields in ten-dimensions, that is (2.16) and (2.17). Our goal is to find the remaining fields with equations similar to (2.18), (2.19) and (2.20), (2.21). Consider the first equation in (3.12). We note that the combination $(\nabla_{\alpha}A_{\underline{a}\bar{\gamma}} - \nabla_{\underline{a}}A_{\alpha\bar{\gamma}})$ satisfies

$$\gamma^{\underline{a}}_{(\alpha\beta}(\nabla_{\gamma)}A_{\underline{a}\bar{\gamma}} - \nabla_{\underline{a}}A_{\gamma)\bar{\gamma}}) = 0. \tag{3.17}$$

The proof is the same as in the flat space case [21], using the general (anti-)commutator of covariant derivatives acting on a superfield A_{CD}

$$[\nabla_A, \nabla_B]A_{CD} = -T_{AB}{}^{E}\nabla_E A_{CD} - R_{AB\,C}{}^{E}A_{ED} - R_{AB\,D}{}^{E}A_{CE}, \tag{3.18}$$

the relevant commutators are the same as in flat space. Therefore we have that

$$\nabla_{\alpha}A_{\underline{a}\bar{\gamma}} - \nabla_{\underline{a}}A_{\alpha\bar{\gamma}} = (\gamma_{\underline{a}})_{\alpha\beta}W^{\beta}{}_{\bar{\gamma}}, \tag{3.19}$$

because the identity $(\gamma^{\underline{a}})_{(\alpha\beta}(\gamma_{\underline{a}})_{\gamma)\delta} = 0$.

The next equation in the chain is for $\nabla_{\alpha}W^{\beta}{}_{\bar{\gamma}}$. From (3.19), we obtain after some covariant derivative algebra and the background values for the curvature

$$\begin{aligned}10\nabla_{\alpha}W^{\beta}{}_{\bar{\gamma}} = (\gamma^{\underline{ab}})_{\alpha}{}^{\beta}F_{\underline{ab}\bar{\gamma}} - (\gamma^{\underline{a}})_{\alpha\rho}(\gamma_{\underline{a}})^{\beta\gamma}\nabla_{\gamma}W^{\rho}{}_{\bar{\gamma}} \\ + (\gamma^{\underline{a}})^{\beta\gamma}\left([\nabla_{\underline{a}}, \nabla_{\gamma}]A_{\alpha\bar{\gamma}} + [\nabla_{\underline{a}}, \nabla_{\alpha}]A_{\gamma\bar{\gamma}}\right),\end{aligned} \tag{3.20}$$

where

$$F_{\underline{ba}\bar{\gamma}} = \nabla_{\underline{b}}A_{\underline{a}\bar{\gamma}} - \nabla_{\underline{a}}A_{\underline{b}\bar{\gamma}}. \tag{3.21}$$

This is again very similar to the flat ten-dimensional superspace [21], the difference is in the terms involving commutators. Using (3.18),

$$[\nabla_{\underline{a}}, \nabla_{\gamma}]A_{\alpha\bar{\gamma}} = -T_{\underline{a}\gamma}{}^{A}\nabla_A A_{\alpha\bar{\gamma}} = \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\gamma}{}^{\bar{\delta}}\nabla_{\bar{\delta}}A_{\alpha\bar{\gamma}}, \tag{3.22}$$

$$[\nabla_{\underline{a}}, \nabla_{\alpha}]A_{\gamma\bar{\gamma}} = -T_{\underline{a}\alpha}{}^{A}\nabla_A A_{\gamma\bar{\gamma}} = \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\alpha}{}^{\bar{\delta}}\nabla_{\bar{\delta}}A_{\gamma\bar{\gamma}}, \tag{3.23}$$

we obtain,

$$\left(\nabla_{\alpha}W^{\beta}{}_{\bar{\gamma}} - \frac{1}{2}\eta^{\beta\bar{\delta}}\nabla_{\bar{\delta}}A_{\alpha\bar{\gamma}}\right) + \frac{1}{10}\gamma^{\underline{a}}_{\alpha\rho}\gamma^{\beta\gamma}_{\underline{a}}\left(\nabla_{\gamma}W^{\rho}{}_{\bar{\gamma}} - \frac{1}{2}\eta^{\rho\bar{\delta}}\nabla_{\bar{\delta}}A_{\gamma\bar{\gamma}}\right) = \frac{1}{10}(\gamma^{\underline{ab}})_{\alpha}{}^{\beta}F_{\underline{ab}\bar{\gamma}}. \tag{3.24}$$

This is solved by

$$\nabla_\alpha W^\beta{}_{\bar\gamma} - \frac{1}{2}\eta^{\beta\bar\delta}\nabla_{\bar\delta}A_{\alpha\bar\gamma} = \frac{1}{4}(\gamma^{\underline{ab}})_\alpha{}^\beta F_{\underline{ab}\bar\gamma}. \tag{3.25}$$

Similarly from the second equation in (3.12), we obtain

$$\nabla_{\bar\alpha}A_{\gamma\underline{a}} - \nabla_{\underline{a}}A_{\gamma\bar\alpha} = (\gamma_{\underline{a}})_{\bar\alpha\bar\beta}W_\gamma{}^{\bar\beta}, \quad \nabla_{\bar\alpha}W_\gamma{}^{\bar\beta} + \frac{1}{2}\eta^{\delta\bar\beta}\nabla_\delta A_{\gamma\bar\alpha} = \frac{1}{4}(\gamma^{\underline{ab}})_{\bar\alpha}{}^{\bar\beta}F_{\gamma\underline{ab}}, \tag{3.26}$$

where $F_{\gamma\underline{ab}} = \nabla_{\underline{a}}A_{\gamma\underline{b}} - \nabla_{\underline{b}}A_{\gamma\underline{a}}$.

Let us summarize our results up to now. The unintegrated vertex operator is $U = \lambda^\alpha\bar\lambda^{\bar\beta}A_{\alpha\bar\beta}$ and its BRST invariance imply a chain of superfields defined by

$$\nabla_{(\alpha}A_{\beta)\bar\gamma} = \gamma^{\underline{a}}_{\alpha\beta}A_{\underline{a}\bar\gamma}, \tag{3.27}$$

$$\nabla_\alpha A_{\underline{a}\bar\gamma} - \nabla_{\underline{a}}A_{\alpha\bar\gamma} = (\gamma_{\underline{a}})_{\alpha\beta}W^\beta{}_{\bar\gamma}, \tag{3.28}$$

$$\nabla_\alpha W^\beta{}_{\bar\gamma} - \frac{1}{2}\eta^{\beta\bar\delta}\nabla_{\bar\delta}A_{\alpha\bar\gamma} = \frac{1}{4}(\gamma^{\underline{ab}})_\alpha{}^\beta F_{\underline{ab}\bar\gamma}, \tag{3.29}$$

$$\nabla_{(\bar\alpha}A_{\gamma\bar\beta)} = \gamma^{\underline{a}}_{\bar\alpha\bar\beta}A_{\gamma\underline{a}}, \tag{3.30}$$

$$\nabla_{\bar\alpha}A_{\gamma\underline{a}} - \nabla_{\underline{a}}A_{\gamma\bar\alpha} = (\gamma_{\underline{a}})_{\bar\alpha\bar\beta}W_\gamma{}^{\bar\beta}, \tag{3.31}$$

$$\nabla_{\bar\alpha}W_\gamma{}^{\bar\beta} + \frac{1}{2}\eta^{\delta\bar\beta}\nabla_\delta A_{\gamma\bar\alpha} = \frac{1}{4}(\gamma^{\underline{ab}})_{\bar\alpha}{}^{\bar\beta}F_{\gamma\underline{ab}}, \tag{3.32}$$

These equations are the analogous to the equations (2.16)–(2.21) of flat superspace.

Before ending this section, we obtain the transformations of the superfields in (3.27)–(3.32) under the gauge transformations of (3.13). Using (3.15), (3.16) and commutation relations in our background we obtain

$$\begin{aligned}
\delta A_{\underline{a}\bar\beta} &= \nabla_{\underline{a}}\bar\Lambda_{\bar\beta} - \nabla_{\bar\beta}\Lambda_{\underline{a}} + \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar\beta}{}^\gamma\Lambda_\gamma,\\
\delta W^\gamma{}_{\bar\beta} &= \frac{1}{2}\eta^{\gamma\bar\delta}\nabla_{\bar\beta}\left(\bar\Lambda_{\bar\delta} + 2\eta_{\alpha\bar\delta}\Lambda^\alpha\right) + \frac{1}{2}(\eta\gamma^{\underline{a}})^\gamma{}_{\bar\beta}(\Lambda_{\underline{a}} - \bar\Lambda_{\underline{a}}),\\
\delta F_{\underline{ab}\bar\beta} &= -\nabla_{\bar\beta}\nabla_{[\underline{a}}\Lambda_{\underline{b}]} + \frac{1}{2}(\gamma_{[\underline{a}}\eta\gamma_{\underline{b}]})_{\bar\beta\alpha}\Lambda^\alpha - \frac{1}{4}R_{\underline{ab}\,\underline{cd}}(\gamma^{\underline{cd}})_{\bar\beta}{}^{\bar\alpha}\bar\Lambda_{\bar\alpha},
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\delta A_{\alpha\underline{a}} &= \nabla_{\underline{a}}\Lambda_\alpha - \nabla_\alpha\bar\Lambda_{\underline{a}} - \frac{1}{2}(\gamma_{\underline{a}}\eta)_\alpha{}^{\bar\beta}\bar\Lambda_{\bar\beta},\\
\delta W_\alpha{}^{\bar\gamma} &= -\frac{1}{2}\eta^{\delta\bar\gamma}\nabla_\alpha\left(\Lambda_\delta - 2\eta_{\delta\bar\alpha}\bar\Lambda^{\bar\alpha}\right) + \frac{1}{2}(\gamma^{\underline{a}}\eta)_\alpha{}^{\bar\gamma}(\Lambda_{\underline{a}} - \bar\Lambda_{\underline{a}}),\\
\delta F_{\alpha\underline{ab}} &= -\nabla_\alpha\nabla_{[\underline{a}}\bar\Lambda_{\underline{b}]} - \frac{1}{2}(\gamma_{[\underline{a}}\eta\gamma_{\underline{b}]})_{\alpha\bar\beta}\bar\Lambda^{\bar\beta} - \frac{1}{4}R_{\underline{ab}\,\underline{cd}}(\gamma^{\underline{cd}})_\alpha{}^\beta\Lambda_\beta.
\end{aligned} \tag{3.34}$$

The gauge transformations for $F_{\underline{ab}\bar\beta}$ and $F_{\alpha\underline{ab}}$ require a separate note. The combination $(\gamma_{[\underline{a}}\eta\gamma_{\underline{b}]})$ vanishes when one indices is in AdS_5 and the other in S^5. Similarly, $R_{\underline{ab}\,\underline{cd}}$ is zero unless all its four indices are in AdS_5 or all four are in S^5. In these cases, the transformations simplify to

$$\delta F_{ab'\bar\beta} = -\nabla_{\bar\beta}\nabla_{[a}\Lambda_{b']}, \quad \delta F_{\alpha ab'} = -\nabla_\alpha\nabla_{[a}\bar\Lambda_{b']}. \tag{3.35}$$

On the other hand, when both indices are in, for example, AdS_5, the transformation for $F_{ab\bar{\beta}}$ is

$$\delta F_{ab\bar{\beta}} = -\nabla_{\bar{\beta}}\nabla_{[a}\Lambda_{b]} + \frac{1}{2}(\gamma_{ab})_{\bar{\beta}}{}^{\bar{\alpha}}(\bar{\Lambda}_{\bar{\alpha}} + 2\eta_{\alpha\bar{\alpha}}\Lambda^{\alpha}). \tag{3.36}$$

Combinations of the types $(\Lambda_{\delta} - 2\eta_{\delta\bar{\alpha}}\bar{\Lambda}^{\bar{\alpha}})$ and $(\bar{\Lambda}_{\bar{\alpha}} + 2\eta_{\alpha\bar{\alpha}}\Lambda^{\alpha})$ will appear again when we discuss the local supersymmetry transformation for the gravitino, they are the local supersymmetry parameters for the $AdS_5 \times S^5$ superspace.

We will use the transformations the to verify that the analogous of (2.26) and (2.32) in $AdS_5 \times S^5$ transform adequately.

4 Integrated vertex operator in type IIB superstring in $AdS_5 \times S^5$

After the initial discussion on the superstring in $AdS_5 \times S^5$ background, the unintegrated vertex operator and some of its consequences, we want to explicitly solve the equations (2.24), (2.25) that define the integrated vertex operator (2.22). Unlike the flat space case, the b ghosts of the $AdS_5 \times S^5$ superstring can be constructed without the introduction of non-minimal fields [22, 23]. Therefore it is guaranteed that ∂U is cohomologically trivial.

As before we will work on-shell. The first equation of motion we will need is the one for $\bar{\lambda}$ obtained from (3.5)

$$\nabla\bar{\lambda}^{\bar{\alpha}} = \frac{1}{4}(\bar{\lambda}\gamma^{[ab]})^{\bar{\alpha}}N_{[ab]}, \tag{4.1}$$

where $A^{[ab]}B_{[ab]} \equiv A^{ab}B_{ab} - A^{a'b'}B_{a'b'}$.

From the same argument as before, working on-shell allows us to chose W to have the following form

$$W = \bar{\lambda}^{\bar{\beta}}\left(J^{\alpha}A_{\alpha\bar{\beta}} + J^{\underline{a}}A_{\underline{a}\bar{\beta}} - 2\eta_{\alpha\bar{\alpha}}J^{\bar{\alpha}}W^{\alpha}{}_{\bar{\beta}} + \frac{1}{2}N^{\underline{ab}}F_{\underline{ab}\bar{\beta}}\right). \tag{4.2}$$

Acting with Q and using the equations (3.12), (3.19) and (3.25) we obtain,

$$\begin{aligned} QW =& \partial U - \lambda^{\alpha}\nabla\bar{\lambda}^{\bar{\beta}}A_{\alpha\bar{\beta}} + \bar{\lambda}^{\bar{\beta}}(\bar{\lambda}\gamma^{\underline{a}})_{\bar{\gamma}}J^{\bar{\gamma}}A_{\underline{a}\bar{\beta}} + \bar{\lambda}^{\bar{\beta}}\bar{\lambda}^{\bar{\gamma}}J^{\underline{a}}\nabla_{\bar{\gamma}}A_{\underline{a}\bar{\beta}} - \frac{1}{2}\bar{\lambda}^{\bar{\beta}}(\bar{\lambda}\gamma_{\underline{a}}\eta)^{\alpha}J^{\underline{a}}A_{\alpha\bar{\beta}} \\ &- 2\eta_{\alpha\bar{\alpha}}\bar{\lambda}^{\bar{\beta}}\nabla\bar{\lambda}^{\bar{\alpha}}W^{\alpha}{}_{\bar{\beta}} + 2\eta_{\alpha\bar{\alpha}}\bar{\lambda}^{\bar{\beta}}J^{\bar{\alpha}}\bar{\lambda}^{\bar{\gamma}}\nabla_{\bar{\gamma}}W^{\alpha}{}_{\bar{\beta}} \\ &+ \frac{1}{2}\bar{\lambda}^{\bar{\beta}}\lambda^{\gamma}N^{\underline{ab}}\nabla_{\gamma}F_{\underline{ab}\bar{\beta}} + \frac{1}{2}\bar{\lambda}^{\bar{\beta}}\bar{\lambda}^{\bar{\gamma}}N^{\underline{ab}}\nabla_{\bar{\gamma}}F_{\underline{ab}\bar{\beta}}. \end{aligned} \tag{4.3}$$

Let us consider first the fourth term above. Using (3.13), (3.18), (3.4) and

$$\gamma_{ab}\gamma_{\underline{cdefg}}\gamma^{ab} - \gamma_{a'b'}\gamma_{\underline{cdefg}}\gamma^{a'b'} = 0, \quad \gamma^{[bc]}\gamma_a\gamma_{[bc]} = 16\gamma_a, \quad \gamma^{[bc]}\gamma_{a'}\gamma_{[bc]} = -16\gamma_{a'}, \tag{4.4}$$

we obtain

$$\bar{\lambda}^{\bar{\beta}}\bar{\lambda}^{\bar{\gamma}}\nabla_{\bar{\beta}}A_{\underline{a}\bar{\gamma}} = \frac{1}{2}\bar{\lambda}^{\bar{\beta}}(\bar{\lambda}\gamma_{\underline{a}}\eta)^{\alpha}A_{\alpha\bar{\beta}}. \tag{4.5}$$

which cancels with the last two terms in the first line of (4.3). Consider now the second term in the second line of (4.3), it contains

$$\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}\nabla_{\bar\beta}W^\alpha{}_{\bar\gamma} = \frac{1}{10}\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}(\gamma^{\underline{a}})^{\alpha\delta}\nabla_{\bar\beta}\left(\nabla_\delta A_{\underline{a}\bar\gamma} - \nabla_{\underline{a}}A_{\delta\bar\gamma}\right) \tag{4.6}$$
$$= \frac{1}{10}\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}(\gamma^{\underline{a}})^{\alpha\delta}\left(\{\nabla_\delta,\nabla_{\bar\beta}\}A_{\underline{a}\bar\gamma} - \nabla_\delta\nabla_{\bar\beta}A_{\underline{a}\bar\gamma} + [\nabla_{\underline{a}},\nabla_{\bar\beta}]A_{\delta\bar\gamma} - \nabla_{\underline{a}}\nabla_{\bar\beta}A_{\delta\bar\gamma}\right).$$

Using (3.18) and (4.5),

$$\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}\nabla_{\bar\beta}W^\alpha{}_{\bar\gamma} = \frac{1}{10}\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}(\gamma^{\underline{a}})^{\alpha\delta}\left(-R_{\delta\bar\beta\underline{a}}{}^{\underline{b}}A_{\underline{b}\bar\gamma} - \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar\beta}{}^{\rho}\nabla_\delta A_{\rho\bar\gamma} - T_{\underline{a}\bar\beta}{}^{\rho}\nabla_\rho A_{\delta\bar\gamma}\right). \tag{4.7}$$

Finally, after using (3.4) we obtain

$$\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}\nabla_{\bar\beta}W^\alpha{}_{\bar\gamma} = -\frac{1}{2}\bar\lambda^{\bar\beta}(\bar\lambda\gamma^{\underline{a}}\eta)^\alpha A_{\underline{a}\bar\gamma}, \tag{4.8}$$

which helps to cancel the second term in the second line with the third term in the first line of (4.3).

Up to now we have

$$QW = \partial U - \lambda^\alpha\nabla\bar\lambda^{\bar\beta}A_{\alpha\bar\beta} - 2\eta_{\alpha\bar\alpha}\bar\lambda^{\bar\beta}\nabla\bar\lambda^{\bar\alpha}W^\alpha{}_{\bar\beta} + \frac{1}{2}\bar\lambda^{\bar\beta}\lambda^\gamma N^{\underline{ab}}\nabla_\gamma F_{\underline{ab}\bar\beta} + \frac{1}{2}\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}N^{\underline{ab}}\nabla_{\bar\gamma}F_{\underline{ab}\bar\beta}. \tag{4.9}$$

The last term can be written as

$$\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}\nabla_{\bar\beta}F_{\underline{ab}\bar\gamma} = \frac{1}{2}\bar\lambda^{\bar\beta}\bar\lambda^{\bar\gamma}(\gamma_{[\underline{a}}\eta\gamma_{\underline{b}]})_{\bar\beta\rho}W^\rho{}_{\bar\gamma}, \tag{4.10}$$

after using the definition of $F_{\underline{ab}\bar\gamma}$ (3.21) and commuting derivatives. Note that $\gamma_{[\underline{a}}\eta\gamma_{\underline{b}]}$ is different from zero only if $(\underline{a},\underline{b}) = (a,b)$ or $(\underline{a},\underline{b}) = (a',b')$, then $\frac{1}{2}N^{\underline{ab}}\gamma_{[\underline{a}}\eta\gamma_{\underline{b}]} = N^{[ab]}\gamma_{[ab]}\eta$. The equation (4.9) reduces to

$$QW = \partial U - \lambda^\alpha\nabla\bar\lambda^{\bar\beta}A_{\alpha\bar\beta} + \frac{1}{2}\bar\lambda^{\bar\beta}\lambda^\gamma N^{\underline{ab}}\nabla_\gamma F_{\underline{ab}\bar\beta}$$
$$- 2\eta_{\alpha\bar\alpha}\bar\lambda^{\bar\beta}\left(\nabla\bar\lambda^{\bar\alpha} - \frac{1}{4}\bar\lambda^{\bar\gamma}N^{[ab]}(\gamma_{[ab]})_{\bar\gamma}{}^{\bar\alpha}\right)W^\alpha{}_{\bar\beta}. \tag{4.11}$$

Note that the last term vanishes on-shell. It remains to compute $\nabla_\gamma F_{\underline{ab}\bar\beta}$. Using (3.19) and commuting derivatives,

$$\nabla_\gamma F_{\underline{ab}\bar\beta} = -(\gamma_{[\underline{a}})_{\gamma\rho}\nabla_{\underline{b}]}W^\rho{}_{\bar\beta} - \frac{1}{2}(\gamma_{[\underline{a}}\eta)_\gamma{}^{\bar\delta}\nabla_{\bar\delta}A_{\underline{b}]\bar\beta} - R_{\underline{ab}\gamma}{}^{\delta}A_{\delta\bar\beta} - R_{\underline{ab}\bar\beta}{}^{\bar\delta}A_{\gamma\bar\delta}. \tag{4.12}$$

From here and using $\lambda^\gamma N^{\underline{ab}}(\gamma_{\underline{a}})_{\gamma\delta} = -\frac{1}{2}(\lambda\omega)(\lambda\gamma^{\underline{b}})_\delta$ we obtain

$$\lambda^\gamma N^{\underline{ab}}\nabla_\gamma F_{\underline{ab}\bar\beta} = (\lambda\omega)(\gamma^{\underline{a}})_{\gamma\rho}\left(\nabla_{\underline{a}}W^\rho{}_{\bar\beta} + \frac{1}{2}\eta^{\rho\bar\delta}\nabla_{\bar\delta}A_{\underline{a}\bar\beta}\right)$$
$$- \lambda^\gamma N^{\underline{ab}}\left(R_{\underline{ab}\gamma}{}^{\delta}A_{\delta\bar\beta} + R_{\underline{ab}\bar\beta}{}^{\bar\delta}A_{\gamma\bar\delta}\right). \tag{4.13}$$

The first term in this expression contains

$$(\gamma^{\underline{a}})_{\alpha\beta}\nabla_{\underline{a}}W^{\beta}{}_{\bar{\gamma}} = \{\nabla_{\alpha},\nabla_{\beta}\}W^{\beta}{}_{\bar{\gamma}}, \tag{4.14}$$

after using (3.25) and (4.12) we can show

$$(\gamma^{\underline{a}})_{\alpha\beta}\left(\nabla_{\underline{a}}W^{\beta}{}_{\bar{\gamma}} + \frac{1}{2}\eta^{\beta\bar{\rho}}\nabla_{\bar{\rho}}A_{\underline{a}\bar{\gamma}}\right) = 0, \tag{4.15}$$

therefore, (4.11) simplifies to

$$\begin{aligned} QW = \partial U - 2\eta_{\alpha\bar{\alpha}}\bar{\lambda}^{\bar{\beta}}\left(\nabla\bar{\lambda}^{\bar{\alpha}} - \frac{1}{4}\bar{\lambda}^{\bar{\gamma}}N^{[ab]}(\gamma_{[ab]})_{\bar{\gamma}}{}^{\bar{\alpha}}\right)W^{\alpha}{}_{\bar{\beta}} - \lambda^{\alpha}\nabla\bar{\lambda}^{\bar{\beta}}A_{\alpha\bar{\beta}} \\ - \frac{1}{2}\bar{\lambda}^{\bar{\beta}}\lambda^{\gamma}N^{\underline{ab}}\left(R_{\underline{ab}\gamma}{}^{\delta}A_{\delta\bar{\beta}} + R_{\underline{ab}\bar{\beta}}{}^{\bar{\delta}}A_{\gamma\bar{\delta}}\right). \end{aligned} \tag{4.16}$$

After using the background values of the curvature (3.4)

$$N^{\underline{ab}}R_{\underline{ab}\alpha}{}^{\beta} = -\frac{1}{2}N^{[ab]}(\gamma_{[ab]})_{\alpha}{}^{\beta}, \quad N^{\underline{ab}}R_{\underline{ab}\bar{\alpha}}{}^{\bar{\beta}} = -\frac{1}{2}N^{[ab]}(\gamma_{[ab]})_{\bar{\alpha}}{}^{\bar{\beta}}, \tag{4.17}$$

we finally obtain

$$QW = \partial U - \left(\nabla\bar{\lambda}^{\bar{\alpha}} - \frac{1}{4}\bar{\lambda}^{\bar{\gamma}}N^{[ab]}(\gamma_{[ab]})_{\bar{\gamma}}{}^{\bar{\alpha}}\right)\left(2\eta_{\alpha\bar{\alpha}}\bar{\lambda}^{\bar{\beta}}W^{\alpha}{}_{\bar{\beta}} + \lambda^{\alpha}A_{\alpha\bar{\alpha}}\right). \tag{4.18}$$

Using the equation of motion (4.1) we have shown that

$$QW = \partial U. \tag{4.19}$$

A similar calculation shows that the operator $\bar{W}$ that satisfies $Q\bar{W} = \bar{\partial}U$ is given by

$$\bar{W} = \lambda^{\alpha}\left(\bar{J}^{\bar{\beta}}A_{\alpha\bar{\beta}} + \bar{J}^{\underline{a}}A_{\alpha\underline{a}} + 2\eta_{\beta\bar{\beta}}\bar{J}^{\beta}W_{\alpha}{}^{\bar{\beta}} + \frac{1}{2}\bar{N}^{\underline{ab}}F_{\alpha\underline{ab}}\right). \tag{4.20}$$

Performing the same steps one can show that

$$Q\bar{W} = \bar{\partial}U - \left(\bar{\nabla}\lambda^{\alpha} - \frac{1}{4}\lambda^{\gamma}\bar{N}^{[ab]}(\gamma_{[ab]})_{\gamma}{}^{\alpha}\right)\left(2\eta_{\alpha\bar{\beta}}\lambda^{\gamma}W_{\gamma}{}^{\bar{\beta}} + \bar{\lambda}^{\bar{\beta}}A_{\alpha\bar{\beta}}\right). \tag{4.21}$$

Note that the second term is proportional to the equation of motion for λ.

In summary, we have found the conformal weights $(1,0)$ and $(0,1)$ superfields

$$W = \bar{\lambda}^{\bar{\beta}}\left(J^{\alpha}A_{\alpha\bar{\beta}} + J^{\underline{a}}A_{\underline{a}\bar{\beta}} - 2\eta_{\alpha\bar{\alpha}}J^{\bar{\alpha}}W^{\alpha}{}_{\bar{\beta}} + \frac{1}{2}N^{\underline{ab}}F_{\underline{ab}\bar{\beta}}\right), \tag{4.22}$$

$$\bar{W} = \lambda^{\alpha}\left(\bar{J}^{\bar{\beta}}A_{\alpha\bar{\beta}} + \bar{J}^{\underline{a}}A_{\alpha\underline{a}} + 2\eta_{\beta\bar{\beta}}\bar{J}^{\beta}W_{\alpha}{}^{\bar{\beta}} + \frac{1}{2}\bar{N}^{\underline{ab}}F_{\alpha\underline{ab}}\right), \tag{4.23}$$

which satisfy $QW = \partial U$ and $Q\bar{W} = \bar{\partial}U$ on-shell.

From the definition of W, is has to transform under (3.13) as

$$\delta W = \partial U - Q(\Psi), \tag{4.24}$$

as in flat space. Using the gauge transformations of the superfields inside W, we obtain that

$$\delta W = \partial(\lambda^\alpha \Lambda_\alpha + \bar{\lambda}^{\bar{\alpha}} \bar{\Lambda}_{\bar{\alpha}}) - Q\left(J^\alpha \Lambda_\alpha + J^{\underline{a}} \Lambda_{\underline{a}} - 2\eta_{\alpha\bar{\alpha}} J^{\bar{\alpha}} \Lambda^\alpha + \frac{1}{2} N^{\underline{ab}} \nabla_{[\underline{a}} \Lambda_{\underline{b}]}\right). \tag{4.25}$$

which fixes Ψ to be

$$\Psi = J^\alpha \Lambda_\alpha + J^{\underline{a}} \Lambda_{\underline{a}} - 2\eta_{\alpha\bar{\alpha}} J^{\bar{\alpha}} \Lambda^\alpha + \frac{1}{2} N^{\underline{ab}} \nabla_{[\underline{a}} \Lambda_{\underline{b}]}. \tag{4.26}$$

An analogous result holds for $\bar{W}$

$$\delta W = \partial(\lambda^\alpha \Lambda_\alpha + \bar{\lambda}^{\bar{\alpha}} \bar{\Lambda}_{\bar{\alpha}}) - Q(\bar{\Psi}), \tag{4.27}$$

with $\bar{\Psi}$ given by

$$\bar{\Psi} = \bar{J}^{\bar{\alpha}} \bar{\Lambda}_{\bar{\alpha}} + \bar{J}^{\underline{a}} \bar{\Lambda}_{\underline{a}} + 2\eta_{\alpha\bar{\alpha}} \bar{J}^\alpha \bar{\Lambda}^{\bar{\alpha}} + \frac{1}{2} \bar{N}^{\underline{ab}} \nabla_{[\underline{a}} \bar{\Lambda}_{\underline{b}]}. \tag{4.28}$$

It is interesting to note that this looks like the vertex operator for $N = 1$ vector multiplet in ten dimensions for an open superstring, but now in an $AdS_5 \times S^5$ background. Perhaps this identification can be made more precise by studying boundary conditions for the pure spinor string in this space. Appropriate boundary conditions will reduce the number of independent components of J^α and $J^{\bar{\alpha}}$.

Now that W and $\bar{W}$ have been constructed, we can find the conformal weight $(1, 1)$ operator V that satisfies $QV = \partial\bar{W} - \bar{\partial}W$. We will need to use the equations of motion of the world-sheet action and some Maurer-Cartan identities. Also, the equations for the pure spinor ghosts imply that the ghost Lorenz currents satisfy

$$\bar{\nabla} N^{ab} = -N^{c[a} \bar{N}_c{}^{b]}, \quad \bar{\nabla} N^{a'b'} = N^{c'[a'} \bar{N}_{c'}{}^{b']}, \quad \bar{\nabla} N^{ab'} = N^{c'a} \bar{N}_{c'}{}^{b'} + \bar{N}^{ca} N_c{}^{b'}, \tag{4.29}$$

$$\nabla \bar{N}^{ab} = N^{c[a} \bar{N}_c{}^{b]}, \quad \nabla \bar{N}^{a'b'} = -N^{c'[a'} \bar{N}_{c'}{}^{b']}, \quad \nabla \bar{N}^{ab'} = N^{ca} \bar{N}_c{}^{b'} + \bar{N}^{c'a} N_{c'}{}^{b'}. \tag{4.30}$$

The equations of morion for the currents J^A for $A = (\underline{a}, \alpha, \bar{\alpha})$ are

$$\begin{aligned}
\bar{\nabla} J^{\bar{\alpha}} &= \frac{1}{4} N^{[ab]} \bar{J}^{\bar{\beta}} (\gamma_{[ab]})_{\bar{\beta}}{}^{\bar{\alpha}} + \frac{1}{4} \bar{N}^{[ab]} J^{\bar{\beta}} (\gamma_{[ab]})_{\bar{\beta}}{}^{\bar{\alpha}}, \\
\nabla \bar{J}^\alpha &= \frac{1}{4} N^{[ab]} \bar{J}^\beta (\gamma_{[ab]})_\beta{}^\alpha + \frac{1}{4} \bar{N}^{[ab]} J^\beta (\gamma_{[ab]})_\beta{}^\alpha, \\
\bar{\nabla} J^{\underline{a}} &= -\gamma^{\underline{a}}_{\bar{\alpha}\bar{\beta}} J^{\bar{\alpha}} \bar{J}^{\bar{\beta}} - \delta^{\underline{a}}_b (N^{bc} \bar{J}_c + \bar{N}^{bc} J_c) + \delta^{\underline{a}}_{b'} (N^{b'c'} \bar{J}_{c'} + \bar{N}^{b'c'} J_{c'}), \\
\nabla \bar{J}^{\underline{a}} &= \gamma^{\underline{a}}_{\alpha\beta} J^\alpha \bar{J}^\beta - \delta^{\underline{a}}_b (N^{bc} \bar{J}_c + \bar{N}^{bc} J_c) + \delta^{\underline{a}}_{b'} (N^{b'c'} \bar{J}_{c'} + \bar{N}^{b'c'} J_{c'}).
\end{aligned} \tag{4.31}$$

They are calculated varying the action with respect to $\delta g = g(\delta X^{\underline{a}} T_{\underline{a}} + \delta X^\alpha T_\alpha + \delta X^{\bar{\alpha}} T_{\bar{\alpha}})$ and using the Maurer-Cartan identities

$$\partial \bar{J}^A - \bar{\partial} J^A - J^B \bar{J}^C f_{CB}{}^A = 0. \tag{4.32}$$

The vertex operator V couples all gauge covariant currents with conformal weight $(1, 0)$ with the ones with weight $(0, 1)$ and it has the form

$$\begin{aligned}
V = {} & 2\eta_{\beta\bar{\gamma}} J^\alpha \bar{J}^\beta W_\alpha{}^{\bar{\gamma}} - 2\eta_{\gamma\bar{\alpha}} J^{\bar{\alpha}} \bar{J}^{\bar{\beta}} W^\gamma{}_{\bar{\beta}} + J^\alpha \bar{J}^{\bar{\beta}} A_{\alpha\bar{\beta}} + J^\alpha \bar{J}^{\underline{a}} A_{\alpha\underline{a}} - J^{\underline{a}} \bar{J}^{\bar{\alpha}} A_{\underline{a}\bar{\alpha}} \\
& + \frac{1}{2} J^\alpha \bar{N}^{\underline{ab}} F_{\alpha\underline{ab}} - \frac{1}{2} N^{\underline{ab}} \bar{J}^{\bar{\alpha}} F_{\underline{ab}\bar{\alpha}} + J^{\bar{\beta}} \bar{J}^\alpha \bar{\mathcal{V}}_{\alpha\bar{\beta}} + J^{\underline{a}} \bar{J}^\alpha \bar{\mathcal{V}}_{\underline{a}\alpha} + J^{\bar{\alpha}} \bar{J}^{\underline{a}} \mathcal{V}_{\underline{a}\bar{\alpha}} + J^{\underline{a}} \bar{J}^{\underline{b}} \mathcal{V}_{\underline{ab}} \\
& + \frac{1}{4} N^{\underline{ab}} \bar{N}^{\underline{cd}} \mathcal{V}_{\underline{ab}\ \underline{cd}} + \frac{1}{2} N^{\underline{ab}} \bar{J}^\alpha \bar{\mathcal{V}}_{\underline{ab}\alpha} + \frac{1}{2} J^{\bar{\alpha}} \bar{N}^{\underline{ab}} \mathcal{V}_{\underline{ab}\bar{\alpha}} + \frac{1}{2} J^{\underline{a}} \bar{N}^{\underline{bc}} \mathcal{V}_{\underline{a}\ \underline{bc}} + \frac{1}{2} N^{\underline{bc}} \bar{J}^{\underline{a}} \bar{\mathcal{V}}_{\underline{a}\ \underline{bc}}.
\end{aligned} \tag{4.33}$$

From the structure of W, $\bar{W}$ and the BRST transformations of the currents J^α and $\bar{J}^{\bar{\alpha}}$ we can immediately identify the first seven superfields in V with the ones that already appeared in W and $\bar{W}$. The remaining superfields, collectively called $\mathcal{V}$, are constrained by $QV = \partial\bar{W} - \bar{\partial}W$.

For example, $\bar{\mathcal{V}}_{\underline{a}\alpha}$ is constrained by the equation

$$\gamma^{\underline{a}}_{\beta\gamma}\bar{\mathcal{V}}_{\underline{a}\alpha} = \gamma^{\underline{a}}_{\alpha(\beta}A_{\gamma)\underline{a}} - 2\eta_{\alpha\bar{\gamma}}\nabla_{(\beta}W_{\gamma)}{}^{\bar{\gamma}}. \tag{4.34}$$

Using the definition (3.31) for $W_\gamma{}^{\bar{\gamma}}$ and (anti-)commuting covariant derivatives, one can obtain that $\eta_{\alpha\bar{\gamma}}\nabla_{(\beta}W_{\gamma)}{}^{\bar{\gamma}} = \frac{1}{2}\gamma^{\underline{a}}_{\alpha(\beta}A_{\gamma)\underline{a}} + \gamma^{\underline{a}}_{\beta\gamma}\mathcal{O}_{\underline{a}\alpha}$ where $\mathcal{O}_{\underline{a}\alpha}$ is a combination of superfields and derivatives. This allows us to find $\bar{\mathcal{V}}_{\underline{a}\alpha}$. The resulting expression contains the superfield $A_{\alpha\bar{\beta}}$ and its derivatives. The full expression for $\bar{\mathcal{V}}_{\underline{a}\alpha}$ is not very illuminating and it turns out to be

$$\begin{aligned}\bar{\mathcal{V}}_{\underline{a}\alpha} = & -\frac{1}{10}(\eta\gamma^{\underline{b}}\gamma_{\underline{a}}\eta)_\alpha{}^\beta A_{\beta\underline{b}} + \frac{1}{40}(\eta\gamma_{\underline{a}})_\alpha{}^{\bar{\beta}}\eta^{\bar{\gamma}\bar{\delta}}\nabla_{\bar{\beta}}A_{\gamma\bar{\delta}} \\ & -\frac{1}{10}(\eta\gamma^{\underline{b}})_\alpha{}^{\bar{\beta}}\left(\nabla_{\underline{b}}A_{\underline{a}\bar{\beta}} + \frac{1}{8}(\eta\gamma_{\underline{a}}\gamma_{\underline{b}})^{\gamma\bar{\delta}}\nabla_{\bar{\beta}}A_{\gamma\bar{\delta}} - \frac{1}{4}\gamma^{\gamma\delta}_{\underline{a}}\nabla_{\bar{\beta}}A_{\gamma\bar{\delta}}\right).\end{aligned} \tag{4.35}$$

All other superfields $\mathcal{V}$ in (4.33) will appear in equations that can be inverted. We will write all their defining equations. Their explicit expressions can be obtained in the same way as the one for $\bar{\mathcal{V}}_{\underline{a}\alpha}$ in (4.35).

The superfields $\mathcal{V}_{\underline{a}\bar{\alpha}}$ satisfies the equation

$$\gamma^{\underline{a}}_{\bar{\beta}\bar{\gamma}}\mathcal{V}_{\underline{a}\bar{\alpha}} = -\gamma^{\underline{a}}_{\bar{\alpha}(\bar{\beta}}A_{\underline{a}\bar{\gamma})} - 2\eta_{\gamma\bar{\alpha}}\nabla_{(\bar{\beta}}W^\gamma{}_{\bar{\gamma})}. \tag{4.36}$$

The superfield $\mathcal{V}_{\underline{ab}}$ which, from its coupling in (4.33), can be identified with fluctuations of the metric plus the NSNS two-form, satisfies the equations

$$\gamma^{\underline{b}}_{\alpha\beta}\mathcal{V}_{\underline{ba}} = \nabla_{(\alpha}A_{\beta)\underline{a}} + \frac{1}{2}(\gamma_{\underline{a}}\eta)_{(\alpha}{}^{\bar{\gamma}}A_{\beta)\bar{\gamma}}, \quad \gamma^{\underline{b}}_{\bar{\alpha}\bar{\beta}}\mathcal{V}_{\underline{ab}} = -\nabla_{(\bar{\alpha}}A_{\underline{a}\bar{\beta})} + \frac{1}{2}(\gamma_{\underline{a}}\eta)_{(\bar{\alpha}}{}^\gamma A_{\gamma\bar{\beta})}, \tag{4.37}$$

which are equivalent when they are expressed in terms of $A_{\alpha\bar{\beta}}$. Note that $\mathcal{V}_{\underline{ab}}$ contains the components $\mathcal{V}_{ab'}$ that twist the AdS_5 and S^5 spaces.

The superfield $\bar{\mathcal{V}}_{\alpha\bar{\beta}}$ satisfies the equations

$$\begin{aligned}&\frac{1}{2}(\gamma_{\underline{a}}\eta)_\alpha{}^{\bar{\gamma}}\bar{\mathcal{V}}_{\beta\bar{\gamma}} + \frac{1}{2}(\gamma_{\underline{a}}\eta)_\beta{}^{\bar{\gamma}}A_{\alpha\bar{\gamma}} + \gamma^{\underline{b}}_{\alpha\beta}\mathcal{V}_{\underline{ab}} - \nabla_\alpha\bar{\mathcal{V}}_{\underline{a}\beta} - 2\eta_{\beta\bar{\gamma}}\nabla_{\underline{a}}W_\alpha{}^{\bar{\gamma}} = 0, \\ &\frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar{\alpha}}{}^\gamma\bar{\mathcal{V}}_{\gamma\bar{\beta}} + \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar{\beta}}{}^\gamma A_{\gamma\bar{\alpha}} + \gamma^{\underline{b}}_{\bar{\alpha}\bar{\beta}}\mathcal{V}_{\underline{ba}} - \nabla_{\bar{\alpha}}\mathcal{V}_{\underline{a}\bar{\beta}} - 2\eta_{\gamma\bar{\beta}}\nabla_{\underline{a}}W^\gamma{}_{\bar{\alpha}} = 0,\end{aligned} \tag{4.38}$$

which are also equivalent when written in terms of $A_{\alpha\bar{\beta}}$.

The next four superfields are related to fluctuations of the connections. First, $\mathcal{V}_{\underline{ab}\bar{\beta}}$ is given by

$$\frac{1}{2}(\gamma^{\underline{ab}}\eta)_{\bar{\alpha}\gamma}\mathcal{V}_{\underline{ab}\bar{\beta}} + \frac{1}{2}(\gamma^{\underline{ab}}\eta)_{\gamma\bar{\beta}}F_{\underline{ab}\bar{\alpha}} - \nabla_{\bar{\alpha}}\bar{\mathcal{V}}_{\gamma\bar{\beta}} + \nabla_{\bar{\beta}}A_{\gamma\bar{\alpha}} - \gamma^{\underline{a}}_{\bar{\alpha}\bar{\beta}}\bar{\mathcal{V}}_{\underline{a}\gamma} = 0, \tag{4.39}$$

and $\bar{\mathcal{V}}_{\underline{ab}\alpha}$ is determined by

$$\frac{1}{2}(\gamma^{\underline{ab}}\eta)_{\alpha\bar{\beta}}\bar{\mathcal{V}}_{\underline{ab}\gamma} + \frac{1}{2}(\eta\gamma^{\underline{ab}})_{\gamma\bar{\beta}}F_{\alpha\underline{ab}} - \nabla_\alpha\bar{\mathcal{V}}_{\gamma\bar{\beta}} + \nabla_\gamma A_{\alpha\bar{\beta}} + \gamma^{\underline{a}}_{\alpha\gamma}\mathcal{V}_{\underline{a}\bar{\beta}} = 0, \tag{4.40}$$

and $\mathcal{V}_{\underline{a}\ \underline{bc}}$ satisfies the equation

$$\frac{1}{2}(\gamma^{\underline{bc}}\eta)_{\bar{\alpha}\beta}\mathcal{V}_{\underline{a}\ \underline{bc}} + \nabla_\beta A_{\underline{a}\bar{\alpha}} - \nabla_{\bar{\alpha}}\bar{\mathcal{V}}_{\underline{a}\beta} - (\gamma_{\underline{a}}\eta)_{\bar{\alpha}}{}^{\gamma}\eta_{\beta\bar{\gamma}}W_{\gamma}{}^{\bar{\gamma}} = 0. \tag{4.41}$$

Finally, the background field $\bar{\mathcal{V}}_{\underline{a}\ \underline{bc}}$ satisfies the equation

$$\frac{1}{2}(\gamma^{\underline{bc}}\eta)_{\alpha\bar{\beta}}\bar{\mathcal{V}}_{\underline{a}\ \underline{bc}} + \nabla_{\bar{\beta}}A_{\alpha\underline{a}} + \nabla_{\alpha}\mathcal{V}_{\underline{a}\bar{\beta}} - (\gamma_{\underline{a}}\eta)_{\alpha}{}^{\bar{\gamma}}\eta_{\gamma\bar{\beta}}W^{\gamma}{}_{\bar{\gamma}} = 0. \tag{4.42}$$

The last superfield to be defined is related to fluctuations of the curvature $\mathcal{V}_{\underline{ab}\ \underline{cd}}$ and it satisfies the equations

$$\begin{aligned}\bar{N}^{\underline{ab}}\left(\frac{1}{2}(\gamma^{\underline{cd}}\eta)_{\alpha\bar{\beta}}\mathcal{V}_{\underline{cd}\ \underline{ab}} + \nabla_{\bar{\beta}}F_{\alpha\underline{ab}} + \nabla_{\alpha}\mathcal{V}_{\underline{ab}\bar{\beta}}\right)&\\ +\frac{1}{2}\bar{N}^{[ab]}\left((\gamma_{[ab]})_{\alpha}{}^{\gamma}\bar{\mathcal{V}}_{\gamma\bar{\beta}} + (\gamma_{[ab]})_{\bar{\beta}}{}^{\bar{\gamma}}A_{\alpha\bar{\gamma}}\right) &= 0,\end{aligned} \tag{4.43}$$

$$\begin{aligned}N^{\underline{ab}}\left(\frac{1}{2}(\gamma^{\underline{cd}}\eta)_{\bar{\alpha}\beta}\mathcal{V}_{\underline{ab}\ \underline{cd}} + \nabla_{\beta}F_{\underline{ab}\bar{\alpha}} - \nabla_{\bar{\alpha}}\bar{\mathcal{V}}_{\underline{ab}\beta}\right)&\\ +\frac{1}{2}N^{[ab]}\left((\gamma_{[ab]})_{\bar{\alpha}}{}^{\bar{\gamma}}\bar{\mathcal{V}}_{\beta\bar{\gamma}} + (\gamma_{[ab]})_{\beta}{}^{\gamma}A_{\gamma\bar{\alpha}}\right) &= 0.\end{aligned} \tag{4.44}$$

As a simple application one can calculate the integrated vertex operator (4.33) for the radius operator found in [22]. The unintegrated vertex operator is $U = \lambda^\alpha\bar{\lambda}^{\bar{\beta}}\eta_{\alpha\bar{\beta}}$ and the integrated vertex operator should be proportional to the lagrangian in (3.5). Using $A_{\alpha\bar{\alpha}} = \eta_{\alpha\bar{\alpha}}$ the solution for (4.34)–(4.44) gives

$$V = J^\alpha\bar{J}^{\bar{\beta}}\eta_{\alpha\bar{\beta}} - 3J^{\bar{\beta}}\bar{J}^{\alpha}\eta_{\alpha\bar{\beta}} + 2J^{\underline{a}}\bar{J}^{\underline{b}}\eta_{\underline{ab}} + N^{ab}\bar{N}_{ab} - N^{a'b'}\bar{N}_{a'b'}. \tag{4.45}$$

Which is proportional to the lagrangian up to the equations of motion for the ghosts.

4.1 Local symmetries and field content

Due to the extra mixing coming from the RR background, non vanishing torsions and curvatures, it is quite involved to get the equations of motion for the superfields defined by V in a way that we can identify what is their lowest component. However, we can use the gauge invariance and look for fields that transform in the expected way. As in flat space, V inherits a gauge transformation from the unintegrated vertex operator $\delta U = Q(\lambda^\alpha\Lambda_\alpha + \bar{\lambda}^{\bar{\alpha}}\bar{\Lambda}_{\bar{\alpha}})$, after finding the gauge transformations of W and $\bar{W}$, we have that V transforms as

$$\begin{aligned}\delta V =& \bar{\partial}\left(J^\alpha\Lambda_\alpha + J^{\underline{a}}\Lambda_{\underline{a}} - 2\eta_{\alpha\bar{\alpha}}J^{\bar{\alpha}}\Lambda^{\alpha} + \frac{1}{2}N^{\underline{ab}}\nabla_{[\underline{a}}\Lambda_{\underline{b}]}\right)\\ &- \partial\left(\bar{J}^{\bar{\alpha}}\bar{\Lambda}_{\bar{\alpha}} + \bar{J}^{\underline{a}}\bar{\Lambda}_{\underline{a}} + 2\eta_{\alpha\bar{\alpha}}\bar{J}^{\alpha}\bar{\Lambda}^{\bar{\alpha}} + \frac{1}{2}\bar{N}^{\underline{ab}}\nabla_{[\underline{a}}\bar{\Lambda}_{\underline{b}]}\right),\end{aligned} \tag{4.46}$$

the integral of vertex operator (4.33) is gauge invariant under (3.13).

The transformation of each field in V (4.33) can be found using the definition above and the equations for the Maurer-Cartan currents and $N^{\underline{ab}}$, $\bar{N}^{\underline{ab}}$. Alternatively, we can use the definition of each $\mathcal{V}$ in terms of the superfields in W and $\bar{W}$ and use (3.13), (3.33) and (3.34).

We can find a very simple expression for the superfield whose lowest component gives the metric plus the NSNS two-form

$$\mathcal{V}_{\underline{ab}} = \frac{1}{16}(\eta\gamma_{\underline{a}}\gamma_{\underline{b}})^{\alpha\bar{\alpha}} A_{\alpha\bar{\alpha}} - \frac{1}{64}\gamma_{\underline{a}}^{\bar{\alpha}\bar{\beta}}\gamma_{\underline{b}}^{\alpha\beta}\nabla_{\bar{\alpha}}\nabla_{\alpha}A_{\beta\bar{\beta}}. \tag{4.47}$$

Its gauge transformation can be computed and it is given by

$$\delta\mathcal{V}_{\underline{ab}} = \nabla_a\Lambda_b - \nabla_a\bar{\Lambda}_b. \tag{4.48}$$

where we can see that $\Lambda_a - \bar{\Lambda}_a$ is related to the diffeomorphism parameter and $\Lambda_a + \bar{\Lambda}_a$ is the parameter of the gauge transformation for the NSNS two-form.

The superfield whose lowest component is the gravitino can be identified by looking for the correct gauge transformation. It turns out that the combination $\bar{\mathcal{V}}_{\underline{a}\alpha} - \frac{1}{4}(\gamma_{\underline{a}}{}^{\underline{b}})_\alpha{}^\beta A_{\beta\underline{b}}$ transforms as

$$\delta\left(\bar{\mathcal{V}}_{\underline{a}\alpha} - \frac{1}{4}(\gamma_{\underline{a}}{}^{\underline{b}})_\alpha{}^\beta A_{\beta\underline{b}}\right) = \frac{1}{2}(\gamma_{\underline{a}}\eta)_\alpha{}^{\bar{\beta}}(\bar{\Lambda}_{\bar{\beta}} + 2\eta_{\delta\bar{\beta}}\Lambda^\delta) + \nabla_{\underline{a}}(\Lambda_\alpha - 2\eta_{\alpha\bar{\gamma}}\bar{\Lambda}^{\bar{\gamma}}). \tag{4.49}$$

From this we will identify one of the gravitini as the lowest component of the superfield

$$\mathcal{E}_{\underline{a}}{}^{\bar{\alpha}} = \eta^{\alpha\bar{\alpha}}\left(\bar{\mathcal{V}}_{\underline{a}\alpha} - \frac{1}{4}(\gamma_{\underline{a}}{}^{\underline{b}})_\alpha{}^\beta A_{\beta\underline{b}}\right), \tag{4.50}$$

which will transform as

$$\delta\mathcal{E}_{\underline{a}}{}^{\bar{\alpha}} = \frac{1}{2}(\eta\gamma_{\underline{a}}\eta)^{\bar{\alpha}\bar{\beta}}(\bar{\Lambda}_{\bar{\beta}} + 2\eta_{\delta\bar{\beta}}\Lambda^\delta) + \nabla_{\underline{a}}(\eta^{\alpha\bar{\alpha}}\Lambda_\alpha - 2\bar{\Lambda}^{\bar{\alpha}}). \tag{4.51}$$

Note that $(\eta\gamma_{\underline{a}}\eta)$ is γ_a and $-\gamma_{a'}$, which is the expected behavior for the term depending on the cosmological constant in the local transformation of the gravitino. Furthermore, it reduces to the usual local supersymmetry transformation in the limit where the RR background goes to zero.

Now we want to find the bi-spinor superfield whose lowest component is bi-spinor RR field-strength. Comparing with the flat superstring we start with $\mathcal{V}_{\alpha\bar{\alpha}}$ and find that it transforms as

$$\delta\mathcal{V}_{\alpha\bar{\alpha}} = -2\eta_{\alpha\bar{\beta}}\nabla_{\bar{\alpha}}\bar{\Lambda}^{\bar{\beta}} + 2\eta_{\beta\bar{\alpha}}\nabla_\alpha\Lambda^\beta. \tag{4.52}$$

After using (3.15) and (3.16) we have that

$$\delta\mathcal{V}_{\alpha\bar{\alpha}} = -\nabla_\alpha\bar{\Lambda}_{\bar{\alpha}} - \nabla_{\bar{\alpha}}\Lambda_\alpha - (\gamma^{\underline{ab}}\eta)_{\alpha\bar{\alpha}}(\nabla_{\underline{a}}\Lambda_{\underline{b}} - \nabla_{\underline{a}}\bar{\Lambda}_{\underline{b}}). \tag{4.53}$$

Since the RR field-strengths should be gauge invariant, the following combination

$$\mathcal{F}_{\alpha\bar{\alpha}} = \mathcal{V}_{\alpha\bar{\alpha}} + A_{\alpha\bar{\alpha}} + (\gamma^{\underline{ab}}\eta)_{\alpha\bar{\alpha}}\mathcal{V}_{\underline{ab}}, \tag{4.54}$$

has the desired property. The usual definition using upper indices can be achieved using the η tensors. The zero mode of the gauge invariant scalar $\phi = \eta^{\alpha\bar{\alpha}}\mathcal{F}_{\alpha\bar{\alpha}}$ is the operator that changes the radius of the $AdS_5 \times S^5$ space. The superfields whose lowest components are the dilatini can be found in similar ways.

5 Conclusion and prospects

In this paper we did the explicit construction of the integrated vertex operator in an $AdS_5 \times S^5$ background using the pure spinor formalism. We have found how all superfields present in the integrated vertex are related to the initial superfield $A_{\alpha\bar{\alpha}}(g)$ in the unintegrated vertex operator. The analysis done here complements the formal construction previously done in [13]. However, the integrated vertex operator found in that work does not depend on the Lorentz ghost currents with mixed indices. It would be interesting to find the origin of this discrepancy.

The final answer resembles the construction of the integrated vertex operator in a flat background, replacing the flat space supersymmetric currents $(\Pi^m, \partial\theta^\alpha, d_\alpha, N^{mn})$ with the the isometry invariant currents $(J^{\underline{a}}, J^\alpha, J^{\bar{\alpha}}, N^{\underline{ab}})$. The superspace description of linearized supergravity found here should be related to the supergravity pre-potential found in [24]. A possible future application is to find explicit expressions for vertex operators for some specific supergravity state defined by its $\mathfrak{psu}(2,2|4)$ labels, in the same way single trace operators of $N = 4$ SYM are defined. Progress in this problem was made in [25] where explicits states were found expanding the vertex operator close to the boundary of AdS_5. These results were later used in [26] to compute amplitudes with one closed string state and open string states. Another known explicit vertex operator is the one corresponding to the beta deformation [19]. A possible way to describe more general explicit states is to use the formalism described in [27].

Another interesting direction is to study vertex operators for open strings in $AdS_5 \times S^5$. As we noticed before, the gauge parameters for the local symmetries of the supergravity states satisfy equations that resembles the equations of motion of an $N = 1$ vector multiplet in ten dimensions. Of course this identification has to be made more precise since it is known that there are no D9-branes in $AdS_5 \times S^5$. The boundary interaction of a D-brane should be an operator of the form

$$I_{\text{boundary}} = \int \left(A_{\underline{a}}(g)\mathsf{J}^{\underline{a}} + A_\alpha(g)\mathsf{J}^\alpha + A_{\bar{\alpha}}(g)\mathsf{J}^{\bar{\alpha}} + F_{\underline{ab}}(g)\mathsf{N}^{\underline{ab}} \right), \tag{5.1}$$

where $(A_{\underline{a}}, A_\alpha, A_{\bar{\alpha}}, F_{\underline{ab}})$ describe the massless states of the D-brane and $(\mathsf{J}^{\underline{a}}, \mathsf{J}^\alpha, \mathsf{J}^{\bar{\alpha}}, \mathsf{N}^{\underline{ab}})$ are appropriate isometry invariant world-sheet one-forms evaluated at the boundary. A BRST analysis of the boundary conditions with the interaction I_{boundary} along the lines of [28] should give the correct physical state conditions for $(A_{\underline{a}}, A_\alpha, A_{\bar{\alpha}}, F_{\underline{ab}})$ and the allowed D-branes.

Perhaps the most important problem is to understand the quantum correction to the physical states conditions. For the case of superstrings in flat space, since it is a free theory, the primary state condition receives only a one loop quantum correction, the anomalous dimension for the exponential operator $e^{ik\cdot X}$. From the BRST cohomology point of view there is no correction at all. This will not be true for fluctuations around a general background since the physical state condition will receive corrections from vertices of the action for both computations. However, for the case of $AdS_5 \times S^5$ the situation for the massless spectrum should be similar to the flat space case. These states are dual to the protected

BPS operators in the $N = 4$ SYM theory. The only quantum correction to the primary state condition should be a one loop correction, proportional to the quadratic Casimir of $\mathfrak{psu}(2,2|4)$ algebra [29]. The BRST cohomology condition should receive no correction. It would be very interesting to understand how all possible quantum corrections to the classical calculations in this work cancel. We plan to address this in a future work.

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A The remaining equations

In this appendix we list the additional equations the superfields $\mathcal{V}$ satisfy. They are all consequences of (2.16) when written in terms of $A_{\alpha\bar\alpha}$. Some of the equations are written without removing the Maurer-Cartan or ghost currents since this is their most compact form. The equations are

$$\nabla_\alpha \mathcal{V}_{\underline{ab}} - \nabla_{\underline{a}} A_{\alpha\underline{b}} + \frac{1}{2}(\gamma_{\underline{a}}\eta)_\alpha{}^{\bar\beta}\mathcal{V}_{\underline{b}\bar\beta} - \frac{1}{2}(\gamma_{\underline{b}}\eta)_\alpha{}^{\bar\beta}A_{\underline{a}\bar\beta} = 0, \tag{A.1}$$

$$\nabla_{\bar\alpha}\mathcal{V}_{\underline{ab}} + \nabla_{\underline{b}}A_{\underline{a}\bar\alpha} - \frac{1}{2}(\gamma_{\underline{a}}\eta)_{\bar\alpha}{}^{\beta}A_{\beta\underline{b}} - \frac{1}{2}(\gamma_{\underline{b}}\eta)_{\bar\alpha}{}^{\beta}\bar{\mathcal{V}}_{\underline{a}\beta} = 0, \tag{A.2}$$

$$\lambda^\alpha N^{\underline{ab}}(\nabla_\alpha\bar{\mathcal{V}}_{\underline{ab}\beta} - \gamma^{\underline{c}}_{\alpha\beta}\bar{\mathcal{V}}_{\underline{cab}}) + \lambda^\alpha N^{[ab]}(\gamma_{[ab]})_{\beta\bar\gamma}W_\alpha{}^{\bar\gamma} = 0, \tag{A.3}$$

$$\bar\lambda^{\bar\alpha}\bar N^{\underline{ab}}(\nabla_{\bar\alpha}\mathcal{V}_{\underline{ab}\bar\beta} - \gamma^{\underline{c}}_{\bar\alpha\bar\beta}\mathcal{V}_{\underline{cab}}) - \bar\lambda^{\bar\alpha}\bar N^{[ab]}(\gamma_{[ab]}\eta)_{\bar\beta\gamma}W^{\gamma}{}_{\bar\alpha} = 0, \tag{A.4}$$

$$N^{\underline{ab}}\left(\nabla_\beta F_{\underline{ab}\bar\alpha} + \frac{1}{2}(\gamma^{\underline{cd}}\eta)_{\bar\alpha\beta}\mathcal{V}_{\underline{abcd}} - \nabla_{\bar\alpha}\bar{\mathcal{V}}_{\underline{ab}\beta}\right) + \frac{1}{2}N^{[ab]}\left((\gamma_{[ab]})_{\bar\alpha}{}^{\bar\gamma}\bar{\mathcal{V}}_{\beta\bar\gamma} + (\gamma_{[ab]})_\beta{}^{\gamma}A_{\gamma\bar\alpha}\right) = 0, \tag{A.5}$$

$$\bar N^{\underline{ab}}\left(\nabla_{\bar\beta}F_{\alpha\underline{ab}} + \frac{1}{2}(\gamma^{\underline{cd}}\eta)_{\alpha\bar\beta}\mathcal{V}_{\underline{cdab}} + \nabla_\alpha\mathcal{V}_{\underline{ab}\bar\beta}\right) + \frac{1}{2}\bar N^{[ab]}\left((\gamma_{[ab]})_\alpha{}^{\gamma}\bar{\mathcal{V}}_{\gamma\bar\beta} + (\gamma_{[ab]})_{\bar\beta}{}^{\bar\gamma}A_{\alpha\bar\gamma}\right) = 0, \tag{A.6}$$

$$\lambda^\alpha N^{\underline{ab}}\nabla_\alpha F_{\underline{ab}\bar\beta} - \frac{1}{2}\lambda^\alpha N^{[ab]}(\gamma_{[ab]})_{\bar\beta}{}^{\bar\gamma}A_{\alpha\bar\gamma} = 0, \tag{A.7}$$

$$\bar\lambda^{\bar\alpha}\bar N^{\underline{ab}}\nabla_{\bar\alpha}F_{\beta\underline{ab}} - \frac{1}{2}\bar\lambda^{\bar\alpha}\bar N^{[ab]}(\gamma_{[ab]})_\beta{}^{\gamma}A_{\gamma\bar\alpha} = 0, \tag{A.8}$$

$$N^{\underline{ab}}(\nabla_{(\bar\alpha}F_{\underline{ab}\bar\beta)} + \gamma^{\underline{c}}_{\bar\alpha\bar\beta}\bar{\mathcal{V}}_{\underline{cab}}) + N^{[ab]}(\gamma_{[ab]}\eta)_{(\bar\alpha\gamma}W^{\gamma}{}_{\bar\beta)} = 0, \tag{A.9}$$

$$\bar N^{\underline{ab}}(\nabla_{(\alpha}F_{\beta)\underline{ab}} - \gamma^{\underline{c}}_{\alpha\beta}\mathcal{V}_{\underline{cab}}) + \bar N^{[ab]}(\gamma_{[ab]}\eta)_{(\alpha\bar\gamma}W_{\beta)}{}^{\bar\gamma} = 0, \tag{A.10}$$

$$\lambda^\alpha N^{\underline{ab}}\bar J^{\underline{c}}\left(-\frac{1}{4}(\gamma_{\underline{c}}\eta)_\alpha{}^{\bar\beta}F_{\underline{ab}\bar\beta} + \frac{1}{2}\nabla_\alpha\bar{\mathcal{V}}_{\underline{cab}}\right) - \lambda^\alpha N^{ab}\bar J_a A_{\alpha b} + \lambda^\alpha N^{a'b'}\bar J_{a'}A_{\alpha b'} = 0, \tag{A.11}$$

$$\bar\lambda^{\bar\alpha}\bar N^{\underline{ab}}J^{\underline{c}}\left(-\frac{1}{4}(\gamma_{\underline{c}}\eta)_{\bar\alpha}{}^{\beta}F_{\beta\underline{ab}} + \frac{1}{2}\nabla_{\bar\alpha}\mathcal{V}_{\underline{cab}}\right) + \bar\lambda^{\bar\alpha}\bar N^{ab}J_a A_{b\bar\alpha} - \bar\lambda^{\bar\alpha}\bar N^{a'b'}J_{a'}A_{b'\bar\alpha} = 0, \tag{A.12}$$

JHEP10(2017)178

$$N^{\underline{ab}}\bar{J}^{\underline{c}}(\nabla_{\bar{\alpha}}\bar{\mathcal{V}}_{\underline{cab}}+\nabla_{\underline{c}}F_{\underline{ab}\bar{\alpha}}-\frac{1}{2}(\gamma_{\underline{c}}\eta)_{\bar{\alpha}}{}^{\beta}\bar{\mathcal{V}}_{\underline{ab}\alpha})$$
$$+\frac{1}{2}N^{[ab]}\bar{J}^{\underline{c}}(\gamma_{[ab]})_{\bar{\alpha}}{}^{\bar{\beta}}\mathcal{V}_{\underline{c}\bar{\beta}}+N^{ab}\bar{J}_{[a}A_{b]\bar{\alpha}}-N^{a'b'}\bar{J}_{[a'}A_{b']\bar{\alpha}}=0, \tag{A.13}$$
$$\bar{N}^{\underline{ab}}J^{\underline{c}}(\nabla_{\alpha}\mathcal{V}_{\underline{cab}}-\nabla_{\underline{c}}F_{\alpha\underline{ab}}+\frac{1}{2}(\gamma_{\underline{c}}\eta)_{\alpha}{}^{\bar{\beta}}\mathcal{V}_{\underline{ab}\bar{\beta}})$$
$$+\frac{1}{2}\bar{N}^{[ab]}J^{\underline{c}}(\gamma_{[ab]})_{\alpha}{}^{\beta}\bar{\mathcal{V}}_{\underline{c}\alpha}-\bar{N}^{ab}J_{[a}A_{\alpha b]}+\bar{N}^{a'b'}J_{[a'}A_{\alpha b']}=0, \tag{A.14}$$
$$\lambda^{\alpha}N^{\underline{ab}}\bar{N}^{\underline{cd}}\nabla_{\alpha}\mathcal{V}_{\underline{abcd}}+\frac{1}{2}\lambda^{\alpha}N^{\underline{ab}}\bar{N}^{[ab]}(\gamma_{[ab]})_{\alpha}{}^{\beta}\bar{\mathcal{V}}_{\underline{ab}\beta}-\lambda^{\alpha}N^{ca}\bar{N}_{c}{}^{b}F_{\alpha ab}$$
$$+\lambda^{\alpha}N^{c'a'}\bar{N}_{c'}{}^{b'}F_{\alpha a'b'}-\lambda^{\alpha}N^{ca}\bar{N}_{c}{}^{b'}F_{\alpha ab'}+\lambda^{\alpha}N_{c'}{}^{b'}\bar{N}^{c'a}F_{\alpha ab'}=0, \tag{A.15}$$
$$\bar{\lambda}^{\bar{\alpha}}N^{\underline{ab}}\bar{N}^{\underline{cd}}\nabla_{\bar{\alpha}}\mathcal{V}_{\underline{abcd}}+\frac{1}{2}\bar{\lambda}^{\bar{\alpha}}N^{[ab]}\bar{N}^{\underline{ab}}(\gamma_{[ab]})_{\bar{\alpha}}{}^{\bar{\beta}}\mathcal{V}_{\underline{ab}\bar{\beta}}-\bar{\lambda}^{\bar{\alpha}}N^{ca}\bar{N}_{c}{}^{b}F_{ab\bar{\alpha}}$$
$$+\bar{\lambda}^{\bar{\alpha}}N^{c'a'}\bar{N}_{c'}{}^{b'}F_{a'b'\bar{\alpha}}+\bar{\lambda}^{\bar{\alpha}}N^{c'a}\bar{N}_{c'}{}^{b'}F_{ab'\bar{\alpha}}+\bar{\lambda}^{\bar{\alpha}}N_{c}{}^{b'}\bar{N}^{ca}F_{\alpha ab'}=0. \tag{A.16}$$

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JHEP10(2017)178