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AGT, N-Burge partitions and $\mathcal{W}_N$ minimal models

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Abstract: Let $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ be a conformal block, with n consecutive channels χ_ι, $\iota = 1, \cdots, n$, in the conformal field theory $\mathcal{M}^{p,p'}_N \times \mathcal{M}^{\mathcal{H}}$, where $\mathcal{M}^{p,p'}_N$ is a $\mathcal{W}_N$ minimal model, generated by chiral spin-2, $\cdots$, spin-N currents, and labeled by two co-prime integers p and p', $1 < p < p'$, while $\mathcal{M}^{\mathcal{H}}$ is a free boson conformal field theory. $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ is the expectation value of vertex operators between an initial and a final state. Each vertex operator is labelled by a charge vector that lives in the weight lattice of the Lie algebra A_{N-1}, spanned by weight vectors $\vec{\omega}_1, \cdots, \vec{\omega}_{N-1}$. We restrict our attention to conformal blocks with vertex operators whose charge vectors point along $\vec{\omega}_1$. The charge vectors that label the initial and final states can point in any direction.

Following the $\mathcal{W}_N$ AGT correspondence, and using Nekrasov's instanton partition functions without modification to compute $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$, leads to ill-defined expressions. We show that restricting the states that flow in the channels χ_ι, $\iota = 1, \cdots, n$, to states labeled by N partitions that we call N-Burge partitions, that satisfy conditions that we call N-Burge conditions, leads to well-defined expressions that we propose to identify with $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$. We check our identification by showing that a non-trivial conformal block that we compute, using the N-Burge conditions satisfies the expected differential equation. Further, we check that the generating functions of triples of Young diagrams that obey 3-Burge conditions coincide with characters of degenerate $\mathcal{W}_3$ irreducible highest weight representations.

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1 Introduction

We propose a modification of the $\mathcal{W}_N$ AGT correspondence so that it applies to $\mathcal{W}_N$ minimal models, and use it compute $\mathcal{W}_N$ minimal model conformal blocks that are expectation values of vertex operators whose charge vectors are vectors in the A_{N-1} weight lattice, that point along the direction of the fundamental weight vector $\vec{\omega}_1$.

1.1 The AGT correspondence

The original AGT correspondence, or simply AGT, is the statement that the instanton partition functions of 4D linear and cyclic U(2) quiver $\mathcal{N}=2$ supersymmetric gauge theories are equal, up to a Heisenberg factor, to Virasoro conformal blocks on the sphere and on the torus, respectively, with non-minimal central charges [1]. It was conjectured by Alday, Gaiotto and Tachikawa in [1], proven in important special cases in [2–6], then proven in full generality by Alba, Fateev, Litvinov and Tarnopolskiy in [7].

1.2 The $\mathcal{W}_N$ AGT correspondence

The correspondence was extended to an identification, also up to a Heisenberg factor, of the instanton partition functions of 4D linear and cyclic U(N) quiver $\mathcal{N}=2$ supersymmetric gauge theories and conformal blocks in $\mathcal{W}_N$ conformal field theories[1] on the sphere and on the torus, with non-minimal central charges [9–11]. However, the latter identification is restricted to a class of $\mathcal{W}_N$ conformal blocks, with non-minimal central charges, characterised by a condition discussed by Fateev and Litvinov [12] and by Wyllard [9].

1.3 The condition of Fateev, Litvinov and Wyllard

Consider a $\mathcal{W}_N$ Toda conformal block that consists of n consecutive channels, that is, the expectation value of $(n+3)$ $\mathcal{W}_N$ vertex operators. Each vertex operator represents a $\mathcal{W}_N$ highest weight state that is labelled by an $(N-1)$-component charge-vector that lives in the A_{N-1} weight lattice spanned by the fundamental weight vectors $\{\vec{\omega}_1, \cdots, \vec{\omega}_{N-1}\}$.

[1] We take $\mathcal{W}_N$ to be the infinite-dimensional algebra generated by chiral spin-2, $\cdots$, spin-N currents, referred to as $\mathcal{W}(2, 3, \cdots, N)$ in [8].

The $\mathcal{W}_N$ AGT correspondence applies to n-channel $\mathcal{W}_N$ conformal blocks that involve $(n+3)$ vertex operators that consist of a $\mathcal{W}_N$ factor and a Heisenberg factor, such that the A_{N-1} charge-vector of two of these operators can point in any direction along the A_{N-1} weight lattice, while the charge vectors of the remaining $(n+1)$ operators are restricted to point along the same direction, for example $\vec{\omega}_1$, or directions that are related to $\vec{\omega}_1$ by Weyl reflections. In the sequel, we refer to this condition as the FLW condition.

1.4 $\mathcal{W}_N$ AGT in non-minimal $\mathcal{W}_N$ models

Applying the AGT prescription to $\mathcal{W}_N$ conformal blocks, that is, identifying U(N) instanton partition functions with conformal blocks, up to Heisenberg factors, makes perfect sense for conformal field theories with non-minimal central charges.

The instanton partition functions are in the form of sums over products of rational functions of the parameters of the theory, as in (3.1). Each sum corresponds to a gauge group in the quiver gauge theory. The terms in a sum are parameterised by the set of all possible N-partitions. There are no conditions on the partitions that we are allowed to sum over.

On the conformal field side of the AGT correspondence, each term in a sum corresponds to a state in a Verma module of the algebra $\mathcal{W}_N \times \mathcal{H}$, where $\mathcal{H}$ is the Heisenberg algebra. The fact that we sum over all possible N partitions corresponds to allowing all possible states that live in a $\mathcal{W}_N$ Verma module, times a Heisenberg module, to flow in the channels of the conformal block.

1.5 $\mathcal{W}_N$ AGT in minimal $\mathcal{W}_N$ models

Choosing the parameters of the instanton partition functions such that one obtains minimal models on the conformal field theory side of AGT leads to zeros in the denominators of the instanton partition functions. These singularities are non-physical and can be traced to the fact that by summing over all possible states in the $\mathcal{W}_N$ modules that flow through the channels of the conformal blocks, one allows for the flow of null states that should decouple.

One approach to remove these non-physical singularities is to enforce the fusion rules at the level of the instanton partition functions. This requires that we analytically continue the instanton partition functions, in a way that preserves the fusion rules, then show that to each zero in the denominator of a summand, there is a higher order zero in the numerator of the same factor, such that the corresponding null state decouples in the appropriate limit.

1.6 Restricting the Young diagrams

In this work, we choose to follow a different approach. That is, we characterise the sets of N partitions that lead to null states and exclude them from the sums. This is the approach that was followed in [13, 14] to obtain conformal blocks in Virasoro minimal models. In [13, 14], this procedure led to well-defined expressions. The proposal of [13, 14] is that these expressions can be identified with minimal model conformal blocks, up to Heisenberg factors, which was checked to be the case in a number of non-trivial examples.

The present work is an extension of the proposal of [13, 14] to U(N) instanton partition functions and $\mathcal{W}_N$ conformal blocks.

1.7 N-Burge partitions

Our goal is to provide AGT relations for $\mathcal{B}_{N,n}^{p,p',\mathcal{H}}$. The difficulty is that naively applying (3.1) to minimal $\mathcal{W}_N$ model, one gets singular expressions, as explained in detail in the context of $\mathcal{W}_2$ in [13, 14]. The origin of these singularities is related to the existence of zero-norm states in the $\mathcal{W}_N$ Verma modules with central charge (2.8) and associated to the vectors (2.15). Summation in (3.1) includes states in a Verma module rather than in an irrep of $\mathcal{W}_N^{p,p'} \times \mathcal{H}$ and therefore containing the contribution of zero-norm states. These states should be removed when computing minimal model conformal blocks.

In this work, as in [13, 14], we avoid these zeros by restricting the summations over the N-partitions that appear in the sum (3.1). We provide an expression of $\mathcal{B}_{N,n}^{p,p',\mathcal{H}}$ in terms of a sum of the type (3.1) that consists of products of factors Z_{bb}^{ι}. Each Z_{bb}^{ι} is an expectation value of an $\mathcal{W}_N \times \mathcal{H}$ vertex operator $\mathcal{O}_\iota$, characterised by a charge vector $\vec{a}_{m_\iota n_\iota}$. This charge vector lives in the A_2 weight lattice and points in the direction of the fundamental weight $\vec{\omega}_1$. The expectation value of $\mathcal{O}_\iota$ is computed between $\mathcal{W}_N \times \mathcal{H}$ basis states, an in-state labeled by a charge vector $\vec{P}_{\vec{r}_{\iota-1}\ \vec{s}_{\iota-1}}$, and N partitions $\vec{Y}^{\iota-1}$, and an out-state labeled by a charge vector $\vec{P}_{\vec{r}_\iota\ \vec{s}_\iota}$, and N partitions $\vec{Y}^{\iota}$.

The charge vectors $\vec{P}_{\vec{r}_{\iota-1}\ \vec{s}_{\iota-1}}$ and $\vec{P}_{\vec{r}_\iota\ \vec{s}_\iota}$ are chosen such that degenerate $\mathcal{W}_N \times \mathcal{H}$ highest weight modules flow in the intermediate channels. Given this choice, Z_{bb}^{ι} which is a rational function of its parameters, can have zeros in the denominators, leading to ill-defined expressions. We characterize the singularities in Z_{bb}^{ι} that lead to ill-defined expressions, and attribute these singularities to zero-norm states that should not be allowed to propagate in the channels of the minimal model conformal blocks. We eliminate these zero-norm states by restricting the N-partitions that label the states that flow in the ι-th channel in a minimal model conformal block, and that appear in (3.1) to N-partitions $\vec{Y} = \{Y_1, \cdots, Y_N\}$, that satisfy the conditions

$$Y_{i,\mathtt{R}} - Y_{i+1,\mathtt{R}+s_i-1} \geqslant -r_i + 1 \tag{1.1}$$

where $Y_{i,\mathtt{R}}$ is the $\mathtt{R}$-row of Y_i, $i = 1, \cdots, N$, r_i and s_i, $i = 1, \cdots, N$, $\sum_{i=1}^{N} r_i = p$, $\sum_{i=1}^{N} s_i = p'$, are parameters that characterise the $\mathcal{W}_N$ irreducible highest weight module that flows in the ι-th channel under consideration, and $Y_{N+1} = Y_1$.[2]

These restricted *N-Burge partitions* were introduced, in case $N = 2$, in [15], further studied in [16] and appeared in full generality in [17, 18]. We show that when used to restrict AGT to compute $\mathcal{B}_{N,n}^{p,p',\mathcal{H}}$, that is when we sum over N-Burge partitions rather than on all possible N-partitions, we obtain

$$\mathcal{B}_{N,n}^{p,p',\mathcal{H}} = \sum_{\vec{Y}^1,\cdots,\vec{Y}^n}' \prod_{\iota=1}^{n+1} q_\iota^{|\vec{Y}^\iota|} Z_{bb}^{\iota} \left(\vec{P}_{\vec{r}_{\iota-1}\ \vec{s}_{\iota-1}}, \vec{Y}^{\iota-1} \mid \vec{a}_{m_\iota n_\iota} \mid \vec{P}_{\vec{r}_\iota\ \vec{s}_\iota}, \vec{Y}^{\iota} \right) \tag{1.2}$$

where $\sum'$ indicates that the sum is restricted to N-partitions that satisfy the N-Burge conditions (1.1), we obtain well-defined expressions. Brief explanations of the meaning of

[2]For explicit examples of partitions that satisfy N-Burge conditions, see subsection 5.5.

the various terms in equation (1.2) were given in earlier paragraphs. More details definitions can be found in section 2.

1.8 Outline of contents

In section 2, we recall the basics of $\mathcal{W}_N$ algebras and conformal field theories, and in 3, we do the same for the original, unmodified $\mathcal{W}_N$ AGT correspondence. In 4, we discuss the restrictions that we need to impose on the N-partitions that are summed over in Nekrasov's instanton partition functions in order to compute conformal blocks in minimal $\mathcal{W}_N$ models. In 5, we recall basic facts related to the $\mathcal{W}_3$ minimal models, then we check the N-Burge conditions obtained in 4, in the context of $\mathcal{W}_3$ minimal models, by considering conformal blocks that satisfy differential equations. In 6, we discuss the derivation of the characters of $\mathcal{W}_3$ minimal models from the 3-Burge partitions. Finally section 7 contains a number of remarks.

2 $\mathcal{W}_N$ algebras and conformal field theories

We recall basic definitions related to $\mathcal{W}_N$ algebras, with focus on $\mathcal{W}_3$, followed by basic definitions related to $\mathcal{W}_N$ conformal field theories, with focus on $\mathcal{W}_3$ minimal models.

2.1 $\mathcal{W}_N$ algebras

The Virasoro algebra is generated by the Laurent components of the holomorphic part of stress-energy tensor $T(z)$ which is a spin-2 chiral field [19]. The $\mathcal{W}$ algebras are extensions of the Virasoro algebra, generated by higher-spin chiral fields. For a comprehensive review, see [8]. In this work, we use $\mathcal{W}_N$ to indicate the infinite-dimensional algebra generated by chiral fields of spin $2, 3, \cdots, N$, referred to as $\mathcal{W}(2, 3, \cdots, N)$ in [8].

2.1.1 The $\mathcal{W}_3$ algebra

The $\mathcal{W}_N$ algebra, for large N, is a complicated object. To be concrete, we choose to work in terms of examples from $\mathcal{W}_3$, which is the simplest $\mathcal{W}_N$ algebra beyond Virasoro, and that has the basic features of higher N $\mathcal{W}_N$ algebras, particularly the fact that it is not a Lie algebra.

$\mathcal{W}_3$ is generated by the Laurent components of the chiral spin-2 stress-energy tensor $T(z)$, L_n, $n \in \mathbb{Z}$, and the Laurent components of a chiral spin-3 current $\mathcal{W}(z)$, W_n, $n \in \mathbb{Z}$ [20]. Following the notation and conventions of [21], the defining relations of the $\mathcal{W}_3$ algebra are

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, W_n] &= (2m-n)\, W_{m+n}, \\
[W_m, W_n] &= (m-n)\left(\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right) L_{m+n} \\
&\quad + \left(\frac{c}{360}\right) m(m^2-4)(m^2-1)\delta_{m+n,0} + \beta\,(m-n)\,\Lambda_{m+n},
\end{aligned} \tag{2.1}$$

where

$$\beta = \frac{16}{22+5c}, \quad \Lambda_m = \sum_{p\geqslant -1} L_{m-p}\, L_p + \sum_{p\leqslant -2} L_p\, L_{m-p} - \frac{3}{10}(m+2)(m+3)L_m, \tag{2.2}$$

and c is the Virasoro central charge.

2.1.2 $\mathcal{W}_3$ heighest weight states and eigenvalues

Consider a vector $\vec{P}$ in the weight lattice of the Lie algebra A_2, spanned by the fundamental weight vectors $\vec{\omega}_1$ and $\vec{\omega}_2$,

$$\vec{P} = P_1\,\vec{\omega}_1 + P_2\,\vec{\omega}_2, \tag{2.3}$$

where $P_1, P_2 \in \mathbb{R}$. The $\mathcal{W}_3$ highest weight state $|\vec{P}\,\rangle$, associated to $\vec{P}$, is defined by

$$L_0|\vec{P}\rangle = \Delta_{\vec{P}}\,|\vec{P}\rangle, \quad W_0|\vec{P}\rangle = w_{\vec{P}}\,|\vec{P}\rangle \tag{2.4}$$

The eigenvalues $\Delta_{\vec{P}}$ and $w_{\vec{P}}$ are given in terms of $\vec{P}$ by

$$\Delta_{\vec{P}} = \frac{c-2}{12} + \frac{1}{2}\vec{P}^2, \quad w_{\vec{P}} = \sqrt{-3\beta}\,\prod_{i=1}^{3}\langle \vec{P}\,|\,\vec{h}_i\rangle, \tag{2.5}$$

where $\vec{h}_i$, $i=1,2,3$, are the weight vectors of the first fundamental representation of the Lie algebra A_2, and $\langle \vec{P}\,|\,\vec{h}_i\rangle$ is a scalar product on the A_2 weight lattice. The vertex operator $\mathcal{O}_{\vec{P}}$ is associated to the $\mathcal{W}_3$ highest weight state $|\vec{P}\rangle$.

Another standard parametrisation of the highest weight vectors, and the corresponding $\mathcal{W}_3$ vertex operators, is in terms of the vector charge $\vec{a}_{\vec{P}}$,

$$\vec{a}_{\vec{P}} = \left(b+b^{-1}\right)\,\left(\vec{\omega}_1+\vec{\omega}_2\right) + \vec{P} \tag{2.6}$$

In the above Toda-like notation, b parametrizes the central charge

$$c = 2 + 24\,\left(b+b^{-1}\right)^2 \tag{2.7}$$

2.1.3 Higher rank $\mathcal{W}_N$ algebras

The $\mathcal{W}_4$, or $\mathcal{W}(2,3,4)$ algebra generated by spin 2, 3, and 4 chiral fields is defined in [22, 23]. The definition of the algebras for higher N is involved. We refer to [8] for a complete discussion.

2.2 $\mathcal{W}_N$ conformal field theories

We take $\mathcal{M}_N \times \mathcal{M}^{\mathcal{H}}$ to be the 2-dimensional conformal field theory that consists of a factor $\mathcal{M}_N$ based on the infinite-dimensional $\mathcal{W}_N$ algebra, and a factor $\mathcal{M}^{\mathcal{H}}$ based on the Heisenberg algebra $\mathcal{H}$. In this work, we focus on minimal models, and write $\mathcal{M}_N = \mathcal{M}_N^{p,p'}$.

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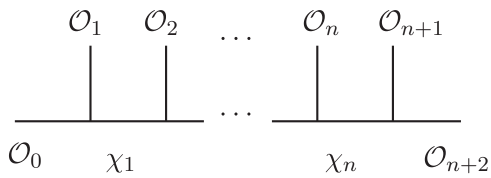

Figure 1. The comb diagram of an n-channel linear conformal block. The initial and final states, $\mathcal{O}_0$ and $\mathcal{O}_{n+2}$, the vertex operators $\mathcal{O}_i$, $i = 1, \cdots, n+1$, and the $\mathcal{W}_N$ irreducible highest weight representation that flows in the ι-th channel χ_ι, $\iota = 1, \cdots, n$, are defined in the text.

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2.2.1 The labels of the minimal $\mathcal{W}_N$ models, the background charge parameter and the screening charge parameters

A minimal $\mathcal{W}_N$ model, $\mathcal{M}_N^{p,p'}$, is labeled by two coprime integers, p and p', $1 < p < p'$. More precisely, the central charge $c_N^{p,\,p'}$, of $\mathcal{M}_N^{p,p'}$, is

$$c_N^{p,\,p'} = (N-1)\left(1 - N(N+1)\,\alpha_0^2\right), \tag{2.8}$$

where α_0 is the background charge parameter

$$\alpha_0 = \alpha_+ + \alpha_-, \tag{2.9}$$

and α_+ and α_- are the screening charge parameters

$$\alpha_+ = \left(\frac{p'}{p}\right)^{\frac{1}{2}}, \quad \alpha_- = -\left(\frac{p}{p'}\right)^{\frac{1}{2}} \tag{2.10}$$

2.2.2 Remarks

1. The definition (2.9) of the background charge parameter α_0, as well as the definition (2.15) of the A_{N-1} charge vector, below, are appropriate for minimal models based on $\mathcal{W}_N$ algebras, such that $N \geqslant 3$, but differ from the definitions used in minimal models based on the Virasoro algebra $\mathcal{W}_2$. In this work, we focus on $\mathcal{W}_N$ minimal models, such that $N \geqslant 3$. *2.* The $\mathcal{W}_N$ AGT correspondence is discussed in [9, 10] in the context of non-minimal $\mathcal{W}_N$ models. These models are not labeled by coprime integers, and their central charges do not satisfy (2.8).

2.2.3 Minimal $\mathcal{W}_N$ conformal blocks $\mathcal{B}_{N,n}^{p,p'}$

We are interested in computing linear conformal blocks $\mathcal{B}_N^{p,p'}$ in $\mathcal{W}_N$ minimal models $\mathcal{M}_N^{p,p'}$ that can be represented schematically as in figure 1.

An n-channel conformal block $\mathcal{B}_{N,n}^{p,p'}$ is an expectation value of $(n+2)$ $\mathcal{W}_N$ vertex operators, $\mathcal{O}_i$, $i = 1, \cdots, n+1$, inserted at different points on a Riemann surface, which in this work we take to be a Riemann sphere,[3] between an initial state and a final state, such

[3]Except in section 6, when we discuss the characters of degenerate $\mathcal{W}_3$ irreducible highest weight representations, which are essentially 0-point conformal blocks on the torus.

that the states that flow in channel χ_ι, $\iota = 1, \cdots, n$, between two consecutive insertions, belong to one and only one $\mathcal{W}_N$ irreducible highest weight representation, $\mathcal{H}_\iota$. In the following, we specify the parameters that label the vertex operators $\mathcal{O}_i$ and the highest weight representations χ_ι.

2.2.4 Labels of vertex operators

A vertex operator $\mathcal{O}_i$ of $\mathcal{M}_N^{p,p'}$ is labelled by two sets of non-zero positive integers $\vec{r} = \{r_1, \cdots, r_{N-1}\}$ and $\vec{s} = \{s_1, \cdots, s_{N-1}\}$, that satisfy

$$1 \leqslant \left(\sum_{i=1}^{N-1} r_i \right) \leqslant p, \quad 1 \leqslant \left(\sum_{i=1}^{N-1} s_i \right) \leqslant p' \tag{2.11}$$

It is useful to define two more non-zero positive integers r_N and s_N, such that

$$\sum_{i=1}^{N} r_i = p, \quad \sum_{i=1}^{N} s_i = p' \tag{2.12}$$

2.2.5 Charge vectors of vertex operators

The vertex operators $\mathcal{O}_i$, $i = 1, \cdots, n+1$, are represented by external vertical line segments in figure 1. Each $\mathcal{O}_i(z_i)$ is labelled by a vector charge $a_{\vec{r}_i\, \vec{s}_i}$, that has $(N-1)$ components, parameterised in terms of the screening charge parameters as

$$\vec{a}_{\vec{r}_i\, \vec{s}_i} = \sum_{i=1}^{N} \left(\left(1 - r_i \right) \alpha_+ + \left(1 - s_i \right) \alpha_- \right) \vec{\omega}_i, \tag{2.13}$$

where the parameters $\vec{r}$ and $\vec{s}$ were discussed in paragraph 2.2.4.[4] We include these details by writing $\mathcal{O}_i$ as $\mathcal{O}_{\vec{r}_i\, \vec{s}_i}(z_i)$. The initial and final states correspond to the vertex operators $\mathcal{O}_0(z_0)$ and $\mathcal{O}_{n+2}(z_{n+2})$. In this work, we take the charge vectors of $\mathcal{O}_0(z_0)$ and $\mathcal{O}_{n+2}(z_{n+2})$ to be arbitrary, the charge vectors of all remaining vertex operators to satisfy the FLW condition, and we write

$$\vec{a}_{\vec{r}_i\, \vec{s}_i} = a_{r_1 s_1} \vec{\omega}_1 = \left(\left(1 - r_1 \right) \alpha_+ + \left(1 - s_1 \right) \alpha_- \right) \vec{\omega}_1, \quad i = 1, \cdots, n+1 \tag{2.14}$$

2.2.6 Charges of the highest weight states that flow in the channels

The channels χ_ι, $\iota = 1, \cdots, n$, are represented by internal line segments in figure 1. In $\mathcal{W}_N$ minimal models, each channel χ_ι carries states that belong to a degenerate $\mathcal{W}_N$ irreducible highest weight representation $\mathcal{H}^{p,p'}_{\vec{r}_\iota\, \vec{s}_\iota}$. Each of these representations consists of a highest weight state and infinitely-many descendents. The highest weight state of $\mathcal{H}^{p,p'}_{\vec{r}_\iota\, \vec{s}_\iota}$ is labelled by a charge that flows between a vertex operator at z_i and a vertex operator at z_{i+1}, $i = 1, \cdots, n$.

[4]The indices i in $\vec{r}_i$ and $\vec{s}_i$ on the left hand side of (2.13) refer to the corresponding vertex operator and are not summed over. The indices i in r_i, s_i and $\vec{\omega}_i$ on the right hand side refer to the fundamental weight vectors of the A_{N-1} and are summed over. The integers r_i and s_i on the right hand side are the components of the vectors $\vec{r}_i$ and $\vec{s}_i$ on the left hand side, respectively.

2.2.7 The charge vectors

The A_{N-1} charge vector $\vec{P}_{\vec{r}\,\vec{s}}$, of a degenerate $\mathcal{W}_N$ irreducible highest weight representation $\mathcal{H}^{p,p'}_{\vec{r}\,\vec{s}}$, is defined as

$$\vec{P}_{\vec{r}\,\vec{s}} = -\sum_{i=1}^{N-1} \Big(r_i \alpha_+ + s_i \alpha_- \Big)\, \vec{\omega}_i, \tag{2.15}$$

where $\vec{\omega}_i$, $i = 1, \cdots, N$, are the A_{N-1} fundamental weight vectors.

2.2.8 The conformal dimensions

The conformal dimension $\Delta_{\vec{r}\,\vec{s}}$ of the vertex operator $\mathcal{O}_{\vec{r}\,\vec{s}}$ that carries a charge vector $\vec{P}_{\vec{r}\,\vec{s}}$ is

$$\Delta_{\vec{r}\,\vec{s}} = \frac{1}{2} \left(\vec{P}_{\vec{r}\,\vec{s}} + \alpha_0\, \vec{\rho} \right) \cdot \left(\vec{P}_{\vec{r}\,\vec{s}} - \alpha_0\, \vec{\rho} \right) = \frac{1}{2} \left(\vec{P}^2_{\vec{r}\,\vec{s}} - \alpha_0^2\, \vec{\rho}^2 \right) \tag{2.16}$$

where the product in the middle term of equation (2.16) is a scalar product of two vectors, $\vec{P}^2$ and $\vec{\rho}^2$ are the squares of the norms of the charge vector $\vec{P}$ and the Weyl vector $\vec{\rho}$,

$$\vec{\rho} = \sum_{i=0}^{n-1} \vec{\omega}_i \tag{2.17}$$

2.2.9 The degenerate irreducible highest weight representations of $\mathcal{M}^{p,p'}_N$

These are obtained from the corresponding Verma modules by factoring out the submodules that consist of zero-norm states and their descendants. It can be shown that in the representation associated to $\mathcal{O}_{\vec{r}\,\vec{s}}$ there are $(N-1)$ zero-norm states of conformal dimensions $\Delta_{\vec{r}\,\vec{s}} + r_i s_i$, $i = 1, \cdots, N-1$.

Following [1, 9], we introduce an auxiliary free boson theory $\mathcal{M}^{\mathcal{H}}$, compute conformal blocks $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ in $\mathcal{M}^{p,p'}_N \times \mathcal{M}^{\mathcal{H}}$, then factor out the Heisenberg contribution $\mathcal{M}^{\mathcal{H}}$, which is known. Before we do that, we need to recall basic definitions related to $\mathcal{B}^{p,p'}_{N,n}$.

2.2.10 $\mathcal{M}^{p,p'}_N$ vertex operators

A vertex operator $\mathcal{O}_{\vec{r}\,\vec{s}}$, in $\mathcal{M}^{p,p'}_N$, located at z on the Riemann sphere, is represented, at operator level, as a vertex operator. In the Coulomb gas representation of $\mathcal{W}_N$ minimal models, a vertex operator is represented as an exponential of $(N-1)$ free bosons, $\phi_i, i = 1, \cdots, N-1$, that live in the A_{N-1} root lattice,

$$\mathcal{O}_{\vec{r},\vec{s}}(z) = e^{i\,\vec{a}\,\cdot\,\vec{\phi}(z)}, \quad \vec{a} = \sum_{i=1}^{N-1} a_i\, \vec{\omega}_i, \quad \vec{\phi}(z) = \sum_{i=1}^{N-1} \phi_i(z)\, \vec{\alpha}_i, \tag{2.18}$$

where $\vec{\alpha}_i$ and $\vec{\omega}_i$, $i = 1, \cdots, N-1$, are the fundamental root and weight vectors of A_{N-1}. In this work, we focus on vertex operators that satisfy the FLW condition discussed in subsection 1.3, that is

$$\vec{a}_{\,FLW} = a_1 \vec{\omega}_1 \tag{2.19}$$

2.2.11 $\mathcal{M}^{\mathcal{H}}$ vertex operators

As we will see in the sequel, a modification of the $\mathcal{W}_N$ AGT prescription, obtained by restricting the Young diagrams that we sum over, will provide us with well-defined expressions that we identify with $\mathcal{W}_N$ conformal blocks that satisfy the FLW condition, in $\mathcal{M}_N^{p,p'} \times \mathcal{M}^{\mathcal{H}}$ conformal field theories. These conformal blocks are expectation values of holomorphic vertex operators that consist of two factors. One factor belongs to $\mathcal{M}_N^{p,p'}$ and was discussed in paragraph 2.2.10. The other factor belongs to $\mathcal{M}^{\mathcal{H}}$ and has, at this stage, the form[5]

$$\mathcal{O}^{\mathcal{H}}(z) = e^{i\left(\alpha_0-\alpha_N\right)\phi_N^+(z)} e^{i\alpha_N\phi_N^-(z)}, \tag{2.20}$$

where ϕ_N^+ and ϕ_N^- are the positive frequency and negative frequency components of the holomorphic factor of an N-th, independent free boson ϕ_N, and the charges of the exponentials of these components are chosen to different, as in equation (2.20), where α_0 is the background charge parameter. This vertex operator, in which no zero-mode appears, first appeared in [24] and was studied further in [7].

2.2.12 $\mathcal{M}_N^{p,p'} \times \mathcal{M}^{\mathcal{H}}$ conformal blocks

A conformal block is an expectation value of holomorphic vertex operators. We use $\mathcal{B}_{N,n}^{p,p'}$, $\mathcal{B}_n^{\mathcal{H}}$ and $\mathcal{B}_{N,n}^{p,p',\mathcal{H}}$, for a linear conformal block, with n consecutive channels in $\mathcal{M}_N^{p,p'}$, $\mathcal{M}^{\mathcal{H}}$, and $\mathcal{M}_N^{p,p'} \times \mathcal{M}^{\mathcal{H}}$, respectively. Only conformal blocks that live on the Riemann sphere, with n consecutive channels, as in figure 1, are considered in this work. The extension to cyclic conformal blocks on the torus is straightforward by allowing the initial and final states to be descendants, identifying them, then summing over all possible descendants.

Our notation is such that an n-channel conformal block $\mathcal{B}_{N,n}^{\text{indices}}$, is the expectation value of $(n+3)$ chiral vertex operators $\mathcal{O}_\iota^{\text{same indices}}(z_\iota)$, $\iota = 0, \cdots, (n+2)$, in $\mathcal{M}^{\text{same indices}}$ and z_ι are the coordinates of the vertex insertions.

We wish to compute $\mathcal{B}_{N,n}^{p,p',\mathcal{H}}$. In this case, each vertex operator is a product of two vertex operators, $\mathcal{O}_\iota^{p,p'}(z_\iota) \times \mathcal{O}^{\mathcal{H}}(z_\iota)$, where $\mathcal{O}_\iota^{p,p'}(z_\iota)$ is in $\mathcal{M}_N^{p,p'}$, $\mathcal{O}^{\mathcal{H}}(z_\iota)$ is in $\mathcal{M}^{\mathcal{H}}$, and the charge of $\mathcal{O}^{\mathcal{H}}(z_\iota)$ is completely determined by that of $\mathcal{O}_\iota^{p,p'}(z_\iota)$, by setting

$$\alpha_N = \alpha_1, \tag{2.21}$$

where α_1 is the charge parameter of $\mathcal{O}_\iota^{p,p'}(z_\iota)$, as in equation (2.18), which satisfies the FLW condition as in equation (2.19), and α_N is the charge parameter of $\mathcal{O}^{\mathcal{H}}(z_\iota)$, as in equation (2.20).

A holomorphic linear conformal block that consists of n consecutive channels is the expectation value of $(n+3)$ holomorphic vertex operators at positions $z_i, i = 0, \cdots, n+3$,

$$\mathcal{B}_{N,n}^{p,p'} = \langle\, \mathcal{O}_{\vec{r}_0\, \vec{s}_0}(z_0)\, \mathcal{O}_{r_1\, s_1}(z_1) \cdots \mathcal{O}_{r_{n+1}\, s_{n+1}}(z_{n+1})\, \mathcal{O}_{\vec{r}_{n+2}\, \vec{s}_{n+2}}(z_{n+2})\,\rangle, \tag{2.22}$$

[5] As we will see shortly, in the case that we are interested in, the charge α_N of the $\mathcal{M}^{\mathcal{H}}$ vertex operator is completely fixed by the charge of the $\mathcal{M}_N^{p,p'}$ vertex operator that will multiply it, at the same point on the Riemann sphere.

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where the vertex operators $\mathcal{O}_{r_i,s_i}$, $i=1,\cdots,n+1$, are specified below. When the positions z_i are generic, global conformal invariance on the sphere can be used to set $z_0=0$, $z_{n+1}=1$, $z_{n+2}=\infty$, then to scale the positions of the remaining points such that[6]

$$q_\iota = \frac{|z_\iota|}{|z_{\iota+1}|} < 1, \quad \iota = 1,\cdots,n \tag{2.23}$$

2.2.13 The parameters that appear in $\mathcal{B}^{p,p'}_{N,n}$

We can summarise the above discussion as follows. $\mathcal{B}^{p,p'}_{N,n}$ depends on three sets of parameters,[7]

$$\mathcal{B}^{p,p'}_{N,n} = \mathcal{B}^{p,p'}_{N,n}\left(q_1,\cdots,q_n \,|\, \vec{P}_{\vec{r}_0\,\vec{s}_0},\cdots,\vec{P}_{\vec{r}_{n+2}\,\vec{s}_{n+2}} \,|\, \vec{a}_{r_1\,s_1},\cdots,\vec{a}_{r_{n+1}\,s_{n+1}}\right) \tag{2.24}$$

The parameters $\{q_1,\cdots,q_n\}$ are the ratios of consecutive positions defined in (2.23). The charge vectors $\vec{P}_{\vec{r}_0\,\vec{s}_0}$ and $\vec{P}_{\vec{r}_{n+2}\,\vec{s}_{n+2}}$ label the $\mathcal{W}_N$ initial and final states. They do not need to satisfy the FLW condition. The charge vectors $\vec{P}_{\vec{r}_\iota\,\vec{s}_\iota}$, $\iota=1,\cdots,n$, label the highest weight states of the $\mathcal{W}_N$ irreducible highest weight representations that flow in the ι-th channel. The charges $\vec{a}_{r_1\,s_1},\cdots,\vec{a}_{r_{n+1}\,s_{n+1}}$ label the vertex operator insertions. They need to satisfy the FLW condition, so as vectors on the A_2 weight lattice, they point in th edirection of the fundamental weight $\vec{\omega}_1$ only.

2.2.14 The Heisenberg factor

The conformal block $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$, which includes the contribution of the Heisenberg algebra, depends on the same parameters as $\mathcal{B}^{p,p'}_{N,n}$, in (2.24). The two expressions are related by

$$\mathcal{B}^{p,p',\mathcal{H}}_{N,n} = \prod_{\iota=1}^{n}\prod_{l=\iota}^{n}\left(1-q_\iota\cdots q_l\right)^{\frac{a_{\iota+1}\left(\alpha_0-a_{l+2}\right)}{N}}\,\mathcal{B}^{p,p'}_{N,n}, \tag{2.25}$$

where the variables q_ι, $\iota=1,\cdots,n$, were defined in (2.23), and a_ι is an abbreviation of $a_{r_\iota s_\iota}$. The factor that multiplies $\mathcal{B}^{p,p'}_{N,n}$, on the right hand side of (2.25), is the Heisenberg factor. It follows directly from the contribution of the Heisenberg vertex operators in (2.20) to the expectation value in (2.22).

3 $\mathcal{W}_N$ AGT correspondence

We recall basic definitions related to partitions, then discuss the $\mathcal{W}_N$ AGT correspondence.

3.1 Partitions

A partition π of an integer $|\pi|$ is a set of non-negative integers $\{\pi_1,\cdots,\pi_p\}$, where p is the number of parts, $\pi_i \geqslant \pi_{i+1}$, and $\sum_{i=1}^p \pi_i = |\pi|$. π is represented by a Young diagram Y, which is a set of p rows $\{Y_1,\cdots,Y_p\}$, such that the $\mathtt{R}$-th row has $Y_\mathtt{R}=\pi_\mathtt{R}$ cells, $Y_\mathtt{R}\geqslant Y_{\mathtt{R}+1}$, and $|Y|=\sum_\mathtt{R} Y_\mathtt{R}=|\pi|$. We use $Y_\mathtt{R}$ for the $\mathtt{R}$-th row as well as for the number of cells in that row. $Y^\intercal$ is the transpose of Y.

[6] We take z_1 to be closest to $z_0=0$, followed by z_2, etc. and z_{n+2} to be farthest.

[7] As pointed out earlier, the subscript i for $\vec{r}_i$ and $\vec{s}_i$ is the position of the corresponding operator insertion, and should not be confused with a component of the vector r_i and s_i.

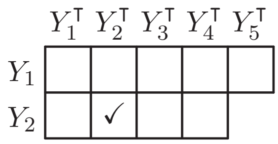

Figure 2. The 2-row Young diagram Y of the partition $5+4$. The rows are numbered from top to bottom. The 5-row transpose Young diagram $Y^{\intercal}$ of the partition $2+2+2+2+1$, which are the columns of Y, are numbered from left to right. From the viewpoint of Y, the marked cell ☑ has $A_{☑,Y} = 2$, $A^{+}_{☑,Y} = 3$, and $L_{☑,Y} = 0$. From the viewpoint of $Y^{\intercal}$, ☑ has $A_{☑,Y^{\intercal}} = 0$, $A^{+}_{☑,Y^{\intercal}} = 1$, and $L_{☑,Y^{\intercal}} = 2$.

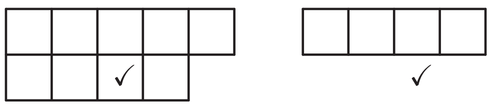

Figure 3. A 2-partition $\{Y_1, Y_2\}$. Y_1 is on the left, Y_2 is on the right. The cell ☑ has coordinates $(2,3)$, $A_{☑,Y_1} = 1$, $A^{+}_{☑,Y_1} = 2$, $L_{☑,Y_1} = 0$, $A_{☑,Y_2} = -3$, $A^{+}_{☑,Y_2} = -2$, and $L_{☑,Y^1} = -1$.

3.1.1 Cells and coordinates

We use $\square$ for a cell in a Young diagram Y, which is a square in the south-east quadrant of the plane, with coordinates $\{\mathtt{R}, \mathtt{C}\}$, such that $\mathtt{R}$ is the row-number, counted from top to bottom, and $\mathtt{C}$ is the column number, counted from left to right.

3.1.2 Arms and legs

$A_{\square,Y_i}$ is the arm of $\square$ in Y_i, that is, the number of cells in the same row as, but to the right of $\square$ in Y_i, and $L_{\square,W_j}$ to be the leg of $\square$ with respect its position in W_j, that is the number of cells in the same column as, but below $\square$ in Y_i. We define $A^{+}_{\square,Y_i} = A_{\square,Y_i} + 1$.

3.1.3 *N*-partitions

The AGT representation of $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ involves a multi-sum over $(n+2)$ N-partitions $\vec{Y}^{\iota}$, $\iota = 0, \cdots, n+1$, where $\vec{Y}^{\iota}$ is a set of of N Young diagrams, $\{Y_1^{\iota}, \cdots, Y_N^{\iota}\}$, and $|\vec{Y}^{\iota}| = |Y_1^{\iota}| + \cdots + |Y_N^{\iota}|$ is the total number of cells in $\vec{Y}^{\iota}$. The N-partitions $\{Y_1^{\iota}, \cdots, Y_N^{\iota}\}$, $\iota \in 1, \cdots, n$, are non-empty Young diagrams, while $\{Y_1^{\iota}, \cdots, Y_N^{\iota}\}$, $\iota = 0, n+1$ are empty,[8] $\vec{Y}^{(0)} = \vec{Y}^{(n+1)} = \vec{\emptyset}$, where $\vec{\emptyset}$ is an N-partition that consists of N empty Young diagrams.

[8] We work in terms of $(n+2)$ linearly-ordered N-partitions. Since we consider conformal blocks of vertex operators, the initial and final N-partitions are always empty, but we prefer to work in terms of $(n+2)$ rather than n non-empty N-partitions to make the notation in the sequel more uniform.

3.2 Extending AGT to $\mathcal{W}_N$

The AGT correspondence of Alday, Gaiotto and Tachikawa [1], extended to $\mathcal{W}_N \oplus \mathcal{H}$ by Mironov and Morozov [10] and Wyllard [9], identifies a class of conformal blocks in $\mathcal{M}_N^{\text{non.min},\mathcal{H}}$, that we specify below, with instanton partition functions in 4-dimensional $\mathcal{N}=2$ supersymmetric quiver U(N) gauge theories [25].

3.2.1 AGT in non-minimal $\mathcal{W}_N$ models

The $\mathcal{W}_N$ AGT expression for a conformal block $\mathcal{B}_{N,n}^{\text{non.min},\mathcal{H}}$, that has n consecutive channels χ_ι, $\iota = 1, \cdots, n$, is an n-fold sum,[9]

$$\mathcal{B}_{N,n}^{\text{non.min},\mathcal{H}} = \sum_{\vec{Y}^1,\cdots,\vec{Y}^n} \prod_{\iota=1}^{n+1} q_\iota^{|\vec{Y}^\iota|} Z_{bb}^\iota \left(\vec{P}_{(\iota-1)}, \vec{Y}^{\iota-1} \mid a_\iota \mid \vec{P}_{(\iota)}, \vec{Y}^\iota \right), \tag{3.1}$$

the factors $q_\iota^{|\vec{Y}^\iota|} Z_{bb}^\iota[\vec{P}^{(\iota-1)}, \vec{Y}^{\iota-1} \mid \mu^\iota \mid \vec{P}^{(\iota)}, \vec{Y}^\iota]$, $\iota = 1, \cdots, n+1$, are defined in subsection 3.2.2. Each factor Z_{bb}^ι is a rational function that depends on two N-partitions of *'unrestricted'* Young diagrams $\{Y_1^{\iota-1}, \cdots, Y_N^{\iota-1}\}$ and $\{Y_1^\iota, \cdots, Y_N^\iota\}$. In other words, there are no conditions on these Young diagrams and all possible N-partitions are allowed. The denominator z_{den}^ι of Z_{bb}^ι is a product of the norms of the states that flow in the preceding channel $\chi^{\iota-1}$ and the subsequent channel χ^ι. Since Z_{bb}^ι is labeled by unrestricted N-partitions, and the sums are over all possible unrestricted N-partitions, the states that flow in each channel belong to a Verma module of $\mathcal{W}_N^{\text{non.min},\mathcal{H}}$.

3.2.2 AGT for non-minimal $\mathcal{W}_N$

The decomposition of conformal blocks in (3.1) follows that in [26] and is represented as a *comb diagram* in figure 1. The function Z_{bb} is

$$Z_{bb}\left(\vec{a}, \vec{Y} \mid \mu \mid \vec{b}, \vec{W}\right) = \frac{z_{\text{num}}\left(\vec{a}, \vec{Y} \mid \mu \mid \vec{b}, \vec{W}\right)}{z_{\text{den}}\left(\vec{a}, \vec{Y} \mid \vec{b}, \vec{W}\right)}, \tag{3.2}$$

and has the following ingredients. N-component vector $\vec{a}^\iota = \{a_1^\iota, \cdots, a_N^\iota\}$, such that $\sum_{i=1}^N a_i^\iota = 0$, is the charge of the highest weight state of the $\mathcal{W}_N$ irrep that flows in the intermediate channel χ_ι. Each of the two N-partition sets $\vec{V}^\iota = \{V_1^\iota, \cdots, V_N^\iota\}$, and $\vec{W}^\iota = \{W_1^\iota, \cdots, W_N^\iota\}$, labels the elements of the special orthogonal basis in the $\mathcal{M}_N \times \mathcal{M}^{\mathcal{H}}$ Verma module associated with the vertex operator in channel χ_ι. In equation 3.1, $\vec{Y}$ and $\vec{W}$ are attached to the line segments on the left and the right of a given vertex, respectively, see figure 1. The scalar μ^ι is the charge of the vertex operator that connects channels χ_ι and $\chi_{\iota+1}$. In the following, we study the structure of the right hand side of (3.2).

[9] Recall that the N-partitions $\vec{Y}^0$ and $\vec{Y}^{n+1}$ are empty.

3.2.3 The numerator

$$z_{\text{num}}\left(\vec{a},\vec{Y}\mid\mu\mid\vec{b},\vec{W}\right)$$
$$=\prod_{i,j=1}^{N}\prod_{\square\in Y_i}\Big(E[a_i-b_j,Y_i,W_j,\square]-\mu\Big)\prod_{\blacksquare\in W_j}\Big(\epsilon_1+\epsilon_2-E[b_j-a_i,W_j,Y_i,\blacksquare]-\mu\Big)\,, \tag{3.3}$$

where the elementary function $E[x,Y_i,W_j,\square]$ is defined as

$$E[x_{ij},Y_i,W_j,\square]=x_{ij}+A^{+}_{\square,Y_i}\epsilon_2-L_{\square,W_j}\epsilon_1, \tag{3.4}$$

x_{ij} is an indeterminate, and $\{\epsilon_1,\epsilon_2\}$ are complex parameters related to the central charge to be specified below.

3.2.4 The denominator

$$z_{\text{den}}\left(\vec{a},\vec{Y}\mid\vec{b},\vec{W}\right)=\left(z_{\text{norm}}\left(\vec{a},\vec{Y}\right)\;z_{\text{norm}}\left(\vec{b},\vec{W}\right)\right)^{\frac{1}{2}}, \tag{3.5}$$
$$z_{\text{norm}}\left(\vec{a},\vec{Y}\right)=z_{\text{num}}\left(\vec{a},\vec{Y}\mid 0\mid\vec{a},\vec{Y}\right) \tag{3.6}$$

In gauge theory, z_{norm} is a normalization factor related to the contribution of the vector multiplets [1]. In conformal field theory, it accounts for the norms of the states that propagate into and out of the vertex operator insertion in Z_{bb}.

4 AGT for minimal models. Finiteness and the N-Burge conditions

We consider the building block partition function introduced in (3.2) and subsequent equations, and set the parameters to those relevant to $\mathcal{M}_N^{p,p',\mathcal{H}}$. We show that by restricting the Young diagrams, we obtain well-defined expressions that we identify with minimal $\mathcal{W}_N$ conformal blocks.

4.1 Minimal model parameters

Since we focus on the minimal models, we choose to work in terms of the screening charge parameters $\{\alpha_-,\alpha_+\}$ rather than Nekrasov's deformation parameters $\{\epsilon_1,\epsilon_2\}$, by setting

$$\alpha_-=\epsilon_1,\quad \alpha_+=\epsilon_2, \tag{4.1}$$

where α_- and α_+ are real and satisfy $\alpha_-<0<\alpha_+$. We write

$$E[x_i,x_j,Y_i,W_j,\square]=x_i-x_j+A^{+}_{\square,i}\,\alpha_+-L_{\square,j}\,\alpha_- \tag{4.2}$$

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4.2 $\mathcal{W}_N$ parameters

The parameters x_i and x_j in (4.2) are scalar components of the vector of gauge theory Coulomb parameters $\{x_1, \cdots, x_N\}$, that satisfy $\sum_{i=1}^{N} x_i = 0$ and $A^{+}_{\Box,i} = A_{\Box,i} + 1$. We identify the Coulomb parameters with the minimal model parameters by setting

$$x_i = x_i^{+} \alpha_{+} + x_i^{-} \alpha_{-}, \quad i = 1, \cdots, N \tag{4.3}$$

and choosing

$$x_i^{+} = - \sum_{j=1}^{N-1} \langle \vec{\omega}_j \,|\, \vec{h}_i \rangle \, r_j, \quad x_i^{-} = - \sum_{j=1}^{N-1} \langle \vec{\omega}_j \,|\, \vec{h}_i \rangle \, s_j \tag{4.4}$$

where $\vec{\omega}_i$, $i = 1, \cdots, N-1$, are the A_{N-1} fundamental weight vectors, $\vec{h}_i$, $i = 1, \cdots, N-1$, are the weight vectors of the first fundamental representation of the Lie algebra A_{N-1}, and $\langle \vec{\omega}_j \,|\, \vec{h}_i \rangle$ is the scalar product of $\vec{\omega}_j$ and $\vec{h}_i$, regarded as N-component vectors in the weight lattice of A_{N-1}. Noting that $\vec{h}_i - \vec{h}_{i+1} = \vec{\alpha}_i$, $i = 1, \cdots, N-1$, where $\vec{\alpha}_i$ are the simple root vectors of A_{N-1}, and that $\langle \vec{\omega}_i | \vec{\alpha}_j \rangle = \delta_{ij}$, where $\delta_{ii} = 1$, and $\delta_{ij} = 0$, for $i \neq j$, the above definitions allow us to write

$$x_i^{+} - x_{i+1}^{+} = -r_i, \quad x_i^{-} - x_{i+1}^{-} = -s_i, \quad i = 1, \cdots, N-1 \tag{4.5}$$

4.3 Scanning products for zeros

Consider the denominator z_{den} of Z_{bb}, defined in 3.2.4. To look for zeros in z_{den}, it is sufficient to look for zeros in $z_{\text{norm}}[\vec{x}, \vec{Y}]$, defined in (3.6). Consider $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ and focus on a channel that carries states that belong to the degenerate $\mathcal{W}_n$ irreducible highest weight representation $\mathcal{H}^{p,p'}_{r,s}$, where p and p' are coprime, $0 < p < p'$, $r = r_1, \cdots, r_{N-1}$, and $s = s_1, \cdots, s_{N-1}$. Recall that we also define $r_N = p - \sum_{i=1}^{N-1} r_i$, and $s_N = p' - \sum_{i=1}^{N-1} s_i$, and that $0 < r_i < p$, and $0 < s_i < p'$, $i = 1, \cdots, N$.

Proposition 4.1 $z_{\text{norm}}[\vec{x}, \vec{Y}] \neq 0$, *if and only if*

$$Y_{i+1,\mathtt{R}} - Y_{i,\mathtt{R}+s_i-1} \geqslant -r_i + 1 \tag{4.6}$$

where $Y_{i,\mathtt{R}}$ is the $\mathtt{R}$-row in Y_i, $i = 1, \cdots, N$, r_i and s_i, $i = 1, \cdots, N$, are the integers that parameterise the degenerate $\mathcal{W}_N$ irreducible highest weight representation that flows in the channel under consideration, $r_N = p - \sum_{i=1}^{N-1} r_i$ and $s_N = p' - \sum_{i=1}^{N-1} s_i$.

4.4 Zero-conditions

The proof of Proposition 4.1 is based on checking the factors that appear in $z_{\text{norm}}[\vec{x}, \vec{Y}]$ for zeros. This requires introducing a number of elementary concepts that were first introduced in [14].

4.4.1 Two zero-conditions

As we will show, a factor in z_{norm} has a zero when an equation of type

$$C_{+} \alpha_{+} + C_{-} \alpha_{-} = 0, \tag{4.7}$$

is satisfied, where C_+ and C_- are non-zero positive integers, and $\alpha_- < 0 < \alpha_+$. Since $\alpha_- = -p/\sqrt{p\,p'}$, $\alpha_+ = p'/\sqrt{p\,p'}$, p and p' are coprime, the same factor in z_{norm} has a zero when the two conditions

$$C_+ = c\, p, \quad C_- = c\, p', \tag{4.8}$$

are satisfied, where c is a proportionality constant that remains to be determined.

4.4.2 From two zero-conditions to one zero-condition

This paragraph contains the core of the proof. Consider the two conditions

$$A^+_{\square,i} = A' \geqslant 1, \quad -L_{\square,j} = L' \geqslant 1 \tag{4.9}$$

where $A^+_{\square,i} = A_{\square,i} + 1$. These conditions are satisfied *if and only if* $\square \in Y_i$, and $\square \notin Y_j$. If $\square$ is in row-$\mathtt{R}$ and column-$\mathtt{C}$ in Y_i, then the first condition in (4.9) implies that there is a cell $\boxplus \in Y_i$, that may be $\square$ or lies to the right of $\square$, with coordinates $\{\mathtt{R}, \mathtt{C} + A' - 1\}$, such that, this cell $\boxplus$ lies on a vertical boundary of Y_i. The latter statement means that, **1.** there are no cells to the right of $\boxplus$, and **2.** there may or may not be cells below $\boxplus$. The latter two statements imply that the $[\mathtt{C} + A' - 1]$-column in Y_i, or equivalently, the $[\mathtt{C} + A' - 1]$-row in Y_i^{T}, has length *at least* $\mathtt{R}$,

$$Y^{\mathsf{T}}_{i,\mathtt{C}+A'-1} \geqslant \mathtt{R} \tag{4.10}$$

Using the definition of $L_{\square,j}$ in 3.1.2, we see that $L_{\square,j} = Y^{\mathsf{T}}_{j,\mathtt{C}} - \mathtt{R}$, and we can write the equality in the second condition in (4.9) as

$$-L_{\square,j} = -Y^{\mathsf{T}}_{j,\mathtt{C}} + \mathtt{R} = L' \tag{4.11}$$

In other words,

$$\mathtt{R} = L' + Y^{\mathsf{T}}_{j,\mathtt{C}}, \tag{4.12}$$

and from (4.10), we obtain

$$Y^{\mathsf{T}}_{i,\mathtt{C}+A'-1} - Y^{\mathsf{T}}_{j,\mathtt{C}} \geqslant L', \tag{4.13}$$

where $L' > 0$, which is one condition that is equivalent to the two conditions in (4.9).

4.5 Non-zero condition

Consider a function $\mathcal{F}[Y_i, Y_j]$, of a pair of Young diagrams $\{Y_i, Y_j\}$, such that $\mathcal{F}[Y_i, Y_j] = 0$, *if and only if* (4.13) is satisfied. This implies that $\mathcal{F}[Y_i, Y_j] \neq 0$, *if and only if* $\{Y_i, Y_j\}$ satisfies the complementary condition

$$Y^{\mathsf{T}}_{i,\mathtt{C}+A'-1} - Y^{\mathsf{T}}_{j,\mathtt{C}} < L', \tag{4.14}$$

which we choose to write as

$$Y^{\mathsf{T}}_{j,\mathtt{C}} - Y^{\mathsf{T}}_{i,\mathtt{C}+A'-1} \geqslant 1 - L' \tag{4.15}$$

which, following [14], can be written in the form[10]

$$Y_{j,\mathtt{R}} - Y_{i,\mathtt{R}+L'-1} \geqslant 1 - A' \tag{4.16}$$

[10]The proof that (4.15) is equivalent to (4.16) is in subsection **4.10** of [14].

4.5.1 Remarks

1. It is useful, for the purposes of the calculations in the sequel, to note that re-writing (4.15) as (4.16) is equivalent to transposing each of the partitions Y_i and Y_j, replacing the shift in the row number of Y_j by the negative of the right hand side, and *vice versa*. **2.** The subscripts $\mathtt{C}$ and $\mathtt{C} + A' - 1$ on the left hand side of (4.15) refer to *the row-numbers* of the Young diagrams $Y_j^{\intercal}$ and $Y_i^{\intercal}$, respectively. **3.** The Young diagram that the cell $\square$ lives in, in this case Y_i, appears with a minus sign in (4.16). We frequently meet such equations in the sequel, and we need such observations to be able to make simple, quick checks of their consistency. **4.** We refer, in the sequel, to equations such as (4.9) and (4.15) as *'zero-conditions'*, and *'non-zero-conditions'*, respectively.

4.6 Products in the denominator

Two types of products appear in z_{norm}. These are **1.** products in the form $\prod_{\square \in Y_i} E[a_i - a_j, Y_i, Y_j, \square]$, that we denote by $\{Y_i, Y_j\}$, and **2.** products in the form $\prod_{\square \in Y_i} [\alpha_+ + \alpha_- - E[a_i - a_j, Y_i, Y_j, \square]]$ that we denote by $\{Y_i, Y_j\}'$.

4.6.1 Remark

As we will show, it is sufficient to consider $\{Y_i, Y_{i+1}\}$, $i = 1, \cdots, N$, where $Y_{N+1} = Y_1$. The conditions that remove these zeros are sufficient to remove the zeros of the other products.

4.6.2 In search of zeros

We plan to proceed as follows. **1.** We consider the products in z_{den}, one at a time, **2.** search for possible zeros, as in subsection 4.4.1, **3.** find the conditions that imposed on the pair $\{Y_i, Y_j\}$ in order to avoid the zeros, and **4.** when there is more than one of condition to avoid a zero, we choose the strongest condition. That is, we choose the condition that eliminates more zeros than any other condition. To do this, we use the fact that r_i, and s_i, $i = 1, \cdots, N$, are non-zero positive integers.

4.6.3 Products that have no zeros

$\{Y_i, Y_i\}$, $i = 1, \cdots, N$, has no zeros since that requires a factor that satisfies

$$E[0, Y_i, Y_i, \square] = A^{+}_{\square,i}\, \alpha_+ - L_{\square,i}\, \alpha_- = 0, \tag{4.17}$$

which is not possible since $\square \in Y_i$, thus $A^{+}_{\square,i} > 0$, $L_{\square,i} \geqslant 0$, $\alpha_+ > 0$, and $\alpha_- < 0$. $\{Y_i, Y_i\}'$, $i = 1, \cdots, N$, has no zeros for the same reason.

4.6.4 Products that have zeros

Next, we consider $\{Y_i, Y_j\}$, such that $i \neq j$, of which there are $N(N-1)$ products. To do that, we need to introduce some simple definitions.

4.6.5 Periodic N-partitions

It is convenient to regard the set of N-partitions $\{Y_1, \cdots, Y_N\}$ as a subset of a set of infinitely-many partitions with periodicity N. More precisely, we consider a set of infinitely-many partitions, Y_i, $i \in \mathbb{Z}$, and define $Y_{i+kN} = Y_i$, $i \in 1, \cdots, N$, $k = \mathbb{Z}$. The N-partitions that we start with correspond to the *'fundamental subset'* Y_i, $i = 1, \cdots, N$.[11]

4.6.6 An N-site circle $\mathcal{C}_N$

Since $Y_i = Y_{i+kN}$, $i \in 1, \cdots, N$, $k \in \mathbb{Z}$, we regard Y_i, $i \in \mathbb{Z}$ as assigned to the sites σ_i, $i = 1, \cdots, N$, of an N-site circle $\mathcal{C}_N$, and assign partition Y_{i+kN} to site $i\sigma_i$.

4.6.7 Periodic x_i^+, x_i^-, r_i and s_i parameters

Similarly to the partitions Y_i, $i \in 1, \cdots, N$, whose definition is extended to all $i \in \mathbb{Z}$, we extend the definition of the parameters x_i^+, x_i^-, r_i and s_i, $i, 1, \cdots, N$, defined in 2.2.4, to x_i^+, x_i^-, r_i and s_i, $i \in \mathbb{Z}$, and define

$$x_{i+kN}^+ = x_i^+, \quad x_{i+kN}^- = x_i^-, \quad r_{i+kN} = r_i, \quad s_{i+kN} = s_i, \quad i = 1, \cdots, N, \quad k \in \mathbb{Z} \tag{4.18}$$

We attach x_{i+kN}^+, x_{i+kN}^-, r_{i+kN} and s_{i+kN}, $i = 1, \cdots, N$, $k \in \mathbb{Z}$, to site σ_i in $\mathcal{C}_N$. The parameters that we start with correspond to those in the *fundamental subset* x_i^+, x_i^-, r_i and s_i, $i = 1, \cdots, N$.

4.7 Conditions from $\{Y_i, Y_{i+1}\}$

Each of these products, for $i = 1, \cdots, N$, vanishes if it contains a factor that satisfies

$$\begin{aligned} E[x_i, x_{i+1}, Y_i, Y_{i+1}, \square] &= \left(x_i^+ - x_{i+1}^+ + A_{\square,i}^+\right) \alpha_+ + \left(x_i^- - x_{i+1}^- - L_{\square,i+1}\right) \alpha_- \\ &= \left(-r_i + A_{\square,i}^+\right) \alpha_+ + \left(-s_i - L_{\square,i+1}\right) \alpha_- = 0, \end{aligned} \tag{4.19}$$

which, using (4.5), leads to the conditions

$$A_{\square,i}^+ = r_i + c\,p, \quad -L_{\square,i+1} = s_i + c\,p' \tag{4.20}$$

where c remains to be determined. Since $A_{\square,i}$, $L_{\square,i+1}$, r_i and s_i, $i = 1, \cdots, N$, are non-zero positive integers, and p and p' are positive co-primes, c must be an integer. Since $r_i < p$, $i = 1, \cdots, N$, if $c < 0$, $A_{\square,i}^+ < 0$, which is not possible, hence $c = 0, 1, 2, \cdots$ In other words, conditions (4.20) are possible for $c = 0, 1, 2, \cdots$, $\square \in Y_i$ and $\square \notin Y_{i+1}$.

4.7.1 From two zero-conditions to one non-zero-condition

Following paragraph 4.4.2 and subsection 4.5, the two zero-conditions in (4.20) can be translated to one non-zero-condition,

$$Y_{i+1,\mathtt{c}}^{\intercal} - Y_{i,\mathtt{c}+\left(r_i-1\right)+c\,p}^{\intercal} \geqslant -\left(s_i - 1\right) - c\,p' \tag{4.21}$$

[11]As explained in paragraph 4.12.2, we actually end up with cylindric partitions [17], since adjacent partitions Y_i and Y_{i+1}, $i \in \mathbb{Z}$, are related by conditions of the type discussed in [17].

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4.7.2 The strongest condition

Since the row-lengths of a partition are weakly decreasing, condition (4.21) is satisfied if

$$Y^{\mathsf{T}}_{i+1,\,\mathtt{C}} - Y^{\mathsf{T}}_{i,\,\mathtt{C}+\left(r_i-1\right)} \geqslant -\left(s_i - 1\right) - c\,p' \tag{4.22}$$

which is the case if

$$Y^{\mathsf{T}}_{i+1,\,\mathtt{C}} - Y^{\mathsf{T}}_{1,\,\mathtt{C}+\left(r_i-1\right)} \geqslant -\left(s_i - 1\right) \tag{4.23}$$

Thus, we should set $c = 0$, which is an allowed value for c, in (4.21), and following [14], re-write it in the simpler form

$$Y_{i+1,\,\mathtt{R}} - Y_{i,\,\mathtt{R}+\left(s_i-1\right)} \geqslant -\left(r_i - 1\right) \tag{4.24}$$

4.8 Conditions from $\{Y_i, Y_{i+1}\}$, $\{Y_{i+1}, Y_{i+2}\}$, $\cdots$, $\{Y_{i+n-1}, Y_{i+n}\}$

Since condition (4.24) relates the partitions Y_i and Y_{i+1}, it also relates, by adjacency, the partitions Y_i and Y_{i+n}, $n \in \mathbb{Z} > 1$. For example, the two conditions

$$Y_{i+2,\,\mathtt{R}} - Y_{i+1,\,\mathtt{R}+s_{i+1}-1} \geqslant -r_{i+1} - 1, \quad Y_{i+1,\,\mathtt{R}} - Y_{i,\,\mathtt{R}+s_i-1} \geqslant -r_i - 1 \tag{4.25}$$

imply

$$Y_{i+2,\,\mathtt{R}} - Y_{i,\,\mathtt{R}+\left(\sum_{j=0}^{1} s_{i+j}\right)-2} \geqslant 2 - \sum_{j=0}^{1} r_{i+j} \tag{4.26}$$

and, in the same way, the n adjacent $\{Y_i, Y_{i+1}\}$ conditions imply

$$Y_{i+n,\,\mathtt{R}} - Y_{i,\,\mathtt{R}-n+\left(\sum_{j=0}^{n-1} s_{i+j}\right)} \geqslant n - \left(\sum_{j=0}^{n-1} r_{i+j}\right) \tag{4.27}$$

We refer to condition (4.27) as an *'n-adjacent'* $\{Y_i, Y_{i+1}\}$ condition, since it comes from n conditions of type $\{Y_i, Y_{i+1}\}$ that involve $(n+1)$ adjacent partitions.

4.8.1 Remarks

1. Note the shift by $-n$ of the row-number of partition Y_i on the left hand side of (4.27), and by n of the term on the right hand side. Condition (4.27) makes sense since $-n + \sum_{j=0}^{n-1} s_{i+j} \geqslant 0$, and $n - \sum_{j=0}^{n-1} r_{i+j} \leqslant 0$. **2.** In the following, we show that it is sufficient to impose condition (4.25) to eliminate the zeros of $\{Y_i, Y_{i+n}\}$, rather than any condition obtained from any another product involving these two partitions. Since condition (4.25) follows from the $\{Y_i, Y_{i+1}\}$ conditions (4.23), the latter are sufficient to eliminate the zeros in $\{Y_i, Y_{i+n}\}$.

4.8.2 A consistency check

$\{Y_i, Y_{i+N}\}$ leads to the condition

$$Y_{i+N,\mathtt{R}} - Y_{i,\mathtt{R}+\sum_{j=0}^{N-1}\left(s_{i+j}-1\right)} \geqslant - \sum_{j=0}^{N-1} \left(r_{i+j} - 1\right) \tag{4.28}$$

which can be written, using (2.12), as

$$Y_{i,\mathtt{R}} - Y_{i,\mathtt{R}+\left(p'-N\right)} \geqslant - \left(p - N\right) \tag{4.29}$$

which are trivial conditions on Y_i, $i = 1, \cdots, N$, since $p' > p \geqslant N$, by definition of the $\mathcal{W}_N$ minimal models. This agrees with the fact that such products do not have zeros, and therefore should not be restricted by any conditions.

4.9 Conditions from $\{Y_i, Y_{i+n}\}$

Each of these products, for $i = 1, \cdots, N$, and $n > 0$, vanishes if it contains a factor that satisfies

$$\begin{aligned} E[x_i, x_{i+n}, Y_i, Y_{i+n}, \square] &= \left(x_i^+ - x_{i+n}^+ + A_{\square,i}^+\right) \alpha_+ + \left(x_i^- - x_{i+n}^- - L_{\square,i+n}\right) \alpha_- \\ &= \left(A_{\square,i}^+ - \sum_{j=0}^{n-1} r_{i+j}\right) \alpha_+ + \left(-L_{\square,i+n} - \sum_{j=0}^{n-1} s_{i+j}\right) \alpha_- = 0, \end{aligned} \tag{4.30}$$

which leads to the conditions

$$A_{\square,i} = c\, p - 1 + \sum_{j=0}^{n-1} r_{i+j}, \quad -L_{\square,i+1} = c\, p' + \sum_{j=0}^{n-1} s_{i+j}, \tag{4.31}$$

where c remains to be determined. Following the same arguments used in 4.7, c must be a non-negative integer. In other words, conditions (4.31) are possible for $c = \{0, 1, \cdots\}$, $\square \in Y_i$ and $\square \notin Y_{i+n}$.

4.9.1 From two zero-conditions to one non-zero-condition

Following paragraphs 4.4.2 and 4.5, the two zero-conditions in (4.31) can be translated to one non-zero-condition,

$$Y^{\mathsf{T}}_{i+n,\mathtt{C}} - Y^{\mathsf{T}}_{i,\mathtt{C}+c\,p-1+\sum_{j=0}^{n-1} r_{i+j}} \geqslant 1 - c\,p' - \sum_{j=0}^{n-1} s_{i+j} \tag{4.32}$$

4.9.2 The strongest condition

Following the arguments in paragraph 4.7.2, the strongest version of condition (4.32) is obtained by setting $c = 0$, which is an allowed value for c. Following [14], the result can be re-written in the simpler form

$$Y_{i+n,\mathtt{R}} - Y_{i,\mathtt{R}-1+\sum_{j=0}^{n-1} s_{i+j}} \geqslant 1 - \sum_{j=0}^{n-1} r_{i+j} \tag{4.33}$$

4.9.3 Comparing the $\{Y_i, Y_{i+n}\}$ conditions and the n adjacent $\{Y_i, Y_{i+1}\}$ conditions

Since the row-lengths of a partition are weakly-decreasing, condition (4.33) is satisfied if

$$Y_{i+n,\,\mathtt{R}} - Y_{i,\,\mathtt{R}-n+\sum_{j=0}^{n-1} s_{i+j}} \geqslant 1 - \sum_{j=0}^{n-1} r_{i+j}, \tag{4.34}$$

where $n > 1$, which is satisfied if

$$Y_{i+n,\,\mathtt{R}} - Y_{i,\,\mathtt{R}-n+\sum_{j=0}^{n-1} s_{i+j}} \geqslant n - \sum_{j=0}^{n-1} r_{i+j}, \tag{4.35}$$

which is condition (4.27). Thus, the n adjacent $\{Y_i, Y_{i+1}\}$ conditions (4.27), which follow from the $\{Y_i, Y_{i+1}\}$ conditions (4.24), are stronger than the $\{Y_i, Y_{i+n}\}$ conditions (4.33), and it is sufficient to impose the underlying $\{Y_i, Y_{i+1}\}$ conditions (4.24) to eliminate the zeros in the $\{Y_i, Y_{i+n}\}$.

4.10 Conditions from $\{Y_i, Y_{i-n}\}$

Each of these products, for $i = 1, \cdots, N$, $n > 0$, vanishes if it contains a factor that satisfies

$$\begin{aligned} E[x_i, x_{i-n}, Y_i, Y_{i-n}, \square] &= \left(x_i^+ - x_{i-n}^+ + A_{\square,i}^+\right) \alpha_+ + \left(x_i^- - x_{i-n}^- - L_{\square,i-n}\right) \alpha_- \\ &= \left(A_{\square,i}^+ + \sum_{j=1}^{n} r_{i-j}\right) \alpha_+ + \left(-L_{\square,i-n} + \sum_{j=1}^{n} s_{i-j}\right) \alpha_- = 0, \end{aligned} \tag{4.36}$$

which leads to the conditions

$$A_{\square,i} = -1 + c\,p - \sum_{j=1}^{n} r_{i-j} \qquad - L_{\square,i-n} = c\,p' - \sum_{j=1}^{n} s_{i-j}, \tag{4.37}$$

where c remains to be determined. Following the same arguments used in 4.7, c must be a non-zero positive integer. In other words, conditions (4.37) are possible for $c = 1, 2, \cdots$, $\square \in Y_i$ and $\square \notin Y_{i-n}$.

4.10.1 From two zero-conditions to one non-zero-condition

Following paragraphs 4.4.2 and 4.5, the two zero-conditions in (4.37) can be translated to one non-zero-condition,

$$Y^{\mathsf{T}}_{i-n,\,\mathtt{C}} - Y^{\mathsf{T}}_{i,\,\mathtt{C}-1+c\,p-\sum_{j=1}^{n} r_{i-j}} \geqslant 1 - c\,p' + \sum_{j=1}^{n} s_{i-j} \tag{4.38}$$

4.10.2 The strongest condition

Following the arguments in paragraph 4.7.2, the strongest version of condition (4.32) is obtained by setting $c = 1$, which is an allowed value for c. Following [14], the result can be re-written in the simpler form

$$Y_{i-n,\,\mathtt{R}} - Y_{i,\,\mathtt{R}-1+p'-\sum_{j=1}^{n} s_{i-j}} \geqslant 1 - p + \sum_{j=1}^{n} r_{i-j} \tag{4.39}$$

4.10.3 Comparing the conditions from $\{Y_i, Y_{i-n}\}$ and from n-adjacent $\{Y_i, Y_{i-1}\}$

Using the N-periodicity of the partitions Y_i, $i \in \mathbb{Z}$, as well as the sum conditions (2.12), we can re-write (4.39) as

$$Y_{i+N-n,\,\mathtt{R}} - Y_{i,\,\left(\mathtt{R}-1+\sum_{j=n+1}^{N} s_{i-j}\right)} \geqslant \left[1 - \sum_{j=n+1}^{N} r_{i-j}\right], \tag{4.40}$$

then using the N-periodicity of the integers r_i and s_i, $i \in \mathbb{Z}$, we re-write (4.40) as

$$Y_{i+N-n,\mathtt{R}} - Y_{i,\,\left(\mathtt{R}-1+\sum_{j=0}^{N-n-1} s_{i+j}\right)} \geqslant \left[1 - \sum_{j=0}^{N-n-1} r_{i+j}\right], \tag{4.41}$$

which is identical to the conditions (4.33), upon a trivial change of labels. Thus, the conditions from $\{Y_i, Y_{i+1}\}$ (4.24) are stronger than the conditions from $\{Y_i, Y_{i+n}\}$ (4.39), and it is sufficient to impose the former to eliminate the zeros in $\{Y_i, Y_{i-n}\}$, $n < 0$.

4.11 Conditions from $\{Y_i, Y_{i+1}\}'$

Each of these products, for $i = 1, \cdots, N$, vanishes if it contains a factor that satisfies

$$\begin{aligned} -\alpha_+ - \alpha_- + E[x_i, x_{i+1}, Y_i, Y_{i+1}, \square] &= \left(x_i^+ - x_{i+1}^+ + A^+_{\square,i} - 1\right)\alpha_+ + \left(x_i^- - x_{i+1}^- - L_{\square,i+1} - 1\right)\alpha_- \\ &= \left(-r_i + A_{\square,i}\right)\alpha_+ + \left(-s_i - L^+_{\square,i+1}\right)\alpha_- = 0, \end{aligned} \tag{4.42}$$

which, using (4.5), leads to the conditions

$$A_{\square,i} = r_i + c\,p, \quad -L_{\square,i+1} = s_i + 1 + c\,p', \tag{4.43}$$

where c remains to be determined. Following the same arguments used in 4.7, c must be a non-negative integer. In other words, conditions (4.43) are possible for $c = 0, 1, 2, \cdots$, $\square \in Y_i$ and $\square \notin Y_{i+1}$.

4.11.1 From two zero-conditions to one non-zero-condition

Following paragraphs 4.4.2 and 4.5, the two zero-conditions in (4.43) can be translated to one non-zero-condition,

$$Y^{\intercal}_{i+1,\,\mathtt{C}} - Y^{\intercal}_{i,\,\mathtt{C}+r_i+c\,p} \geqslant -s_i - c\,p' \tag{4.44}$$

4.11.2 The stronger condition

Since the row-lengths of a partition are weakly decreasing, condition (4.44) is satisfied if

$$Y^{\intercal}_{i+1,\,\mathtt{C}} - Y^{\intercal}_{i,\,\mathtt{C}+r_i} \geqslant -s_i - c\,p', \tag{4.45}$$

which is the case if

$$Y^{\intercal}_{i+1,\,\mathtt{C}} - Y^{\intercal}_{1,\,\mathtt{C}+r_i} \geqslant -s_i \tag{4.46}$$

Thus, we should set $c = 0$, which is an allowed value for c, in (4.44), and following [14], re-write it in the simpler form

$$Y_{i+1,\,\mathtt{R}} - Y_{i,\,\mathtt{R}+s_i} \geqslant -r_i \tag{4.47}$$

4.11.3 Comparing conditions

Using the arguments of paragraph 4.7.2, one finds that conditions (4.24) are stronger than conditions (4.47). Thus conditions (4.24) that eliminate the zeros in $\{Y_i, Y_{i+1}\}$ are sufficient to eliminate the zeros in $\{Y_i, Y_{i+1}\}'$.

4.12 Conjugate products leads to weaker conditions

It is straightforward to see that all remaining conjugate products lead to conditions that are weaker than those of the corresponding products. The reason is that every $\{Y_i, Y_j\}'$ is related to the corresponding product $\{Y_i, Y_j\}$ by replacing each elementary factor $E[x_i, x_j, Y_i, Y_j, \square]$ in $\{Y_i, Y_j\}$ by a factor $-\alpha_+ - \alpha_- + E[x_i, x_j, Y_i, Y_j, \square]$, up to an overall minus sign. As can be seen, by comparing the expressions in subsection 4.7 to the corresponding expressions in this subsection, this amounts to changing

$$A^+_{\square,i} \to A_{\square,i}, \quad L_{\square,j} \to L^+_{\square,j}, \tag{4.48}$$

where we have used Y_i and Y_j for generality. This leads to changing the final expressions for the non-zero conditions,

$$r_i \to r_i + 1, \quad s_i \to s_i + 1, \tag{4.49}$$

which, following the arguments in paragraph 4.7.2, leads to weaker conditions. In particular, the conditions obtained from $\{Y_i, Y_{i+n}\}'$ and from $\{Y_i, Y_{i-n}\}'$ are weaker than those discussed in subsections 4.9 and 4.10, respectively.

4.12.1 The conditions from $\{Y_i, Y_{i+1}\}$ are sufficient

From the above, we conclude that the N-Burge conditions (4.24), which we recall for convenience,

$$Y_{i+1,\mathtt{R}} - Y_{i,\mathtt{R}+\left(s_i-1\right)} \geqslant -\left(r_i - 1\right) \tag{4.50}$$

are sufficient to eliminate all zeros in all products in z_{den}.

4.12.2 Cylindric partitions. N-Burge conditions

The conditions (4.24) form a special case of those that were introduced and studied in [17]. A set of N partitions that satisfy such conditions are called *'cylindric partitions'* in [17]. They have appeared in this specific form in [18].

5 $\mathcal{W}_3$ minimal model conformal blocks from AGT with restricted Young diagrams

We check the validity of the expression in equation (1.2) by computing a non-trivial conformal block in a $\mathcal{W}_3$ minimal model. To do that, we consider a $\mathcal{W}_3$ conformal block that is known to satisfy a third-order ordinary differential equation of Pochammer type, that is solved in terms of ${}_3F_2$ Hypergeometric functions [27]. To be reasonably self-contained, we outline the derivation of this differential equation in some detail.

5.1 A family of holomorphic 4-point functions

Consider the $\mathcal{W}_3$ holomorphic 4-point functions

$$F\{z\} = \langle \prod_{i=0}^{3} \mathcal{O}_{\vec{P}_i}(z_i) \rangle, \tag{5.1}$$

where $\{z\} = \{z_0, \cdots, z_3\}$ is a set of four points on the Riemann sphere, $\mathcal{O}_{\vec{P}_i}(z_i)$, $i = 0, \cdots, 3$, is a vertex operator insertion at z_i. $\mathcal{O}_{\vec{P}_i}$ is a vertex operator that inserts a $\mathcal{W}_3$ highest weight state labeled by the charge vector $\vec{P}_i$. At this point, the charge vectors $\vec{P}_i$, $i = 0, \cdots, 3$, could be any vectors in the weight lattice of the Lie algebra A_2, spanned by the fundamental weight vectors $(\vec{\omega}_1, \vec{\omega}_2)$,

$$\vec{P}_0 = c_{1,i}\, \vec{\omega}_1 + c_{2,i}\, \vec{\omega}_2, \tag{5.2}$$

where $c_{1,i}, c_{2,i} \in \mathbb{R}$.

5.2 Specialising the 4-point functions

For the purposes of this section, we chose to keep $\vec{P}_0$ and $\vec{P}_3$ arbitrary, and set $\vec{P}_1$ and $\vec{P}_2$ to point in the direction of $\vec{\omega}_1$ only, such that

$$\vec{P}_1 = -b\,\vec{\omega}_1, \quad \vec{P}_2 = a\,\vec{\omega}_1, \tag{5.3}$$

where b is the parameter that determines the Virasoro central charge, see equation (2.7), while $a \in \mathbb{R}$ is arbitrary. Using global conformal invariance [28], the holomorphic 4-point function in equation (5.1), with the charge vectors chosen as in equations (5.2) and (5.3), can be written in the form[12]

$$\begin{aligned} F\{z\} = z_{31}^{-2\Delta_1}\, z_{30}^{\left(\Delta_1+\Delta_2-\Delta_3-\Delta_0\right)}\, z_{32}^{\left(\Delta_1+\Delta_0-\Delta_2-\Delta_3\right)}\, z_{20}^{\left(\Delta_3-\Delta_1-\Delta_2-\Delta_0\right)} \\ \mathcal{B}\left(z|\vec{P}_0, \vec{P}_{int}, \vec{P}_3| - b, a\right), \end{aligned} \tag{5.4}$$

where $\mathcal{B}$ is the factor in $F\{z\}$ that depends only on z,[13] the projective invariant cross-ratio of the coordinates,

$$z = \frac{z_{10}z_{23}}{z_{13}z_{20}}, \quad z_{ij} = z_i - z_j, \tag{5.5}$$

and Δ_i is the conformal dimension of $\mathcal{O}_{\vec{P}_i}(z_i)$, see equation (2.5).

The relevant factor on the right hand side of equation (5.4) is $\mathcal{B}[z|\vec{P}_0, \vec{P}_{int}, \vec{P}_3| - b, a]$, all parameters of which are initial data that specify the 4-point function that we wish to compute, except $\vec{P}_{int}$ which remains to be determined. Following [10, 27], $\mathcal{B}[z|\vec{P}_0, \vec{P}_{\text{int}}, \vec{P}_3|-$

[12] The subscripts $\{0, 12, 3\}$ that we use to label the points of the 4-point function on the Riemann sphere, correspond to the subscripts $\{2, 1, 3, 4\}$ used to label the same points in [28].

[13] At this stage, we are working in the context of generic $\mathcal{W}_N$ models.In particular, we did not choose the parameter b in equation (5.3) such that we obtain a minimal $\mathcal{W}_N$ model. Since we have specialised to $\mathcal{W}_3$, that is $N = 3$, and we focus on 4-point functions, that is $n = 1$, once we choose the parameters in $F\{z\}$ to be those of a minimal model labeled by the coprimes p and p', $\mathcal{B}$ in equation (5.4) becomes $\mathcal{B}_{3,1}^{p,p'}$ in the notation of equation (2.24).

$b, a]$ satisfies a third-order ordinary differential equation with respect to z. Requiring that $\mathcal{B}$ satisfies this differential equation, determines the three possible values of the charge vector $\vec{P}_{\text{int}}$ of the $\mathcal{W}_3$ highest weight representation that flows in the internal channel of the conformal block. In the rest of this section, we derive, this equation in six steps.

5.2.1 Step 1

The $\mathcal{W}_3$ Verma module of highest weight vector $(-2b\,\vec{\omega}_1 - b^{-1}\,\vec{\omega}_2)$, associated to the vertex operator $\mathcal{O}_{\vec{P}_1}$, contains a $\mathcal{W}_3$ null-state at level 3. This implies that $F\{z\}$ in equation (5.4) satisfies the null state condition

$$\left(W_{-3}^{(1)} - \frac{16 w_1}{\Delta_1(\Delta_1+1)(5\Delta_1+1)} \left(L_{-1}^{(1)} \right)^3 + \right. \\ \left. \frac{12 w_1}{\Delta_1(5\Delta_1+1)} L_{-1}^{(1)} L_{-2}^{(1)} + \frac{3 w_1(\Delta_1-3)}{2\Delta_1(5\Delta_1+1)} L_{-3}^{(1)} \right) F\{z\} = 0, \quad (5.6)$$

where the generators $L_{-m}^{(1)}$, $m = 1, 2, 3$, and $W_{-3}^{(1)}$ act on $\mathcal{O}_{\vec{P}_1}(z_1)$. We need to express this action in terms of the differential operator action.

5.2.2 Step 2

We use the $\mathcal{W}_3$ Ward identity

$$W_{-3}^{(1)} F\{z\} = \sum_{\substack{j=0 \\ j\neq 1}}^{3} \left(\frac{w_j}{(z_i - z_j)^3} + \frac{W_{-1}^{(j)}}{(z_i - z_j)^2} + \frac{W_{-2}^{(j)}}{(z_i - z_j)} \right) F\{z\}, \quad (5.7)$$

to express the action of $W_{-3}^{(1)}$ on $F\{z\}$ in equation (5.6), in terms of the action of the six lower-degree generators $W_{-1}^{(j)}$, and $W_{-2}^{(j)}$, $j = 0, 2, 3$, which act on the other three vertex operators in $F\{z\}$.

5.2.3 Step 3

We use the five $\mathcal{W}_3$ Ward identities [27],

$$\sum_{j=0}^{3} W_{-2}^{(j)} = 0, \quad (5.8)$$

$$\sum_{j=0}^{3} \left(z_j\, W_{-2}^{(j)} + W_{-1}^{(j)} \right) = 0, \quad (5.9)$$

$$\sum_{j=0}^{3} \left(z_j^2\, W_{-2}^{(j)} + 2 z_j\, W_{-1}^{(j)} + w_j \right) = 0, \quad (5.10)$$

$$\sum_{j=0}^{3} \left(z_j^3\, W_{-2}^{(j)} + 3 z_j^2\, W_{-1}^{(j)} + 3 z_j w_j \right) = 0, \quad (5.11)$$

$$\sum_{j=0}^{3} \left(z_j^4 W_{-2}^{(j)} + 4 z_j^3 W_{-1}^{(j)} + 6 z_j^2 w_j \right) = 0. \quad (5.12)$$

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to express the five generators $W^{(0)}_{-1}$, $W^{(0)}_{-2}$, $W^{(3)}_{-1}$, $W^{(3)}_{-2}$ and $W^{(2)}_{-2}$ in terms of the three generators $W^{(1)}_{-1}$, $W^{(1)}_{-2}$, and $W^{(2)}_{-1}$.

5.2.4 Step 4

We use the fact that there are null-states at level-1 and level-2 in the Verma module with highest weight vector $(-2b\ \vec{\omega}_1 - b^{-1}\ \vec{\omega}_2)$, and that there is a null-state at level-1 in the Verma module with highest weight state vector $(-(b+a)\ \vec{\omega}_1 - b^{-1}\ \vec{\omega}_2)$, to obtain the relations

$$W^{(1)}_{-1} F\{z\} = \frac{3w_1}{2\Delta_1} \partial_{z_1} F\{z\} \tag{5.13}$$

$$W^{(1)}_{-2} F\{z\} = \left(\frac{12 w_1}{\Delta_1(5\Delta_1+1)} \partial^2_{z_1} - \frac{6w_1(\Delta_1+1)}{\Delta_1(5\Delta_1+1)} L^{(1)}_{-2} \right) F\{z\} \tag{5.14}$$

$$W^{(2)}_{-1} F\{z\} = \frac{3w_2}{2\Delta_2} \partial_{z_2} F\{z\}, \tag{5.15}$$

5.2.5 Step 5

To express the action of the Virasoro generators $L^{(1)}_{-m}$, $m = 1, 2, 3$, in above equations, in terms of differential operators, we use the conformal Ward identity

$$L^{(i)}_{-m} F\{z\} = -\sum_{\substack{j=0 \\ j\neq i}}^{3} \left(\frac{(m-1)\Delta_j}{(z_i - z_j)^m} + \frac{\partial_{z_j}}{(z_i - z_j)} \right) F\{z\} \tag{5.16}$$

5.2.6 Step 6

To obtain an ordinary differential equation for the 4-point function $F\{z\}$, it is convenient to work in terms of $\mathcal{B}^{\,\mathrm{simple}}$, that is defined in terms of $\mathcal{B}$, the projective-invariant component of $F\{z\}$ in equation (5.4),

$$\mathcal{B}^{\,\mathrm{simple}} \left(z|\vec{P}_0, \vec{P}_{\mathrm{int}}, \vec{P}_3| -b, a \right) = \left(z^{-\frac{b(2\ell_1+\ell_2)}{3}} (1-z)^{-\frac{3+3b^2-ba}{3}} \right) \mathcal{B} \left(z|\vec{P}_0, \vec{P}_{\mathrm{int}}, \vec{P}_3| -b, a \right), \tag{5.17}$$

$$\ell_i = \langle \vec{P}_0 | \vec{\alpha}_i \rangle + \left(b + \frac{1}{b} \right), \quad i = 1, 2 \tag{5.18}$$

5.3 The differential equation

Following [27], we combine the above equations and find that $\mathcal{B}^{\,\mathrm{simple}}$ satisfies the Pochhammer generalised hypergeometric differential equation

$$\mathcal{D}_z \mathcal{B}^{\,\mathrm{simple}} \Big(z|\vec{P}_0, \vec{P}_{\mathrm{int}}, \vec{P}_3| -b, a \Big) = 0, \tag{5.19}$$

$$\mathcal{D}_z \equiv z\Big(\partial_z + A_1\Big)\Big(\partial_z + A_2\Big)\Big(\partial_z + A_3\Big) - \Big(\partial_z + B_1 - 1\Big)\Big(\partial_z + B_2 - 1\Big)\partial_z, \tag{5.20}$$

$$A_i = \frac{b^2 + 3 - b\,a}{3} + b\,\langle \vec{P}_0 | \vec{h}_1 \rangle + b\,\langle \vec{P}_3 | \vec{h}_i \rangle, \quad i = 1, 2, 3, \tag{5.21}$$

$$B_1 = 1 + b\,\langle \vec{P}_0 | \vec{\alpha}_1 \rangle, \quad B_2 = 1 + b\,\langle \vec{P}_0 | \vec{\alpha}_1 + \vec{\alpha}_2 \rangle$$

Recalling that $\vec{P}_{\rm int}$ is the only undetermined parameter in $\mathcal{B}^{\,\rm simple}$, and requiring that $\mathcal{B}^{\,\rm simple}$ satisfies equation (5.19), implies that there are at most three possible values for $\vec{P}_{\rm int}$, and correspondingly, at most three possible $\mathcal{W}_3$ highest weight modules are allowed to propagate in the intermediate channel of the 4-point function. The solution of equation (5.19) is known to be a hypergeometric function of type ${}_3F_2$ [27].

The differential equation that $\mathcal{B}$ satisfies is obtained by composing the factor on the right hand side of equation (5.17) and $\mathcal{D}_z$ in equation (5.19),

$$\left(\mathcal{D}_z \circ \left(z^{-\frac{b(2\ell_1+\ell_2)}{3}} (1-z)^{-\frac{3+3b^2-ba}{3}} \right) \right) \mathcal{B} \left(z | \vec{P}_0, \vec{P}_{\rm int}, \vec{P}_3 | -b, a \right) = 0 \tag{5.22}$$

Proposing the z expansion

$$\mathcal{B} \left(z | \vec{P}_0, \vec{P}_{\rm int}, \vec{P}_3 | -b, a \right) = z^{\gamma} \left(1 + \mathcal{O}(z) \right) , \tag{5.23}$$

one looks for the possible values of γ that satisfy equation (5.22) to leading order. The values γ_i, $i = 1, 2, 3$, that solve the third-order algebraic characteristic equation can be written as

$$\gamma_i = \Delta_{\vec{P}_{{\rm int},i}} - \Delta_0 - \Delta_1, \tag{5.24}$$

where $\vec{P}_{{\rm int},i}$, $i = 1, 2, 3$, are the charge vectors of the $\mathcal{W}_3$ highest weight modules that are allowed to floow in the internal channel. The values of $\vec{P}_{{\rm int},i}$ that we obtain are

$$\vec{P}_{int,1} = \vec{P}_0 - b\,\vec{\omega}_1, \quad \vec{P}_{int,2} = \vec{P}_0 + b\,\vec{\omega}_2, \quad \vec{P}_{int,3} = \vec{P}_0 + b\,\vec{\omega}_1 - b\,\vec{\omega}_2, \tag{5.25}$$

where the charge vector $\vec{P}_0$, which is an arbitrary vector in the A_2 weight lattice, the parameter b that determines the Virasoro central charge, and the arbitrary real parameter a, are the external data that specify $\mathcal{B}$.

In the following, we focus on the solution of (5.22) that corresponds to the internal channel that carries the $\mathcal{W}_3$ module with highest weight vector $\vec{P}_{{\rm int},1}$. In this specific case, we obtain

$$\mathcal{B} \left(z | \vec{P}_0, \vec{P}_0 - b, \vec{P}_3 | -b, a \right) = \left(z^{\frac{b(2\ell_1+\ell_2)}{3}} (1-z)^{\frac{3+3b^2-ba}{3}} \right) {}_3F_2 \left(A_1, A_2, A_3; B_1, B_2; z \right) \tag{5.26}$$

5.4 Minimal $\mathcal{M}_3^{p,p'}$ models

The above results, obtained from general properties of $\mathcal{W}_3$ algebra, are valid for all values of the Virasoro central charge c. To check equation (1.2), we specialize to the $\mathcal{W}_3$ minimal models $\mathcal{M}_3^{p,p'}$, where p and p' are coprime integers that satisfy $3 \leqslant p < p'$. To do this, we set

$$b \to i\alpha_+, \quad b^{-1} \to i\alpha_-, \quad \alpha_+ = \left(\frac{p'}{p} \right)^{\frac{1}{2}}, \quad \alpha_- = -\left(\frac{p}{p'} \right)^{\frac{1}{2}} \tag{5.27}$$

so that, from equation (2.8), we obtain the Virasoro central charge

$$c_3^{p,p'} = 2 \left(1 - 12\,\alpha_0^2 \right) , \quad \alpha_0 = \alpha_+ + \alpha_- \tag{5.28}$$

Further, we associate each vertex operator $\mathcal{O}_{\vec{P}}$ to a highest weight vector $\vec{P}_{\vec{r}\,\vec{s}}$, where $\vec{r} = \{r_1, r_2\}$, $\vec{s} = \{s_1, s_2\}$, such that r_1, r_2, s_1 and s_2 are integers that satisfy

$$1 \leqslant \left(\sum_{i=1}^{2} r_i \right) \leqslant p, \quad 1 \leqslant \left(\sum_{i=1}^{2} s_i \right) \leqslant p' \tag{5.29}$$

In the sequel, we simplify the notation by writing the charge vector $\vec{P}_{\vec{r},\,\vec{s}}$ as $\vec{P}_{r_1, r_2; s_1, s_2}$, and the corresponding Virasoro and $\mathcal{W}_3$ eigenvalues $\Delta_{\vec{P}_{\vec{r},\,\vec{s}}}$ and $w_{\vec{P}_{\vec{r},\,\vec{s}}}$ as $\Delta_{\vec{r},\,\vec{s}}$ and $w_{\vec{r},\,\vec{s}}$.

5.5 Checking the modified AGT expression

We want to check equation (1.2) for the non-trivial conformal block, computed in (5.26), after specializing the parameters to minimal model ones. We choose $N = 3$, $p = 8$ and $p' = 9$, and consider the function (5.26) for the unitary minimal model $\mathcal{M}_3^{8,\,9}$, with

$$\vec{P}_0 = \vec{P}_{11;12}, \quad \vec{P}_3 = \vec{P}_{11;21}, \quad a = -b, \quad \vec{P}_{\text{int},1} = \vec{P}_0 - b\,\vec{\omega}_1 = \vec{P}_{21;12} \tag{5.30}$$

and compare with the result obtained by applying equation (1.2) to $\mathcal{B}_{3,1}^{8,\,9}[z|\vec{P}_0\,\vec{P}_{\text{int},1}\,\vec{P}_3| - b, -b]$, see equation (2.24). The $\mathcal{W}_3$ irreducible highest weight module that flows in the intermediate channel in this case is characterised by $\{p, p', r_1, r_2, s_1, s_2\} = \{8, 9, 2, 1, 1, 2\}$, and the triples of Young diagrams that are allowed by the 3-Burge conditions, in this case, for $|Y| = 0, 1, 2, 3$ and 4, where $|Y|$ is

$$|Y| = \sum_{i=1}^{3} |Y_i| \tag{5.31}$$

and $|Y_i|$, $i = 1, 2, 3$, is the number of cells in the i-th Young diagram, are

$$\begin{aligned}
&|Y| = 0: \left(\varnothing, \varnothing, \varnothing\right), \quad |Y| = 1: \left(\varnothing, \varnothing, \ydiagram{1}\right), \left(\varnothing, \ydiagram{1}, \varnothing\right), \left(\ydiagram{1}, \varnothing, \varnothing\right) \\
&|Y| = 2: \left(\varnothing, \varnothing, \ydiagram{2}\right), \left(\varnothing, \ydiagram{1}, \ydiagram{1}\right), \left(\varnothing, \ydiagram{2}, \varnothing\right), \left(\ydiagram{1}, \varnothing, \ydiagram{1}\right), \left(\ydiagram{1}, \ydiagram{1}, \varnothing\right), \\
&\qquad\qquad \left(\ydiagram{2}, \varnothing, \varnothing\right), \left(\ydiagram{1,1}, \varnothing, \varnothing\right) \\
&|Y| = 3: \left(\varnothing, \varnothing, \ydiagram{3}\right), \left(\varnothing, \ydiagram{1}, \ydiagram{2}\right), \left(\varnothing, \ydiagram{1}, \ydiagram{1,1}\right), \left(\varnothing, \ydiagram{2}, \ydiagram{1}\right), \left(\varnothing, \ydiagram{3}, \varnothing\right), \\
&\qquad\qquad \left(\ydiagram{1}, \varnothing, \ydiagram{2}\right), \left(\ydiagram{1}, \ydiagram{1}, \ydiagram{1}\right), \left(\ydiagram{1}, \ydiagram{2}, \varnothing\right), \left(\ydiagram{1}, \ydiagram{1,1}, \varnothing\right), \left(\ydiagram{2}, \varnothing, \ydiagram{1}\right), \\
&\qquad\qquad \left(\ydiagram{1,1}, \varnothing, \ydiagram{1}\right), \left(\ydiagram{2}, \ydiagram{1}, \varnothing\right), \left(\ydiagram{1,1}, \ydiagram{1}, \varnothing\right), \left(\ydiagram{3}, \varnothing, \varnothing\right), \left(\ydiagram{2,1}, \varnothing, \varnothing\right), \\
&\qquad\qquad \left(\ydiagram{1,1,1}, \varnothing, \varnothing\right)
\end{aligned}$$

$$
\begin{aligned}
|Y| = 4 :\ & \left(\varnothing, \varnothing, \yng(4)\right), \left(\varnothing, \yng(1), \yng(3)\right), \left(\varnothing, \yng(1), \yng(2,1)\right), \left(\varnothing, \yng(2), \yng(2)\right), \left(\varnothing, \yng(2), \yng(1,1)\right), \\
& \left(\varnothing, \yng(3), \yng(1)\right), \left(\varnothing, \yng(4), \varnothing\right), \left(\yng(1), \varnothing, \yng(3)\right), \left(\yng(1), \yng(1), \yng(2)\right), \left(\yng(1), \yng(1), \yng(1,1)\right), \\
& \left(\yng(1), \yng(2), \yng(1)\right), \left(\yng(1), \yng(1,1), \yng(1)\right), \left(\yng(1), \yng(3), \varnothing\right), \left(\yng(1), \yng(2,1), \varnothing\right), \left(\yng(2), \varnothing, \yng(2)\right), \\
& \left(\yng(1,1), \varnothing, \yng(2)\right), \left(\yng(2), \yng(1), \yng(1)\right), \left(\yng(1,1), \yng(1), \yng(1)\right), \left(\yng(2), \yng(2), \varnothing\right), \left(\yng(2), \yng(1,1), \varnothing\right), \\
& \left(\yng(1,1), \yng(2), \varnothing\right), \left(\yng(1,1), \yng(1,1), \varnothing\right), \left(\yng(3), \varnothing, \yng(1)\right), \left(\yng(2,1), \varnothing, \yng(1)\right), \left(\yng(1,1,1), \varnothing, \yng(1)\right), \\
& \left(\yng(3), \yng(1), \varnothing\right), \left(\yng(2,1), \yng(1), \varnothing\right), \left(\yng(1,1,1), \yng(1), \varnothing\right), \left(\yng(4), \varnothing, \varnothing\right), \left(\yng(3,1), \varnothing, \varnothing\right), \\
& \left(\yng(2,2), \varnothing, \varnothing\right), \left(\yng(2,1,1), \varnothing, \varnothing\right), \left(\yng(1,1,1,1), \varnothing, \varnothing\right)
\end{aligned}
$$

Considering the contribution of the allowed Young diagrams only, we obtain

$$
\begin{aligned}
\mathcal{B}^{8,9}_{3,1}\left(z|\vec{P}_0 \vec{P}^{(1)}_1 \vec{P}_3| - b, -b\right) = 1 &+ \frac{32}{135}z + \frac{101}{729}z^2 + \frac{64576}{649539}z^3 + \frac{124748}{1594323}z^4 \\
&+ \frac{30730880}{473513931}z^5 + \frac{970725028}{17433922005}z^6 + \frac{4604400320}{94143178827}z^7 + \mathcal{O}(z^8),
\end{aligned}
\tag{5.32}
$$

which coincides with ${}_3F_2$ on the right hand side of equation (5.26). In other words, $\mathcal{B}^{8,9}_{3,1}$ coincides with $\mathcal{B}$ in equation (5.4) up to the normalisation factor $z^{-\frac{b(2\ell_1+\ell_2)}{3}}$ and the Heisenberg factor. $(1-z)^{-\frac{3+3b^2-ba}{3}}$.

6 $\mathcal{W}_3$ minimal model characters from 3-Burge partitions

We compare the characters of degenerate $\mathcal{W}_3$ irreducible highest weight representations with the generating functions of triples of Young diagrams that obey 3-Burge conditions.

6.1 $\mathcal{W}_3$ minimal model characters

Expressions for the characters of degenerate irreducible $\mathcal{W}_N$ highest weight representations were computed in [29]. In the following, we specialise these expressions to the $N = 3$ case, explain what the various terms are, how to evaluate them, then compute examples of the characters in q-series form.[14]

The $\mathcal{W}_3$ minimal model character, labeled by two coprime integers p, p', such that $2 < p < p'$, and dominant integral weight vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, of level-[p-3] and level-[p'-3],

[14]The notation used in this section is close to that in [30].

respectively is

$$\chi^{p,p'}_{\boldsymbol{\eta},\boldsymbol{\xi}}(q) = \frac{1}{\eta(q)^2}\sum_{\sigma\in S_3}(-1)^{L_\sigma}\sum_{r,s=-\infty}^{\infty} q^{\frac{pp'}{2}\left\langle r\alpha_1+s\alpha_2,\ r\alpha_1+s\alpha_2\right\rangle}$$
$$\times\, q^{\left\langle p'\sigma\left(\sum_{i=0}^{2} n_i\vec{\omega}_i\right) - p\left(\sum_{i=0}^{2} m_i\vec{\omega}_i\right),\ r\alpha_1+s\alpha_2\right\rangle + \left\langle \sum_{i=0}^{2} n_i\vec{\omega}_i - \sigma\left(\sum_{i=0}^{2} n_i\vec{\omega}_i\right), \sum_{i=0}^{2} m_i\vec{\omega}_i\right\rangle} \tag{6.1}$$

We need to explain what the various terms in equation (6.1) stand for, and how to compute them. As mentioned above, p and p' are coprime integers that satisfy $2 < p < p'$. $\boldsymbol{\eta}$ is a level-[p-3] dominant integral weight vector, and $\boldsymbol{\xi}$ is a level-[p'-3] dominant integral weight vector. They are defined as

$$\boldsymbol{\eta} = (n_0-1)\vec{\omega}_0 + (n_1-1)\vec{\omega}_1 + (n_2-1)\vec{\omega}_2, \qquad n_0+n_1+n_2 = p, \tag{6.2}$$
$$\boldsymbol{\xi} = (m_0-1)\vec{\omega}_0 + (m_1-1)\vec{\omega}_1 + (m_2-1)\vec{\omega}_2, \qquad m_0+m_1+m_2 = p' \tag{6.3}$$

q is an indeterminate, and $\eta(q)$ is the Dedekind function

$$\eta(q) = q^{1/24}\prod_{i=1}^{\infty}(1-q^i) \tag{6.4}$$

S_3 is the symmetric group of degree 3, generated by the permutation operators s_1 and s_2,

$$S_3 = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \tag{6.5}$$

L_σ is the length function of a permutation σ, that is, the minimal number of S_3 generators required to generate σ. Denoting the integral vector $\sum_{i=0}^{2} n_i\vec{\omega}_i$, $n_i \in \mathbb{N}$, $i = 0,1,2$, by $[n_0,n_1,n_2]$, the action of σ on integral vectors in the $\widehat{A}_2$ weight lattice is

$$\begin{aligned}
1[n_0,n_1,n_2] &= [n_0,n_1,n_2], \\
s_1[n_0,n_1,n_2] &= [n_0+n_1, -n_1, n_1+n_2], \\
s_2[n_0,n_1,n_2] &= [n_0+n_2, n_1+n_2, -n_2], \\
s_1s_2[n_0,n_1,n_2] &= [n_0+2n_2+n_1, -n_1-n_2, n_1], \\
s_2s_1[n_0,n_1,n_2] &= [n_0+2n_1+n_2, n_2, -n_1-n_2], \\
s_1s_2s_1[n_0,n_1,n_2] &= [n_0+2n_1+2n_2, -n_2, -n_1]
\end{aligned} \tag{6.6}$$

The $\widehat{A}_2$ simple root vectors satisfy

$$\langle\vec{\alpha}_1,\vec{\alpha}_1\rangle = \langle\vec{\alpha}_2,\vec{\alpha}_2\rangle = 2, \langle\vec{\alpha}_1,\vec{\alpha}_2\rangle = -1 \tag{6.7}$$

The $\widehat{A}_2$ fundamental weight vectors satisfy

$$\langle\vec{\omega}_0,\vec{\omega}_0\rangle = \langle\vec{\omega}_0,\vec{\omega}_1\rangle = \langle\vec{\omega}_0,\vec{\omega}_2\rangle = 0, \langle\vec{\omega}_1,\vec{\omega}_1\rangle = \langle\vec{\omega}_2,\vec{\omega}_2\rangle = \frac{2}{3}, \langle\vec{\omega}_1,\vec{\omega}_2\rangle = \frac{1}{3} \tag{6.8}$$

From

$$\alpha_1 = -\vec{\omega}_0 + 2\vec{\omega}_1 - \vec{\omega}_2, \quad \alpha_2 = -\vec{\omega}_0 - \vec{\omega}_1 + 2\vec{\omega}_2, \tag{6.9}$$

we have

$$\langle \vec{\omega}_1, \alpha_1 \rangle = \langle \vec{\omega}_2, \alpha_2 \rangle = 1, \quad \langle \vec{\omega}_1, \alpha_2 \rangle = \langle \vec{\omega}_2, \alpha_1 \rangle = 0 \tag{6.10}$$

From the above equations, it is straightforward to show that

$$\begin{aligned} \chi^{p,p'}_{\boldsymbol{\eta},\boldsymbol{\xi}}(q) = F &\Big(n_1, m_1 | n_2, m_2\Big) - q^{n_1 m_1} F \Big(n_1, m_1 | n_1 + n_2, m_2\Big) \\ &- q^{n_2 m_2} F \Big(n_1 + n_2, m_1 | -n_2, m_2\Big) + q^{(n_1+n_2)m_1 + n_2 m_2} F \Big(n_1 - n_2, m_1 | n_1, m_2\Big) \\ &+ q^{(n_1+n_2)m_2 + n_1 m_1} F\Big(n_2, m_1 | -n_1 - n_2, m_2\Big) - q^{(n_1+n_2)(m_1+m_2)} F\Big(n_2, m_1 | -n_1, m_2\Big) \end{aligned} \tag{6.11}$$

$$F\Big(x_1, x_2 | y_1, y_2\Big) = \frac{1}{\eta(q)^2} \sum_{r,s \in \mathbb{Z}} q^{pp'(r^2+s^2-rs)+(p'x_1-py_1)r+(p'x_2-py_2)s}, \tag{6.12}$$

6.2 Examples

We find the following q-series expansions

$$\begin{aligned} \chi^{3,7}_{11|11} &= 1 + q^2 + 2q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 11q^8 + 14q^9 + 20q^{10} + \cdots, \\ \chi^{3,7}_{21|11} &= \chi^{3,7}_{11|21} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 14q^7 + 21q^8 + 28q^9 + 39q^{10} + \cdots, \\ \chi^{3,7}_{21|21} &= 1 + 2q + 3q^2 + 5q^3 + 8q^4 + 11q^5 + 17q^6 + 24q^7 + 34q^8 + 47q^9 + 64q^{10} + \cdots, \\ \chi^{3,7}_{31|11} &= \chi^{3,7}_{11|31} = 1 + q + 3q^2 + 3q^3 + 6q^4 + 8q^5 + 13q^6 + 17q^7 + 25q^8 + 33q^9 + 47q^{10} + \cdots \end{aligned} \tag{6.13}$$

Comparing the above expressions with those obtained from counting triples of Young diagrams that satisfy the 3-Burge conditions, we find that they coincide.

7 Summary and comments

We propose a modified $\mathcal{W}_N$ AGT prescription to allow one to compute conformal blocks $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$, from which one can extract $\mathcal{W}_N$ minimal model conformal blocks $\mathcal{B}^{p,p'}_{N,n}$.

7.1 $\mathcal{W}_N$ AGT leads to ill-defined expressions in minimal model conformal blocks

Applying the original, unmodified $\mathcal{W}_N$ AGT correspondence to the minimal $\mathcal{W}_N$ models, times contributions from a free boson, by setting the gauge theory mass and Coulomb parameters to minimal $\mathcal{W}_N$ model values, and *leaving all else the same*, leads to ill-defined expressions in the form of zero divided by zero. The origin of these ill-defined expressions was explained in the context of $\mathcal{W}_2$ in [13, 14]. We review it below.

7.1.1 Norms and couplings of states that flow in channels

The original prescription allows for all states in a specific $\mathcal{W}_N$ Verma module to flow in each specific channel. The norms of these states appear in the denominators of Nekrasov's instanton partition function. Their coupling to other states are given by matrix elements of the $\mathcal{W}_N \times \mathcal{H}$ algebra. These matrix elements appear as the factors in the numerators. When the central charge is non-minimal, there are no zero-norm states in the Verma module, all terms in the denominators are non-vanishing, the expressions are well defined regardless of whether the corresponding matrix elements that describe the couplings to other states are zero or not, and one obtains the correct result. When the central charge is minimal, the situation is drastically different.

7.1.2 The zeros in the denominators

When the central charge is minimal, there are zero-norm states in the Verma module. Including these states in the sums, one obtains zeros in the denominators. This is the origin of the zeros that appear in the denominators of Nekrasov's partition functions if we apply the AGT prescription without modification. They indicate that we have included zero-norm states among the states that flow in the channels of the conformal blocks.

7.1.3 The zeros in the numerators

These zeros are due to the vanishing of the coupling of the zero-norm states and all other states. In [14], it was shown, in the case of Virasoro minimal models, that for every zero in a denominator, there is a zero in the numerator, but the reverse is not true. In other words, the set of terms that contain a zero in the denominator is a proper subset of the set of terms that contain a zero in the numerator. We have not shown that this is the case here, since we do not need it for the purposes of this paper, but it is a straightforward, albeit tedious exercise to show that this is the case. This ensures that one never has terms in the form of a finite number divided by zero, that are strictly infinite, but that one has ill-defined terms in the form of zero divided by zero.

7.1.4 Resolving the ambiguities

Assuming that for every zero in a denominator, there is a higher-degree zero in the numerator, one way to avoid the ill-defined expressions described above is to deform the conformal field theory away from minimality, such that all denominators become non-zero, then carefully prove that the minimal limit exists, presumably by showing that the numerators are always zero, or always vanish faster than the denominators. We are able to do this in simple, specific examples, but we have no proof that this is always the case. In this work, we pursue a different approach.

7.2 Modifying $\mathcal{W}_N$ AGT to apply to minimal model conformal blocks

In this work, as in [13, 14], we avoid the ill-defined expressions by restricting the summations over the N-partitions that appear in the sum (3.1). We start from the original expression for $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ in terms of sums of type (3.1), each of which is a product of building blocks Z^{ι}_{bb}. We characterize the singularities in Z^{ι}_{bb} that lead to ill-defined expressions, and eliminate these zero-norm states by restricting the N-partitions that appear in (3.1) to N-partitions $\vec{Y} = \{Y_1, \cdots, Y_N\}$, that satisfy the conditions N-Burge conditions, which we recall here,

$$Y_{i,\mathtt{R}} - Y_{i+1,\mathtt{R}+s_i-1} \geqslant -r_i + 1 \tag{7.1}$$

where $Y_{i,\mathtt{R}}$ is the $\mathtt{R}$-row of $Y_i, i = 1, \cdots, N$, r_ι and s_ι are parameters that characterise the $\mathcal{W}_N$ irreducible highest weight module that flows in a channel in a minimal model conformal block, and satisfy equation (2.12), and $Y_{N+1} = Y_1$. *Note that we characterise the Young diagrams that do not lead to zeros, and only these. In other words, the Burge conditions are sufficient and necessary conditions for the procedure to work.* This is the reason why in section 6, we obtain the correct character expressions.

For $N = 2$, the *N-Burge partitions* were introduced in [15], and further studied in [16]. They appeared in full generality in [17], and in the form used in this work in [18]. We have shown that when used to restrict AGT to compute $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$, we obtain the expressions which we recall here,

$$\mathcal{B}^{p,p',\mathcal{H}}_{N,n} = \sum_{\vec{Y}^1,\cdots,\vec{Y}^n}{}^{'} \prod_{\iota=1}^{n+1} q_\iota^{|\vec{Y}^\iota|} Z^\iota_{bb} \left(\vec{P}_{\vec{r}_{\iota-1}\ \vec{s}_{\iota-1}}, \vec{Y}^{\iota-1} \mid a_{m_\iota n_\iota} \mid \vec{P}_{\vec{r}_\iota\ \vec{s}_\iota}, \vec{Y}^\iota \right) \tag{7.2}$$

where $\sum'$ indicates that the sum is restricted to N-partitions that satisfy the N-Burge conditions (7.1), which are well-defined expressions that we identify with $\mathcal{W}_N$ minimal model conformal blocks, times Heisenberg factors. We check our identification in a non-trivial case, and show that it produces the correct 0-point conformal blocks on the torus, in specific cases.

7.3 Related works

1. In [31], Santachiara and Tanzini apply AGT to compute conformal blocks of $\{r, s\} = \{1, 2\}$ and $\{2, 1\}$ vertex operators in Virasoro minimal models. The ill-defined expressions were circumvented using an analytic continuation scheme that was tested to low orders in the combinatorial expansion of the instanton partition functions.

   If one can extend the analytic continuation scheme used in [31] to the full instanton partition functions of the most general conformal blocks, and obtain the same result as in the present work, then this would amount to a proof that the proposed modified AGT expression for $\mathcal{B}^{p,p',\mathcal{H}}_{N,n}$ in equation (1.2) is indeed the required minimal model conformal block up to a Heisenberg factor.

2. In [32], Estienne, Pasquier, Santachiara and Serban study conformal blocks of vertex operators such that $r_1 = 2$, and $r_i = 1$, $i = 2, \cdots, N-1$, and $s_i = 1$, $i = 1, \cdots, N$, or $r_i = 1$, $i = 1, \cdots, N$, $s_1 = 2$, and $s_i = 1$, $i = 2, \cdots, N-1$. in $W_N^{p,p'} \oplus \mathcal{H}$ minimal models. From the null-state conditions of these vertex operators, Estienne et al. show that these specific conformal blocks are labeled by N-partitions that satisfy specific conditions. While the notation used in [32] is different from that in this work, one can check, in simple cases, that their N-partitions are equivalent to those that appear in this work.

3. In [33], Fucito, Morales and Poghossian show that $\mathcal{N} = 2$ supersymmetric Yang-Mills gauge theories on the squashed S^4, with rational deformation parameters, are dual to Virasoro minimal models. Ill-defined expressions are handled using a deformation

scheme, akin to that used in [31], and rested to low orders in the combinatorial expansion of the instanton partition functions.

4. In [13, 14], as outlined in section 1, Virasoro minimal model conformal blocks are derived, via a modification of the AGT prescription, from the instanton partition functions of $\mathcal{N}=2$ supersymmetric U(2) quiver gauge theories. In [34], the building block of the instanton partition functions that appeared in [13, 14] is derived by gluing four copies of refined topological vertices [35] to form the partition function of a *strip geometry*, then choosing the gluing parameters and the partitions that label the unglued external legs of the strip appropriately. In [36], the building block of the instanton partition function that is used in the present work to generate $\mathcal{W}_N$ minimal model conformal block is derived from refined topological vertices, using vertex operator methods, along the lines of [37, 38].

5. In [43], Fukuda, Nakamura, Matsup and Zhu studied the representation theory of SH^c, the central extension of the degenerate double affine Hecke algebra [6, 26] in the context of the minimal $\mathcal{W}_N$ models. They found, among other results, that the states are labelled by N-partitions that satisfy the N-Burge conditions discussed in this work.

7.4 Open problems

1. This work may be regarded as an attempt to understand W_N minimal model conformal blocks, that is, expectation values of degenerate $\mathcal{W}_N$ vertex operators, in 2D conformal field theories, in terms of instanton partition functions in 4D $\mathcal{N}=2$ supersymmetric gauge theories. However, the meaning of the choice of gauge theory parameters that lead to minimal $\mathcal{W}_N$ theories, as well the interpretation of the N-Burge conditions at the level of 4D gauge theories remains unclear. One way to address these issues is to use the interpretation of the 2D degenerate vertex operators in terms of 4D surface operators along the lines of [39, 40], where the expectation value of an elementary surface operator, in a 4D $\mathcal{N}=2$ supersymmetric gauge theory, is shown to be equal to the expectation value of vertex operators in a 2D Liouville conformal field theory, in the presence of a degenerate vertex operator of type $\mathcal{O}_{2,1}(z)$.

   The literature on the 2D degenerate vertex operator/4D surface operator connection is extensive, and beyond the limited scope of this work, see [41] for a review. But we expect that the adaptation of 2D degenerate operator/4D surface operator connection to AGT in the context of minimal models will help clarify the issues outlined above.

2. In the present work, we have restricted our attention to W_N conformal blocks that satisfy the FLW conditions of subsection 1.3. In [42], Gomis and LeFloch propose that one can obtain $\mathcal{W}_N$ Toda conformal blocks that are expectation values of vertex operators that include degenerate $\mathcal{W}_N$ vertex operators that do not satisfy the FLW conditions, in addition to non-degenerate vertex operators, and interpret the degenerate operators at the gauge theory level as surface operator insertions. More precisely, the proposal of Gomis and LeFloch is that one can obtain the degenerate

vertex operator insertions that do not satisfy the FLW conditions by starting from vertex operators insertions that satisfy the conditions, then bringing the latter together in a form of operator product expansion. While formally plausible, it is not clear to us at this stage whether the proposal of Gomis and LeFloch leads to tractable results along the lines of the $\mathcal{W}_N$ AGT results presented in this work.

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