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Matching 3d $N = 2$ vortices and monopole operators

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Abstract: In earlier work with N. Seiberg, we explored connections between monopole operators, the Coulomb branch modulus, and vortices for 3d, $\mathcal{N} = 2$ supersymmetric, $U(1)_k$ Chern-Simons matter theories. We here extend the monopole / vortex matching analysis, to theories with general matter electric charges. We verify, for general matter content, that the spin and other quantum numbers of the chiral monopole operators match those of corresponding BPS vortex states, at the top and bottom of the tower associated with quantizing the vortices' Fermion zero modes. There are associated subtleties from non-normalizable Fermi zero modes, which contribute non-trivially to the BPS vortex spectrum and monopole operator matching; a proposed interpretation is further discussed here.

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1 Introduction

Three-dimensional U(1) gauge theories exhibit IR-interesting phenomena and phases, with qualitative similarities to 4d non-Abelian gauge theories. For example, electric-magnetic dualities can be explored in this context, and the U(1) gauge group makes it easier to make the duality more precise, and potentially construct the duality-map between fields. This is particularly true for 3d theories with $\mathcal{N} \geq 2$ supersymmetry, where magnetically charged, BPS vortex solitons can be regarded as giving the dual quanta in terms of the electric variables, with corresponding chiral superfield monopole operators.

Building on [1], we here consider 3d, $\mathcal{N} = 2$ supersymmetric, compact[1] $U(1)_k$ gauge theory (k is the Chern-Simons coefficient), with matter chiral superfields Q_i, with general

[1] I.e. gauge transformations are $A_\mu \to A_\mu + \partial_\mu f$, with $f \sim f + 2\pi$, which requires $n_i \in \mathbb{Z}$ and $q_J \in \mathbb{Z}$. However, $U(1) \not\subset SU(2)$, so there is no instanton sum. The monopole operators here are of singular, Dirac-type, with unobservable string thanks to the quantization conditions.

electric charges $n_i \in \mathbb{Z}$. A key aspect is that the theory has an exact,[2] conserved global $\mathrm{U}(1)_J$ topological symmetry, with current $j_J^\mu = \epsilon^{\mu\rho\sigma}F_{\rho\sigma}/4\pi$, and associated charge

$$\mathrm{U}(1)_J: \qquad q_J = \int \frac{F_{12}}{2\pi} \in \mathbb{Z}. \tag{1.1}$$

The theory contains local operators, and particle states, with $q_J \neq 0$, despite the fact that the photon and Q_i have $q_J = 0$. There are three distinct, related ways to get $q_J \neq 0$:

1. Monopole operators: disorder the gauge field, with q_J units of magnetic flux, around a point x_0^μ in spacetime [4–6]. It is a local, chiral $\mathcal{N}=2$ operator (the 3d reduction of 4d 't Hooft line operators). This short-distance definition of the operator is independent of IR data, e.g. the particular vacua, or the spacetime geometry. The chiral condition implies that the real scalar $\sigma = \Sigma|$ of the $\mathcal{N}=2$ photon linear multiplet has [1, 6]

$$\sigma(x) \to \frac{q_J}{r_{3d}}, \qquad \text{where} \qquad r_{3d} \equiv ||x^\mu - x_0^\mu||_{\text{Euclidean}}. \tag{1.2}$$

2. On the $\sigma \neq 0$ Coulomb branch, if it exists, $\mathrm{U}(1)_J$ is spontaneously broken and the associated, compact NG boson, $a \sim a + 2\pi$, can be identified with the dualized photon [3]. The gauge field linear multiplet $\Sigma = -\frac{i}{2}\overline{D}DV$ can be dualized to chiral superfields [7, 8], and exponentiated to obtain chiral operators [9] with $\mathrm{U}(1)_J$ charge $q_J = \pm 1$:

$$X_\pm \sim e^{\pm(2\pi\sigma/e^2_{\text{eff}} + ia)}. \tag{1.3}$$

The microscopic, monopole disorder operator of the theory at the origin is also denoted as $X_\pm$, with (1.3) its low-energy effective description. The $\mathrm{U}(1)_J$ charge q_J chiral operator is $X_+^{q_J}$ for $q_J > 0$, or $X_-^{|q_J|}$ for $q_J < 0$. The X_+ or X_- monopole operator is only $\mathrm{U}(1)_{\text{gauge}}$ neutral if the corresponding Coulomb branch exists.

3. BPS vortex particle field configurations exist in certain Higgs vacua, $\langle Q_i \rangle \neq 0$, when the FI parameter $\zeta \neq 0$. Their BPS mass is $m = |Z| = |\zeta q_J|$. Using $z = x + iy$ for the 2d spatial plane, the gauge field $A_z \equiv \frac{1}{2}(A_x - iA_y)$ and matter wind at infinity as

$$A_z \to \frac{q_J}{2iz} + \ldots, \qquad \text{for} \quad |z| \to \infty \tag{1.4}$$

$$Q_i \to e^{-in_i q_J \theta}\left(Q_i^{\text{vac}} + \frac{\rho_i}{|z|} + \ldots\right) \qquad \text{for} \quad |z| \to \infty. \tag{1.5}$$

Upon taking $\zeta \to 0$, all $Q_i^{\text{vac}} \to 0$, the BPS magnetic vortices become massless, and can potentially condense and give dual Higgs description of the Coulomb branch [9], in the sense of 3d mirror symmetry's exchange of the electric and magnetic Higgs and Coulomb branches [10]. See also [11, 12] for vortices and partition functions.

[2] In other theories, $\mathrm{U}(1)_J$ can be explicitly broken by short-distance physics, which can add monopole operators to $\mathcal{L}_{\text{eff}}$; then $\mathrm{U}(1)_J$ is at best an accidental, approximate symmetry, if those operators are irrelevant, or a fine-tuning if they are not. E.g. if $\mathrm{U}(1) \subset \mathrm{SU}(2)$, instantons in the UV $\mathrm{SU}(2)/\mathrm{U}(1)$ explicitly break $\mathrm{U}(1)_J$ [2, 3]. In our susy context, the monopole operators are chiral superfields, and holomorphy constrains their possible appearance in the superpotential.

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Connections and distinctions between monopole operators, vortices, and the Coulomb branch, for the theories in flat space, were studied in [1, 13], and will be further explored here. We determine, and match, the gauge and global charges of monopole operators and the vortices. For the monopole operators $X_\pm$, the charges are simply, and exactly, obtained by a one-loop calculation of induced Chern-Simons terms [1, 9, 14, 15] to be

	$\mathrm{U}(1)_{\mathrm{gauge}}$	$\mathrm{U}(1)_j$	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_J$
Q_i	n_i	δ_{ij}	0	0
$X_\pm$	$-(k_c \pm k)$	$-\frac{1}{2}\lvert n_j\rvert$	$\frac{1}{2}\sum_i \lvert n_i\rvert$	± 1

(1.6)

with $k_c \equiv \frac{1}{2}\sum_i n_i|n_i|$ (see section 2). The operators $X_\pm$ in (1.6) exist as gauge invariant operators[3] only if $k = \mp k_c$; this is the condition for the $\langle X_\pm \rangle$ Coulomb branch to exist.

The corresponding charges of BPS vortices arise in a seemingly different way, from quantizing the vortex Fermion zero modes,[4] Ψ_A, with $A = 1 \dots N_z$, i.e. from

$$\{\Psi_A, \Psi_B^\dagger\} = \delta_{AB}, \qquad A, B = 1 \dots N_z. \tag{1.7}$$

This formally gives a tower of 2^{N_z} degenerate states: treating the Ψ_A ($\Psi_A^\dagger$) as raising (lowering) operators, the top and bottom vortex states in this tower are

$$|\Omega_\pm\rangle_{q_J}, \qquad \text{with} \qquad \Psi_A^\dagger|\Omega_+\rangle_{q_J} = \Psi_A|\Omega_-\rangle_{q_J} = 0, \tag{1.8}$$

$$|\Omega_+\rangle_{q_J} \sim \prod_A \Psi_A^\dagger |\Omega_-\rangle_{q_J}, \qquad \text{and} \qquad |\Omega_-\rangle_{q_J} \sim \prod_A \Psi_A |\Omega_+\rangle_{q_J}. \tag{1.9}$$

Writing "$|0\rangle$"$_{q_J}$ as the naive (ignoring zero modes) groundstate for $q_J \neq 0$,

$$|\Omega_\pm\rangle_{q_J} \sim \left(\prod_A \Psi_A\right)^{\mp\frac{1}{2}} \text{“}|0\rangle\text{”}_{q_J}. \tag{1.10}$$

We identify the $X_\pm$ quanta with the top and bottom vortex states:

$$|\Omega_+\rangle_{q_J=\pm 1} \sim X_\pm|0\rangle \qquad \text{and} \qquad |\Omega_-\rangle_{q_J=\pm 1} \sim X_\mp^\dagger|0\rangle, \tag{1.11}$$

with $|0\rangle$ the $q_J = 0$ vacuum. We verify that the vortex charges, computed from (1.10), are indeed compatible with (1.11) and the $X_\pm$ charges in (1.6).

This matching was verified in [1] for theories with N matter fields Q_i, with all $n_i = 1$. The $N = 1$ case is the classic $\mathcal{N} = 2$ susy Abelian Higgs model,[5] and its vortices and zero modes have been studied in e.g. [18–26]. Its $|q_J| = 1$ vortex has one complex Fermion zero mode, Ψ_1 and (1.7) leads to the BPS or anti-BPS doublet, $|\Omega_\pm\rangle_{q_J}$. For $N > 1$, there is

[3]The superconformal $\mathrm{U}(1)_{R_*}$ of the $\mathcal{N} = 2$ SCFT at $Q_i = X_\pm = 0$, is a linear combination of those in (1.6), $\mathrm{U}(1)_{R_*} = \mathrm{U}(1)_R + \sum_j R_j \mathrm{U}(1)_j$, so $\Delta(Q_i) = R_i$, and $\Delta(X_\pm) = R(X_\pm) = \frac{1}{2}\sum_i |n_i|(1 - R_i)$, with R_i determined by F-extremization [16] (or τ_{RR} minimization [14, 17]).

[4]Vs on $S^2 \times \mathbb{R}$, where the $\sigma \neq 0$ in (1.2) lifts all of the monopole operator's Fermi zero modes [6]. This fits with the radial quantization map between energy on $S^2 \times \mathbb{R}$ and operator dimension.

[5]Though the Chern-Simons term must be included, $k \in \mathbb{Z} + \frac{1}{2}$, reflecting the parity anomaly.

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a subtlety [1]: the extra matter fields lead to $\sim 1/|z|$ non-normalizable (log-IR-divergent) Bose and Fermi zero modes, generalizing those found in [27–30]. The ρ_i terms in (1.5), allowed for matter with $Q_i^{\rm vac}=0$, are examples; the interpretation of [1] is that they are actually *vacuum* parameters. The matching (1.11) requires that the non-normalizable Fermi zero modes nevertheless be included among the quantized Ψ_A in (1.7) and (1.10).

We here extend the analysis to theories with general matter charges n_i. We find that, in the $q_J=\pm1$ vortex background (for $\zeta>0$), the Fermion component of Q_i leads to $|n_i|$ zero modes, $\Psi_{i,p=1\ldots|n_i|}$, with charges[6] and spin given by (again, $k_c\equiv\frac{1}{2}\sum_i n_i|n_i|$):

	$\mathrm{U}(1)_{\rm gauge}$	$\mathrm{U}(1)_{\rm spin}$	$\mathrm{U}(1)_j$	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_J$
"$\lvert 0\rangle$"$_{q_J=\pm1}$	$\mp k$	$-\frac{1}{2}k$	0	0	±1
$\Psi^{(q_J=\pm1)}_{i,p=1\ldots\lvert n_i\rvert}$	n_i	$\pm\frac{n_i}{\lvert n_i\rvert}(p-\frac{1}{2})$	δ_{ij}	-1	0
$\prod_{i,p}\Psi^{(q_J=\pm1)}_{i,p}$	$2k_c$	$\pm k_c$	$\lvert n_j\rvert$	$-\sum_i\lvert n_i\rvert$	0
$\lvert\Omega_\pm\rangle_{q_J=1}$	$\mp k_c-k$	$\mp\frac{1}{2}k_c-\frac{1}{2}k$	$\mp\frac{1}{2}\lvert n_j\rvert$	$\pm\frac{1}{2}\sum_i\lvert n_i\rvert$	1

(1.12)

Quantizing the $\Psi_{i,p}$ gives a tower of $2^{\sum|n_i|}$ degenerate vortex states. The top and bottom states $|\Omega_\pm\rangle_{q_J}$, as in (1.8), have quantum numbers that follow from (1.12) and (1.10); this gives the charges of $|\Omega_\pm\rangle_{q_J=1}$ in (1.12). These $|\Omega_\pm\rangle_{q_J=1}$ charges indeed agree with those of X_+ and $X_-^\dagger$ in (1.6), fitting with the proposed operator / state map in (1.11).

As we will see, the $|q_J|=1$ Fermi zero modes in (1.12) have large z behavior (from (1.4)) $|\Psi_{i,p}|\sim|z|^{p-1-|n_i|}$, and the $p=|n_i|$ case is non-normalizable, for every matter field. As in [1], we quantize all Fermi zero modes as in (1.7), including the non-normalizable ones, and interpret the non-normalizable Fermi zero modes as mapping between different Hilbert spaces. But some additional discussion is required here, particularly for theories with $k=k_c=0$. Then both X_+ and X_- exist in the same theory, corresponding to the two Coulomb branches. Fitting with (1.11), both $|\Omega_+\rangle_{q_J=1}$ and $|\Omega_-\rangle_{q_J=1}$ in (1.12) have $\mathrm{U}(1)_{\rm spin}$ zero, and can condense to give the X_+ or X_- branches. But $|\Omega_+\rangle_{q_J=1}$ and $|\Omega_-\rangle_{q_J=1}$ are are related via non-normalizable Fermi zero modes. The BPS quanta created by X_+ and $X_-^\dagger$ evidently must reside in different Hilbert spaces, which seems puzzling.

Our (tentative) interpretation is that this reflects the fact that X_+ and X_- label two disconnected branches of the moduli space of vacua, i.e. that $X_+X_-\sim0$ in the chiral ring. Quantum field theories typically do not have a Hilbert space of single-particle states, with a mapping between them via normalizable zero modes. To the extent that it can happen for BPS states relies on the x-independence of the chiral ring OPE. If a product of chiral operators is zero in the chiral ring, the associated BPS states can appear to reside in different Hilbert spaces. We discuss this further in section 5, e.g. for $N_f=1$ SQED, and its $W=MX_+X_-$ dual. It would be good to have a more complete understanding.

The outline of the remaining sections is as follows. Section 2 briefly reviews some of the basic points, and sets up our notation and conventions; a few more details are in an

[6]Here $\mathrm{U}(1)_{\rm gauge}$ is Higgsed, so the $\mathrm{U}(1)_{\rm gauge}$ charges given here are screened by the $\langle Q_i\rangle$.

appendix. Section 3 broadly discusses the BPS vortices, and their zero modes, for the general $\mathcal{N}=2$ susy, $\mathrm{U}(1)_k$ charge n_i matter theories. Section 4 discusses vortices and zero modes in general cases with a vev $\langle Q_i\rangle \propto \delta_{i,1}$, with Q_1 of charge $n_1=1$. Section 5 considers theories with $N_\pm$ matter fields of charge $n_i=\pm1$, e.g. $\mathcal{N}=2$ SQED with $N_+=N_-=N_f$ flavors. Section 6 discusses cases where $\langle Q_i\rangle\neq 0$ for matter with charge $n_i\neq 1$, where there can be an unbroken $\mathbb{Z}_{|n_i|}$ discrete gauge symmetry, i.e. an orbifold.

One could generalize to non-Abelian gauge theories; it will not be considered here.

2 A few preliminaries (see also the appendix)

2.1 Lagrangian and effective Chern-Simons terms

The $\mathrm{U}(1)_k$ gauge theory, with matter fields Q_i of charges n_i, has classical Lagrangian

$$\mathcal{L}_{\mathrm{cl}} \supset \int d^4\theta\Big(-\frac{1}{e^2}\Sigma^2 - \frac{k}{4\pi}\Sigma V - \frac{\zeta}{2\pi}V + \sum_j Q_j^\dagger e^{2n_jV+2im_j\theta\bar\theta}Q_j\Big). \tag{2.1}$$

We will set the real masses $m_i=0$, and take $W_{\mathrm{tree}}=0$. Dirac-quantization for monopole operators implies that the Chern-Simons coefficient k is quantized as

$$k+\frac{1}{2}\sum_i n_i^2\in\mathbb{Z};\qquad \text{equivalently,}\qquad k+\frac{1}{2}\sum_i n_i\in\mathbb{Z}\ \ . \tag{2.2}$$

The supersymmetric vacua have expectation values of the Coulomb modulus $\sigma=\Sigma|$, or the matter fields $Q_i=Q_i|$, subject to the conditions $D=0$ and $m_j(\sigma)Q_j=0$, where

$$D=-e^2\Big(\sum_i n_i|Q_i|^2-\frac{\zeta_{\mathrm{eff}}}{2\pi}-\frac{k_{\mathrm{eff}}}{2\pi}\sigma\Big), \tag{2.3}$$

and $m_i(\sigma)\equiv m_i+n_i\sigma$. The effective FI parameter ζ_{eff}, and Chern-Simons coefficient k_{eff} in (2.3) are shifted by integrating out massive matter, with $\zeta_{\mathrm{eff}}=\zeta$ for $m_i=0$ and

$$k_{\mathrm{eff}}(\sigma)=k+\frac{1}{2}\sum_i n_i^2\mathrm{sign}(m_i(\sigma))\in\mathbb{Z}. \tag{2.4}$$

"Higgs" susy vacua have $(Q_i\neq0,\ \sigma=0)$, while "Coulomb" vacua have $(Q_i=0,\ \sigma\neq0)$ and $k_{\mathrm{eff}}=\zeta_{\mathrm{eff}}=0$. The asymptotic values of k_{eff} for $\sigma\to\pm\infty$ are

$$k_{\mathrm{eff}}(\sigma=\pm\infty)=k\pm k_c,\qquad k_c\equiv\frac{1}{2}\sum_i n_i|n_i|. \tag{2.5}$$

So the $\sigma\to\pm\infty$ asymptotic regions of the Coulomb branch only exist if (2.5) vanishes, i.e. if $k=\mp k_c$, respectively. For non-zero k_{eff} and ζ_{eff}, there are also isolated "topological vacua", with $Q_i=0$ and $\sigma=-\zeta_{\mathrm{eff}}/k_{\mathrm{eff}}$; those vacua will not enter in our discussion.

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2.2 Chern-Simons contribution to Gauss' law, and charges and spin from q_J

The Chern-Simons term affects Gauss' law (the A_0 EOM), as

$$-\frac{1}{e^2}\partial_i F_{0i} = \rho_{\rm matter} - \frac{k}{2\pi}F_{12}, \tag{2.6}$$

with $\rho_{\rm matter} \equiv \frac{\delta \mathcal{L}_{\rm matter}}{\delta A^0}$ the matter contribution to the electric charge density (see the appendix for our sign conventions). The "k" in (2.6) is the classical value when we consider the theory at $\sigma = 0$, where the matter Fermions are massless and kept in the low-energy theory. On the other hand, for $\sigma \neq 0$, the matter Fermions are massive and can be integrated out, and then we should replace k in (2.6) with $k_{\rm eff}$, as in (2.4).

The Chern-Simons contribution in (2.6) implies that operators or states with $q_J \neq 0$ acquire an associated electric charge, and a related contribution to their spin [31–33]

$$q_{\rm elec} = -kq_J, \qquad \Delta s = -\frac{1}{2}kq_J^2; \tag{2.7}$$

with $k \to k_{\rm eff}$ in (2.7) if the Fermions are massive and integrated out. For vortices, if $k \neq 0$, the last term in (2.6) leads to $A_0 \neq 0$, which complicates the equations of motion.

The gauge and global charges of the $X_\pm$ operators in (1.6) follow from (2.7), and its analogs for mixed gauge-flavor Chern-Simons coefficients. Since $X_\pm$ extend to $\sigma = \pm\infty$, we replace $k \to k_{\rm eff}(\sigma = \pm\infty) = k \pm k_c$ (2.5), and use $q_J \to \pm 1$ in (2.7) to obtain the $U(1)_{\rm gauge}$ charges of $X_\pm$ in (1.6). The $\sigma \to \pm\infty$ Coulomb branch only exists if $k_{\rm eff} = 0$, which is the condition for $X_\pm$ to be a gauge invariant, scalar operator:

$$\text{If } k = \mp k_c, \text{ then the } X_\pm \text{ Coulomb branch exists.} \tag{2.8}$$

The $U(1)_i$ and $U(1)_R$ global charges of $X_\pm$ in (1.6) likewise follow immediately from the one-loop induced, mixed Chern-Simons terms between the gauge field and background gauge fields coupled to the global currents [1, 9, 14, 15]. Integrating out the matter Fermion components of Q_i in (1.6), of mass $m_i(\sigma) = n_i\sigma$, gives mixed CS terms $k_{\rm eff}^{gauge,U(1)_j} = \frac{1}{2}n_j{\rm sign}(n_j\sigma)$ and $k_{\rm eff}^{gauge,U(1)_R} = -\frac{1}{2}\sum_i n_i{\rm sign}(n_i\sigma)$. Taking $\sigma \to \pm\infty$ for $q_J = \pm 1$, the analog (2.7) for the global charges then gives the corresponding charges in (1.6).

2.3 BPS and anti-BPS particles

Particle states can be labelled by their $U(1)_{\rm spin}$, s, and it is convenient to convert the spinors to a rotational spin-diagonal basis ($s = 1$ for $z = x^1 + ix^2$ and $\partial_{\bar z} = \frac{1}{2}(\partial_{x^1} + i\partial_{x^2})$). For the supercharges, we define (fixing a minor notational issue vs [1])

$$Q_\pm \equiv \frac{1}{2}(Q_1 \mp iQ_2), \qquad \overline{Q}_\pm \equiv \overline{Q_\mp} = \frac{1}{2}(\overline{Q}_1 \mp i\overline{Q}_2), \tag{2.9}$$

so $Q_\pm$ and $\overline{Q}_\pm$ have spin $s = \pm\frac{1}{2}$. In terms of these, the $\mathcal{N} = 2$ algebra is

$$\{Q_\pm, \overline{Q}_\pm\} = \mp i(P_1 \pm iP_2), \qquad \{Q_\pm, \overline{Q}_\mp\} = P^0 \pm Z. \tag{2.10}$$

A BPS particle, with $m = Z$, has

$$Z > 0: \qquad Q_-|BPS\rangle = \overline{Q}_+|BPS\rangle = 0, \tag{2.11}$$

and the remaining two supercharges make a two-dimensional representation

$$Z > 0: \qquad |BPS\rangle = \begin{pmatrix} |a\rangle \\ |b\rangle \end{pmatrix}, \qquad \overline{Q}_-|a\rangle = 0, \qquad |b\rangle = Q_+|a\rangle. \tag{2.12}$$

Likewise, an anti-BPS particle has $m = -Z > 0$, and is annihilated by Q_+ and $\overline{Q}_-$. Every BPS state has a CPT conjugate anti-BPS state, with opposite global charges and Z, but with the same $\mathrm{U}(1)_{\mathrm{spin}}$ spin s. The R-charges and spins of these states are [1]

	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_{\mathrm{spin}}$	Z
$\|a\rangle$	r	s	> 0
$\|b\rangle$	$r-1$	$s+\frac{1}{2}$	> 0
$\|\bar{a}\rangle$	$-r$	s	< 0
$\|\bar{b}\rangle$	$-r+1$	$s+\frac{1}{2}$	< 0

(2.13)

3 BPS and anti-BPS vortices

The central term of the supersymmetry algebra (setting real masses $m_i = 0$) is

$$Z = \zeta q_J. \tag{3.1}$$

For $Z > 0$, the vortex can be BPS, annihilated by Q_- and $\overline{Q}_+$ (2.11). For $Z < 0$, the vortex is anti-BPS, annihilated by $\overline{Q}_-$ and Q_+. The condition that these supercharges annihilate the background implies the BPS equations for a *static* (all $\partial_t \to 0$) vortex with $Z > 0$ (resp, a $Z < 0$ anti-BPS vortex) are (with $D_z^{(n_j)} \equiv \frac{1}{2}(D_1 - iD_2)^{(n_j)} \equiv \partial_z + in_jA_z$)

$$\sigma = \pm A_0, \tag{3.2}$$

$$F_{12} = \pm D, \tag{3.3}$$

$$D_z^{(n_j)} Q_j = 0, \qquad \text{resp} \qquad D_{\overline{z}}^{(n_j)} Q_j = 0, \tag{3.4}$$

with D given by (2.3). One must also impose Gauss' law (2.6). In our conventions, the chiral superfields, Q_i, of a $Z > 0$ BPS vortex are *anti*[7]-holomorphic (resp holomorphic for a $Z < 0$ anti-BPS vortex). We will here be particularly interested in the zero modes.

The vortex's Fermi zero modes are the static $\partial_t \to 0$ solutions of the Fermion equations of motion, from (2.1) with $m_i = 0$, in the background of the static vortex's Bosonic fields:

$$\begin{pmatrix} i\frac{k}{4\pi} & 2e^{-2}\partial_{\overline{z}} \\ -2e^{-2}\partial_z & i\frac{k}{4\pi} \end{pmatrix} \begin{pmatrix} \overline{\lambda}_\uparrow \\ \overline{\lambda}_\downarrow \end{pmatrix} - \sqrt{2}\sum_j n_j Q_j^* \begin{pmatrix} \psi_{j\uparrow} \\ \psi_{j\downarrow} \end{pmatrix} = 0, \tag{3.5}$$

$$\begin{pmatrix} in_j(A_0 - \sigma) & 2D_{\overline{z}}^{(n_j)} \\ -2D_z^{(n_j)} & -in_j(A_0 + \sigma) \end{pmatrix} \begin{pmatrix} \psi_{j\uparrow} \\ \psi_{j\downarrow} \end{pmatrix} - \sqrt{2} n_j Q_j \begin{pmatrix} \overline{\lambda}_\uparrow \\ \overline{\lambda}_\downarrow \end{pmatrix} = 0, \tag{3.6}$$

[7]This (unfortunately) is due to following [34] 's sign convention for A_μ; see the appendix. Fitting with (2.10) and (2.11), $\{Q_-, \overline{Q}_-\} = 2iP_z$ annihilates the BPS chiral field configuration, since chiral fields are annihilated by $\overline{Q}_\pm$, and BPS configurations by Q_- and $\overline{Q}_+$. Compared to e.g. [18–20], $(A_\mu, \sigma, \lambda_\alpha, \overline{\lambda}_\alpha)^{\text{here}} = -e(A_\mu, N, \lambda_\alpha, \chi_\alpha)^{\text{there}}$, $q_J^{\text{here}} = -n_{\text{there}}$.

where $\psi_{i\uparrow,\downarrow}$ and $\overline{\lambda}_{\uparrow,\downarrow}$ have spin $\pm\frac{1}{2}$, and $U(1)_R$ charge -1. As we discuss in section 4, the number of solutions of (3.5) and (3.6), and their quantum numbers, are as in (1.12): each matter field contributes $|n_i|$ Fermi zero modes, with spin correlated to the sign of n_i.

3.1 Review of the minimal matter example: a single matter field Q_1 of charge $n_1 = 1$

This is the basic $\mathcal{N} = 2$ Abelian Higgs model, and its BPS vortices have been discussed e.g. in [18, 20–22, 32, 33]. We here review the discussion from [1]. By (2.2), here $k \in \mathbb{Z} + \frac{1}{2}$, and the theory has $\text{Tr}(-1)^F = |k| + \frac{1}{2}$ vacua [1]; we here discuss the BPS vortices of the theory in the Higgs[8] vacuum of the theory with FI parameter $\zeta > 0$, i.e. $\langle Q_1 \rangle = \sqrt{\zeta/2\pi}$.

The solution $A_\mu^{\text{vortex}}(z, \overline{z})$, $Q_1^{\text{vortex}}(z, \overline{z})$, $\sigma^{\text{vortex}}(z, \overline{z})$ of the BPS field equations (3.2), (3.3), (3.4), is not analytically known, nor is it needed: knowing its existence and number of zero modes suffices. The vortex with $U(1)_J$ charge q_J has $|q_J|$ complex Bosonic zero modes, and $|q_J|$ spin $+\frac{1}{2}$ Fermionic zero modes. The $q_J = 1$ vortex has one complex zero mode z_1, the translational invariance zero mode of the BPS vortex core location, and one complex spin $\frac{1}{2}$ Fermionic zero mode [20, 21], Ψ_1, a combination of the photino and the matter fermion that solves (3.5) and (3.6). The Bosonic field configuration is annihilated by Q_- and $\overline{Q}_+$ (2.11), while the other two supercharges give the Fermi zero mode, $\Psi_1 \sim Q_+$, and complex conjugate $\Psi_1^\dagger \sim \overline{Q}_-$, i.e. the photino and matter Fermi field configuration of Ψ_1 follows from acting with Q_+ on $F_{\mu\nu}^{\text{vortex}}(z, \overline{z})$ and $Q_1^{\text{vortex}}(z, \overline{z})$.

Quantizing the $q_J = 1$ vortex Ψ_1 Fermi zero mode, $\{\Psi_1, \Psi_1^\dagger\} = 1$ (so $\Psi_1 \to Q_+/\sqrt{2E}$) yields a BPS doublet (2.12); adding the $q_J = -1$, anti-BPS, CPT conjugate states gives one copy of the spectrum (2.13). The $U(1)_R$ and $U(1)_{\text{spin}}$ quantum numbers there are found as in (1.10) from those of Ψ_1, $|\Omega_\pm\rangle_{q_J=1} \sim \Psi_1^{\mp\frac{1}{2}} \text{“}|0\rangle\text{”}_{q_J=1}$ with “$|0\rangle$”$_{q_J=1}$ assigned spin $-\frac{1}{2}k$ as in (2.7). This gives $r = \frac{1}{2}$ and $s = -\frac{1}{2}(k + \frac{1}{2})$ [1], as in (1.12) with $k_c = \frac{1}{2}\sum_i n_i|n_i| = \frac{1}{2}$. The $k = \mp\frac{1}{2}$ theory is dual to a theory of a free chiral superfield, $X_\pm$ [35]. The FI parameter ζ maps to a real mass m_X in the dual. BPS vortices map to X-particle states.

3.2 Cases with multiple matter fields Q_i: the (anti)-BPS equations for the bosonic fields

By (3.4), the vortex gauge field configuration is completely determined by that of *any* non-zero matter field Q_i:

$$A_z = \frac{i}{n_i}\partial_z \ln Q_i, \qquad \text{resp} \qquad A_{\overline{z}} = \frac{i}{n_i}\partial_{\overline{z}} \ln Q_i, \qquad \text{for any} \quad Q_i \neq 0. \tag{3.7}$$

The condition that the gauge field (3.7) be smooth, with winding number q_J (1.4), implies [36] that a charge $n_i = 1$ matter field has $Q_i(z)$ with $|q_J|$ zeros, at the vortex core locations, $z = z_{i=1\ldots|q_J|}$. For $|q_J| = 1$, a charge n_i matter field with $Q_i^{\text{vac}} \neq 0$ can have an order $|n_i|$ zero at the location z_1 of the BPS (resp. anti-BPS) vortex core

$$Q_i^{\text{vac}} \neq 0: \qquad Q_i = (\overline{z} - \overline{z}_1)^{|n_i|} f_i, \qquad \text{resp} \qquad Q_i = (z - z_1)^{|n_i|} f_i, \tag{3.8}$$

[8] For $|k| > \frac{1}{2}$, one could consider vortices in the other vacua, with $\langle Q_1 \rangle = 0$ and $\langle \sigma \rangle \neq 0$, and domain walls between the vacua, as in [33], but we will not consider such configurations here.

with $f_i \equiv f_i(z, \bar{z})$ non-vanishing. Turning on Bosonic zero modes can resolve the zeros in (3.8) or, with multiple matter fields, eliminate the zeros, as in the examples of [28–30].

Using (3.7), the BPS equations (3.4) can be rewritten in terms of ordinary derivatives and $U(1)_{\text{gauge}}$ neutral ratios of fields, where we divide by any Q_i with $Q_i^{\text{vac}} \neq 0$:

$$\text{(BPS): } \partial_z \left(\frac{Q_j}{Q_i^{n_j/n_i}} \right) = 0, \qquad \text{resp} \qquad \text{(anti-BPS): } \partial_{\bar{z}} \left(\frac{Q_j}{Q_i^{n_j/n_i}} \right) = 0. \tag{3.9}$$

3.3 Vanishing theorem and its consequences

The non-zero solutions of (3.4) are restricted by a vanishing theorem: "*a line bundle of negative degree cannot have a non-zero holomorphic section*"; see e.g. [37] for a nice discussion in the similar context of 2d instantons. With our conventions, this implies

$$\begin{aligned} \text{BPS}: \quad & Q_i = 0 \quad \text{unless} \quad \text{sign}(n_i) = \text{sign}(q_J) \\ \text{anti-BPS}: \quad & Q_i = 0 \quad \text{unless} \quad \text{sign}(n_i) = -\text{sign}(q_J). \end{aligned} \tag{3.10}$$

This can be seen from the identity (writing $x^\mu = (t, \vec{x})$ and $D_\mu^{(n_j)} \equiv (D_0^{(n_j)}, \vec{D}^{(n_j)})$)

$$\int d^2\vec{x} |\vec{D}^{(n_j)} Q_j|^2 = \int d^2\vec{x} \left(|2D_{z,\bar{z}}^{(n_j)} Q_j|^2 \pm n_j |Q_j|^2 F_{12} \right); \tag{3.11}$$

with $[D_{\bar{z}}^{(n_j)}, D_z^{(n_j)}] = \frac{1}{2} n_j F_{12}$. Since the l.h.s. of (3.11) is non-negative, equations (3.4) have a $Q_j \neq 0$ solution only if the second term on the r.h.s. of (3.11) has the correct sign. By (3.1), the $q_J \neq 0$ BPS vacua have $\text{sign}(q_J) = \text{sign}(\zeta)$ and the anti-BPS vacua have $\text{sign}(q_J) = -\text{sign}(\zeta)$. So (3.10) implies, for both BPS and anti-BPS configurations

$$Q_i = 0 \qquad \text{if} \qquad \text{sign}(n_i) = -\text{sign}(\zeta). \tag{3.12}$$

An immediate corollary is that there are only BPS vortices in Higgs vacua where Q_i^{vac} satisfy (3.12), i.e. we solve $D = 0$ (2.3) with $Q_i^{\text{vac}} \neq 0$ only for matter with $\text{sign}(n_i) = \text{sign}(\zeta)$. So, in theories with matter fields with n_i of both signs, all gauge-invariant products, i.e. the Higgs branch moduli, must be set to zero, e.g. the meson fields $M_{i\tilde{j}} = Q_i \widetilde{Q}_{\tilde{j}} = 0$ in a theory with vector-like matter. As discussed in [9], the fact that BPS vortices require $M_{i\tilde{j}} = 0$ can have a simple dual perspective, e.g. for $N_f = 1$ SQED it is clear from the $W = MX_+X_-$ dual that the $X_\pm$ quanta are only BPS for $M = 0$. See [38, 39] for other, dynamical arguments leading to the same conclusion.

3.4 Bosonic zero modes of $|q_J| = 1$ BPS vortices with multiple matter fields

Each matter field with $\text{sign}(n_i) = \text{sign}(\zeta)$ has $|n_i|$ complex Bosonic zero modes, one of which is the vortex core location, z_1 in (3.8). Since matter fields with $\text{sign}(n_i) = -\text{sign}(\zeta)$ are set to zero (3.12), they do not yield Bosonic zero modes. Consider (3.9), taking say Q_1 and Q_j to have $\text{sign}(n_1) = \text{sign}(n_j) = \text{sign}(\zeta)$, and suppose that $Q_1^{\text{vac}} \neq 0$ and $Q_j^{\text{vac}} = 0$. The general solution of (3.9) for a $q_J = 1$ BPS (or $q_J = -1$ anti-BPS) vortex is then

$$\frac{Q_j(z, \bar{z})}{Q_1(z, \bar{z})^{n_j/n_1}} = \frac{\overline{P}_j(\bar{z})}{(\bar{z} - \bar{z}_1)^{|n_j|}}, \qquad \text{resp} \qquad \frac{Q_j(z, \bar{z})}{Q_1(z, \bar{z})^{n_j/n_1}} = \frac{P_j(z)}{(z - z_1)^{|n_j|}}, \tag{3.13}$$

where the denominators are determined by the $z \to z_1$ vanishing degree of Q_1 in (3.8), (which is the only singularity of the ratio) and the numerators by (anti) holomorphy and the condition that the ratio approaches the vacuum value, i.e. zero, for $|z| \to \infty$:

$$\overline{P}_j(\bar{z}) \equiv \sum_{p=1}^{|n_j|} \bar{c}_{j,p} \bar{z}^{p-1}, \qquad \text{resp} \qquad P_j(z) \equiv \sum_{p=1}^{|n_j|} c_{j,p} z^{p-1}. \tag{3.14}$$

The $|n_j|$ coefficients $\bar{c}_{j,p}$ (or $c_{j,p}$) in (3.14) are the Bosonic zero modes for matter field Q_j with $Q_j^{\rm vac} = 0$ in a BPS (or anti-BPS) $q_J = 1$ vortex. Matter field(s) Q_i with $Q_i^{\rm vac} \neq 0$ also yield $|n_i|$ Bosonic zero modes, one of which is the translational zero mode z_1.

3.5 Normalizable vs non-normalizable zero modes

The Bosonic or Fermionic zero modes of the static vortex are replaced with dynamical variables on the vortex worldline theory, *if* the associated induced kinetic term is normalizable. Non-normalizable zero modes, on the other hand, are frozen parameters. For example, the translational zero mode of a $|q_J| = 1$ vortex is quantized as $z_1 \to z_1(t)$, which is normalizable, with finite induced kinetic term $\int d^2z\mathcal{L} \to \frac{1}{2}m_{\rm BPS}|\dot{z}_1|^2$. Considering the $c_{j,p}$ or $\bar{c}_{j,p}$ term in (3.13) for large $|z|$ gives $|Q_j| \sim |c_{j,p}||z|^{p-1-|n_j|}$, so the induced coefficient of a $|\dot{c}_{j,p}|^2$ term involves $\sim \int d^2z|z|^{2(p-1-|n_j|)}$, i.e. $c_{j,p}$ and $\bar{c}_{j,p}$ are normalizable for $1 \leq p < |n_j|$ (requiring $|n_j| > 1$) and log-IR-divergent non-normalizable for $p = |n_j|$.

The non-normalizable $\rho_j \equiv \bar{c}_{j,p=|n_j|}$ or $\rho_j \equiv c_{j,p=|n_j|}$ zero modes in (3.13) generalize the non-normalizable zero modes of "semi-local vortices" [27–29]- [30]. As found there, turning on $\rho_i \neq 0$ dramatically changes the character of the vortex solution, removing the zero in (3.8) at the vortex core, and changing the flux F_{12} in (3.3) from having the usual $\sim e^{-cm_\gamma|z|}$ exponential falloff for large $|z|$ (with m_γ the Higgsed photon mass) into a diffuse, power-law falloff. In our general n_i case, each matter field with $\text{sign}(n_i) = \text{sign}(\zeta)$ and $Q_i^{\rm vac} = 0$ yields one-such non-normalizable ρ_i bosonic zero mode. If $|n_j| > 1$, there are also $|n_j| - 1$ additional normalizable, and hence dynamical, zero modes $\bar{c}_{j,p<|n_j|}$ or $c_{j,p<|n_j|}$.

The bosonic non-normalizable zero modes, ρ_i, are interpreted, as in [1], as superselection parameters already of the $q_J = 0$ *vacuum*, even before adding the vortex: taking $Q_i \sim \rho_i/|z|$ for large $|z|$ has finite energy, with ρ_i non-normalizable, so unchanging in time. Likewise, Fermi zero modes are either normalizable, if $\Psi_A < \mathcal{O}(1/|z|)$ for large $|z|$, or non-normalizable if $\Psi_A = \mathcal{O}(1/|z|)$. As in [1], we quantize *all* the Fermion zero modes as in (1.7), including the non-normalizable ones. The tower of 2^{N_z} states discussed around (1.7) includes states in different Hilbert spaces, if related by a non-normalizable Fermi zero mode. The charges of the states, and in particular the states $|\Omega_\pm\rangle_{q_J}$ at the top and bottom of the tower, are affected by all the Fermi zero modes, with the product in (1.10) including all normalizable and also non-normalizable Fermi zero modes.

4 Fermi zero modes of BPS vortices for somewhat general cases.

We will consider $|q_J| = 1$ BPS and anti-BPS vortices, taking $\zeta/n_1 > 0$, in the vacuum with $\langle\sigma\rangle = 0$ and non-zero expectation value for only Q_1:

$$Q_i^{\rm vac} = \sqrt{\frac{\zeta}{2\pi n_1}}\delta_{i,1}. \tag{4.1}$$

For the rest of this section, we assume that $n_1 = 1$, though we allow for general charges n_j for the other $Q_{j>1}$ matter fields in (4.1). We will discuss the $n_1 \neq 1$ case in section 6.

Each Q_i matter field with $n_i > 0$ has n_i Bosonic zero modes, while Q_i with $n_i < 0$ have none. The Q_1 Bosonic zero mode is the normalizable, translational zero mode, z_1. For the matter fields $Q_{j\neq 1}$, with $n_j > 0$, the Bosonic zero modes are the $\overline{c}_{j,p}$ or $c_{j,p}$ in (3.14), with $p = |n_j|$ non-normalizable. Non-zero time derivatives of the normalizable $c_{j,p}$ and $\overline{c}_{j,p}$ can contribute to the vortex's energy, momentum, and spin angular momentum.

We now consider the Fermi zero modes of the $q_J = 1$ BPS vortex or $q_J = -1$ anti-BPS vortex in the vacuum (4.1). Since the counting and quantum numbers of Fermi zero modes cannot depend on continuous variables, we can simplify things by setting all Bosonic zero modes to zero, in which case

$$Q_i^{\text{vortex}}(z, \overline{z}) = Q_1^{\text{vortex}}(z, \overline{z})\delta_{i,1}. \tag{4.2}$$

Here Q_1^{vortex} coincides with that of $\mathrm{U}(1)_k$ with only the matter field Q_1; the $Q_{i\neq 1}$ matter fields do not affect the solution. Likewise, the Fermi zero mode equations (3.5) and (3.6) involving $\overline{\lambda}_\pm$ and $\psi_{1\pm}$ decouple from those for the $Q_{j>1}$ matter Fermions. The solution for the zero modes from $\overline{\lambda}_{\uparrow,\downarrow}$ and $\psi_{1\uparrow,\downarrow}$ is the same as that of the minimal matter theory reviewed in section 3.1: for $|q_J| = 1$ it gives one Fermion zero mode, Ψ_1, and conjugate $\Psi_1^\dagger$, corresponding to the non-trivial supercharges Q_+ and $\overline{Q}_-$ in (2.12).

Now consider the decoupled equations (3.6) for the $Q_{j>1}$ matter Fermi zero modes:

$$\begin{aligned} \text{BPS}(j\neq 1): \quad & D_{\overline{z}}^{(n_j)}\psi_{j\downarrow} = 0, \qquad D_z^{(n_j)}\psi_{j\uparrow} = -in_jA_0\psi_{j\downarrow}; \\ \text{anti-BPS}(j\neq 1): \quad & D_z^{(n_j)}\psi_{j\uparrow} = 0, \qquad D_{\overline{z}}^{(n_j)}\psi_{j\downarrow} = -in_jA_0\psi_{j\uparrow}. \end{aligned} \tag{4.3}$$

For $k = 0$, it is possible to set $\sigma = A_0 = 0$, and we obtain the simpler version

$$(j \neq 1 \text{ simple version}) \qquad D_z^{(n_j)}\psi_{j,\uparrow} = 0, \qquad \text{and} \qquad D_{\overline{z}}^{(n_j)}\psi_{j,\downarrow} = 0. \tag{4.4}$$

If $k \neq 0$, Gauss' law (2.6) implies that $A_0 = \pm\sigma$ is a complicated function. Fortunately, for any value of k, (4.3) and the simpler version (4.4) have the same number of zero mode solutions, with the same spins. Indeed, using (3.11) and (3.10), it follows that

$$\begin{cases} D_{\overline{z}}^{(n_j)}\psi_{j,\downarrow} = 0 \to \psi_{j\downarrow} = 0 & \text{if} \quad n_jq_J > 0 \\ D_z^{(n_j)}\psi_{j,\uparrow} = 0 \to \psi_{j,\uparrow} = 0 & \text{if} \quad n_jq_J < 0 \end{cases}. \tag{4.5}$$

Consider the case of a $q_J = +1$ BPS vortex; the anti-BPS case is analogous. For matter with $n_j > 0$, (4.5) gives $\psi_{j\downarrow} = 0$ and (4.3) reduces to (4.4). For $n_j < 0$ matter, $\psi_{j\downarrow}$ is non-trivial and satisfies the same equation in (4.3) and (4.4). The difference between the $\psi_{j\uparrow}$ equations in (4.3) and (4.4) for $n_j < 0$ is immaterial in terms of counting solutions: the solution for $\psi_{j\uparrow}$ in either equation is uniquely determined, as $D_z^{(n_i)}$ has trivial kernel for $n_i < 0$ (4.5). So we can always count Fermi zero modes via (4.4).

Using (3.7) to eliminate the gauge field in favor of Q_1, the equations (4.4) become

$$\text{if} \quad n_j > 0: \quad \partial_z\left(\frac{\psi_{j,\uparrow}}{Q_1^{n_j}}\right) = 0; \qquad \text{if} \quad n_j < 0: \quad \partial_{\overline{z}}\left(\frac{\psi_{j,\downarrow}}{(Q_1^\dagger)^{|n_j|}}\right) = 0. \tag{4.6}$$

Since, for $q_J = 1$, Q_1 has a degree one zero at z_1, this gives (similar to (3.13))

$$n_j > 0 \quad (q_J = 1): \qquad \psi_{j,\uparrow} = \frac{Q_1^{n_j}}{(\bar{z} - \bar{z}_0)^{n_j}} \sum_{p=1}^{n_j} \overline{u}_{j,p} \bar{z}^{p-1}, \tag{4.7}$$

with the n_j coefficients, $\overline{u}_{j,p=1,\dots n_j}$, Fermionic zero modes of spin $p - \frac{1}{2}$. Likewise,

$$n_j < 0 \quad (q_J = 1): \qquad \psi_{j,\downarrow} = \frac{(Q_1^\dagger)^{|n_j|}}{(z - z_0)^{|n_j|}} \sum_{p=1}^{|n_j|} d_{j,p} z^{p-1}, \tag{4.8}$$

with the $|n_j|$ coefficients, $d_{j,p}$, Fermionic zero modes of spin $-(p - \frac{1}{2})$. As in the bosonic case, for either (4.7) or (4.8), the $p = |n_j|$ Fermi zero mode is non-normalizable. The spins of $\overline{u}_{j,p}$ and $d_{j,p}$ follow from constructing the angular momentum generator, much as in [40], assigning spin $+1$ to z, and spin $+\frac{1}{2}$ to $\psi_{j,\uparrow}$ in (4.7). By (1.5), $Q_1^{n_j}/(\bar{z}-\bar{z}_0)^{n_j}$ is θ independent for large $|z|$, so we assign spin $+\frac{1}{2}$ to each term $\overline{u}_{j,p}\bar{z}^{p-1}$ in (4.7), and, likewise, spin $-\frac{1}{2}$ to all $d_{j,p}z^{p-1}$ in (4.8). So $\overline{u}_{j,p}$ has spin $p-\frac{1}{2}$ and $d_{j,p}$ has spin $-(p-\frac{1}{2})$. In sum, the $q_J = 1$ vortex has the $\Psi^{(q_J=1)}_{n_j,p}$ in (1.12): $|n_j|$ Fermion zero modes, of spins $\mathrm{sign}(n_j)(p - \frac{1}{2})$, for $p = 1 \dots |n_j|$. The $q_J = -1$ vortex is similar. The other quantum numbers likewise follow from those of $\psi_{j,\uparrow,\downarrow}$, and are as given in (1.12). We assign $\mathrm{U}(1)_{\mathrm{gauge}}$ charges in (1.12), even though $\mathrm{U}(1)_{\mathrm{gauge}}$ is spontaneously broken (screened) by (4.1).

The Bose and Fermi zero modes form supermultiplets of a 1d worldline theory with two unbroken supercharges (see e.g. [25, 41, 42] for some examples), as in the 1d reduction of a 2d $\mathcal{N} = (2,0)$ worldsheet theory of BPS vortex strings in 4d $\mathcal{N} = 1$ theories [43–45]. The zero modes of a matter field Q_i are in $|n_i|$ different $\mathcal{N} = (2,0)$ chiral multiplets (i.e. a complex Boson and a complex Fermion) if $\mathrm{sign}(n_i) = \mathrm{sign}(\zeta)$, or $|n_i|$ $\mathcal{N} = (2,0)$ chiral Fermi multiplets (i.e. a complex Fermion and an auxiliary field) if $\mathrm{sign}(n_i) = -\mathrm{sign}(\zeta)$.

All the Fermi zero modes are quantized, as in (1.7) and (1.8), giving $2^{\sum |n_i|}$ states. The $\Psi^{(q_J=1)}_1 \sim Q_+ Q^{\mathrm{vortex}}_{i=1}$ zero mode should be regarded as Q_+, i.e. neutral under $\mathrm{U}(1)_{\mathrm{gauge}}$ and the non-R-symmetry global symmetries; quantizing this zero mode yields BPS doublets (2.13). Including all zero modes yields $2^{\sum |n_i| - 1}$ BPS doublets.

Consider a theory with vector-like, charge-conjugation symmetric matter content, with pairs Q_i and $\widetilde{Q}_i$, of charges $\pm n_i$. Then $k_c = \frac{1}{2}\sum_i n_i |n_i| = 0$ in (2.5), and the $k = 0$ theory with $\zeta = 0$ has asymptotic Coulomb branches $X_\pm$. The theory respects P and T if $k = 0$, and it respects C if $\zeta = 0$. For every Fermi zero mode $\Psi_{n_j,p}$, there is a Fermi zero mode $\widetilde{\Psi}_{-n_j,p}$ of opposite spin, so the $\prod_A \Psi_A$ appearing in (1.10) has spin $s = 0$, and the top and bottom states $|\Omega_\pm\rangle_{q_J=1}$ have $s = -\frac{1}{2}k$, so spin 0 for $k = 0$, This fits with (1.11): these states map to the quanta of $X_\pm$, $|\Omega_+\rangle_{q_J=\pm 1} \sim X_\pm |0\rangle$ and $|\Omega_-\rangle_{q_J=\pm 1} \sim X_\mp^\dagger |0\rangle$, with $X_\pm$ a gauge invariant operator for $k = 0$.

5 Examples: theories with $N_\pm$ matter fields of charge $n_i = \pm 1$

We denote the matter as $Q_{i=1\dots N_+}$, with $n_i = +1$, and $\widetilde{Q}_{\tilde{i}=1\dots N_-}$, with $n_{\tilde{i}} = -1$. The $\mathrm{U}(1)_j$ global symmetries in (1.6) enhance to $\mathrm{SU}(N_+) \times \mathrm{SU}(N_-) \times \mathrm{U}(1)_A$, where the $\mathrm{U}(1)_A$ charge is $+1$ for all Q_i and $\widetilde{Q}_{\tilde{i}}$. We take $N_+ > 0$, and $\zeta > 0$, and then (4.1) is the general vacuum with BPS vortices; it spontaneously breaks $\mathrm{SU}(N_+) \to \mathrm{SU}(N_+ - 1) \times \mathrm{U}(1)$, so for $N_+ > 1$ the vacua contain the NG bosons $\cong CP^{N_+-1}$. For $N_+N_- \neq 0$, the vacua also include non-compact directions, given by the mesons $M_{i\tilde{j}} = Q_i \widetilde{Q}_{\tilde{j}}$, with $M_{i\tilde{j}}$ of rank 1, but as in (3.12) BPS or anti-BPS vortices require $\widetilde{Q}_{\tilde{i}} = 0$, so $M_{i\tilde{j}} = 0$. The Chern-Simons quantization condition (2.2) gives $k + \frac{1}{2}\Delta N \in \mathbb{Z}$, with $\Delta N \equiv N_+ - N_-$; also, $k_c = \frac{1}{2}\Delta N$.

The cases $(N_+, N_-) = (N, 0)$ were discussed in [1]. The minimal matter case, $N = 1$, was reviewed in section 3.1. The vortices of the $N > 1$ case is the $\mathcal{N} = 2$ version of the "semi-local" vortices of [27, 28]–[29], allowing also for Chern-Simons terms. Our present discussion in this section also includes cases with both $N_+N_- \neq 0$; we did not find much discussion of vortices in such theories in the literature, aside from some brief comments in [23, 24].

For general (N_+, N_-), a $q_J = 1$ BPS vortex has N_+ complex bosonic zero modes. One is the normalizable, translational zero mode, z_1, corresponding to the vortex core location. The remaining $N_+ - 1$ bosonic zero modes are the non-normalizable $\overline{\rho}_i$ parameters in

$$\frac{Q_{i\neq 1}}{Q_1} = \frac{\overline{\rho}_i}{\overline{z}}. \tag{5.1}$$

The N_- negatively charged matter fields $\widetilde{Q}_{\tilde{i}}$ must identically vanish (3.12) in a BPS configuration, so they do not yield bosonic zero modes.

Now consider the Fermi zero modes of the $q_J = 1$ BPS vortex. Again, the counting is independent of the ρ_i in (5.1) (though $\rho_i \neq 0$ does dramatically affect the shape of the solutions) so we set $\rho_i = 0$ for simplicity. As discussed after (4.2), the Fermi zero mode equations (3.5) and (3.6) then decouple among the matter flavors. The photino and $\psi_{j=1}$ matter Fermion have the same solution as the minimal matter $(N_+, N_-) = (1, 0)$ theory, giving the normalizable, complex Fermi zero mode, $\Psi_1 \sim Q_- Q_1^{\mathrm{vortex}}$. The remaining Fermion zero modes solve (4.3), i.e.

$$\begin{pmatrix} 0 & D_{\overline{z}} \\ D_z & iA_0 \end{pmatrix} \begin{pmatrix} \psi_{j\uparrow} \\ \psi_{j\downarrow} \end{pmatrix} = 0, \qquad \text{and} \qquad \begin{pmatrix} 0 & D_{\overline{z}} \\ D_z & -iA_0 \end{pmatrix} \begin{pmatrix} \widetilde{\psi}_{\tilde{j}\uparrow} \\ \widetilde{\psi}_{\tilde{j}\downarrow} \end{pmatrix} = 0, \tag{5.2}$$

with $j = 2 \ldots N_+$, and $\tilde{j} = 1 \ldots N_-$. For each such j and $\tilde{j}$, (5.2) has one zero mode solution, with spin $\frac{1}{2}\mathrm{sign}(n_i)$. As we have argued, for counting solutions and spins, we can replace (5.2) with the simpler version (4.4), whose solutions are as in (4.5), (4.7) and (4.8):

$$\frac{\psi_{j>1,\uparrow}}{Q_1(z,\overline{z})} = \frac{\overline{u}_j}{\overline{z} - \overline{z}_0}, \qquad \text{and} \qquad \frac{\widetilde{\psi}_{\tilde{j},\downarrow}}{Q_1^\dagger(z,\overline{z})} = \frac{d_{\tilde{j}}}{z - z_0}. \tag{5.3}$$

The spinors $\overline{u}_{j>1}$ and $d_{\tilde{j}}$ give $N_+ + N_- - 1$ Fermi zero modes $\Psi_{j>1}$ and $\Psi_{\tilde{j}}$; all are non-normalizable, since all $\lim_{|z|\to\infty} |\psi| \sim 1/|z|$ in (5.3). The charges are as in (1.12):

	$\mathrm{U}(1)_{\mathrm{spin}}$	$\mathrm{SU}(N_+) \times \mathrm{SU}(N_-)$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_J$
Ψ_i	$\frac{1}{2}$	$(N_+, 1)$	1	-1	0
$\Psi_{\tilde{i}}$	$-\frac{1}{2}$	$(1, N_-)$	1	-1	0
$\prod_A \Psi_A$	$\frac{1}{2}\Delta N$	$(1,1)$	N_{tot}	$-N_{\mathrm{tot}}$	0
$\lvert\Omega_\pm\rangle_{q_J=1}$	$-(k \pm \frac{1}{2}\Delta N)$	$(1,1)$	$\mp\frac{1}{2}N_{\mathrm{tot}}$	$\pm\frac{1}{2}N_{\mathrm{tot}}$	1

(5.4)

with $\Delta N \equiv N_+ - N_-$ and $N_{\rm tot} \equiv N_+ + N_-$. To save space, we here formally lump together Ψ_1 and $\Psi_{i>1}$, even though they are different, e.g. Ψ_1 is normalizable and $\Psi_{j>1}$ are not normalizable, consistent with the fact that $SU(N_+)$ is broken by (4.1) to $SU(N_+-1)\times U(1)$.

As discussed [1] and section 3.4, we quantize all $N_+ + N_-$ Fermi zero modes, including the non-normalizable ones. This leads to a tower of $2^{N_+ + N_-}$ vortex states, with the top and bottom states $|\Omega_\pm\rangle_{q_J=1}$, with quantum numbers as in (5.4). The normalizable zero mode, Ψ_1, is identified with Q_-, so the states form $2^{N_+ + N_- - 1}$ BPS doublets (2.12). These come from quantizing the non-normalizable $\Psi_{j>1}$ and $\Psi_{\widetilde{i}}$ Fermi zero modes:

$$|a_{p,\widetilde{p}}\rangle \equiv [\Psi_{j>1}]^p [\Psi_{\widetilde{i}}]^{\widetilde{p}} |\Omega_+\rangle, \qquad |b_{p,\widetilde{p}}\rangle = Q_- |a_{p,\widetilde{p}}\rangle \tag{5.5}$$

meaning to fully antisymmetrize in p different $\Psi_{j>1}$ and $\widetilde{p}$ different $\Psi_{\widetilde{i}}$, with $p = 0 \ldots N_+ - 1$ and $\widetilde{p} = 0 \ldots N_-$. The states (5.5) all have $q_J = 1$, with other quantum numbers

	$U(1)_{\rm spin}$	$SU(N_+-1)$	$SU(N_-)$	$U(1)_R$
$\lvert a_{p,\widetilde{p}}\rangle$	$\frac{1}{2}(p - \widetilde{p} - \Delta N) - k$	$\binom{N_+-1}{p}$	$\binom{N_-}{\widetilde{p}}$	$-(p+\widetilde{p}) + \frac{1}{2}N_{\rm tot}$
$\lvert b_{p,\widetilde{p}}\rangle$	$\frac{1}{2}(p + 1 - \widetilde{p} - \Delta N) - k$	$\binom{N_+-1}{p}$	$\binom{N_-}{\widetilde{p}}$	$-(p+1+\widetilde{p}) + \frac{1}{2}N_{\rm tot}$

(5.6)

The omitted $U(1)_{\rm gauge}$ charge is screened by $Q_1^{\rm vac} \neq 0$, and we omit $U(1)_A$. The SU flavor singlets are $(p, \widetilde{p}) = (0,0)$, $(N_+ - 1, 0)$, $(0, N_-)$, $(N_+ - 1, N_-)$, with $|\Omega_-\rangle = |b_{N_+-1,N_-}\rangle$.

If $k = \mp k_c \equiv \mp\frac{1}{2}\Delta N$, the $X_\pm$ Coulomb branch exists, and $|\Omega_\pm\rangle$ has spin 0, and is an $SU(N_+-1)\times SU(N_-)$ singlet, consistent with (1.11) and interpreting $X_\pm$ as a condensate of these states. (By choice of k, other $SU(N_+-1)\times SU(N_-)$ singlets states can have spin 0; e.g. $|a_{0,N_-}\rangle$ has spin 0 if $k = -\frac{1}{2}N_+$, for all N_-.)

Consider the $(N_+, N_-) = (1,1)$ theory, i.e. $N_f = 1$ SQED, with $k \in \mathbb{Z}$. Taking $\zeta > 0$, there are BPS vortices in the $M = Q\widetilde{Q} = 0$ vacuum (4.1). The $q_J = 1$ BPS vortex has two Fermi zero modes (plus complex conjugates): the $\Psi_1 \sim Q_+ Q_1^{\rm vortex}$ zero mode has spin $+\frac{1}{2}$ and is normalizable, and the $\Psi_{\widetilde{1}} \equiv \Psi_2$ zero mode has spin $-\frac{1}{2}$ and is not normalizable. Quantizing Ψ_1 and Ψ_2 (1.7) gives two BPS doublets:

$$q_J = 1: \qquad \begin{pmatrix} |\Omega_+\rangle \\ Q_+|\Omega_+\rangle \end{pmatrix}, \qquad \begin{pmatrix} \Psi_{\widetilde{1}}|\Omega_+\rangle \\ Q_+\Psi_{\widetilde{1}}|\Omega_+\rangle, \end{pmatrix} \tag{5.7}$$

with $\Psi_{\widetilde{1}}|\Omega_+\rangle \sim \overline{Q}_-|\Omega_-\rangle$ and $|\Omega_-\rangle \sim Q_+\Psi_{\widetilde{1}}|\Omega_+\rangle$. Since $\Psi_1 \sim Q_+ Q_1^{\rm vortex}$, Q_+ in (5.7) has the charges of $\Psi_1 Q_1^*$. The charges of the states are then, as in (1.12) and (2.13)

	$\mathrm{U}(1)_{\mathrm{spin}}$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_J$	$\mathrm{U}(1)_R$
$\lvert\Omega_+\rangle_{q_J=1}$	$-\frac{1}{2}k$	-1	1	1
$Q_+\lvert\Omega_+\rangle$	$-\frac{1}{2}k+\frac{1}{2}$	-1	1	0

(5.8)

	$\mathrm{U}(1)_{\mathrm{spin}}$	$\mathrm{U}(1)_A$	$\mathrm{U}(1)_J$	$\mathrm{U}(1)_R$
$\overline{Q}_-\lvert\Omega_-\rangle$	$-\frac{1}{2}k-\frac{1}{2}$	1	1	0
$\lvert\Omega_-\rangle_{q_J=1}$	$-\frac{1}{2}k$	1	1	-1

(5.9)

The two BPS doublets in (5.8) and (5.9) reside in different Hilbert spaces, since they are connected via the non-normalizable Ψ_2 Fermi zero mode from $\psi_{\widetilde{Q}}$. For $k=0$, both $|\Omega_\pm\rangle_{q_J=1}$ have spin 0, and quantum numbers consistent with (1.11): $|\Omega_+\rangle_{q_J=1}\sim X_+|0\rangle$ and $|\Omega_-\rangle_{q_J=1}\sim X_-^\dagger|0\rangle$. We interpret $|\Omega_\pm\rangle$ in different Hilbert spaces as corresponding to $X_+X_-\sim 0$ in the chiral ring, and the disconnected $X_\pm$ branches of the $\zeta=0$ theory.[9]

The $W=MX_+X_-$ dual [9] must have the same structure: the map from the $X_+|0\rangle$ to the $X_-^\dagger|0\rangle$ BPS state must involve (in addition to the normalizable Q_+ zero mode), a $\sim 1/|z|$ non-normalizable $\psi_M=Q\widetilde{\psi}_{\widetilde{Q}}$ zero mode. Again, we propose that this reflects that the map from $X_-^\dagger|0\rangle$ to $X_+|0\rangle$ is via $X_+X_-\sim\overline{F_M}\sim\{\overline{Q}^\alpha,[\overline{Q}_\alpha,\overline{M}]\}\sim 0$ in the chiral ring. This tentative interpretation should be further clarified, perhaps in future work.

6 Cases with $Q_i^{\mathrm{vac}}\neq 0$ for matter with $n_i\neq 1$

If a matter field Q_1, with $n_1>1$, has an expectation value (4.1) (negative n_1 can be obtained via charge conjugation of the present discussion), $Q_1^{\mathrm{vac}}\neq 0$ breaks $\mathrm{U}(1)_{\mathrm{gauge}}\to\mathbb{Z}_{n_1}$, a discrete gauge symmetry, a.k.a. a $\mathbb{Z}_{n_1}$ orbifold. See [46], and references cited therein, for more about $\mathbb{Z}_{n_1}$ gauge theory. Before the $\mathbb{Z}_{n_1}$ orbifold projection, the Fermion zero modes are essentially the same as in section 4, with $|n_i|$ Fermion zero modes $\Psi_{i=1,p=1\ldots|n_i|}$ for each matter field Q_i, and charges as in (1.12). This includes n_1 Fermi zero modes (one is the supercharge) coming from matter field Q_1 and the photino, from eqs. (3.5), (3.6). The Fermi zero modes are quantized as in (1.7), giving a tower of $2^{\sum_i|n_i|}$ states, and one then projections to $\mathbb{Z}_{n_1}$ gauge invariant states. The top and bottom states $|\Omega_\pm\rangle_{q_J=1}$ (1.8) survive the $\mathbb{Z}_{n_1}$ projection, with quantum numbers again matching with X_+ and $X_-^\dagger$.

As a special case, recall from [1] that if the charges all have a common integer factor, $n_i=n\widetilde{n}_i$, with n and $\widetilde{n}_i$ integer, the theory is simply a $\mathbb{Z}_n$ orbifold of a rescaled theory:

$$n_i\to\widetilde{n}_i\equiv n_i/n,\qquad q_J\to\widetilde{q}_J=nq_J,\qquad \widetilde{a}\to a/n. \tag{6.1}$$

[9] Parity is a symmetry for $k=0$ and maps $X_+\leftrightarrow X_-$. We can turn on a (P odd) real mass m_Q for Q and $\widetilde{Q}$ and then there is only one Coulomb branch, $X_\pm$ if $m(X_\pm)=-m_Q\pm\zeta=0$; $m_Q\neq 0$ also eliminates the non-normalizable $\psi_{\widetilde{Q}}$ zero mode. There is then a BPS state matching either $X_+|0\rangle$, or $X_-^\dagger|0\rangle$, depending on sign$(m_Q\zeta)$. Taking $m_Q\to 0$ requires both doublets in (5.7).

Note that $q_J \in \mathbb{Z}$, while $\widetilde{q}_J \in n\mathbb{Z}$, and a has periodicity $a \sim a + 2\pi$, while $a \sim a + 2\pi/n$. Consider e.g. the theory of a single matter field, Q_1, with charge $n_1 > 1$, which is equivalent to a $\mathbb{Z}_{n_1}$ orbifold of the rescaled theory with matter of charge $\widetilde{n}_1 = 1$. Since the $q_J = 1$ vortex of the original theory maps (6.1) to a $\widetilde{q}_J = n_1$ vortex of the rescaled theory, it has n_1 complex Bosonic zero modes (the locations $z_1, \ldots, z_{n_1}$ of the individual vortex cores in the rescaled theory), and n_1 Fermionic zero modes, $\Psi_1, \ldots, \Psi_{n_1}$, prior to the $\mathbb{Z}_{n_1}$ orbifolding. Quantizing the $\Psi_{A=1\ldots n_1}$ as in (1.7), gives a tower of 2^{n_1} states. The top and bottom states, $|\Omega_\pm\rangle_{q_J=1}$, have charges as given by (1.10) and (1.12), here with $k_c = \frac{1}{2}n_1^2$. These states are $\mathbb{Z}_{n_1}$ invariant, and their charges match those of X_+ and $X_-^\dagger$ in (1.6). For $k = \mp k_c$, the operator $X = X_\pm$ is $\mathrm{U}(1)_{\mathrm{gauge}}$ neutral, with spin 0, and labels a half-Coulomb branch. This theory is a $\mathbb{Z}_{n_1}$ orbifold of a free field theory [1], with X^{1/n_1} the free field.

We can also consider BPS vortices in vacua with $Q_i^{\mathrm{vac}} \neq 0$ for multiple fields, of different charges n_i, with all $\mathrm{sign}(n_i) = \mathrm{sign}(\zeta)$ (3.10) , i.e. a weighted projective space, with weights n_i. The Fermi zero mode analysis for the general case is then complicated by the couplings among flavors in (3.6). In any case, the counting and charges of the Fermi zero modes cannot be affected by continuous moduli, so they must again be as as (1.12).

In conclusion, in all cases the BPS vortex states $|\Omega_\pm\rangle_{q_J=1}$ have quantum numbers compatible with (1.11). For $k = \mp k_c$, it is a spin 0 BPS state, which becomes massless for $\zeta \to 0$ and can condense to give a dual Higgs description of the $X_\pm$ Coulomb branch.

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A Additional details, conventions, and notation

In components, the lagrangian (2.1) is

$$\begin{aligned}\mathcal{L}_{\mathrm{cl}} = &-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{2e^2}(\partial\sigma)^2 - \sum_i |D_\mu^{(n_j)}Q_i|^2 - V_{\mathrm{cl}} \\ &-\overline{\lambda}\left(\frac{i}{e^2}\partial\!\!\!/ + \frac{k}{4\pi}\right)\lambda - \sum_j \overline{\psi}_j(iD\!\!\!\!/^{(n_j)} + m_j(\sigma))\psi_j + i\sqrt{2}\sum_j n_j(Q_j^*\lambda\psi_j - Q_j\overline{\lambda\psi}_j).\end{aligned} \tag{A.1}$$

We use [34] conventions[10] (reduced from 4d to 3d along the $x^{\mu=2}$ direction, see [47]), though this introduces an unfortunate, non-standard sign convention[11] for the gauge field. The gauge supermultiplet fields (A_μ, λ, σ) have 4d mass dimensions, e.g. $[\lambda] = 3/2$, with $[e^2] = 1$, while $[Q_i] = 1/2$ and $[\psi_i] = 1$ for the matter. The scalar potential $V_{\rm cl}$ in (A.1) is

$$V_{\rm cl} = \frac{e^2}{32\pi^2}\left(\sum_i 2\pi n_i |Q_i|^2 - \zeta - k\sigma\right)^2 + \sum_i (m_i + n_i\sigma)^2 |Q_i|^2 \quad . \tag{A.2}$$

In a configuration where the fields asymptote to a zero of (A.2), the total energy of (A.1) (with $m_i = 0$) can be written (using (3.11) and (2.6)), as (with $F_{12} \equiv F_{12}^{W\&B}$)

$$\begin{aligned} E &= \pm\zeta q_J + \frac{1}{2e^2}\int d^2x \left((F_{12} \mp D)^2 + (F_{i0} \mp \partial_i\sigma)^2 + (\partial_0\sigma)^2\right) \\ &\quad + \int d^2x \sum_j \left(|(D_0 \mp i n_j \sigma) Q_j \sigma|^2 + |(D_1 \mp i D_2)^{(n_j)} Q_j|^2\right) \geq \pm\zeta q_J, \end{aligned} \tag{A.3}$$

with D as in (2.3). The BPS (resp. anti-BPS) configurations saturates the inequality for upper (resp. lower) sign choice and $\zeta q_J > 0$ (resp. $\zeta q_J < 0$).

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[10] So $(-++)$ signature and $\gamma^{\mu=0,1,2}_{\alpha\beta} = (-\mathbf{1}, \sigma^1, \sigma^3)$, i.e. $(\gamma^\mu)_\alpha{}^\beta = \gamma^\mu_{\alpha\lambda}\epsilon^{\lambda\beta} = (-i\sigma^2, -\sigma_3, -\sigma_1)$.

[11] Since [34] uses $D_\mu^{(n_j)} \equiv \partial_\mu + i n_j A_\mu^{W\&B}$ in mostly plus signature, $A_\mu^{W\&B} = -A_\mu^{\rm usual}$; this is also apparent from their $\mathcal{L} \subset -j^\mu A_\mu^{W\&B}$. Consequently, $[D_1^{(n_j)}, D_2^{(n_j)}] = i n_j F_{12}^{W\&B} = -i n_j F_{12}^{(\rm usual)}$, which changes the names of BPS vs anti-BPS with respect to much of the vortex literature. This could be fixed by introducing a minus sign in the definition (1.1) of q_J, but that introduces sign differences with other literature, e.g. the definitions of $X_\pm$ in [1, 9], so we will not do that here.

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