Published for SISSA by Springer

Received: *September 7, 2015*
Accepted: *September 21, 2015*
Published: *October 22, 2015*

Erratum: Stability and symmetry breaking in the general three-Higgs-doublet model

M. Maniatis[a] **and O. Nachtmann**[b]

[a] *Departamento de Ciencias Básicas, Universidad del Bío Bío, Casilla 447, Chillán, Chile*
[b] *Institut für Theoretische Physik, Philosophenweg 16, 69120 Heidelberg, Germany*

E-mail: MManiatis@ubiobio.cl, O.Nachtmann@thphys.uni-heidelberg.de

Erratum to: JHEP02(2015)058

ArXiv ePrint: 1408.6833

JHEP10(2015)149

In section 4 of the original article the method employed to obtain the stationary points of $J_4(\boldsymbol{k})$ on the boundary $\partial\mathcal{D}_{\boldsymbol{k}}$ is incorrect. That is, (4.10) is incorrect. The same applies in section 6 to the method for finding the stationarity matrices $\underline{K}$ of rank 1. Thus, (6.1) and (6.2) are incorrect. In section 7 the following equations are incorrect: (7.4), (7.12)–(7.14). In appendix B (B7) is incorrect. The correct methods and equations are given in the following. Confer also [1].

doi:10.1007/JHEP10(2015)149

Abstract: Stability, electroweak symmetry breaking, and the stationarity equations of the general three-Higgs-doublet model (3HDM) where all doublets carry the same hypercharge are discussed in detail. Employing the bilinear formalism the study of the 3HDM potential turns out to be straightforward. For the case that the potential leads to the physically relevant electroweak symmetry breaking we present explicit formulae for the masses of the physical Higgs bosons.

Contents

1 Introduction

T.D. Lee has shown decades ago that in the general two-Higgs-doublet model (THDM) CP violation is possible in the Higgs sector [2]. Meanwhile a lot of effort has been spent to investigate the THDM; see for instance the review [3] and references therein. In particular, some progress could be made employing the bilinear approach. The bilinears appear naturally in the Higgs potential in any n-Higgs doublet model (nHDM), since only the gauge-invariant scalar products of the Higgs-boson doublet fields may appear in the potential. The bilinear formalism was developed in detail in [4, 5] and independently in [6].

Initiated by these works, many aspects of the THDM and the general nHDM were considered within this formalism. For instance, CP-violation properties of the THDM were presented in [6, 7]. Different symmetries of the THDM and the general nHDM were considered in some detail employing bilinears; see for instance [8–14]. Relations between vacua of different properties in multi-Higgs-doublet models were derived in [15].

In this work we will focus on the three-Higgs-doublet model (3HDM). Many of the properties of this model are direct generalizations of the THDM, but there appear also new aspects. As we will see in detail, the space of Higgs-boson doublets does, in terms of bilinears, not correspond to the forward light cone space, as in case of the THDM [5], but to a certain subspace; see [6, 7, 16]. Driven mainly by the quark- and neutrino mixing data, several 3HDM's have been proposed; see for instance [17–20]

JHEP10(2015)149

In an analogous way to the study of the THDM in [5] we will discuss in the following, in sections 2 to 7, stability, electroweak symmetry breaking, and the stationarity points of the potential for any 3HDM. In section 7 we discuss the potential after symmetry breaking, section 8 presents our conclusion. Throughout the study we will illustrate the general results by two simple illustrative 3HDM examples. In appendix A we give mathematical relations concerning bilinears. In appendix B we discuss an explicit non-trivial example of a 3HDM, based on an $O(2) \times \mathbb{Z}_2$ symmetry [21].

2 Bilinears

We consider the tree-level Higgs potential of models with three Higgs-boson doublets satisfying $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ electroweak gauge symmetry. This is a generalization of the case of two Higgs-boson doublets which were discussed in detail in [5].

We assume that all doublets carry hypercharge $y = +1/2$ and denote the complex doublet fields by

$$\varphi_i(x) = \begin{pmatrix} \varphi_i^+(x) \\ \varphi_i^0(x) \end{pmatrix}; \qquad i = 1, 2, 3. \tag{2.1}$$

In the most general $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge invariant Higgs potential the Higgs-boson doublets enter solely via products of the following form:

$$\varphi_i(x)^\dagger \varphi_j(x), \qquad i, j \in \{1, 2, 3\}. \tag{2.2}$$

It is convenient to discuss the properties of the Higgs potential such as its stability and its stationary points in terms of gauge invariant bilinears.

First we introduce the 3×2 matrix of the Higgs-boson fields in the following way,

$$\phi = \begin{pmatrix} \varphi_1^+ & \varphi_1^0 \\ \varphi_2^+ & \varphi_2^0 \\ \varphi_3^+ & \varphi_3^0 \end{pmatrix} = \begin{pmatrix} \varphi_1^{\mathrm{T}} \\ \varphi_2^{\mathrm{T}} \\ \varphi_3^{\mathrm{T}} \end{pmatrix}. \tag{2.3}$$

We arrange all possible $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ invariant scalar products into the hermitian 3×3 matrix

$$\underline{K} = \phi\phi^\dagger = \begin{pmatrix} \varphi_1^\dagger\varphi_1 & \varphi_2^\dagger\varphi_1 & \varphi_3^\dagger\varphi_1 \\ \varphi_1^\dagger\varphi_2 & \varphi_2^\dagger\varphi_2 & \varphi_3^\dagger\varphi_2 \\ \varphi_1^\dagger\varphi_3 & \varphi_2^\dagger\varphi_3 & \varphi_3^\dagger\varphi_3 \end{pmatrix}. \tag{2.4}$$

A basis for the 3×3 matrices is given by

$$\lambda_\alpha, \quad \alpha = 0, 1, \ldots, 8 \tag{2.5}$$

where

$$\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{1}_3 \tag{2.6}$$

JHEP10(2015)149

and λ_a, $a = 1, \ldots, 8$, are the Gell-Mann matrices. Here and in the following greek indices $(\alpha, \beta, \ldots)$ run from 0 to 8 and latin indices $(a, b, \ldots)$ from 1 to 8. We have

$$\mathrm{tr}(\lambda_\alpha \lambda_\beta) = 2\delta_{\alpha\beta}, \qquad \mathrm{tr}(\lambda_\alpha) = \sqrt{6}\,\delta_{\alpha 0}. \tag{2.7}$$

The decomposition of $\underline{K}$ (2.4) reads now

$$\underline{K} = \frac{1}{2} K_\alpha \lambda_\alpha \tag{2.8}$$

where the real coefficients K_α are given by

$$K_\alpha = K_\alpha^* = \mathrm{tr}(\underline{K}\lambda_\alpha). \tag{2.9}$$

With the matrix $\underline{K}$, as defined in terms of the doublets in (2.4), as well as the decomposition (2.8), (2.9), we immediately express the scalar products in terms of the bilinears,

$$\begin{aligned}
\varphi_1^\dagger\varphi_1 &= \frac{K_0}{\sqrt{6}} + \frac{K_3}{2} + \frac{K_8}{2\sqrt{3}}, \qquad & \varphi_1^\dagger\varphi_2 &= \frac{1}{2}\left(K_1 + iK_2\right), \qquad & \varphi_1^\dagger\varphi_3 &= \frac{1}{2}\left(K_4 + iK_5\right), \\
\varphi_2^\dagger\varphi_2 &= \frac{K_0}{\sqrt{6}} - \frac{K_3}{2} + \frac{K_8}{2\sqrt{3}}, \qquad & \varphi_2^\dagger\varphi_3 &= \frac{1}{2}\left(K_6 + iK_7\right), \qquad & \varphi_3^\dagger\varphi_3 &= \frac{K_0}{\sqrt{6}} - \frac{K_8}{\sqrt{3}}.
\end{aligned} \tag{2.10}$$

In the following we shall frequently use also

$$\begin{aligned}
K_+ &= \sqrt{\frac{2}{3}} K_0 + \sqrt{\frac{1}{3}} K_8 = \varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2, \\
K_- &= \sqrt{\frac{1}{3}} K_0 - \sqrt{\frac{2}{3}} K_8 = \sqrt{2}\varphi_3^\dagger\varphi_3.
\end{aligned} \tag{2.11}$$

From (2.10) follows

$$K_3 = \varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2, \tag{2.12}$$

therefore, we have the inequalities

$$K_+ \geq |K_3| \geq 0, \qquad K_- \geq 0. \tag{2.13}$$

Furthermore, we see from (2.11) that $K_+ = 0$ implies $\varphi_1 = \varphi_2 = 0$ which gives with (2.10)

$$K_1 = K_2 = \ldots = K_7 = 0. \tag{2.14}$$

Further discussion of the basis change from $\alpha = 0, \ldots, 8$ to $+, 1, \ldots, 7, -$ is given in appendix A.

The matrix $\underline{K}$ (2.4) is positive semidefinite which follows directly from its definition. This in turn gives

$$\sqrt{\frac{3}{2}} K_0 = \mathrm{tr}(\underline{K}) \geq 0, \qquad \det(\underline{K}) \geq 0\,. \tag{2.15}$$

The hermitian matrix $\underline{K}$ (2.4) is constructed from the Higgs field matrix, $\underline{K} = \phi\phi^\dagger$. Therefore, the nine coefficients K_α of its decomposition (2.8) are completely fixed given the Higgs-boson fields.

JHEP10(2015)149

Since the 3×2 matrix ϕ has trivially rank smaller or equal 2, this holds also for the matrix $\underline{K}$. On the other hand, any hermitian 3×3 matrix with rank equal or smaller than 2 which clearly has then vanishing determinant, $\det(\underline{K}) = 0$, determines the Higgs-boson fields φ_i, $i = 1, 2, 3$ uniquely, up to a gauge transformation. This was shown in detail in [5] in their theorem 5 for the general case of n-Higgs-boson doublets. In appendix A we show that the gauge orbits of the three Higgs fields (2.1) are characterised by the following set in the 9-dimensional space of $(K_0, \ldots, K_8)$:

$$\begin{aligned} K_0 &\geq 0, \\ (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) = K_0^2 - \frac{1}{2} K_a K_a &\geq 0, \\ \det(\underline{K}) = \frac{1}{12} G_{\alpha\beta\gamma} K_\alpha K_\beta K_\gamma &= 0. \end{aligned} \tag{2.16}$$

Here $G_{\alpha\beta\gamma}$ are completely symmetric constants defined in (A.31), (A.32). That is, to every gauge orbit of the Higgs-boson fields corresponds exactly one vector (K_α) satisfying (2.16) and vice versa. The first two relations of (2.16) are analogous to the *light cone* conditions of the THDM; see (36) of [5]. The determinant relation, trilinear in the K_α, is specific for the 3HDM. Further discussions of the matrices $\underline{K}$ with rank 0, 1, 2 are presented in appendix A.

Based on the bilinears we shall in the following discuss the potential, basis transformations, stability, minimization, and electroweak symmetry breaking of the general 3HDM.

3 The 3HDM potential and basis transformations

In terms of the bilinear coefficients, K_0, K_a, $a = 1, \ldots, 8$ we can write the general 3HDM potential in the form

$$V = \xi_0 K_0 + \xi_a K_a + \eta_{00} K_0^2 + 2K_0 \eta_a K_a + K_a \eta_{ab} K_b, \tag{3.1}$$

where the 54 parameters ξ_0, ξ_a, η_{00}, η_a and $\eta_{ab} = \eta_{ba}$ are real. The potential (3.1) contains all possible linear and quadratic terms of the bilinears — corresponding to all gauge invariant quadratic and quartic terms of the Higgs-boson doublets. Terms higher than quadratic in the bilinears should not appear in the potential with view of renormalizability. Any constant term in the potential can be dropped and therefore (3.1) is the most general 3HDM potential. We also introduce the notation

$$\boldsymbol{K} = (K_1, \ldots, K_8)^{\mathrm{T}}, \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_8)^{\mathrm{T}}, \quad \boldsymbol{\eta} = (\eta_1, \ldots, \eta_8)^{\mathrm{T}}, \quad E = (\eta_{ab}), \quad (\tilde{E}_{\alpha\beta}) = \begin{pmatrix} \eta_{00} & \eta_b \\ \eta_a & \eta_{ab} \end{pmatrix}. \tag{3.2}$$

We can then write the potential (3.1) in the compact form

$$V = \xi_\alpha K_\alpha + K_\alpha \tilde{E}_{\alpha\beta} K_\beta. \tag{3.3}$$

Let us now consider a change of basis of the Higgs-boson fields, $\varphi_i(x) \to \varphi'_i(x)$, where

$$\begin{pmatrix} \varphi'_1(x)^{\mathrm{T}} \\ \varphi'_2(x)^{\mathrm{T}} \\ \varphi'_3(x)^{\mathrm{T}} \end{pmatrix} = U \begin{pmatrix} \varphi_1(x)^{\mathrm{T}} \\ \varphi_2(x)^{\mathrm{T}} \\ \varphi_3(x)^{\mathrm{T}} \end{pmatrix}, \tag{3.4}$$

with $U \in \mathrm{U}(3)$ a 3×3 unitary transformation, that is, $U^\dagger U = \mathbb{1}_3$. From (3.4) we have $\phi'(x) = U\phi(x)$, for the matrix $\underline{K}$ (2.4)

$$\underline{K}'(x) = U\underline{K}(x)U^\dagger, \tag{3.5}$$

and for the bilinears

$$K_0'(x) = K_0(x), \qquad K_a'(x) = R_{ab}(U)K_b(x). \tag{3.6}$$

Here $R_{ab}(U)$ is defined by

$$U^\dagger \lambda_a U = R_{ab}(U)\,\lambda_b. \tag{3.7}$$

The matrix $R(U)$ has the properties

$$R^*(U) = R(U), \quad R^{\mathrm{T}}(U)\,R(U) = \mathbb{1}_8, \quad \det R(U) = 1, \tag{3.8}$$

that is, $R(U) \in \mathrm{SO}(8)$. But the $R(U)$ form only a subset of $\mathrm{SO}(8)$.

For the bilinears a pure phase transformation, $U = \exp(i\alpha)\mathbb{1}_3$, plays no role. We shall, therefore, consider here only transformations (3.4) with $U \in \mathrm{SU}(3)$. In the transformation of the bilinears (3.6) $R_{ab}(U)$ is then the 8×8 matrix corresponding to U in the adjoint representation of $\mathrm{SU}(3)$.

The Higgs potential (3.1) remains unchanged under the replacement (3.6) if we perform an appropriate transformation of the parameters

$$\begin{aligned} &\xi_0' = \xi_0, \qquad \boldsymbol{\xi}' = R(U)\,\boldsymbol{\xi}, \\ &\eta_{00}' = \eta_{00}, \qquad \boldsymbol{\eta}' = R(U)\,\boldsymbol{\eta}, \qquad E' = R(U)\,E\,R^{\mathrm{T}}(U). \end{aligned} \tag{3.9}$$

In the pure 3HDM potential, that is the model without fermions, we can use (3.9) to bring e.g. $\boldsymbol{\xi}$ to a standard form. Consider the hermitian matrix

$$\underline{\Lambda}_\xi = \xi_a \lambda_a. \tag{3.10}$$

Applying a transformation $U \in \mathrm{SU}(3)$ we get with (3.7)–(3.9)

$$U\underline{\Lambda}_\xi U^\dagger = R_{ba}(U)\xi_a\lambda_b = \xi_b'\lambda_b \equiv \underline{\Lambda}_{\xi'} \tag{3.11}$$

With a suitable transformation U we can, therefore, diagonalise $\underline{\Lambda}_\xi$. That is, we always can achieve the form

$$\underline{\Lambda}_{\xi'} = \xi_3'\lambda_3 + \xi_8'\lambda_8, \qquad \boldsymbol{\xi}' = \left(0, 0, \xi_3', 0, 0, 0, 0, \xi_8'\right)^{\mathrm{T}}. \tag{3.12}$$

The number of relevant parameters of the general 3HDM potential is, therefore,

$$54 - 6 = 48. \tag{3.13}$$

Note that instead of $\boldsymbol{\xi}$ we could have chosen $\boldsymbol{\eta}$ in the above argument. Note also the slick proof of (3.12) and (3.13) employing the bilinear formalism.

Let us remark on the basis transformations with respect to the 3-Higgs-doublet model. In a realistic model we have to consider, besides the Higgs potential, kinetic terms for the Higgs-boson doublet fields as well as Yukawa terms which provide couplings of the Higgs-boson doublets to fermions. Under a basis transformation, that is, a transformation of the Higgs-boson doublets of the form (3.4), or equivalently, in terms of the bilinears, a transformation of the form (3.6), the kinetic terms of the Higgs doublets will remain invariant. However, we emphasize that, in general, the Yukawa couplings are *not* invariant under such a change of basis.

In order to illustrate the use of the bilinears we will consider two simple examples of explicit 3HDM Higgs potentials,

$$\text{Example } I \qquad V_I = -\mu^2 \varphi_3^\dagger \varphi_3 + \lambda(\varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2 + \varphi_3^\dagger \varphi_3)^2. \tag{3.14}$$

Here $\mu^2 > 0$ is a parameter of dimension mass squared and $\lambda > 0$ is dimensionless. Employing (2.10) we write this potential in terms of the bilinears as

$$V_I = -\frac{\mu^2}{\sqrt{6}} K_0 + \frac{\mu^2}{\sqrt{3}} K_8 + \frac{3}{2}\lambda K_0^2 = -\frac{\mu^2}{\sqrt{2}} K_- + \lambda \Big(K_+ + \frac{1}{\sqrt{2}} K_- \Big)^2. \tag{3.15}$$

This corresponds to the general form (3.1) with parameters,

$$\xi_0 = -\frac{\mu^2}{\sqrt{6}}, \qquad \boldsymbol{\xi} = \mu^2 (0,0,0,0,0,0,0,\frac{1}{\sqrt{3}})^{\mathrm{T}}, \qquad \eta_{00} = \frac{3}{2}\lambda, \qquad \boldsymbol{\eta} = 0, \qquad E = 0. \tag{3.16}$$

In the basis $+, 1, \dots, 7, -$ (see (A.34) to (A.38)) this gives for ξ and $\tilde{E}$,

$$\begin{aligned} &\xi_- = -\frac{\mu^2}{\sqrt{2}}, \quad \xi_+ = \xi_1 = \ldots = \xi_7 = 0, \\ &\tilde{E}_{++} = \lambda, \quad \tilde{E}_{+-} = \tilde{E}_{-+} = \frac{\lambda}{\sqrt{2}}, \quad \tilde{E}_{--} = \frac{\lambda}{2}, \text{ and all other elements zero.} \end{aligned} \tag{3.17}$$

$$\begin{aligned} \text{Example } II \qquad V_{II} &= m_1^2 \varphi_1^\dagger \varphi_1 + m_2^2 \varphi_2^\dagger \varphi_2 - \mu^2 \varphi_3^\dagger \varphi_3 + \lambda (\varphi_3^\dagger \varphi_3)^2 \\ &= \frac{1}{2} m_1^2 \Big(\sqrt{\frac{2}{3}} K_0 + \sqrt{\frac{1}{3}} K_8 + K_3 \Big) + \frac{1}{2} m_2^2 \Big(\sqrt{\frac{2}{3}} K_0 + \sqrt{\frac{1}{3}} K_8 - K_3 \Big) \\ &\quad - \frac{1}{\sqrt{2}} \mu^2 \Big(\sqrt{\frac{1}{3}} K_0 - \sqrt{\frac{2}{3}} K_8 \Big) + \frac{1}{2} \lambda \Big(\sqrt{\frac{1}{3}} K_0 - \sqrt{\frac{2}{3}} K_8 \Big)^2 \\ &= \frac{1}{2} m_1^2 (K_+ + K_3) + \frac{1}{2} m_2^2 (K_+ - K_3) - \frac{1}{\sqrt{2}} \mu^2 K_- + \frac{1}{2} \lambda K_-^2, \end{aligned} \tag{3.18}$$

where we require

$$m_1^2 > 0, \quad m_2^2 > 0, \quad \mu^2 > 0, \quad \lambda > 0. \tag{3.19}$$

Here, in the basis $+, 1, \dots, 7, -$ (see (A.34) to (A.38)) only the following elements of ξ and $\tilde{E}$ are non zero

$$\xi_+ = \frac{1}{2}(m_1^2 + m_2^2), \quad \xi_3 = \frac{1}{2}(m_1^2 - m_2^2), \quad \xi_- = -\frac{1}{\sqrt{2}} \mu^2, \quad \tilde{E}_{--} = \frac{1}{2}\lambda. \tag{3.20}$$

JHEP10(2015)149

4 Stability of the 3HDM

Let us now analyse stability of the general 3HDM potential (3.1), given in terms of the bilinears K_0 and $\boldsymbol{K}$ on the domain determined by (2.16). This can be done in an analogous way to the THDM; see [5]. The case $\sqrt{3/2}K_0 = \varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2 + \varphi_3^\dagger\varphi_3 = 0$ corresponds to vanishing Higgs-boson fields and $V = 0$. For $K_0 > 0$ we define

$$\boldsymbol{k} = \frac{\boldsymbol{K}}{K_0} = \left(\frac{K_a}{K_0}\right). \tag{4.1}$$

Due to (2.16) we have for $\boldsymbol{k}$ the domain $\mathcal{D}_{\boldsymbol{k}}$:

$$\begin{aligned} &2 - \boldsymbol{k}^2 \geq 0, \\ &\det(\sqrt{2/3}\mathbb{1}_3 + k_a\lambda_a) = 0. \end{aligned} \tag{4.2}$$

The domain boundary, $\partial\mathcal{D}_{\boldsymbol{k}}$, is characterised by

$$2 - \boldsymbol{k}^2 = 0. \tag{4.3}$$

From (3.1) and (4.1) we obtain, for $K_0 > 0$, $V = V_2 + V_4$ with

$$V_2 = K_0\, J_2(\boldsymbol{k}), \qquad J_2(\boldsymbol{k}) := \xi_0 + \boldsymbol{\xi}^\mathrm{T}\boldsymbol{k}, \tag{4.4}$$

$$V_4 = K_0^2\, J_4(\boldsymbol{k}), \qquad J_4(\boldsymbol{k}) := \eta_{00} + 2\boldsymbol{\eta}^\mathrm{T}\boldsymbol{k} + \boldsymbol{k}^\mathrm{T} E \boldsymbol{k} \tag{4.5}$$

where we introduce the functions $J_2(\boldsymbol{k})$ and $J_4(\boldsymbol{k})$ on the domain (4.2).

A stable potential means that it is bounded from below. The stability is determined by the behaviour of V in the limit $K_0 \to \infty$, that is, by the signs of $J_4(\boldsymbol{k})$ and $J_2(\boldsymbol{k})$ in (4.4), (4.5). For a model to be at least *marginally* stable, the conditions

$$\begin{aligned} &J_4(\boldsymbol{k}) > 0 \quad \text{or} \\ &J_4(\boldsymbol{k}) = 0 \quad \text{and} \quad J_2(\boldsymbol{k}) \geq 0 \end{aligned} \tag{4.6}$$

for all $\boldsymbol{k} \in \mathcal{D}_{\boldsymbol{k}}$, that is, all $\boldsymbol{k}$ satisfying (4.2) are necessary and sufficient, since this is equivalent to $V \geq 0$ for $K_0 \to \infty$ in all possible allowed directions $\boldsymbol{k}$. The more strict stability property $V \to \infty$ for $K_0 \to \infty$ and any allowed $\boldsymbol{k}$ requires V to be stable either in the strong or the weak sense. For strong stability we require

$$J_4(\boldsymbol{k}) > 0 \tag{4.7}$$

for all $\boldsymbol{k} \in \mathcal{D}_{\boldsymbol{k}}$; see (4.2). For stability in the weak sense we require for all $\boldsymbol{k} \in \mathcal{D}_{\boldsymbol{k}}$

$$\begin{aligned} J_4(\boldsymbol{k}) &\geq 0, \\ J_2(\boldsymbol{k}) &> 0 \text{ for all } \boldsymbol{k} \text{ where } J_4(\boldsymbol{k}) = 0. \end{aligned} \tag{4.8}$$

To check that $J_4(\boldsymbol{k})$ is positive (semi-)definite, it is sufficient to consider its value for all stationary points on the domain $\mathcal{D}_{\boldsymbol{k}}$. This holds because the global minimum of the continuous function $J_4(\boldsymbol{k})$ is reached on the compact domain $\mathcal{D}_{\boldsymbol{k}}$, and the global minimum is among those stationary points.

To obtain the stationary points of $J_4(\boldsymbol{k})$ in the interior of the domain $\mathcal{D}_{\boldsymbol{k}}$ we add to $J_4(\boldsymbol{k})$ the second condition in (4.2) with a Lagrange multiplier u. The stationary points are then obtained from

$$\begin{aligned} \nabla_{k_1,\ldots,k_8}\Big[J_4(\boldsymbol{k}) - u\cdot g(\boldsymbol{k})\Big] &= 0,\\ g(\boldsymbol{k}) = \det(\sqrt{2/3}\mathbb{1}_3 + k_a\lambda_a) &= 0,\\ 2-\boldsymbol{k}^2 &> 0, \end{aligned} \tag{4.9}$$

provided the gradient matrix of the constraint equation has rank 1. This can easily be checked. With $k_0 = 1$ we have from (A.30) and (A.33)

$$\frac{1}{K_0}\underline{K} = \frac{1}{2}k_\alpha\lambda_\alpha, \tag{4.10}$$

$$g(\boldsymbol{k}) = \det\left(2\frac{\underline{K}}{K_0}\right) = \frac{2}{3}G_{\alpha\beta\gamma}k_\alpha k_\beta k_\gamma, \tag{4.11}$$

$$\frac{\partial g(\boldsymbol{k})}{\partial k_a} = 2G_{a\beta\gamma}k_\beta k_\gamma = 2\frac{M_a}{K_0^2}. \tag{4.12}$$

In our case the corresponding matrix $\underline{M} = M_\alpha\lambda_\alpha/2$ has rank 1, see (A.26), therefore, M_0 and at least one element M_a with $a\in\{1,\ldots,8\}$ have to be non zero. That is, $\big(\partial g(\boldsymbol{k})/\partial k_a\big)$ has rank 1 as required.

For the stationary points on the boundary $\partial\mathcal{D}_{\boldsymbol{k}}$ we have two constraints, see (4.2), (4.3),

$$g_1(\boldsymbol{k}) = \det(\sqrt{2/3}\mathbb{1}_3 + k_a\lambda_a) = 0, \qquad g_2(\boldsymbol{k}) = 2 - k_a k_a = 0. \tag{4.13}$$

Here the gradient matrix reads

$$\begin{pmatrix} \frac{\partial g_1(\boldsymbol{k})}{\partial k_a} \\ \frac{\partial g_2(\boldsymbol{k})}{\partial k_a} \end{pmatrix} = \begin{pmatrix} 2\frac{M_a}{K_0^2} \\ -2k_a \end{pmatrix}. \tag{4.14}$$

For $\underline{K}$ as in (4.10) but now of rank 1 we find from (A.26), $\underline{M} = 0$ and, therefore, $M_a = 0$, $a = 1,\ldots,8$. Thus, the gradient matrix has here only rank 1 and not the required rank 2 for the application of the Lagrange multiplier method in an analogous way to (4.9). We turn, therefore, to the parametrization of the rank 1 matrices $\underline{K}$ of (A.14) to (A.16):

$$\frac{\underline{K}}{K_0} = \sqrt{\frac{3}{2}}\boldsymbol{w}\boldsymbol{w}^\dagger \tag{4.15}$$

where

$$\boldsymbol{w}^\dagger\boldsymbol{w} = 1. \tag{4.16}$$

This gives, according to (2.9) and (4.1), (4.5),

$$\begin{aligned} k_a \equiv k_a(\boldsymbol{w}^\dagger,\boldsymbol{w}) &= \mathrm{tr}\left(\frac{1}{K_0}\underline{K}\lambda_a\right) = \sqrt{\frac{3}{2}}\boldsymbol{w}^\dagger\lambda_a\boldsymbol{w},\\ J_4(\boldsymbol{k}) \equiv J_4(\boldsymbol{w}^\dagger,\boldsymbol{w}) &= \eta_{00} + 2\eta_a\sqrt{\frac{3}{2}}\boldsymbol{w}^\dagger\lambda_a\boldsymbol{w} + \frac{3}{2}\big(\boldsymbol{w}^\dagger\lambda_a\boldsymbol{w}\big)E_{ab}\big(\boldsymbol{w}^\dagger\lambda_b\boldsymbol{w}\big). \end{aligned} \tag{4.17}$$

Now we can determine the stationary points of $J_4(\boldsymbol{w}^\dagger, \boldsymbol{w})$ subject to the constraint (4.16). With u a Lagrange multiplier we get

$$\begin{aligned} \nabla_{\boldsymbol{w}^\dagger}\left[J_4(\boldsymbol{w}^\dagger, \boldsymbol{w}) - u(\boldsymbol{w}^\dagger\boldsymbol{w} - 1)\right] = 0, \\ \boldsymbol{w}^\dagger\boldsymbol{w} - 1 = 0. \end{aligned} \tag{4.18}$$

Here the gradient matrix of the constraint is of rank 1 as required and we get explicitly

$$\begin{aligned} \left[\sqrt{6}\eta_a\lambda_a + 3E_{ab}(\boldsymbol{w}^\dagger\lambda_b\boldsymbol{w})\lambda_a - u\right]\boldsymbol{w} = 0, \\ \boldsymbol{w}^\dagger\boldsymbol{w} - 1 = 0. \end{aligned} \tag{4.19}$$

All stationary points obtained from (4.9) and (4.19) have to fulfill the condition $J_4(\boldsymbol{k}) > 0$ for stability in the strong sense. If for all stationary points we have $J_4(\boldsymbol{k}) \geq 0$, then for every solution $\boldsymbol{k}$ with $J_4(\boldsymbol{k}) = 0$ we have to have $J_2(\boldsymbol{k}) > 0$ for stability in the weak sense, or at least $J_2(\boldsymbol{k}) = 0$ for *marginal* stability. If none of these conditions is fulfilled, that is, if we find at least one stationary direction $\boldsymbol{k}$ with $J_4(\boldsymbol{k}) < 0$ or $J_4(\boldsymbol{k}) = 0$ but $J_2(\boldsymbol{k}) < 0$, the potential is unstable.

In our explicit example I, with V_I from (3.15), the functions $J_2(\boldsymbol{k})$ and $J_4(\boldsymbol{k})$ read

$$J_2(\boldsymbol{k}) = \left(-\frac{1}{\sqrt{6}} + \frac{k_8}{\sqrt{3}}\right)\mu^2, \qquad J_4(\boldsymbol{k}) = \frac{3}{2}\lambda. \tag{4.20}$$

Obviously, $J_4(\boldsymbol{k})$ is always positive for $\lambda > 0$ in any direction $\boldsymbol{k}$, therefore, the potential is stable in the strong sense. That is, stability is here guarantied by the quartic terms of the potential alone.

For example II from (3.18), (3.19), we get

$$V_2 = \frac{1}{2}m_1^2(K_+ + K_3) + \frac{1}{2}m_2^2(K_+ - K_3) - \frac{1}{\sqrt{2}}\mu^2 K_-, \qquad V_4 = \frac{1}{2}\lambda K_-^2. \tag{4.21}$$

We have $V_4 > 0$ for $K_- > 0$ but $V_4 = 0$ for $K_- = 0$. Thus, we have to investigate V_2 for $K_- = 0$:

$$V_2\Big|_{K_-=0} = \frac{1}{2}m_1^2(K_+ + K_3) + \frac{1}{2}m_2^2(K_+ - K_3). \tag{4.22}$$

Due to (2.13) and (2.14) we have

$$V_2\Big|_{K_-=0} \geq 0 \tag{4.23}$$

where $V_2\Big|_{K_-=0} = 0$ only holds if

$$K_+ + K_3 = 0, \qquad K_+ - K_3 = 0, \tag{4.24}$$

that is, for $K_+ = 0$. But this implies $\underline{K} = 0$. Thus, the potential V_{II} from (3.18), (3.19) is stable in the weak sense.

5 Electroweak symmetry breaking of the 3HDM

Suppose now that the 3HDM potential is stable, that is, bounded from below. Then the global minimum will be among the stationary points of V. In the following the different types of minima with respect to electroweak symmetry breaking are discussed and the corresponding stationarity equations are presented.

As we have discussed in section 2, the space of the Higgs-boson doublets is determined, up to electroweak gauge transformations, by the space of the hermitian 3×3 matrices $\underline{K}$ with rank smaller or equal 2. Since the rank of the matrix $\underline{K}$ is equal to the rank of the Higgs-boson field matrix ϕ (2.3) we can distinguish the different types of minima with respect to electroweak symmetry breaking as follows. At the global minimum, that is, the vacuum configuration, we write the 3×2 matrix of the Higgs-boson fields as

$$\langle \phi \rangle = \begin{pmatrix} v_1^+ & v_1^0 \\ v_2^+ & v_2^0 \\ v_3^+ & v_3^0 \end{pmatrix}. \tag{5.1}$$

In the case this matrix has rank 2, we cannot, by a $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ transformation, achieve a form with all charged components v_i^+, $i = 1, 2, 3$ vanishing. This means that the full $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is broken. In case we have at the minimum a matrix $\langle \phi \rangle$ with rank one, we can, by a $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ transformation, achieve a form with all charged components v_i^+ vanishing. The unbroken U(1) gauge group can then be identified with the electromagnetic gauge group. Therefore, a minimum with rank one corresponds to the electroweak-symmetry breaking $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \ \to \mathrm{U}(1)_{em}$. Eventually, a vanishing matrix at the minimum, $\langle \phi \rangle = 0$, corresponds to an unbroken electroweak symmetry. Of course, only a minimum with a partially broken electroweak symmetry is physically acceptable.

We study now the matrix $\underline{K}_v$ corresponding to $\langle \phi \rangle$ (5.1)

$$\underline{K}_v = \langle \phi \rangle \langle \phi \rangle^\dagger = \frac{1}{2} K_{v\alpha} \lambda_\alpha. \tag{5.2}$$

For an acceptable vacuum $\langle \phi \rangle$, $\underline{K}_v$ must have rank 1. From (A.20) we see that $\underline{K}_v$ has rank 1 and is positive semidefinite if and only if

$$\begin{aligned} \mathrm{tr}\, \underline{K}_v &= \sqrt{\frac{3}{2}} K_{v0} > 0, \\ 2K_{v0}^2 - K_{va} K_{va} &= 0, \\ \det(\underline{K}_v) &= 0. \end{aligned} \tag{5.3}$$

By a suitable U(3) transformation (3.4) we can bring the vacuum value $\langle \phi \rangle$ of rank 1 to the form

$$\langle \phi \rangle = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & v_0/\sqrt{2} \end{pmatrix}, \qquad v_0 > 0. \tag{5.4}$$

In a realistic model v_0 must be the usual Higgs-boson vacuum expectation value,

$$v_0 \approx 246 \text{ GeV}. \tag{5.5}$$

With (5.4) we get in this basis a particularly simple form for $\underline{K}_v$ respectively $K_{v\alpha}$:

$$\begin{aligned}
\underline{K}_v &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v_0^2 \end{pmatrix} = \frac{1}{2} K_{v\alpha}\lambda_\alpha, \\
(K_{v\alpha}) &= \frac{v_0^2}{\sqrt{6}}\left(1,\, 0,\, \ldots,\, 0,\, -\sqrt{2}\right)^{\mathrm{T}} \\
K_{v+} &= 0, \quad K_{v-} = \frac{1}{\sqrt{2}} v_0^2.
\end{aligned} \tag{5.6}$$

Another possible choice for the vacuum expectation value, obtainable by a suitable transformation (3.4) from (5.4) is

$$\langle \phi \rangle = \begin{pmatrix} 0 & v_0/\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad v_0 > 0. \tag{5.7}$$

Here we get

$$\begin{aligned}
\underline{K}_v =& \frac{1}{2}\begin{pmatrix} v_0^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(K_{v\alpha}) =& v_0^2\left(\tfrac{1}{\sqrt{6}},\, 0,\, 0,\, \tfrac{1}{2},\, 0,\, 0,\, 0,\, 0,\, \tfrac{1}{2\sqrt{3}}\right)^{\mathrm{T}}.
\end{aligned} \tag{5.8}$$

In the cases where $\langle \phi \rangle$ of (5.1) has rank 2 or rank 0 also the matrix $\underline{K}_v$, (5.2), has rank 2 or zero, respectively. The corresponding conditions for $\underline{K}_v$ are given explicitly in (A.19) and (A.21), respectively. We can, therefore, summarise our findings for the vacuum values to a given potential V as follows.

Let $\langle \phi \rangle$ be the vacuum expectation value of the Higgs-boson field matrix to a given, stable, potential V and $\underline{K}_v = \langle \phi \rangle \langle \phi \rangle^\dagger = K_{v\alpha}\lambda_\alpha/2$. The gauge symmetry $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is fully broken by the vacuum if and only if

$$K_{v0} > 0, \qquad 2K_{v0}^2 - K_{va}K_{va} > 0. \tag{5.9}$$

We have the breaking $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{em}$ if and only if

$$K_{v0} > 0, \qquad 2K_{v0}^2 - K_{va}K_{va} = 0. \tag{5.10}$$

We have no breaking of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ if and only if

$$K_{v\alpha} = 0. \tag{5.11}$$

Of course, we always have

$$\det \underline{K}_v = \frac{1}{12} G_{\alpha\beta\gamma} K_{v\alpha} K_{v\beta} K_{v\gamma} = 0 \tag{5.12}$$

with $G_{\alpha\beta\gamma}$ defined in (A.31).

JHEP10(2015)149

6 Stationary points

Following the study of stability and electroweak symmetry breaking in the last two sections we shall now present the stationarity equations. We suppose again that the potential is stable. Then the global minimum is among the stationary points of V.

We classify the stationary points by the rank of the stationarity matrix $\underline{K}$. In the following we use the conditions for $\underline{K}$ having rank 0, 1, 2, or 3 as given in appendix A; see (A.18) – (A.21).

The matrix $\underline{K} = 0$, respectively $K_\alpha = 0$, $\alpha = 0, \ldots, 8$, always corresponds to a stationary point of V with value $V(K_\alpha) = 0$.

All stationarity matrices $\underline{K} = K_\alpha \lambda_\alpha/2$ of rank 2 are obtained from the following system of equations where u is a Lagrange multiplier:

$$\begin{aligned} \nabla_{K_0,\ldots,K_8}\Big[V(K_0,\ldots,K_8) - u\ \det(\underline{K})\Big] &= 0,\\ 2K_0^2 - K_a K_a &> 0,\\ \det(\underline{K}) &= 0,\\ K_0 &> 0. \end{aligned} \tag{6.1}$$

Explicitly we get here, using (3.3),

$$\begin{aligned} \xi_\alpha + 2\tilde{E}_{\alpha\beta}K_\beta - \frac{u}{4}G_{\alpha\beta\gamma}K_\beta K_\gamma &= 0,\\ (3\delta_{\alpha 0}\delta_{\beta 0} - \delta_{\alpha\beta})K_\alpha K_\beta &> 0,\\ G_{\alpha\beta\gamma}K_\alpha K_\beta K_\gamma &= 0,\\ K_0 &> 0. \end{aligned} \tag{6.2}$$

The gradient matrix of the constraint is given by (see (A.30) and (A.33))

$$\left(\nabla_{K_\alpha} \det(\underline{K})\right) = \left(\nabla_{K_\alpha}\left(\frac{1}{12}G_{\alpha'\beta\gamma}K_{\alpha'}K_\beta K_\gamma\right)\right) = \left(\frac{1}{4}G_{\alpha\beta\gamma}K_\beta K_\gamma\right) = \left(\frac{1}{4}M_\alpha\right) \tag{6.3}$$

and has rank 1 as required. This holds since for $\underline{K}$ of rank 2, $\underline{M}$ has rank 1, as we see from (A.26), implying, for instance, $M_0 = \mathrm{tr}(\underline{M}\lambda_0) > 0$.

For the stationarity matrices $\underline{K} = K_\alpha\lambda_\alpha/2$ of rank 1 we cannot use the Lagrange multiplier method in an analogous way to (6.1). The two constraints for rank 1, $g_1 = 2K_0^2 - K_a K_a = 0$ and $g_2 = \det(\underline{K}) = 0$ yield a gradient matrix of rank 1 for $\underline{K}$ of rank 1. However, the Langrange multiplier method requires that this matrix has rank 2 in this case. The stationarity equations of the potential $V(\underline{K})$ for $\underline{K}$ of rank 1 follow from a parametrization of the matrix $\underline{K}$ of rank 1, as given in (A.16):

$$\underline{K}(K_0, \boldsymbol{w}^\dagger, \boldsymbol{w}) = K_0\sqrt{\frac{3}{2}}\boldsymbol{w}\boldsymbol{w}^\dagger \tag{6.4}$$

where

$$K_0 > 0, \qquad \boldsymbol{w}^\dagger\boldsymbol{w} - 1 = 0. \tag{6.5}$$

This gives

$$K_\alpha(K_0, \boldsymbol{w}^\dagger, \boldsymbol{w}) = \mathrm{tr}\left(\underline{K}(K_0, \boldsymbol{w}^\dagger, \boldsymbol{w})\lambda_\alpha\right) = K_0\sqrt{\frac{3}{2}}\boldsymbol{w}^\dagger\lambda_\alpha\boldsymbol{w}. \tag{6.6}$$

Taking into account the constraint (6.5) with a Lagrange multiplier u we have to determine the stationary points of

$$V(K_\alpha(K_0, \boldsymbol{w}^\dagger, \boldsymbol{w})) - u(\boldsymbol{w}^\dagger\boldsymbol{w} - 1) \tag{6.7}$$

under variation of K_0, and $\boldsymbol{w}^\dagger$, $\boldsymbol{w}$. The gradient matrix of the constraint has rank 1 as required and we get with (3.3) the following system of equations

$$\begin{aligned} \left[K_0\sqrt{\frac{3}{2}}\xi_\alpha\lambda_\alpha + 3K_0^2\tilde{E}_{\alpha\beta}(\boldsymbol{w}^\dagger\lambda_\beta\boldsymbol{w})\lambda_\alpha - u\right]\boldsymbol{w} &= 0,\\ \sqrt{\frac{3}{2}}\xi_\alpha(\boldsymbol{w}^\dagger\lambda_\alpha\boldsymbol{w}) + 3K_0(\boldsymbol{w}^\dagger\lambda_\alpha\boldsymbol{w})\tilde{E}_{\alpha\beta}(\boldsymbol{w}^\dagger\lambda_\beta\boldsymbol{w}) &= 0,\\ \boldsymbol{w}^\dagger\boldsymbol{w} - 1 &= 0,\\ K_0 &> 0. \end{aligned} \tag{6.8}$$

The stationarity matrix $\underline{K} = K_\alpha\lambda_\alpha/2$ with the lowest value of $V(K_0, \ldots, K_8)$ gives the global minimum $\underline{K}_v$ of the potential. Note that in general there may be degenerate global minima with the same potential value. Systems of equations of the kind (6.2), (6.8) can be solved via the Groebner-basis approach or homotopy continuation; see for instance [22, 23].

7 The potential after symmetry breaking

In this section we discuss the potential after symmetry breaking and the procedure to calculate the physical Higgs-boson masses and self couplings in the 3HDM. We will assume that the potential is stable and leads to the desired electroweak symmetry breaking $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{em}$. In particular, the global minimum is then a solution of the set of equations (6.8). In this case we can, in the unitary gauge, by an electroweak gauge transformation and a U(3) rotation (3.4) always achieve the form (5.4) for the vacuum expectation value of the Higgs-field matrix. For the original Higgs fields expressed in terms of the physical fields we get then

$$\varphi_{1/2}(x) = \begin{pmatrix} H^+_{1/2}(x) \\ \frac{1}{\sqrt{2}}\left(H^0_{1/2}(x) + iA^0_{1/2}(x)\right)\end{pmatrix}, \qquad \varphi_3(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_0 + h_0(x)\end{pmatrix}, \tag{7.1}$$

with v_0 real and positive, neutral fields $H^0_1(x)$, $A^0_1(x)$, $H^0_2(x)$, $A^0_2(x)$, $h_0(x)$, as well as the complex charged fields $H^+_1(x)$ and $H^+_2(x)$. The negatively charged Higgs-boson fields are defined by $H^-_{1/2}(x) = \left(H^+_{1/2}(x)\right)^\dagger$. Thus, we have in the 3HDM the following physical fields

$$\begin{aligned} &\text{five neutral fields:} \quad H^0_1(x), A^0_1(x), H^0_2(x), A^0_2(x), h_0(x)\\ &\text{two charged fields:} \quad H^+_1(x), H^+_2(x). \end{aligned} \tag{7.2}$$

In general, however, the physical fields of definite mass are linear combinations of the fields in (7.2). Obviously, the 3 original complex doublets of any 3HDM, corresponding to 12 real degrees of freedom, yield 5 real fields and 2 complex fields, with the 3 remaining degrees of freedom absorbed via the mechanism of electroweak symmetry breaking.

Throughout this section we shall work in a basis where $\underline{K}_v$ has the form (5.6). Representing $\underline{K}$ as in (6.4) we get

$$\underline{K}_v = \frac{v_0^2}{2}\boldsymbol{e}_3\boldsymbol{e}_3^\dagger, \qquad \boldsymbol{w} = \boldsymbol{e}_3, \tag{7.3}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are the three-dimensional Cartesian unit vectors. That is, $K_0 = v_0^2/\sqrt{6}$ and $\boldsymbol{w} = \boldsymbol{e}_3$ must be solutions of (6.8). We get

$$\boldsymbol{e}_3^\dagger \lambda_\alpha \boldsymbol{e}_3 = \sqrt{\frac{2}{3}}(\delta_{\alpha 0} - \sqrt{2}\delta_{\alpha 8}) \tag{7.4}$$

and define for $\alpha = 0, \dots 8$

$$\begin{aligned}
\zeta_\alpha &= \xi_\alpha + 2\tilde{E}_{\alpha\beta}K_{v\beta} = \xi_\alpha + 2\tilde{E}_{\alpha -}K_{v-}, \\
\zeta_+ &= \xi_+ + 2\tilde{E}_{+-}K_{v-}, \\
\zeta_- &= \xi_- + 2\tilde{E}_{--}K_{v-}
\end{aligned} \tag{7.5}$$

where the $\pm$ components are defined in (A.34) ff. Inserting all this in (6.8) with $\boldsymbol{w} = \boldsymbol{e}_3$ we get from the first equation there

$$\left[\frac{v_0^2}{2}\zeta_\alpha\lambda_\alpha - u\right]\boldsymbol{e}_3 = 0 \tag{7.6}$$

that is,

$$\frac{v_0^2}{2}\Big(\boldsymbol{e}_1(\zeta_4 - i\zeta_5) + \boldsymbol{e}_2(\zeta_6 - i\zeta_7)\Big) + \boldsymbol{e}_3\left(\frac{v_0^2}{\sqrt{2}}\zeta_- - u\right) = 0. \tag{7.7}$$

The ζ_α are all real, therefore, we have from (7.7)

$$\zeta_4 = \zeta_5 = \zeta_6 = \zeta_7 = 0, \qquad \frac{v_0^2}{\sqrt{2}}\zeta_- = u. \tag{7.8}$$

The second equation of (6.8) gives

$$\zeta_- = 0 \tag{7.9}$$

and thus, from (7.8), $u = 0$.

To summarise, in the basis where $\langle\phi\rangle$ and $\underline{K}_v$ have the forms (5.4) and (5.6), respectively, we have

$$\zeta_\alpha = 0 \quad \text{for } \alpha = 4, 5, 6, 7, -. \tag{7.10}$$

The next task is to expand ϕ, $\underline{K}$, and V in terms of the physical fields (7.2). For ϕ we write

$$\begin{aligned}
\phi(x) &= \langle\phi\rangle + \phi^{(1)}(x), \\
\phi^{(1)}(x) &= \langle\phi\rangle\frac{h_0(x)}{v_0} + \phi'^{(1)}(x), \\
\phi'^{(1)}(x) &= \begin{pmatrix} H_1^+(x) & \frac{1}{\sqrt{2}}(H_1^0(x) + iA_1^0(x)) \\ H_2^+(x) & \frac{1}{\sqrt{2}}(H_2^0(x) + iA_2^0(x)) \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{7.11}$$

From this we get for $\underline{K}(x)$ and $K_\alpha(x)$ the following with $\underline{K}_v$ and $K_{v\alpha}$ given in (5.6)

$$\begin{aligned}
\underline{K}(x) &= \underline{K}_v + \underline{K}^{(1)}(x) + \underline{K}^{(2)}(x), &(7.12)\\
\underline{K}^{(1)}(x) &= \frac{2h_0(x)}{v_0}\underline{K}_v + \underline{K}'^{(1)}(x), &(7.13)\\
\underline{K}'^{(1)}(x) &= \frac{v_0}{2}\Big(H_1^0(x)\lambda_4 - A_1^0(x)\lambda_5 + H_2^0(x)\lambda_6 - A_2^0(x)\lambda_7\Big), &(7.14)\\
\underline{K}^{(2)}(x) &= \phi^{(1)}(x)\phi^{(1)\dagger}(x). &(7.15)\\
\big(K_\alpha^{(1)}(x)\big) &= v_0\Big(\sqrt{\frac{2}{3}}h_0(x), 0, 0, 0, H_1^0(x), -A_1^0(x), H_2^0(x), -A_2^0(x), -\frac{2}{\sqrt{3}}h_0(x)\Big)^{\mathrm{T}}, &(7.16)\\
K_+^{(1)}(x) &= 0, \qquad K_-^{(1)}(x) = \sqrt{2}v_0h_0(x),\\
K_0^{(2)}(x) &= \sqrt{\frac{2}{3}}\Big[H_1^-(x)H_1^+(x) + H_2^-(x)H_2^+(x) + \frac{1}{2}\big((H_1^0(x))^2 + (A_1^0(x))^2\\
&\qquad + (H_2^0(x))^2 + (A_2^0(x))^2 + (h_0(x))^2\big)\Big],\\
K_1^{(2)}(x) &= H_1^+(x)H_2^-(x) + H_1^-(x)H_2^+(x) + H_1^0(x)H_2^0(x) + A_1^0(x)A_2^0(x),\\
K_2^{(2)}(x) &= i(H_1^+(x)H_2^-(x) - H_1^-(x)H_2^+(x)) + H_1^0(x)A_2^0(x) - A_1^0(x)H_2^0(x),\\
K_3^{(2)}(x) &= H_1^-(x)H_1^+(x) - H_2^-(x)H_2^+(x) + \frac{1}{2}\big((H_1^0(x))^2 + (A_1^0(x))^2 - (H_2^0(x))^2 - (A_2^0(x))^2\big),\\
K_4^{(2)}(x) &= H_1^0(x)h_0(x),\\
K_5^{(2)}(x) &= -A_1^0(x)h_0(x),\\
K_6^{(2)}(x) &= H_2^0(x)h_0(x),\\
K_7^{(2)}(x) &= -A_2^0(x)h_0(x),\\
K_8^{(2)}(x) &= \frac{1}{\sqrt{3}}\Big[H_1^-(x)H_1^+(x) + H_2^-(x)H_2^+(x) + \frac{1}{2}\big((H_1^0(x))^2 + (A_1^0(x))^2 + (H_2^0(x))^2\\
&\qquad + (A_2^0(x))^2\big) - (h_0(x))^2\Big],\\
K_+^{(2)}(x) &= H_1^-(x)H_1^+(x) + H_2^-(x)H_2^+(x) + \frac{1}{2}\big((H_1^0(x))^2 + (A_1^0(x))^2 + (H_2^0(x))^2 + (A_2^0(x))^2\big),\\
K_-^{(2)}(x) &= \frac{1}{\sqrt{2}}\,(h_0(x))^2\,. &(7.17)
\end{aligned}$$

For the potential we have the following expansion in the order of the physical fields

$$\begin{aligned}
V &= V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} + V^{(4)}. &(7.18)\\
V^{(0)} &= K_{v\alpha}\xi_\alpha + K_{v\alpha}\tilde{E}_{\alpha\beta}K_{v\beta},\\
V^{(1)} &= K_\alpha^{(1)}(x)\xi_\alpha + 2K_\alpha^{(1)}(x)\tilde{E}_{\alpha\beta}K_{v\beta}, &(7.19)\\
V^{(2)} &= K_\alpha^{(2)}(x)\xi_\alpha + 2K_\alpha^{(2)}(x)\tilde{E}_{\alpha\beta}K_{v\beta} + K_\alpha^{(1)}(x)\tilde{E}_{\alpha\beta}K_\beta^{(1)}(x),\\
V^{(3)} &= 2K_\alpha^{(2)}(x)\tilde{E}_{\alpha\beta}K_\beta^{(1)}(x),\\
V^{(4)} &= K_\alpha^{(2)}(x)\tilde{E}_{\alpha\beta}K_\beta^{(2)}(x).
\end{aligned}$$

We shall now discuss $V^{(0)}$, $V^{(1)}$, and $V^{(2)}$, where it is convenient to use the basis $(+, 1, \ldots, 7, -)$; see (A.34) ff. For $V^{(0)}$ we find with (5.6), (7.5), and (7.10),

$$V^{(0)} = K_{v-}(\xi_- + \tilde{E}_{--}K_{v-}) = \frac{1}{2}K_{v-}(\xi_- + \zeta_-) = \frac{1}{2}K_{v-}\xi_- = \frac{v_0^2}{2\sqrt{6}}(\xi_0 - \sqrt{2}\xi_8). \tag{7.20}$$

For $V^{(1)}$ we get from (5.6), (7.5), (7.10), and (7.16),

$$V^{(1)} = K_+^{(1)}(x)\zeta_+ + \sum_{\alpha=1}^{7} K_\alpha^{(1)}(x)\zeta_\alpha + K_-^{(1)}(x)\zeta_- = 0. \tag{7.21}$$

This must be so, since we are expanding around the global minimum. From $V^{(2)}$ we get the mass matrices squared for the charged and neutral physical fields:

$$\begin{aligned} V^{(2)} = {} & \left(H_1^-(x),\, H_2^-(x)\right) \mathscr{M}_{ch}^2 \begin{pmatrix} H_1^+(x) \\ H_2^+(x) \end{pmatrix} \\ & + \left(H_1^0(x),\, A_1^0(x),\, H_2^0(x),\, A_2^0(x),\, h_0(x)\right) \frac{1}{2}\mathscr{M}_{\mathrm{n}}^2 \begin{pmatrix} H_1^0(x) \\ A_1^0(x) \\ H_2^0(x) \\ A_2^0(x) \\ h_0(x) \end{pmatrix}. \end{aligned} \tag{7.22}$$

With (5.6), (7.10), (7.16), and (7.17) we get

$$\mathscr{M}_{ch}^2 = \begin{pmatrix} \zeta_+ + \zeta_3 & \zeta_1 - i\zeta_2 \\ \zeta_1 + i\zeta_2 & \zeta_+ - \zeta_3 \end{pmatrix}, \tag{7.23}$$

$$\mathscr{M}_{\mathrm{n}}^2 = \begin{pmatrix} \zeta_+ + \zeta_3 + 2v_0^2\tilde{E}_{44} & -2v_0^2\tilde{E}_{45} & \zeta_1 + 2v_0^2\tilde{E}_{46} & \zeta_2 - 2v_0^2\tilde{E}_{47} & 2\sqrt{2}v_0^2\tilde{E}_{4-} \\ -2v_0^2\tilde{E}_{54} & \zeta_+ + \zeta_3 + 2v_0^2\tilde{E}_{55} & -\zeta_2 - 2v_0^2\tilde{E}_{56} & \zeta_1 + 2v_0^2\tilde{E}_{57} & -2\sqrt{2}v_0^2\tilde{E}_{5-} \\ \zeta_1 + 2v_0^2\tilde{E}_{64} & -\zeta_2 - 2v_0^2\tilde{E}_{65} & \zeta_+ - \zeta_3 + 2v_0^2\tilde{E}_{66} & -2v_0^2\tilde{E}_{67} & 2\sqrt{2}v_0^2\tilde{E}_{6-} \\ \zeta_2 - 2v_0^2\tilde{E}_{74} & \zeta_1 + 2v_0^2\tilde{E}_{75} & -2v_0^2\tilde{E}_{76} & \zeta_+ - \zeta_3 + 2v_0^2\tilde{E}_{77} & -2\sqrt{2}v_0^2\tilde{E}_{7-} \\ 2\sqrt{2}v_0^2\tilde{E}_{-4} & -2\sqrt{2}v_0^2\tilde{E}_{-5} & 2\sqrt{2}v_0^2\tilde{E}_{-6} & -2\sqrt{2}v_0^2\tilde{E}_{-7} & 4v_0^2\tilde{E}_{--} \end{pmatrix}. \tag{7.24}$$

Note that from (5.6), (7.5), and (7.10) we get

$$\sqrt{2}v_0^2\tilde{E}_{\alpha-} = -\xi_\alpha \text{ for } \alpha = 4, 5, 6, 7, -. \tag{7.25}$$

Since we are expanding around the global minimum we must have that $V^{(0)}$ is below or at most equal to $V(K_\alpha = 0) = 0$. From (7.20) this implies

$$V^{(0)} = \frac{1}{2\sqrt{2}}v_0^2\xi_- = \frac{1}{2\sqrt{6}}v_0^2(\xi_0 - \sqrt{2}\xi_8) \leq 0. \tag{7.26}$$

Furthermore, the mass squared matrices $\mathscr{M}_{ch}^2$ and $\mathscr{M}_{\mathrm{n}}^2$ must be positive semidefinite. This implies, for instance, from (7.23) that we must have

$$\zeta_+ \geq \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}. \tag{7.27}$$

To obtain the physical Higgs bosons of definite mass, the matrices (7.23) and (7.24) have to be diagonalised. We note that the field $h_0(x)$ is a mass eigenstate if

$$\tilde{E}_{\alpha -} = 0 \qquad \text{for } \alpha = 4, 5, 6.7. \tag{7.28}$$

In this case $h_0(x)$ is what is called *aligned* with the vacuum expectation value and its mass squared is given by

$$m_{h_0}^2 = 4v_0^2 \tilde{E}_{--} = -2\sqrt{2}\xi_- = -\frac{8}{v_0^2} V^{(0)}. \tag{7.29}$$

In our example I, the 3HDM Higgs potential (3.15), we find stationary points for vanishing fields, corresponding to an unbroken EW symmetry, from the set (6.1) we get no solution with $K_0 > 0$, and from the set (6.8) we get one solution with

$$\boldsymbol{w} = \boldsymbol{e}_3, \qquad K_0 = \frac{1}{\sqrt{6}} \frac{\mu^2}{\lambda}. \tag{7.30}$$

The corresponding potential value is $V^{(0)} = -1/4 \cdot (\mu^2)^2/\lambda$ and is the deepest stationary point and therefore the global minimum. From (5.6) we see that the global minimum corresponds to a vacuum expectation value $v_0 = \sqrt{\mu^2/\lambda}$. For the mass matrices we get from (7.23) and (7.24)

$$\mathscr{M}_{ch}^2 = \text{diag}(\lambda v_0^2, \lambda v_0^2), \quad \mathscr{M}_{\text{n}}^2 = \text{diag}(\lambda v_0^2, \lambda v_0^2, \lambda v_0^2, \lambda v_0^2, 2\lambda v_0^2). \tag{7.31}$$

Turning to example II, V_{II} of (3.18), we have as stationary points the trivial one, $K_\alpha = 0$ with $V_{II}(0) = 0$, no stationary point from (6.1) and one point from (6.8). The latter is obtained again for

$$\boldsymbol{w} = \boldsymbol{e}_3, \qquad K_0 = \frac{1}{\sqrt{6}} \frac{\mu^2}{\lambda}. \tag{7.32}$$

Here we get from (5.6), (7.23), and (7.24)

$$\begin{gathered} v_0^2 = \frac{\mu^2}{\lambda}, \qquad \left(K_{v+},\, K_{v1},\, \dots\,, K_{v7},\, K_{v-}\right) = \left(0,\, \dots,\, 0,\, \tfrac{1}{\sqrt{2}} v_0^2\right), \\ \mathscr{M}_{ch}^2 = \text{diag}(m_1^2, m_2^2), \qquad \mathscr{M}_{\text{n}}^2 = \text{diag}(m_1^2, m_1^2, m_2^2, m_2^2, 2\lambda v_0^2). \end{gathered} \tag{7.33}$$

This simple example shows that the squared masses of the charged physical Higgs bosons need not be degenerate in a 3HDM having the correct electroweak symmetry breaking. In appendix B a nontrivial example of a 3HDM is discussed.

8 Conclusion

The three-Higgs-doublet model has been studied as a generalization of the THDM. Stability, electroweak symmetry breaking, and the types of stationary points of the potential have been investigated. Explicit sets of equations have been presented which allow to determine the stability of any 3HDM and, in case of a stable potential, to find the global minimum or the degenerate global minima in case the potential has such. For the case that the 3HDM has the physically relevant electroweak symmetry breaking $\text{SU}(2)_L \times \text{U}(1)_Y \to \text{U}(1)_{em}$ we have given explicit expressions for the mass squared matrices of the charged and neutral physical Higgs bosons. The use of bilinears turns out to be very helpful: in particular, irrelevant gauge degrees of freedom are avoided and the degree of the polynomial equations which are to be solved is reduced in this formalism. In general, the sets of equations which determine stability and the stationary points are rather involved. However, approaches like the Groebner-basis approach or homotopy continuation may be applied to solve these systems of equations in an efficient way. This has been demonstrated for a 3HDM based on a $O(2) \times \mathbb{Z}_2$ symmetry.

Acknowledgments

The work of M.M. was supported, in part, by Fondecyt (Chile) Grant No. 1140568.

A Properties of the matrix $\underline{\mathbf{K}}$

Here we want to discuss the properties of the matrix $\underline{K}$ (2.4) with respect to its rank.

First we note that the 3×3 matrix $\underline{K}$ is hermitian and positive semidefinite. Hence, we can, by a unitary transformation, diagonalise this matrix,

$$U\underline{K}U^\dagger = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \tag{A.1}$$

with all $\kappa_i \geq 0$. In particular, we have,

$$\begin{aligned} \mathrm{tr}(\underline{K}) &= \kappa_1 + \kappa_2 + \kappa_3, \\ (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) &= 2\kappa_1\kappa_2 + 2\kappa_2\kappa_3 + 2\kappa_1\kappa_3, \\ \det(\underline{K}) &= \kappa_1\kappa_2\kappa_3. \end{aligned} \tag{A.2}$$

Employing the properties of the Gell-Mann matrices (2.7) we can write the second trace condition in the form

$$(\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) = K_0^2 - \frac{1}{2}K_aK_a. \tag{A.3}$$

Suppose now that the matrix $\underline{K}$ has rank 3, then, we have to have for all three κ_i

$$\kappa_i > 0. \tag{A.4}$$

It follows immediately from (A.2)

$$\mathrm{tr}(\underline{K}) > 0, \quad (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) > 0, \quad \det(\underline{K}) > 0. \tag{A.5}$$

If, for the reverse, we have for a hermitian matrix $\underline{K}$ the conditions (A.5) fulfilled, then, using (A.2) we find that we must have all $\kappa_i > 0$. That is, $\underline{K}$ has rank 3 and is positive definite.

Suppose the matrix $\underline{K}$ has rank 2, then, without loss of generality, we can assume

$$\kappa_1 > 0, \quad \kappa_2 > 0, \quad \kappa_3 = 0. \tag{A.6}$$

It follows immediately from (A.2) that

$$\mathrm{tr}(\underline{K}) > 0, \quad (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) > 0, \quad \det(\underline{K}) = 0. \tag{A.7}$$

If, for the reverse, we have for a hermitian matrix $\underline{K}$ the conditions (A.7) fulfilled, then, from the last equation in (A.2) at least one $\kappa_i = 0$. Without loss of generality we can suppose $\kappa_3 = 0$. We have then

$$\begin{aligned} \mathrm{tr}(\underline{K}) &= \kappa_1 + \kappa_2 > 0, \\ (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) &= 2\kappa_1\kappa_2 > 0 \end{aligned} \tag{A.8}$$

which implies $\kappa_1 > 0$ and $\kappa_2 > 0$. That is, $\underline{K}$ has rank 2 and is positive semidefinite.

Another way to characterise the positive semidefinite matrices of rank 2 is as follows. We set

$$\kappa_1 = \sqrt{\frac{3}{2}} K_0 \sin^2(\chi), \qquad \kappa_2 = \sqrt{\frac{3}{2}} K_0 \cos^2(\chi), \qquad \kappa_3 = 0, \qquad K_0 > 0, \quad 0 < \chi \leq \frac{\pi}{4}. \tag{A.9}$$

Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ be orthonormal eigenvectors of $\underline{K}$ to κ_1 and κ_2, respectively, then we have

$$\underline{K} = K_0 \sqrt{\frac{3}{2}} \Big(\sin^2(\chi) \boldsymbol{w}_1 \boldsymbol{w}_1^\dagger + \cos^2(\chi) \boldsymbol{w}_2 \boldsymbol{w}_2^\dagger \Big), \tag{A.10}$$

where

$$\boldsymbol{w}_i^\dagger \boldsymbol{w}_j = \delta_{ij}. \tag{A.11}$$

For $0 < \chi < \pi/4$ the $\boldsymbol{w}_i$ are fixed up to phases, for $\chi = \pi/4$ we may make arbitrary U(2) rotations of $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Clearly, every positive semidefinite matrix $\underline{K}$ of the form (A.10) has rank 2 and every positive semidefinte matrix $\underline{K}$ of rank 2 can be written in the form (A.10).

Now, let us suppose the matrix $\underline{K}$ has rank 1, then, without loss of generality, we can assume

$$\kappa_1 > 0, \quad \kappa_2 = 0, \quad \kappa_3 = 0. \tag{A.12}$$

It follows immediately from (A.2)

$$\mathrm{tr}(\underline{K}) > 0, \quad (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) = 0, \qquad \det(\underline{K}) = 0. \tag{A.13}$$

On the other hand, having the conditions (A.13) for a hermitian matrix $\underline{K}$ fulfilled, employing (A.2), the determinant condition requires that at least one κ_i vanishes, for instance $\kappa_3 = 0$ without loss of generality. Then the second condition requires that another eigenvalue has to vanish, for instance $\kappa_2 = 0$. Eventually, the first condition then dictates that the remaining $\kappa_1 > 0$. Hence, $\underline{K}$ has rank 1 and is positive semidefinite.

Let now $\boldsymbol{w}$ be eigenvector of $\underline{K}$ to the eigenvalue κ_1 with

$$\boldsymbol{w}^\dagger \boldsymbol{w} = 1. \tag{A.14}$$

We set

$$\kappa_1 = K_0 \sqrt{\frac{3}{2}}, \qquad K_0 > 0. \tag{A.15}$$

Then, any positive semidefinite matrix $\underline{K}$ of rank 1 can be represented as

$$\underline{K} = K_0\sqrt{\frac{3}{2}}\boldsymbol{w}\boldsymbol{w}^\dagger. \tag{A.16}$$

Conversely, any matrix of the form (A.16) with $K_0 > 0$ and $\boldsymbol{w}^\dagger\boldsymbol{w} = 1$ is a positive semidefinite matrix of rank 1. Clearly, $\boldsymbol{w}$ is fixed up to a phase transformation.

Finally, suppose the matrix $\underline{K}$ has rank 0, then, clearly, all κ_i have to vanish, corresponding to

$$\mathrm{tr}(\underline{K}) = 0, \quad (\mathrm{tr}(\underline{K}))^2 - \mathrm{tr}(\underline{K}^2) = 0, \quad \det(\underline{K}) = 0. \tag{A.17}$$

Vice versa, starting with the conditions (A.17) for a hermitian matrix $\underline{K}$, the determinant condition requires that one eigenvalue, for instance $\kappa_3 = 0$ has to vanish, the second condition in turn requires that another, say $\kappa_2 = 0$, and the first trace condition that also the third $\kappa_1 = 0$. This means $\underline{K} = 0$. Therefore, we have shown the following theorem.

Theorem 1. *: Let $\underline{K} = K_\alpha\lambda_\alpha/2$ be a hermitian matrix. $\underline{K}$ has rank 3 and is positive definite if and only if*

$$\begin{aligned} \mathrm{tr}(\underline{K}) = \sqrt{\frac{3}{2}}K_0 &> 0,\\ 2K_0^2 - K_aK_a &> 0,\\ \det(\underline{K}) &> 0. \end{aligned} \tag{A.18}$$

$\underline{K}$ has rank 2 and is positive semidefinite if and only if

$$\begin{aligned} \mathrm{tr}(\underline{K}) = \sqrt{\frac{3}{2}}K_0 &> 0,\\ 2K_0^2 - K_aK_a &> 0,\\ \det(\underline{K}) &= 0. \end{aligned} \tag{A.19}$$

$\underline{K}$ has rank 1 and is positive semidefinite if and only if

$$\begin{aligned} \mathrm{tr}(\underline{K}) = \sqrt{\frac{3}{2}}K_0 &> 0,\\ 2K_0^2 - K_aK_a &= 0,\\ \det(\underline{K}) &= 0. \end{aligned} \tag{A.20}$$

$\underline{K} = 0$ if and only if

$$\begin{aligned} \mathrm{tr}(\underline{K}) = \sqrt{\frac{3}{2}}K_0 &= 0,\\ 2K_0^2 - K_aK_a &= 0,\\ \det(\underline{K}) &= 0. \end{aligned} \tag{A.21}$$

With this theorem we have expressed the properties of the matrix $\underline{K}$ in terms of the expansion coefficients K_α, $\alpha = 0, \ldots, 8$. The conditions explicitly written in terms of K_0

and K_a in (A.18) to (A.21) are of the type of *light-cone* conditions familiar from the two-Higgs-doublet model; see (36) of [5]. But the determinant condition, trilinear in K_α, is specific for the 3HDM.

To express also $\det(\underline{K})$ in terms of the expansion coefficients K_α, $\alpha = 0, \dots, 8$, we proceed as follows (see also [14]). We introduce, along with the matrix $\underline{K}$, a matrix $\underline{M} = (M_{ij})$:

$$M_{ij} = \epsilon_{ikl}\epsilon_{jmn}K_{mk}K_{nl}. \tag{A.22}$$

For a hermitian matrix $\underline{K}$ also $\underline{M}$ is hermitian. For any $U \in \mathrm{U}(3)$ we have the relation

$$\epsilon_{ijk}U_{ii'}U_{jj'}U_{kk'} = \epsilon_{i'j'k'}\det(U). \tag{A.23}$$

Using this we find easily that under a transformation (3.5) of $\underline{K}$ we get also for $\underline{M}$

$$\underline{M}' = U\ \underline{M}\ U^\dagger. \tag{A.24}$$

Furthermore we find

$$\det(\underline{K}) = \frac{1}{3!}\operatorname{tr}(\underline{K}\underline{M}). \tag{A.25}$$

Consider now a unitary transformation U which diagonalises $\underline{K}$; see (A.1).

We find then from (A.22)

$$U\underline{M}U^\dagger = \begin{pmatrix} 2\kappa_2\kappa_3 & 0 & 0 \\ 0 & 2\kappa_1\kappa_3 & 0 \\ 0 & 0 & 2\kappa_1\kappa_2 \end{pmatrix}, \tag{A.26}$$

and

$$\det(\underline{K}) = \frac{1}{3!}\operatorname{tr}(\underline{K}\underline{M}) = \kappa_1\kappa_2\kappa_3, \tag{A.27}$$

$$\operatorname{tr}(\underline{M}) = (\operatorname{tr}(\underline{K}))^2 - \operatorname{tr}(\underline{K}^2). \tag{A.28}$$

As for $\underline{K}$ in (2.8) we can expand $\underline{M}$ in terms of λ_α,

$$\underline{M} = \frac{1}{2}M_\alpha\lambda_\alpha, \qquad M_\alpha = \operatorname{tr}(\underline{M}\lambda_\alpha). \tag{A.29}$$

Inserting here (A.22) we get the expression of M_α in terms of the K_β (2.9) as follows:

$$M_\alpha = G_{\alpha\beta\gamma}K_\beta K_\gamma \tag{A.30}$$

where

$$G_{\alpha\beta\gamma} = \frac{1}{4}\Big\{ \operatorname{tr}(\lambda_\alpha)\operatorname{tr}(\lambda_\beta)\operatorname{tr}(\lambda_\gamma) + \operatorname{tr}(\lambda_\alpha\lambda_\beta\lambda_\gamma + \lambda_\alpha\lambda_\gamma\lambda_\beta) - \operatorname{tr}(\lambda_\alpha)\operatorname{tr}(\lambda_\beta\lambda_\gamma) - \operatorname{tr}(\lambda_\beta)\operatorname{tr}(\lambda_\gamma\lambda_\alpha) - \operatorname{tr}(\lambda_\gamma)\operatorname{tr}(\lambda_\alpha\lambda_\beta)\Big\}. \tag{A.31}$$

Clearly, $G_{\alpha\beta\gamma}$ is completely symmetric in α, β, γ. Explicitly we get

$$G_{0\beta\gamma} = \sqrt{\frac{3}{2}}\delta_{\beta 0}\delta_{\gamma 0} - \frac{1}{\sqrt{6}}\delta_{\beta\gamma}, \qquad G_{abc} = d_{abc} \tag{A.32}$$

with d_{abc} the usual symmetric constants of SU(3); see, for instance, appendix C of [24]. From (A.25), (A.29), and (A.30) we find

$$\det \underline{K} = \frac{1}{12} K_\alpha M_\alpha = \frac{1}{12} G_{\alpha\beta\gamma} K_\alpha K_\beta K_\gamma. \tag{A.33}$$

This is the desired expression of $\det(\underline{K})$ in terms of the K_α.

Finally we discuss the transformation from the basis $\alpha = 0, 1, \ldots, 7, 8$ to $+, 1, \ldots, 7, -$. This is achieved by the matrix

$$S = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ 0 & \mathbb{1}_7 & 0 \\ \sqrt{\frac{1}{3}} & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} S_{+0} & 0 & S_{+8} \\ 0 & \mathbb{1}_7 & 0 \\ S_{-0} & 0 & S_{-8} \end{pmatrix}. \tag{A.34}$$

We have then, in accord with (2.11),

$$\begin{aligned} K_+ &= S_{+0} K_0 + S_{+8} K_8 = \sqrt{\frac{2}{3}} K_0 + \sqrt{\frac{1}{3}} K_8, \\ K_- &= S_{-0} K_0 + S_{-8} K_8 = \sqrt{\frac{1}{3}} K_0 - \sqrt{\frac{2}{3}} K_8, \\ K_a, &\quad a = 1, \ldots, 7 \quad \text{unchanged}. \end{aligned} \tag{A.35}$$

The matrix S satisfies

$$S S^{\mathrm{T}} = \mathbb{1}_9, \qquad S = S^{\mathrm{T}}. \tag{A.36}$$

The basis change for (ξ_α) and $\tilde{E} = (\tilde{E}_{\alpha\beta})$ is then done in an analogous way:

$$\begin{pmatrix} \xi_+ \\ \xi_a \\ \xi_- \end{pmatrix} = S \begin{pmatrix} \xi_0 \\ \xi_a \\ \xi_8 \end{pmatrix}, \quad \begin{pmatrix} \tilde{E}_{++} & \tilde{E}_{+b} & \tilde{E}_{+-} \\ \tilde{E}_{a+} & \tilde{E}_{ab} & \tilde{E}_{a-} \\ \tilde{E}_{-+} & \tilde{E}_{-b} & \tilde{E}_{--} \end{pmatrix} = S (\tilde{E}_{\alpha\beta}) S^{\mathrm{T}}, \quad a, b \in \{1, \ldots, 7\}, \quad \alpha, \beta \in \{0, \ldots, 8\}. \tag{A.37}$$

We have, due to (A.36), for instance,

$$\xi_\alpha K_\alpha = \xi_+ K_+ + \sum_{a=1}^{7} \xi_a K_a + \xi_- K_-. \tag{A.38}$$

B Example of a 3HDM Higgs potential

Let us apply the developed formalism to a non-trivial 3HDM potential. We emphasize that any specific 3HDM can be treated along the following lines. We will apply the homotopy continuation approach to solve the systems of polynomial equations allowing us to discuss stability and the stationarity points of the model. Of course, other methods may be applied, like the Groebner-basis approach. These methods were successfully applied to Higgs potentials in the past; see for instance [22, 23]. In these works brief introductions to Groebner-bases and homotopy continuation can also be found.

The model we want to study was presented in [21] and is based on a $O(2) \times \mathbb{Z}_2$ symmetry involving three Higgs-boson doublets. All the elementary particles and in particular the

JHEP10(2015)149

	s	U(1)	$\mathbb{Z}_2$
$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}e^{2i\theta} & 0\\ 0 & e^{-2i\theta}\end{pmatrix}$	$\mathbb{1}_2$
φ_3	1	1	-1

Table 1. Assignments of the transformation behaviour of the Higgs-boson doublets under the symmetries s, U(1), $\mathbb{Z}_2$.

three Higgs-boson doublets are assigned to irreducible representations of the $O(2) \times \mathbb{Z}_2$ symmetry. For the three Higgs-boson doublets the assignments were chosen as given in table 1.

Here, the group $O(2)$ is decomposed into unitary rotations U(1) and reflections s.

The general 3HDM Higgs potential, symmetric under $O(2) \times \mathbb{Z}_2$ except for the term proportional to μ_m reads

$$
\begin{aligned}
V_{O(2)\times\mathbb{Z}_2} = &\ \mu_0\varphi_3^\dagger\varphi_3 + \mu_{12}\left(\varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2\right) + \mu_m\left(\varphi_1^\dagger\varphi_2 + \varphi_2^\dagger\varphi_1\right) \\
&+a_1(\varphi_3^\dagger\varphi_3)^2 + a_2\varphi_3^\dagger\varphi_3\left(\varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2\right) + a_3\left(\varphi_3^\dagger\varphi_1\cdot\varphi_1^\dagger\varphi_3 + \varphi_3^\dagger\varphi_2\cdot\varphi_2^\dagger\varphi_3\right) \\
&+a_4\varphi_3^\dagger\varphi_1\cdot\varphi_3^\dagger\varphi_2 + a_4^*\varphi_1^\dagger\varphi_3\cdot\varphi_2^\dagger\varphi_3 \\
&+a_5\left((\varphi_1^\dagger\varphi_1)^2 + (\varphi_2^\dagger\varphi_2)^2\right) + a_6\varphi_1^\dagger\varphi_1\cdot\varphi_2^\dagger\varphi_2 + a_7\varphi_1^\dagger\varphi_2\cdot\varphi_2^\dagger\varphi_1.
\end{aligned}
\tag{B.1}
$$

The term $\mu_m\left(\varphi_1^\dagger\varphi_2 + \varphi_2^\dagger\varphi_1\right)$ breaks the U(1) symmetry softly (for details we refer to [21]). This model has nine real parameters and one complex parameter a_4, corresponding to eleven real parameters in total.

With the help of (2.10) we write the potential in terms of bilinears. We identify the parameters of the potential (B.1), but written in the form (3.1), as

$$
\begin{aligned}
&\xi_0 = \frac{1}{\sqrt{6}}(\mu_0 + 2\mu_{12}), \quad \boldsymbol{\xi} = \left(\mu_m, 0, 0, 0, 0, 0, 0, \frac{1}{\sqrt{3}}(\mu_{12} - \mu_0)\right)^{\mathrm{T}}, \quad \eta_{00} = \frac{1}{6}(a_1 + 2a_2 + 2a_5 + a_6), \\
&\boldsymbol{\eta} = \left(0, 0, 0, 0, 0, 0, 0, \frac{\sqrt{2}}{6}(-a_1 - a_2/2 + a_5 + a_6/2)\right)^{\mathrm{T}}, \\
&E = \frac{1}{4}\begin{pmatrix}
a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_7 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2a_5 - a_6 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_3 & 0 & \mathrm{Re}(a_4) & \mathrm{Im}(a_4) & 0\\
0 & 0 & 0 & 0 & a_3 & \mathrm{Im}(a_4) & -\mathrm{Re}(a_4) & 0\\
0 & 0 & 0 & \mathrm{Re}(a_4) & \mathrm{Im}(a_4) & a_3 & 0 & 0\\
0 & 0 & 0 & \mathrm{Im}(a_4) & -\mathrm{Re}(a_4) & 0 & a_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4/3a_1 - 4/3a_2 + 2/3a_5 + 1/3a_6
\end{pmatrix}.
\end{aligned}
\tag{B.2}
$$

Obviously, all parameters are real in terms of bilinears.

We choose as an explicit numerical example the following values for the parameters, where we take only the quartic couplings from the reference point in [21]:

$$
\begin{aligned}
&a_1 = 2.5, \quad a_2 = 3, \qquad a_3 = -5, \quad a_4 = -0.0474041, \quad a_5 = 1.5, \quad a_6 = 2, \qquad a_7 = 3,\\
&\mu_0 = -90,774 \text{ GeV}^2, \quad \mu_{12} = -75,645 \text{ GeV}^2, \qquad\qquad \mu_m = -45,387 \text{ GeV}^2.
\end{aligned}
\tag{B.3}
$$

Actually, we start with all quartic parameters as given in (B.3) and then fix the quadratic parameters by the condition of a vanishing gradient of the potential, employing $K_{v0} = v_0^2/\sqrt{6}$ (see (5.6) with v_0 as given in (5.5)). In this way we ensure that there is at least one stationary solution which corresponds to the correct vacuum expectation value. Let us note that this procedure by no means guarantees that the corresponding potential is stable and has a global minimum with the correct partially broken electroweak symmetry — as we will see below.

The stability and stationarity equations are polynomial systems of equations as given in (4.9), (4.19) and (6.1), (6.8), respectively. In this example we apply for all the polynomial systems of equations the homotopy continuation approach as implemented in the PHCpack package [25]. We first look for solutions disregarding the inequalities and then select by hand all solutions which fulfill them. Technically, for the real indeterminants we take into account solutions with an imaginary part smaller than 0.001. With respect to our computations we remark that for the most involved cases of systems of equations we encounter about a minute of time consumption on an ordinary PC.

We start with studying stability of the potential; see section 4. To this end we separate the quadratic and the quartic terms of the potential. Inserting the parameters (B.2) into (4.4), (4.5), yields

$$
\begin{aligned}
J_2(\boldsymbol{k}) =& \frac{\mu_0+2\mu_{12}}{\sqrt{6}} + \left(\frac{\mu_{12}-\mu_0}{\sqrt{3}}\right) k_8 + \mu_m k_1,\\
J_4(\boldsymbol{k}) =& \frac{1}{6}(a_1+2a_2+2a_5+a_6) + \frac{1}{3\sqrt{2}}(-2a_1-a_2+2a_5+a_6)k_8 + \frac{a_7}{4}(k_1^2+k_2^2)\\
&+\frac{1}{4}(2a_5-a_6)k_3^2 + \frac{a_3}{4}(k_4^2+k_5^2+k_6^2+k_7^2) + \frac{\mathrm{Re}(a_4)}{2}(k_4k_6-k_5k_7)\\
&+\frac{\mathrm{Im}(a_4)}{2}(k_4k_7+k_5k_6) + \frac{1}{12}(4a_1-4a_2+2a_5+a_6)k_8^2
\end{aligned}
\tag{B.4}
$$

with the parameters given in (B.3). Now we have to find the stationary points of $J_4(\boldsymbol{k})$, that is, we have to solve the systems of equations (4.9) and (4.19), respectively.

In case of (4.9) the invariants are the eight components of the vector $\boldsymbol{k}$ and one Lagrange multiplier u. in case of (4.19) the invariants are the vector components of $\boldsymbol{w}$, $\boldsymbol{w}^\dagger$, (the eigenvector of the matrix $\underline{K}$, see (A.16)), as well as one Lagrange multiplier u. We decompose the vector components of $\boldsymbol{w}$, $\boldsymbol{w}^\dagger$ into real and imaginary parts such that in this form all indeterminants, that is, $\mathrm{Re}(w_1)$, $\mathrm{Im}(w_1)$, $\mathrm{Re}(w_2)$, $\mathrm{Im}(w_2)$, $\mathrm{Re}(w_3)$, $\mathrm{Im}(w_3)$, u, are real. Then we split the system of equations (4.19) into its real and imaginary parts. Note that the last equation $\boldsymbol{w}^\dagger\boldsymbol{w} - 1 = 0$ has only a trivial imaginary part. Eventually, we end up with seven equations in seven indeterminants in this case.

JHEP10(2015)149

With respect to the system (4.9) we detect four solutions $\boldsymbol{k}$ and u fulfilling the inequality $2 - \boldsymbol{k}^2 > 0$. Plugging these solutions into $J_4(\boldsymbol{k})$ in (B.4) we find solely positive values. With respect to the systems (4.19) we find solely solutions which give, with respect to $J_4(\boldsymbol{k})$, also positive values. Stability in general requires that there is no stationary direction $\boldsymbol{k}$ with $J_4(\boldsymbol{k}) < 0$ or $J_4(\boldsymbol{k}) = 0$ but $J_2(\boldsymbol{k}) < 0$. In our example $J_4(\boldsymbol{k})$ is positive for all stationary directions $\boldsymbol{k}$, therefore, the potential is stable in the strong sense for the chosen parameters.

Since the potential with parameters (B.3) is stable we proceed by studying the stationary points; see section 6. To this end we plug the parameters (B.2) into the potential (3.1). This gives

$$\begin{aligned} V =& \frac{\mu_0 + 2\mu_{12}}{\sqrt{6}} K_0 + \left(\frac{\mu_{12} - \mu_0}{\sqrt{3}} \right) K_8 + \mu_m K_1 \\ &+ \frac{1}{6}(a_1 + 2a_2 + 2a_5 + a_6) K_0^2 + \frac{a_7}{4} K_1^2 + \frac{a_7}{4} K_2^2 + \frac{1}{4}(2a_5 - a_6) K_3^2 \\ &+ \frac{a_3}{4}(K_4^2 + K_5^2 + K_6^2 + K_7^2) + \frac{\mathrm{Re}(a_4)}{2}(K_4 K_6 - K_5 K_7) + \frac{\mathrm{Im}(a_4)}{2}(K_4 K_7 + K_5 K_6) \\ &+ \frac{1}{3\sqrt{2}}(-2a_1 - a_2 + 2a_5 + a_6) K_0 K_8 + \frac{1}{12}(4a_1 - 4a_2 + 2a_5 + a_6) K_8^2. \end{aligned} \tag{B.5}$$

Now we have to solve the systems of polynomial equations (6.1) (corresponding to solutions which break electroweak symmetry fully) and (6.8) (corresponding to solutions with break electroweak symmetry partially leaving the electromagnetic U(1) symmetry intact).

For the set of equality equations (6.1) we find one real solution fulfilling the inequalities $2K_0^2 - K_a K_a > 0$ and $K_0 > 0$,

$$K_0 = 24,705.6 \text{ GeV}^2, \; K_1 = 30,258 \text{ GeV}^2, \; K_8 = 17,469.5 \text{ GeV}^2, \; K_{2/3/4/5/6/7} = 0. \tag{B.6}$$

This solution corresponds to a potential value of $V(K) = -1.83109 \cdot 10^9$ GeV^4. Since this solution originates from the set (6.1) it corresponds to a stationary point with fully broken electroweak symmetry.

Eventually, we write the set (6.8) again with explicit real and imaginary parts of the vectors $\boldsymbol{w}$, $\boldsymbol{w}^\dagger$ and subsequently separate real and imaginary parts of the equations. We have here eight equations in eight indeterminants, K_0, $\mathrm{Re}(w_1)$, $\mathrm{Im}(w_1)$, $\mathrm{Re}(w_2)$, $\mathrm{Im}(w_2)$, $\mathrm{Re}(w_3)$, $\mathrm{Im}(w_3)$, u. In terms of bilinears, employing (A.16), we encounter three real solutions fulfilling the inequality $K_0 > 0$:

$$\begin{aligned} &K_0 = 14,823.3 \text{ GeV}^2, && K_8 = -20,963.4 \text{ GeV}^2, && K_{1/\dots/7} = 0, \\ &K_0 = 39,367.3 \text{ GeV}^2, && K_1 = -21,327.1 \text{ GeV}^2, && K_5 = \pm 33,865.6 \text{ GeV}^2, \\ &K_7 = -K_5, && K_8 = -18,734.2 \text{ GeV}^2, && K_{2/3/4/6} = 0. \end{aligned} \tag{B.7}$$

The first solution in (B.7) corresponds to a potential value of $V(K) = -8.24 \cdot 10^8$ GeV^4 and the other two solutions to $V(K) = -1.54 \cdot 10^9$ GeV^4. These solutions correspond to the correct electroweak symmetry breaking.

JHEP10(2015)149

In addition, we always have the trivial solution with vanishing bilinears, corresponding to a vanishing potential. This solution corresponds to an unbroken electroweak symmetry.

The global minimum is given by the stationary point corresponding to the deepest potential value. In this example the deepest stationary point is given by (B.6) and corresponds to a fully broken electroweak symmetry which is physically not acceptable.

Our analysis clearly shows that requiring the potential to have a stationary point giving the desired electroweak symmetry breaking and vacuum expectation value does not guarantee that one has a viable model. In contrast, the study of stability and *all* stationary points reveals where the true global minimum of the potential is. Then one has to check if, at this global minimum, one has the desired partial electroweak symmetry breaking. As we have seen, our methods employing bilinears allow to perform these investigations in an efficient way.

References

[1] M. Maniatis and O. Nachtmann, *Stability and symmetry breaking in the general n-Higgs-doublet model*, *Phys. Rev. D* in press, arXiv:1504.01736 [INSPIRE].

[2] T.D. Lee, *A Theory of Spontaneous T Violation*, *Phys. Rev.* **D 8** (1973) 1226 [INSPIRE].

[3] G.C. Branco, P.M. Ferreira, L. Lavoura, M.N. Rebelo, M. Sher and J.P. Silva, *Theory and phenomenology of two-Higgs-doublet models*, *Phys. Rept.* **516** (2012) 1 [arXiv:1106.0034] [INSPIRE].

[4] F. Nagel, *New aspects of gauge-boson couplings and the Higgs sector*, PhD-thesis, Heidelberg University (2004), http://www.ub.uni-heidelberg.de/archiv/4803.

[5] M. Maniatis, A. von Manteuffel, O. Nachtmann and F. Nagel, *Stability and symmetry breaking in the general two-Higgs-doublet model*, *Eur. Phys. J.* **C 48** (2006) 805 [hep-ph/0605184] [INSPIRE].

[6] C.C. Nishi, *CP violation conditions in N-Higgs-doublet potentials*, *Phys. Rev.* **D 74** (2006) 036003 [*Erratum ibid.* **D 76** (2007) 119901] [hep-ph/0605153] [INSPIRE].

[7] M. Maniatis, A. von Manteuffel and O. Nachtmann, *CP violation in the general two-Higgs-doublet model: A Geometric view*, *Eur. Phys. J.* **C 57** (2008) 719 [arXiv:0707.3344] [INSPIRE].

[8] E. Ma and M. Maniatis, *Symbiotic Symmetries of the Two-Higgs-Doublet Model*, *Phys. Lett.* **B 683** (2010) 33 [arXiv:0909.2855] [INSPIRE].

[9] B. Grzadkowski, M. Maniatis and J. Wudka, *The bilinear formalism and the custodial symmetry in the two-Higgs-doublet model*, *JHEP* **11** (2011) 030 [arXiv:1011.5228] [INSPIRE].

[10] P.M. Ferreira, H.E. Haber, M. Maniatis, O. Nachtmann and J.P. Silva, *Geometric picture of generalized-CP and Higgs-family transformations in the two-Higgs-doublet model*, *Int. J. Mod. Phys.* **A 26** (2011) 769 [arXiv:1010.0935] [INSPIRE].

JHEP10(2015)149

[11] C.C. Nishi, *The Structure of potentials with N Higgs doublets*, Phys. Rev. **D 76** (2007) 055013 [arXiv:0706.2685] [INSPIRE].

[12] I.P. Ivanov, *Properties of the general NHDM. II. Higgs potential and its symmetries*, JHEP **07** (2010) 020 [arXiv:1004.1802] [INSPIRE].

[13] V. Keus, S.F. King and S. Moretti, *Three-Higgs-doublet models: symmetries, potentials and Higgs boson masses*, JHEP **01** (2014) 052 [arXiv:1310.8253] [INSPIRE].

[14] I.P. Ivanov and C.C. Nishi, *Properties of the general NHDM. I. The Orbit space*, Phys. Rev. **D 82** (2010) 015014 [arXiv:1004.1799] [INSPIRE].

[15] A. Barroso, P.M. Ferreira, R. Santos and J.P. Silva, *Stability of the normal vacuum in multi-Higgs-doublet models*, Phys. Rev. **D 74** (2006) 085016 [hep-ph/0608282] [INSPIRE].

[16] I.P. Ivanov and E. Vdovin, *Classification of finite reparametrization symmetry groups in the three-Higgs-doublet model*, Eur. Phys. J. **C 73** (2013) 2309 [arXiv:1210.6553] [INSPIRE].

[17] S.-L. Chen, M. Frigerio and E. Ma, *Large neutrino mixing and normal mass hierarchy: A Discrete understanding*, Phys. Rev. **D 70** (2004) 073008 [*Erratum ibid.* **D 70** (2004) 079905] [hep-ph/0404084] [INSPIRE].

[18] L. Lavoura and H. Kühböck, *A_4 model for the quark mass matrices*, Eur. Phys. J. **C 55** (2008) 303 [arXiv:0711.0670] [INSPIRE].

[19] A. Aranda, C. Bonilla, F. de Anda, A. Delgado and J. Hernandez-Sanchez, *Higgs decay into two photons from a 3HDM with flavor symmetry*, Phys. Lett. **B 725** (2013) 97 [arXiv:1302.1060] [INSPIRE].

[20] A. Aranda, C. Bonilla and J.L. Diaz-Cruz, *Three generations of Higgses and the cyclic groups*, Phys. Lett. **B 717** (2012) 248 [arXiv:1204.5558] [INSPIRE].

[21] W. Grimus, L. Lavoura and D. Neubauer, *A Light pseudoscalar in a model with lepton family symmetry $O(2)$*, JHEP **07** (2008) 051 [arXiv:0805.1175] [INSPIRE].

[22] M. Maniatis, A. von Manteuffel and O. Nachtmann, *Determining the global minimum of Higgs potentials via Groebner bases: Applied to the NMSSM*, Eur. Phys. J. **C 49** (2007) 1067 [hep-ph/0608314] [INSPIRE].

[23] M. Maniatis and D. Mehta, *Minimizing Higgs Potentials via Numerical Polynomial Homotopy Continuation*, Eur. Phys. J. Plus **127** (2012) 91 [arXiv:1203.0409] [INSPIRE].

[24] O. Nachtmann, *Elementary Particle Physics, Concepts and Phenomena*, Springer, Berlin, Heidelberg (1990).

[25] J. Verschelde, *Algorithm 795: PHCpack: A General Purpose Solver for Polynomial systems by Homotopy Continuation*, *ACM Trans. Math. Software* **25** (1999) 2.

JHEP10(2015)149