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Quivers for 3-manifolds: the correspondence, BPS states, and 3d $\mathcal{N}=2$ theories

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Abstract: We introduce and explore the relation between quivers and 3-manifolds with the topology of the knot complement. This idea can be viewed as an adaptation of the knots-quivers correspondence to Gukov-Manolescu invariants of knot complements (also known as F_K or $\hat{Z}$). Apart from assigning quivers to complements of $T^{(2,2p+1)}$ torus knots, we study the physical interpretation in terms of the BPS spectrum and general structure of 3d $\mathcal{N}=2$ theories associated to both sides of the correspondence. We also make a step towards categorification by proposing a t-deformation of all objects mentioned above.

Keywords: Chern-Simons Theories, M-Theory, Quantum Groups, Topological Strings

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1 Introduction

In recent years many interesting relations between quantum field theory, string theory, low-dimensional topology, and representation theory have been studied. In this paper we focus on the knots-quivers correspondence [1, 2] and the Gukov-Manolescu invariants of knot complements [3]. (We will abbreviate them to GM invariants and denote F_K — often this symbol is also used as their name.)

The core of the knots-quivers correspondence is an equality (after appropriate change of variables) between the generating series of symmetrically coloured HOMFLY-PT polynomials of a knot and a certain motivic generating series associated to the quiver. This implies that Labastida-Mariño-Ooguri-Vafa (LMOV) invariants of knots [4–6] can be expressed in terms of motivic Donaldson-Thomas (DT) invariants of associated quivers [7, 8], which are known to be integer [9]. In consequence, by assigning a quiver to the knot, we automatically prove the integrality of LMOV invariants, which is the statement of the LMOV conjecture [4–6]. This was done for all knots up to 6 crossings and infinite families of $T^{(2,2p+1)}$

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torus knots and twist knots in [2]. More systematic approach in terms of tangles provided a proof of the knots-quivers correspondence for all two-bridge knots [10] and recently for all arborescent knots [11], however finding a general proof remains to be an open problem.

Geometric and physical interpretation of the knots-quivers correspondence given in [12, 13] connects it with Gromov-Witten invariants and counts of holomorphic curves, showing that the quiver encodes the construction of BPS spectrum from the basic states. This spectrum is shared by dual 3d $\mathcal{N}=2$ theories associated to the knot complement and the quiver, and respecitve counts of BPS states are given by LMOV and DT invariants. The moduli spaces of vacua of these theories are encoded in the graphs of A-polynomials [14–18] and their quiver versions [12, 13, 19, 20]. Another recent results on the knots-quivers correspondence include a relation to combinatorics of counting paths [19], a consideration of more general case of topological strings on various Calabi-Yau manifolds [20], and a connection with the topological recursion [21].

The origins of GM invariants lie in attempts of categorification of the Witten-Reshetikhin-Turaev (WRT) invariants of 3-manifolds. In order to solve the problem of non-integrality of WRT invariants Gukov, Pei, Putrov, and Vafa introduced new invariants of 3-manifolds [22, 23] (we will call them GPPV invariants, they are denoted by $\hat{Z}$ and often this symbol is also used as their name). The GPPV invariant is a series in q with integer coefficients and the WRT invariant can be recovered from the $q \to e^{\frac{2\pi i}{k}}$ limit (see [22–27] for details and generalisations). Physical origins of GPPV invariants lie in the 3d-3d correspondence [28–30]: $\hat{Z}$ is a supersymmetrix index of 3d $\mathcal{N}=2$ theory with 2d $\mathcal{N}=(0,2)$ boundary condition studied first in [31]. Detailed analysis of this interpretation, as well as the application of resurgence, can be found in [22–24], whereas [32, 33] contain some recent results on GPPV invariants coming from the study of 3d-3d corresponcendce. Another interpretation of GPPV invariants as characters of 2d logarithmic vertex operator algebras was proposed in [25]. On the other hand, this work — together with [34–36] — initiated an exploration of the intruguing modular properties of $\hat{Z}$.

GM invariants can be treated as knot complement versions of GPPV invariants, $F_K = \hat{Z}\left(S^3 \backslash K\right)$ [3] (because of that, sometimes GM invariants are denoted and called $\hat{Z}$, descending it from GPPV invariants). From the physical point of view, the GM invariant arises from the reduction of 6d $\mathcal{N}=(0,2)$ theory describing M5-branes on the 3-manifold with the topology of the knot complement. Important properties analysed in [3] include the behaviour under surgeries, the agreement of the asymptotic expansion of F_K with the Melvin-Morton-Rozansky expansion of the coloured Jones polynomials [37–40], relations to the Alexander polynomials, and the annihilation by the quantum A-polynomials introduced in [41, 42]. These relations suggest an interesting geometric interpretation of GM invariants in terms of the annuli counting presented in [43] and based on [44]. Another recent developments include a generalisation to arbitrary gauge group [45], an adaptation of the large colour R-matrix approach to GM invariants [46], and a relation to Akutsu-Deguchi-Ohtsuki invariant for $q = e^{\frac{2\pi i}{k}}$ [47].

From our perspective, the most important new results are closed form expressions for a-deformed GM invariants provided in [43]. They reduce to the initial F_K for $a = q^2$ and turn out to be closely related to HOMFLY-PT polynomials. This allows us to propose

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a correspondence between knot complements and quivers via GM invariants and quiver motivic generating series. Moreover, since HOMFLY-PT polynomials admit t-deformation in terms of superpolynomials [48], we can follow [43] and conjecture results for a, t-deformed GM invariants. We also study the physical intepretation of the new correspondence (for clarity we will refer to the knots-quivers correspondence of [1, 2] as the standard one), generalising the results of [12, 13]. It includes an exploration of the duality between 3d $\mathcal{N} = 2$ theories associated to the knot complement and the quiver, as well as a study of their BPS states, which allows us to introduce knot complement analogs of LMOV invariants via DT invariants.

The rest of the paper is organised as follows. Section 2 provides a short introduction to the knots-quivers correspondence and GM invariants of knot complements. In section 3 we state our main conjecture, assigning quivers to knot complements, and study consequences for 3d $\mathcal{N} = 2$ theories and their BPS spectra. In section 4 we show how these ideas can be applied and checked on concrete examples of unknot, trefoil, cinquefoil, and general $T^{(2,2p+1)}$ torus knots. Section 5 contains a t-deformation of previous results inspired by the categorification of HOMFLY-PT polynomials. In section 6 we conclude with a discussion of interesting open problems.

2 Prerequisites

In this section we recall relevant aspects of the knots-quivers correspondence, its physical interpretation, and GM invariants of knot complements.

2.1 Knot invariants

If $K \subset S^3$ is a knot, then its HOMFLY-PT polynomial $P_K(a, q)$ [49, 50] is a topological invariant which can be calculated via the skein relation. More generally, coloured HOMFLY-PT polynomials $P_{K,R}(a, q)$ are similar polynomial knot invariants depending also on a representation R of the Lie algebra $\mathfrak{u}(N)$. In this setting, the original HOMFLY-PT corresponds to the fundamental representation. From the physical point of view, $P_{K,R}(a, q)$ is the expectation value of the knot viewed as a Wilson line in $\mathrm{U}(N)$ Chern-Simons gauge theory [51].

In the context of the knots-quivers correspondence, we are interested in the HOMFLY-PT generating series:

$$P_K(\lambda, a, q) = \sum_{r=0}^{\infty} P_{K,r}(a, q)\lambda^{-r}, \tag{2.1}$$

where $P_{K,r}(a, q)$ are HOMFLY-PT polynomials coloured by the totally symmetric representations S^r (with r boxes in one row of the Young diagram), which for brevity we will call simply HOMFLY-PT polynomials. The unusual expansion variable with the negative power comes from the necessity of resolving the clash of four different conventions present in the literature (and avoiding the confusion with the quiver variables):

- KRSS convention from [1, 2, 52, 53],
- FGS convention from [3, 14, 16, 18, 45],
- EKL convention from [12, 13],
- EGGKPS convention from [43].

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The dictionary is given by

$$\begin{aligned}\lambda &= x_{\mathrm{KRSS}}^{-1} = x_{\mathrm{EKL}}^{-1} = y_{\mathrm{FGS}} = y_{\mathrm{EGGKPS}}, \quad \mu = y_{\mathrm{KRSS}} = y_{\mathrm{EKL}}^{1/2} = x_{\mathrm{FGS}} q_{\mathrm{FGS}}^{-1} = x_{\mathrm{EGGKPS}}, \\ a &= a_{\mathrm{KRSS}}^{2} = a_{\mathrm{EKL}}^{2} = a_{\mathrm{FGS}} = a_{\mathrm{EGGKPS}}, \quad q = q_{\mathrm{KRSS}}^{2} = q_{\mathrm{EKL}}^{2} = q_{\mathrm{FGS}} = q_{\mathrm{EGGKPS}}. \end{aligned} \tag{2.2}$$

HOMFLY-PT polynomials satisfy recurrence relations encoded in the quantum a-deformed A-polynomials [14–18]:

$$\hat{A}(\hat{\mu}, \hat{\lambda}, a, q) P_{K,r}(a, q) = 0, \tag{2.3}$$

where

$$\hat{\mu} P_{K,r}(a, q) = q^r P_{K,r}(a, q), \qquad \hat{\lambda} P_{K,r}(a, q) = P_{K,r+1}(a, q). \tag{2.4}$$

For simplicity, in the remaining part of the paper we will drop "a-deformed" and call $\hat{A}(\hat{\mu}, \hat{\lambda}, a, q)$ the quantum A-polynomials.

The LMOV invariants [4–6] are numbers assembled into the LMOV generating function $N(\lambda, a, q) = \sum_{r,i,j} N_{r,i,j} \lambda^{-r} a^i q^j$ that gives the following expression for the HOMFLY-PT generating series:

$$P_K(\lambda, a, q) = \mathrm{Exp}\left[\frac{N(\lambda, a, q)}{1-q}\right]. \tag{2.5}$$

Exp is the plethystic exponential: if $f(t) = \sum_n a_n t^n$ and $a_0 = 0$, then

$$\mathrm{Exp}\,[f(t)] = \exp\left[\sum_k \frac{1}{k} f(t^k)\right] = \prod_n (1-t^n)^{a_n}. \tag{2.6}$$

LMOV invariants can be extracted also from the A-polynomials, see [52, 53].

2.2 Quivers and their representations

A quiver Q is an oriented graph, i.e. a pair (Q_0, Q_1) where Q_0 is a finite set of vertices and Q_1 is a finite set of arrows between them. We number the vertices by $1, 2, \ldots, m = |Q_0|$. An adjacency matrix of Q is the $m \times m$ integer matrix with entries C_{ij} equal to the number of arrows from i to j. If $C_{ij} = C_{ji}$, we call the quiver symmetric.

A quiver representation with dimension vector $\boldsymbol{d} = (d_1, \ldots, d_m)$ is the assignment of a vector space of dimension d_i to the node $i \in Q_0$ and of a linear map $\gamma_{ij} : \mathbb{C}^{d_i} \to \mathbb{C}^{d_j}$ to each arrow from vertex i to vertex j. Quiver representation theory studies moduli spaces of stable quiver representations. While explicit expressions for invariants describing those spaces are hard to find in general, they are quite well understood in the case of symmetric quivers [7–9, 54, 55]. Important information about the moduli space of representations of a symmetric quiver is encoded in the motivic generating series defined as

$$P_Q(\boldsymbol{x}, q) = \sum_{d_1,\ldots,d_m \geq 0} (-q^{1/2})^{\sum_{i,j} C_{ij} d_i d_j} \prod_{i=1}^{m} \frac{x_i^{d_i}}{(q;q)_{d_i}}, \tag{2.7}$$

where the denominator is the q-Pochhammer symbol:

$$(z;q)_n = \prod_{k=0}^{n-1} (1 - z q^k). \tag{2.8}$$

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If we write

$$P_Q(\boldsymbol{x},q) = \mathrm{Exp}\left[\frac{\Omega(\boldsymbol{x},q)}{1-q}\right], \tag{2.9}$$

we obtain the generating series of motivic Donaldson-Thomas (DT) invariants $\Omega_{\boldsymbol{d},s}$ [7, 8]:

$$\Omega(\boldsymbol{x},q) = \sum_{\boldsymbol{d},s} \Omega_{\boldsymbol{d},s}\boldsymbol{x}^{\boldsymbol{d}}q^s = \sum_{\boldsymbol{d},s} \Omega_{(d_1,\ldots,d_m),s}\left(\prod_i x_i^{d_i}\right)q^s. \tag{2.10}$$

The DT invariants have two geometric interpretations, either as the intersection homology Betti numbers of the moduli space of all semi-simple representations of Q of dimension vector $\boldsymbol{d}$, or as the Chow-Betti numbers of the moduli space of all simple representations of Q of dimension vector $\boldsymbol{d}$, see [54, 55]. [9] provides a proof of integrality of DT invariants for the symmetric quivers.

2.3 Knots-quivers correspondence

The knots-quivers correspondence [1, 2] is a conjecture that for each knot K there exist a quiver Q and integers n_i, a_i, l_i, $i \in Q_0$, such that

$$P_K(\lambda,a,q) = P_Q(\boldsymbol{x},q)\big|_{x_i=\lambda^{n_i}a^{a_i}q^{l_i}}. \tag{2.11}$$

If we substitute (2.5) and (2.9), we obtain the knots-quivers correspondence at the level of LMOV and DT invariants:

$$N(\lambda,a,q) = \Omega(\boldsymbol{x},q)\big|_{x_i=\lambda^{n_i}a^{a_i}q^{l_i}}. \tag{2.12}$$

Since DT invariants are integer, this equation implies integrality of $N_{r,i,j}$, which is known as the LMOV conjecture.

From the physical point of view, DT and LMOV invariants count BPS states in 3d $\mathcal{N}=2$ theories denoted by $T[L_K]$ and $T[Q_{L_K}]$ [12, 13]. $T[L_K]$ is the effective 3d $\mathcal{N}=2$ theory on the world-volume of M5-brane wrapped on the knot conormal inside the resolved conifold:

$$\begin{array}{rl} \text{space-time}: & \mathbb{R}^4 \times S^1 \times X \\ & \quad\cup \qquad\quad \cup \\ \text{M5-brane}: & \mathbb{R}^2 \times S^1 \times L_K. \end{array}$$

The structure of $T[L_K]$ can be read from the semiclassical limit ($q = e^{\hbar} \to 1$) of the HOMFLY-PT generating series:

$$\sum_{r=0}^{\infty} P_{K,r}(a,q)\lambda^{-r} \underset{\hbar\to 0}{\overset{q^r=\mu}{\longrightarrow}} \int \frac{d\mu}{\mu}\prod_i \frac{dz_i}{z_i}\exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[L_K]}(\mu,\lambda,a,z_i) + \mathcal{O}(\hbar^0)\right], \tag{2.13}$$

where the integral $\int \frac{d\mu}{\mu}\prod_i \frac{dz_i}{z_i}$ corresponds to the gauge group $\mathrm{U}(1)_M \times \mathrm{U}(1)_{Z_1} \times \ldots \times \mathrm{U}(1)_{Z_k}$ (we single out $\mathrm{U}(1)_M$ and its fugacity μ because for the knot complement theory it becomes

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a global symmetry). $\widetilde{\mathcal{W}}_{T[L_K]}(\mu,\lambda,a,z_i)$ is the twisted superpotential with two typical kinds of contributions:

$$\begin{aligned} \mathrm{Li}_2\left(a^{n_Q}\mu^{n_M}z_i^{n_{Z_i}}\right) &\quad\longleftrightarrow\quad (\text{chiral field})\,, \\ \frac{\kappa_{ij}}{2}\log\zeta_i\cdot\log\zeta_j &\quad\longleftrightarrow\quad (\text{Chern-Simons coupling})\,. \end{aligned} \tag{2.14}$$

Each dilogarithm is interpreted as the one-loop contribution of a chiral field with charges (n_Q, n_M, n_{Z_i}) under the global symmetry $\mathrm{U}(1)_Q$ (arising from the internal 2-cycle in the resolved conifold geometry) and the gauge group $\mathrm{U}(1)_M\times\mathrm{U}(1)_{Z_1}\times\ldots\times\mathrm{U}(1)_{Z_k}$. Quadratic-logarithmic terms are identified with Chern-Simons couplings among the various U(1) gauge and global symmetries, with ζ_i denoting the respective fugacities. For more details see [12, 16, 18, 28, 29, 56, 57].

Integrating over the gauge fugacities z_i using a saddle-point approximation gives the effective twisted superpotential of the theory:

$$\widetilde{\mathcal{W}}^{\text{eff}}_{T[L_K]}(\mu,\lambda,a)=\widetilde{\mathcal{W}}_{T[L_K]}(\mu,\lambda,a,z_i^*),\quad \text{where}\quad \left.\frac{\partial\widetilde{\mathcal{W}}_{T[L_K]}(\mu,\lambda,a,z_i)}{\partial z_i}\right|_{z_i=z_i^*}=0. \tag{2.15}$$

The moduli space of vacua of $T[L_K]$, given by the extremal points of the effective twisted superpotential, coincides with the graph of the classical A-polynomial:

$$\begin{aligned} \frac{\partial\widetilde{\mathcal{W}}^{\text{eff}}_{T[L_K]}(\mu,\lambda,a)}{\partial\log\lambda}=0 \qquad\Leftrightarrow\qquad & A(\mu,\lambda,a)=0, \\ & A(\mu,\lambda,a)=\lim_{q\to1}\hat{A}(\hat{\mu},\hat{\lambda},a,q). \end{aligned} \tag{2.16}$$

In analogy to $T[L_K]$, the structure of $T[Q_{L_K}]$ is encoded in the semiclassical limit of the motivic generating series [12]:

$$\begin{aligned} P_Q(\boldsymbol{x},q) &\xrightarrow[\hbar\to0]{q^{d_i}=y_i}\int\prod_i\frac{dy_i}{y_i}\exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[Q_{L_K}]}(\boldsymbol{x},\boldsymbol{y})+\mathcal{O}(\hbar^0)\right], \\ \widetilde{\mathcal{W}}_{T[Q_{L_K}]}(\boldsymbol{x},\boldsymbol{y}) &=\sum_i\mathrm{Li}_2(y_i)+\log\left((-1)^{C_{ii}}x_i\right)\,\log y_i+\sum_{i,j}\frac{C_{ij}}{2}\log y_i\,\log y_j. \end{aligned} \tag{2.17}$$

Using the dictionary (2.13)–(2.14), we can interpret the elements of (2.17) in the following way:

- The integral $\int\prod_i\frac{dy_i}{y_i}$ corresponds to having the gauge group $\mathrm{U}(1)^{(1)}\times\cdots\times\mathrm{U}(1)^{(m)}$,
- $\mathrm{Li}_2(y_i)$ represents the chiral field with charge 1 under $\mathrm{U}(1)^{(i)}$,
- $\frac{C_{ij}}{2}\log y_i\,\log y_j$ corresponds to the gauge Chern-Simons couplings, $\kappa^{\text{eff}}_{ij}=C_{ij}$,
- $\log\left((-1)^{C_{ii}}x_i\right)\,\log y_i$ represents the Chern-Simons coupling between a gauge symmetry and its dual topological symmetry (the Fayet-Iliopoulos coupling).

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The saddle point of the twisted superpotential encodes the moduli space of vacua of $T[Q_{L_K}]$ and defines the quiver A-polynomials [12, 13, 19, 20]:

$$\frac{\partial \widetilde{\mathcal{W}}_{T[Q_{L_K}]}(\boldsymbol{x},\boldsymbol{y})}{\partial \log y_i} = 0 \qquad \Leftrightarrow \qquad A_i(\boldsymbol{x},\boldsymbol{y}) = 1 - y_i - x_i(-y_i)^{C_{ii}} \prod_{j\neq i} y_j^{C_{ij}} = 0. \tag{2.18}$$

$A_i(\boldsymbol{x},\boldsymbol{y})$ is a classical limit of the quantum quiver A-polynomial, which annihilates the motivic generating series:

$$\begin{aligned} \hat{A}_i(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q)P_Q(\boldsymbol{x},q) &= 0, \\ \hat{x}_i P_Q(x_1,\dots,x_i,\dots,x_m,q) &= x_i P_Q(x_1,\dots,x_i,\dots,x_m,q), \\ \hat{y}_i P_Q(x_1,\dots,x_i,\dots,x_m,q) &= P_Q(x_1,\dots,qx_i,\dots,x_m,q). \end{aligned} \tag{2.19}$$

The general formula for the quantum quiver A-polynomial corresponding to the quiver with adjacency matrix C is given by [13]

$$\hat{A}_i(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q) = 1 - \hat{y}_i - \hat{x}_i(-q^{1/2}\hat{y}_i)^{C_{ii}} \prod_{j\neq i} \hat{y}_j^{C_{ij}} \tag{2.20}$$

and we can see that

$$A_i(\boldsymbol{x},\boldsymbol{y}) = \lim_{q\to 1} \hat{A}_i(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q). \tag{2.21}$$

2.4 GM invariants

GM invariants $F_K = \hat{Z}\left(S^3\backslash K\right)$ implicitly depend on the gauge group and results of [3] correspond to the simplest nontrivial case of SU(2). General case is analysed in [45], but it is often very involved from computational point of view. One of the goals of [43] was overcoming these difficulties and studying the large-N behaviour of $F_K^{\mathrm{SU}(N)}(\mu,q)$ — the GM invariant corresponding to the symmetric representations of SU(N). It turns out that we can introduce a-deformed GM invariants $F_K(\mu,a,q)$ which capture all N by the following relation:

$$F_K(\mu,a=q^N,q) = F_K^{\mathrm{SU}(N)}(\mu,q). \tag{2.22}$$

$F_K(\mu,a,q)$ is annihilated by the quantum a-deformed A-polynomials:

$$\hat{A}(\hat{\mu},\hat{\lambda},a,q)F_K(\mu,a,q) = 0, \tag{2.23}$$

where

$$\hat{\mu}F_K(\mu,a,q) = \mu F_K(\mu,a,q), \qquad \hat{\lambda}F_K(\mu,a,q) = F_K(q\mu,a,q). \tag{2.24}$$

Since we adapt the convention that the series expansion of F_K starts from 1, as in [43], sometimes we have to rescale $\hat{\lambda}$ in order to compare to the literature.

From our point of view, the crucial result of [43] is the connection between a-deformed GM invariants and HOMFLY-PT polynomials. In general, we expect it to be some version of the Fourier transform, but it is difficult to define it properly. However, in some cases it reduces to a simple substitution:

$$F_K(\mu,a,q) = \left. P_{K,r}(a,q)\right|_{q^r=\mu}. \tag{2.25}$$

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It is not yet known what conditions are sufficient and which are necessary for this equation to hold. [43] contains an explicit check for the trefoil by comparing with $F_{3_1}^{\mathrm{SU}(N)}(\mu, q)$ from [3, 45]. Unfortunately, already for the figure-eight knot the substitution $q^r = \mu$ leads to ill-defined series in both μ and μ^{-1} and this situtation is ubiquitous. Nevertheless, solving equation (2.23) order by order in μ works well for 4_1, which suggests that there exists a well-defined $F_{4_1}(\mu, a, q)$, however not equal to $P_{4_1,r}(a, q)|_{q^r=\mu}$. On the other hand, there is a class of knots for which we know general expressions for $P_{K,r}(a, q)$ and the substitution $q^r = \mu$ leads to well-defined series. They are $T^{(2,2p+1)}$ torus knots and for them (2.25) is conjectured to hold [43]. Closed form expressions for F_K are essential for finding quivers, so $T^{(2,2p+1)}$ torus knots will be main focus of our interest.

Since there exists a t-deformation of HOMFLY-PT polynomials provided by the super-polynomials [48], it is natural to consider the following t-deformation of GM invariants [43]:

$$F_K(\mu, a, q, t) = \mathcal{P}_{K,r}(a, q, t)|_{q^r=\mu}\,. \tag{2.26}$$

We will study this deformation for $T^{(2,2p+1)}$ torus knots in section 5.

For simplicity, in the remaining part of the paper we will refer to $F_K^{\mathrm{SU}(2)}(\mu, q)$, $F_K^{\mathrm{SU}(N)}(\mu, q)$, $F_K(\mu, a, q)$, and $F_K(\mu, a, q, t)$ broadly as GM invariants, specyfying the case explicitly only when it is not obvious from the context.

3 Quivers for GM invariants

In this section we introduce quivers for GM invariants and study their physical interpretation in terms of BPS state counts and 3d $\mathcal{N} = 2$ effective theories.

3.1 Main conjecture

The connection between GM invariants and HOMFLY-PT polynomials, together with closed form expressions for $F_{T^{(2,2p+1)}}(\mu, a, q)$ found in [43], suggest that the idea of knots-quivers correspondence can be reformulated for GM invariants. In other words, we expect the following:

Conjecture 1 *For a given knot complement $M_K = S^3\backslash K$, the GM invariant $F_K(\mu, a, q)$ can be written in the form*

$$F_K(\mu, a, q) = \sum_{d_1,\ldots,d_m\geq 0} (-q^{1/2})^{\sum_{i,j=1}^m C_{ij}d_id_j} \prod_{i=1}^m \frac{\mu^{n_id_i}a^{a_id_i}q^{l_id_i}}{(q;q)_{d_i}}, \tag{3.1}$$

where C is a symmetric $m \times m$ matrix and n_i, a_i, l_i are fixed integers. In consequence, there exist a quiver Q which adjacency matrix is equal to C and the motivic generating series

$$F_Q(\boldsymbol{x}, q) = \sum_{d_1,\ldots,d_m\geq 0} (-q^{1/2})^{\sum_{i,j=1}^m C_{ij}d_id_j} \prod_{i=1}^m \frac{x_i^{d_i}}{(q;q)_{d_i}} \tag{3.2}$$

reduces to the GM invariant after the change of variables $x_i = \mu^{n_i}a^{a_i}q^{l_i}$:

$$F_K(\mu, a, q) = F_Q(\boldsymbol{x}, q)|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}}\,. \tag{3.3}$$

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Let us stress that we may encounter negative entries of the quiver adjacency matrix: $C_{ij} < 0$, so we can either accept having two types of arrows (ordinary arrows and "antiarrows" which annihilate each other) or use the change of framing to shift all entries (for details see [2]). Moreover, there is a possiblity of the sign difference in the knot complement and quiver variables which would lead to more complicated change of variables $x_i = (-1)^{j_i}\mu^{n_i}a^{a_i}q^{l_i}$, but for all analysed examples it was not the case.

We can also obtain quivers for SU(N) (and more specifically initial SU(2)) GM invariants by substituting $a = q^N$ in (3.3), which leads to

$$F_K^{\mathrm{SU}(N)}(\mu, q) = F_Q(\boldsymbol{x}, q)\big|_{x_i=\mu^{n_i}q^{l_i+Na_i}} . \tag{3.4}$$

3.2 Corollary — BPS states and 3d $\mathcal{N} = 2$ effective theories

If there exist a quiver Q corresponding to the GM invariant of M_K, we can compute the DT invariants using

$$F_Q(\boldsymbol{x}, q) = \mathrm{Exp}\left[\frac{\Omega(\boldsymbol{x}, q)}{1-q}\right] . \tag{3.5}$$

Since Q is symmetric, we immediately know that DT invariants (which are coefficients of $\Omega(\boldsymbol{x}, q)$) are integer numbers [9].

We can also make a step back and apply the change of variables $x_i = \mu^{n_i}a^{a_i}q^{l_i}$ to (3.5). This leads us to the definition of the knot complement analogs of LMOV invariants:

$$N(\mu, a, q) = \sum_{r,i,j} N_{r,i,j}\mu^r a^i q^j = \Omega(\boldsymbol{x}, q)\big|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}} , \tag{3.6}$$

which implies

$$\begin{aligned} F_K(\mu, a, q) &= \mathrm{Exp}\left[\frac{N(\mu, a, q)}{1-q}\right] = \exp\left[\sum_{n,r,i,j}\frac{N_{r,i,j}\mu^{rn}a^{in}q^{jn}}{1-q^n}\right] \\ &= \prod_{r,i,j,l}\left(1-\mu^r a^i q^{j+l}\right)^{-N_{r,i,j}} = \prod_{r,i,j}(\mu^r a^i q^j; q)_\infty^{-N_{r,i,j}} \end{aligned} \tag{3.7}$$

Since DT invariants are integer numbers, so are the knot complement analogs of LMOV invariants. This is consistent with the physical interpretation given in [12, 13]: we expect that $\Omega_{\boldsymbol{d},s}$ and $N_{r,i,j}$ invariants count the number of BPS states in a dual 3d $\mathcal{N} = 2$ theories $T[Q_{M_K}]$ and $T[M_K]$.

We identify $T[M_K]$ with the effective 3d $\mathcal{N} = 2$ theory on $\mathbb{R}^2 \times S^1$ which can be engineered in two equivalent ways. One is the compactification of N M5-branes on the knot complement:

$$\begin{array}{rccccc} \text{space-time}: & \mathbb{R}^4 & \times & S^1 & \times & T^*M_K \\ & \cup & & & & \cup \\ N \text{ M5-branes}: & \mathbb{R}^2 & \times & S^1 & \times & M_K. \end{array}$$

When $N \to \infty$, many protected quantities — such as the twisted superpotential — depend on the number of M5-branes only via the combination q^N which can be treated as a separate

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variable a. The second way comes from the large-N transition from the deformed to the resolved conifold [4, 58]:

$$\begin{array}{rccc} \text{space-time}: & \mathbb{R}^4 \times S^1 \times T^*S^3 & & \text{space-time}: \ \mathbb{R}^4 \times S^1 \times X \\ & \cup \qquad\quad \cup & \longleftrightarrow & \cup \qquad\quad \cup \\ N \text{ M5-branes}: & \mathbb{R}^2 \times S^1 \times S^3 & & \text{M5-brane}: \ \mathbb{R}^2 \times S^1 \times L_K. \\ \text{M5-brane}: & \mathbb{R}^2 \times S^1 \times L_K & & \end{array}$$

Then $\log a$ is a complexified Kähler parameter of X. More details, together with a description of a third way of engineering $T[M_K]$, are available in [43].

The structure of $T[M_K]$ can be read from the semiclassical limit of the GM invariant:

$$F_K(\mu, a, q) \underset{\hbar\to 0}{\to} \int \prod_i \frac{dz_i}{z_i} \exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[M_K]}(\mu, a, z_i) + \mathcal{O}(\hbar^0)\right]. \tag{3.8}$$

Recalling the relation (2.25), we can see that $\widetilde{\mathcal{W}}_{T[M_K]}$ is the same as the effective twisted superpotential of the 3d $\mathcal{N}=2$ theory analysed in [16, 18]. On the other hand, the perspective of the large-N transition explains why $\widetilde{\mathcal{W}}_{T[M_K]}$ is a Legendre transform of the superpotential of the theory $T[L_K]$ discussed in section 2.3:

$$\widetilde{\mathcal{W}}_{T[L_K]}(\mu, \lambda, a, z_i) = \widetilde{\mathcal{W}}_{T[M_K]}(\mu, a, z_i) - \log\mu\log\lambda. \tag{3.9}$$

We can also consider the effective twisted superpotential

$$\widetilde{\mathcal{W}}^{\text{eff}}_{T[M_K]}(\mu, a) = \widetilde{\mathcal{W}}_{T[M_K]}(\mu, a, z_i^*), \qquad \text{where} \qquad \left.\frac{\partial\widetilde{\mathcal{W}}_{T[M_K]}(\mu, a, z_i)}{\partial z_i}\right|_{z_i = z_i^*} = 0,$$

and introduce λ back as the variable dual to μ, which is equivalent to the saddle point equation for $\widetilde{\mathcal{W}}^{\text{eff}}_{T[L_K]}$ and the vanishing of the A-polynomial:

$$\frac{\partial\widetilde{\mathcal{W}}^{\text{eff}}_{T[M_K]}(\mu, a)}{\partial\log\mu} = \log\lambda \quad\Leftrightarrow\quad \frac{\partial\widetilde{\mathcal{W}}^{\text{eff}}_{T[L_K]}(\mu, \lambda, a)}{\partial\log\lambda} = 0 \quad\Leftrightarrow\quad A(\mu, \lambda, a) = 0. \tag{3.10}$$

In consenquence $T[M_K]$ have the same moduli space of vacua as $T[L_K]$ and both are described by the A-polynomial of K.

In analogy to (2.17), the structure of $T[Q_{M_K}]$ is encoded in the semiclassical limit of $F_Q(\boldsymbol{x}, q)$:

$$\begin{aligned} F_Q(\boldsymbol{x}, q) &\xrightarrow[\hbar\to 0]{q^{d_i} = y_i} \int \prod_i \frac{dy_i}{y_i} \exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[Q_{M_K}]}(\boldsymbol{x}, \boldsymbol{y}) + \mathcal{O}(\hbar^0)\right], \\ \widetilde{\mathcal{W}}_{T[Q_{M_K}]}(\boldsymbol{x}, \boldsymbol{y}) &= \sum_i \mathrm{Li}_2(y_i) + \log\left((-1)^{C_{ii}} x_i\right)\log y_i + \sum_{i,j} \frac{C_{ij}}{2}\log y_i \log y_j. \end{aligned} \tag{3.11}$$

$\widetilde{\mathcal{W}}_{T[Q_{M_K}]}(\boldsymbol{x}, \boldsymbol{y})$ is a twisted superpotential of the theory $T[Q_{M_K}]$ and each of its terms can be interpreted according to the dictionary described in section 2.3. The saddle point of

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the twisted superpotential encodes the moduli space of vacua of $T[Q_{M_K}]$ and defines the quiver A-polynomials:

$$\frac{\partial \widetilde{\mathcal{W}}_{T[Q_{M_K}]}(\boldsymbol{x},\boldsymbol{y})}{\partial \log y_i} = 0 \quad \Leftrightarrow \quad A_i(\boldsymbol{x},\boldsymbol{y}) = 0. \tag{3.12}$$

$A_i(\boldsymbol{x},\boldsymbol{y})$ is a classical limit of the quantum quiver A-polynomial, which annihilates the motivic generating series $F_Q(\boldsymbol{x},q)$:

$$A(\boldsymbol{x},\boldsymbol{y}) = \lim_{q\to 1} \hat{A}_i(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q), \qquad \hat{A}_i(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q) = 1 - \hat{y}_i - \hat{x}_i(-q^{1/2}\hat{y}_i)^{C_{ii}} \prod_{j\neq i} \hat{y}_j^{C_{ij}}. \tag{3.13}$$

Applying the change of variables $x_i = \mu^{n_i} a^{a_i}$ $(q \to 1)$ together with $\prod_i y_i = \lambda$, we can transform $A_i(\boldsymbol{x},\boldsymbol{y})$ into the classical A-polynomial $A(\mu,\lambda,a)$ given by (3.10).

4 Examples

This section is devoted to the explicit results: computations of quivers, BPS state counts, and 3d $\mathcal{N}=2$ theories for different GM invariants of knot complements. The choice of examples on which we check ideas from section 3 is determined by the fact that the closed form formulas for $F_K(\mu,a,q)$ are completely new results, so far available only for the $T^{(2,2p+1)}$ torus knot complements [43].

We start from the unknot complement in the unreduced normalisation. For all other cases we use the reduced one, which corresponds to the division by the unknot factor. For more details see [43].[1]

4.1 Unknot complement

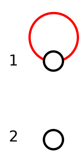
The simplest example is the unknot complement, for which the GM invariant is given by [43]:

$$F_{0_1}(\mu,a,q) = \frac{(\mu q;q)_\infty}{(\mu a;q)_\infty}. \tag{4.1}$$

Using formulas for quantum dilogarithms we can write

$$\begin{aligned} F_{0_1}(\mu,a,q) &= F_Q(x_1,x_2,q)\big|_{x_1=\mu q^{1/2},\, x_2=\mu a}, \\ F_Q(x_1,x_2,q) &= \sum_{d_1,d_2\geq 0} (-q^{1/2})^{d_1^2} \frac{x_1^{d_1} x_2^{d_2}}{(q;q)_{d_1}(q;q)_{d_2}}. \end{aligned} \tag{4.2}$$

We can see that the quiver for the unknot complement is given by

$Q =$  $\iff \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$ (4.3)

which is the same as in the standard knots-quivers correspondence for the unknot. However, now the variable a appears in the change of variables for the node without the loop.

[1] Note that what we call unreduced normalisation is called fully unreduced in [43].

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For the SU(N) GM invariants we obtain:

$$F_{0_1}^{\mathrm{SU}(N)}(\mu,q)=\frac{(\mu q;q)_\infty}{(\mu q^N;q)_\infty}=F_Q(x_1,x_2,q)|_{x_1=\mu q^{1/2},\,x_2=\mu q^N}\,. \tag{4.4}$$

In case of original SU(2) GM invariants, we still have same quiver (4.3) and motivic generating series (4.2), but the change of variables reduces to $x_1=\mu q^{1/2},\ x_2=\mu q^2$.

Since

$$F_Q(x_1,x_2,q)=\mathrm{Exp}\left[\frac{\Omega(\boldsymbol{x},q)}{1-q}\right]=\mathrm{Exp}\left[\frac{\sum_{\boldsymbol{d},s}\Omega_{\boldsymbol{d},s}\boldsymbol{x}^{\boldsymbol{d}}q^s}{1-q}\right]=\mathrm{Exp}\left[\frac{-q^{1/2}x_1+x_2}{1-q}\right], \tag{4.5}$$

we have only two nonzero DT invariants:

$$\Omega_{(1,0),1/2}=-1,\qquad \Omega_{(0,1),0}=1. \tag{4.6}$$

By definition, they lead to two nonzero knot complement analogs of LMOV invariants, exactly as in the case of the standard LMOV invariants for the unknot [4]:

$$\begin{aligned}N(\mu,a,q)&=\Omega(\boldsymbol{x},q)|_{x_1=\mu q^{1/2},\,x_2=\mu a}\\ \sum_{r,i,j}N_{r,i,j}\mu^r a^i q^j&=-\mu q+\mu a\end{aligned}\qquad\Longrightarrow\qquad\begin{cases}N_{1,0,1}=-1\\ N_{1,1,0}=1.\end{cases} \tag{4.7}$$

Alternatively, we could have obtained this result by a direct comparison between (4.1) and (3.7).

As discussed in section 3, we can use the semiclassical limit of the motivic generating series and the GM invariant to obtain effective twisted superpotentials of theories $T[Q_{M_{0_1}}]$ and $T[M_{0_1}]$, whose BPS states are counted by DT invariants and knot complement analogs of LMOV invariants respectively. The limit (3.11) for the unknot complement quiver is given by

$$\begin{aligned}F_Q(\boldsymbol{x},q)&\overset{q^{d_i}=y_i}{\underset{\hbar\to0}{\longrightarrow}}\int\frac{dy_1}{y_1}\frac{dy_2}{y_2}\exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[Q_{M_{0_1}}]}(\boldsymbol{x},\boldsymbol{y})+\mathcal{O}(\hbar^0)\right],\\ \widetilde{\mathcal{W}}_{T[Q_{M_{0_1}}]}(\boldsymbol{x},\boldsymbol{y})&=\mathrm{Li}_2(y_1)+\mathrm{Li}_2(y_2)+\log(-x_1)\log y_1+\log x_2\log y_2+\frac{1}{2}\log y_1\log y_1.\end{aligned} \tag{4.8}$$

We can see that $T[Q_{M_{0_1}}]$ is a $\mathrm{U}(1)^{(1)}\times\mathrm{U}(1)^{(2)}$ gauge theory with one chiral field for each group and effective Chern-Simons level one for $\mathrm{U}(1)^{(1)}$, which is the same as $T[Q_{L_{0_1}}]$ [12]. According to (3.12), the critical point of quiver twisted superpotential given by

$$\begin{aligned}0&=\frac{\partial\widetilde{\mathcal{W}}_{T[Q_{M_{0_1}}]}}{\partial\log y_1}=\log(-x_1)+\log y_1-\log(1-y_1)\,,\\ 0&=\frac{\partial\widetilde{\mathcal{W}}_{T[Q_{M_{0_1}}]}}{\partial\log y_2}=\log x_2+\log y_2-\log(1-y_2)\end{aligned} \tag{4.9}$$

defines the quiver A-polynomials

$$A_1(\boldsymbol{x},\boldsymbol{y})=1-y_1+x_1y_1,\qquad\qquad A_2(\boldsymbol{x},\boldsymbol{y})=1-y_2-x_2. \tag{4.10}$$

A_1 and A_2 are decoupled, which originates from the lack of arrows between vertices 1 and 2 in Q. The quantum quiver A-polynomial, which annihilates $F_Q(\boldsymbol{x}, q)$ and reduces to (4.10) for $q \to 1$, is given by

$$\hat{A}_1(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, q) = 1 - \hat{y}_1 + \hat{x}_1(-q^{1/2}\hat{y}_1), \qquad \hat{A}_2(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, q) = 1 - \hat{y}_2 - \hat{x}_2. \tag{4.11}$$

Applying the change of variables $x_1 = \mu$, $x_2 = \mu a$, $y_1 y_2 = \lambda$ to (4.10), we recover the classical A-polynomial of the unknot[2] [16]:

$$A_{0_1}(\mu, \lambda, a) = 1 - a\mu - \lambda + \mu\lambda, \tag{4.12}$$

which is in line with section 3.

We also expect that the A-polynomial of the unknot encodes the moduli space of vacua of $T[M_{0_1}]$ — the theory read from the semiclassical limit of $F_{0_1}(\mu, a, q)$:

$$\begin{aligned} F_{0_1}(\mu, a, q) &\underset{\hbar \to 0}{\longrightarrow} \exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[M_{0_1}]}(\mu, a) + \mathcal{O}(\hbar^0)\right], \\ \widetilde{\mathcal{W}}_{T[M_{0_1}]}(\mu, a) &= \mathrm{Li}_2(\mu) - \mathrm{Li}_2(a\mu). \end{aligned} \tag{4.13}$$

Solving

$$\log \lambda = \frac{\partial \widetilde{\mathcal{W}}_{T[M_{0_1}]}(\mu, a)}{\partial \log \mu} = \log(1 - a\mu) - \log(1 - \mu) \tag{4.14}$$

we indeed find an agreement with $A_{0_1}(\mu, \lambda, a) = 0$.

4.2 Trefoil knot complement

Let us move to the trefoil knot complement and start from the closed form expression found in [43]:

$$F_{3_1}(\mu, a, q) = \sum_{k_1=0}^{\infty} (\mu q)^{k_1} \frac{(\mu; q^{-1})_{k_1} (a q^{-1}; q)_{k_1}}{(q; q)_{k_1}}. \tag{4.15}$$

Using the formula

$$(\xi; q)_k = \sum_{l=0}^{k} \left(-q^{1/2}\right)^{l^2} \xi^l q^{-\frac{l}{2}} \frac{(q; q)_k}{(q; q)_l (q; q)_{k-l}} \tag{4.16}$$

for $(\mu q^{-1}; q^{-1})_k$ and the identity

$$\frac{(q^{-1}; q^{-1})_k}{(q^{-1}; q^{-1})_l (q^{-1}; q^{-1})_{k-l}} = \left(-q^{1/2}\right)^{l^2 + (k-l)^2 - k^2} \frac{(q; q)_k}{(q; q)_l (q; q)_{k-l}}, \tag{4.17}$$

we can write

$$F_{3_1}(\mu, a, q) = \sum_{k_1=0}^{\infty} \sum_{l_1=0}^{k_1} \mu^{k_1 + l_1} q^{k_1 + \frac{l_1}{2}} \left(-q^{1/2}\right)^{(k_1 - l_1)^2 - k_1^2} \frac{(a q^{-1}; q)_{k_1}}{(q; q)_{l_1} (q; q)_{k_1 - l_1}}. \tag{4.18}$$

[2]We have to take into account the rescaling of λ by $a^{1/2}$ with respect to y_{FGS} due to the fact the we start $F_{0_1}(\mu, a, q)$ from 1.

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Then we use

$$\frac{(\xi;q)_k}{(q;q)_l(q;q)_{k-l}} = \sum_{\substack{\alpha+\beta=l\\ \gamma+\delta=k-l}} \xi^{\alpha+\gamma} q^{-\frac{1}{2}(\alpha+\gamma)} \left(-q^{1/2}\right)^{\alpha^2+\gamma^2+2\gamma l} \tag{4.19}$$

to get

$$F_{3_1}(\mu,a,q) = \sum_{k_1=0}^{\infty}\sum_{l_1=0}^{k_1} \sum_{\substack{\alpha_1+\beta_1=l_1\\ \gamma_1+\delta_1=k_1-l_1}} \left(-q^{1/2}\right)^{(k_1-l_1)^2-k_1^2+\alpha_1^2+\gamma_1^2+2\gamma_1 l_1} \frac{\mu^{k_1+l_1} a^{\alpha_1+\gamma_1} q^{k_1+\frac{l_1}{2}-\frac{3}{2}(\alpha_1+\gamma_1)}}{(q;q)_{\alpha_1}(q;q)_{\beta_1}(q;q)_{\gamma_1}(q;q)_{\delta_1}}. \tag{4.20}$$

After the change of variables

$$\begin{gathered} \alpha_1 = d_1, \quad \beta_1 = d_2, \quad \gamma_1 = d_3, \quad \delta_1 = d_4 \\ l_1 = d_1 + d_2, \qquad k_1 = d_1 + d_2 + d_3 + d_4 \end{gathered} \tag{4.21}$$

we have

$$\begin{aligned} F_{3_1}(\mu,a,q) &= F_Q(\boldsymbol{x},q)\big|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}}\,, \\ F_Q(\boldsymbol{x},q) &= \sum_{d_1,\ldots,d_4\geq 0} (-q^{1/2})^{-d_2^2+d_3^2-2d_1d_2-2d_1d_4-2d_2d_4} \prod_{i=1}^{4} \frac{x_i^{d_i}}{(q;q)_{d_i}}, \end{aligned} \tag{4.22}$$

with $x_i = \mu^{n_i}a^{a_i}q^{l_i}$ given by

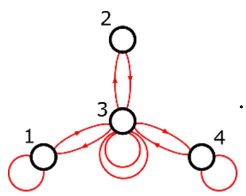
$$x_1 = \mu^2 a, \qquad x_2 = \mu^2 q^{3/2}, \qquad x_3 = \mu a q^{-1/2}, \qquad x_4 = \mu q. \tag{4.23}$$

The quiver adjacency matrix reads:

$$C = \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}. \tag{4.24}$$

This form will be useful for obtaining quivers for all $T^{(2,2p+1)}$ torus knots, but we can shift the framing by 1 to have only the non-negative entries:

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \Longleftrightarrow \quad Q = \tag{4.25}$$



For the trefoil the DT spectrum is infinite, but we can find the invariants order by order in x_i. Let us focus on the linear one:

$$\begin{gathered} F_Q(\boldsymbol{x},q) = \mathrm{Exp}\left[\frac{\Omega(\boldsymbol{x},q)}{1-q}\right] \\ \sum_{d_1,\ldots,d_4\geq 0} (-q^{1/2})^{-d_2^2+d_3^2-2d_1d_2-2d_1d_4-2d_2d_4} \prod_{i=1}^{4} \frac{x_i^{d_i}}{(q;q)_{d_i}} = \exp\left[\sum_{n,\boldsymbol{d},s} \frac{\Omega_{\boldsymbol{d},s}\boldsymbol{x}^{\boldsymbol{d}n}q^{sn}}{1-q^n}\right] \\ 1 + \frac{x_1 + (-q^{1/2})^{-1}x_2 + (-q^{1/2})x_3 + x_4}{1-q} + \mathcal{O}(x_i^2) = 1 + \frac{\sum_{\boldsymbol{d},s}\Omega_{\boldsymbol{d},s}\boldsymbol{x}^{\boldsymbol{d}}q^s}{1-q} + \mathcal{O}(x_i^2). \end{gathered} \tag{4.26}$$

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In consequence

$$\Omega_{(1,0,0,0),0}=1,\quad \Omega_{(0,1,0,0),-1/2}=-1,\quad \Omega_{(0,0,1,0),1/2}=-1,\quad \Omega_{(0,0,0,1),0}=1. \tag{4.27}$$

Applying the change of variables $x_1=\mu^2 a$, $x_2=\mu^2 q^{3/2}$, $x_3=\mu a q^{-1/2}$, $x_4=\mu q$, we obtain the (first four) knot complement analogs of LMOV invariants for the trefoil knot complement:

$$\begin{aligned} N(\mu,a,q) &= \Omega(\boldsymbol{x},q)|_{(4.23)} \\ \sum_{r,i,j} N_{r,i,j}\mu^r a^i q^j &= \mu^2 a-\mu^2 q-\mu a+\mu q \end{aligned} \quad\Longrightarrow\quad \begin{cases} N_{1,0,1}=1, & N_{2,0,1}=-1, \\ N_{1,1,0}=-1, & N_{2,1,0}=1. \end{cases} \tag{4.28}$$

$\Omega_{\boldsymbol{d},s}$ and $N_{r,i,j}$ count the BPS states in 3d $\mathcal{N}=2$ theories $T[Q_{M_{3_1}}]$ and $T[M_{3_1}]$ respectively. The structure of the first theory can be read off from the semiclassical limit of $F_Q(\boldsymbol{x},q)$:

$$F_Q(\boldsymbol{x},q) \underset{\hbar\to 0}{\overset{q^{d_i}=y_i}{\longrightarrow}} \int \prod_{i=1}^{4} \frac{dy_i}{y_i} \exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[Q_{M_{3_1}}]}(\boldsymbol{x},\boldsymbol{y})+\mathcal{O}(\hbar^0)\right], \tag{4.29}$$

$$\begin{aligned} \widetilde{\mathcal{W}}_{T[Q_{M_{3_1}}]}(\boldsymbol{x},\boldsymbol{y}) = &\sum_{i=1}^{4}\mathrm{Li}_2(y_i)+\log x_1\log y_1+\log(-x_2)\log y_2+\log(-x_3)\log y_3+\log x_4\log y_4 \\ &-\log y_1\log y_2-\frac{1}{2}\log y_2\log y_2+\frac{1}{2}\log y_3\log y_3-\log y_1\log y_4-\log y_2\log y_4. \end{aligned}$$

Using the dictionary (2.13)–(2.14), we can see that the gauge group of $T[Q_{M_{3_1}}]$ is $\mathrm{U}(1)^{(1)}\times \mathrm{U}(1)^{(2)}\times \mathrm{U}(1)^{(3)}\times \mathrm{U}(1)^{(4)}$ and we have four chiral fields ϕ_i with charges δ_{ij} under $\mathrm{U}(1)^{(j)}$. We also have gauge Chern-Simons couplings determined by the quiver adjacency matrix: $\kappa_{ij}^{\mathrm{eff}}=C_{ij}$, and Fayet-Iliopoulos couplings between each gauge symmetry and its dual topological symmetry. Comparing with [12], we can see that the structure of $T[Q_{M_{3_1}}]$ is different than $T[Q_{L_{3_1}}]$. The moduli space of vacua of $T[Q_{M_{3_1}}]$ is given by the zero locus of the following quiver A-polynomials:

$$\begin{aligned} A_1(\boldsymbol{x},\boldsymbol{y}) &= 1-y_1-x_1y_2^{-1}y_4^{-1}, & A_2(\boldsymbol{x},\boldsymbol{y}) &= 1-y_2+x_2y_1^{-1}y_2^{-1}y_4^{-1}, \\ A_3(\boldsymbol{x},\boldsymbol{y}) &= 1-y_3+x_3y_3, & A_4(\boldsymbol{x},\boldsymbol{y}) &= 1-y_4-x_4y_1^{-1}y_2^{-1}, \end{aligned} \tag{4.30}$$

which are the classical limits of the annihilators of $F_Q(\boldsymbol{x},q)$:

$$\begin{aligned} \hat{A}_1(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q) &= 1-\hat{y}_1-\hat{x}_1\hat{y}_2^{-1}\hat{y}_4^{-1}, & \hat{A}_2(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q) &= 1-\hat{y}_2+q^{-1/2}\hat{x}_2\hat{y}_1^{-1}\hat{y}_2^{-1}\hat{y}_4^{-1}, \\ \hat{A}_3(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q) &= 1-\hat{y}_3+q^{1/2}\hat{x}_3\hat{y}_3, & \hat{A}_4(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},q) &= 1-\hat{y}_4-\hat{x}_4\hat{y}_1^{-1}\hat{y}_2^{-1}. \end{aligned} \tag{4.31}$$

The structure of the theory $T[M_{3_1}]$ is encoded in the semiclassical limit of $F_{3_1}(\mu,a,q)$:

$$\begin{aligned} F_{3_1}(\mu,a,q) &\underset{\hbar\to 0}{\longrightarrow} \exp\int\frac{dz}{z}\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[M_{3_1}]}(\mu,a,z)+\mathcal{O}(\hbar^0)\right], \\ \widetilde{\mathcal{W}}_{T[M_{3_1}]}(\mu,a,z) &= \log\mu\log z-\mathrm{Li}_2(\mu)+\mathrm{Li}_2(\mu z^{-1})+\mathrm{Li}_2(a)-\mathrm{Li}_2(az)+\mathrm{Li}_2(z). \end{aligned} \tag{4.32}$$

Extremalisation with respect to z and the introduction of the variable λ dual to μ leads to

$$\begin{cases} 0 = \frac{\partial \widetilde{\mathcal{W}}_{T[M_{3_1}]}(\mu,a,z)}{\partial z} \\ \log\lambda = \frac{\partial \widetilde{\mathcal{W}}_{T[M_{3_1}]}(\mu,a,z)}{\partial \log \mu} \end{cases} \implies \begin{cases} 1 = \frac{\mu\left(1-\mu(z^*)^{-1}\right)(1-az^*)}{(1-z^*)} \\ \lambda = \frac{z^*(1-\mu)}{1-\mu(z^*)^{-1}}. \end{cases} \tag{4.33}$$

Eliminating z^* we obtain the zero locus of the A-polynomial

$$A_{3_1}(\mu,\lambda,a) = (\mu-1)\mu^3 - \left(1-\mu+2(1-a)\mu^2 - a\mu^3 + a^2\mu^4\right)\lambda + (1-a\mu)\lambda^2, \tag{4.34}$$

which agrees with [16] after taking into account the rescaling of λ by a^{-1}.

4.3 Cinquefoil knot complement

For more complicated knot complements all expressions become more involved, so we focus on finding the corresponding quivers, having in mind that the analysis of BPS states and 3d $\mathcal{N}=2$ theories can be done analogously to the unknot and trefoil complement case. In this section we concentrate on the $5_1 = T^{(2,5)}$ knot complement with a plan of generalisation to all $T^{(2,2p+1)}$ torus knot complements.

We start from the formula for GM invariant given in [43]:

$$F_{5_1}(\mu,a,q) = \sum_{0\le k_2\le k_1} \mu^{k_1+2k_2} q^{(k_1+k_2)-k_1k_2} \frac{(aq^{-1};q)_{k_1}(\mu;q^{-1})_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \tag{4.35}$$

where we use the q-binomial:

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{(q;q)_k}{(q;q)_l(q;q)_{k-l}}. \tag{4.36}$$

Following the steps from section 4.2, we obtain

$$\begin{aligned} F_{5_1}(\mu,a,q) = \sum_{0\le k_2\le k_1} \sum_{l_1=0}^{k_1} \sum_{\substack{\alpha_1+\beta_1=l_1 \\ \gamma_1+\delta_1=k_1-l_1}} &\left(-q^{1/2}\right)^{-2k_1k_2+(k_1-l_1)^2-k_1^2+\alpha_1^2+\gamma_1^2+2\gamma_1 l_1} \\ &\times \frac{\mu^{k_1+2k_2+l_1} a^{\alpha_1+\gamma_1} q^{k_1+k_2+\frac{1}{2}l_1-\frac{3}{2}(\alpha_1+\gamma_1)}}{(q;q)_{\alpha_1}(q;q)_{\beta_1}(q;q)_{\gamma_1}(q;q)_{\delta_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}. \end{aligned} \tag{4.37}$$

Now we use the formula

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \sum_{m_2=0}^{n_2} q^{(m_1-m_2)(n_2-m_2)} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} n_1-m_1 \\ n_2-m_2 \end{bmatrix} \tag{4.38}$$

in two iterations:

$$\begin{aligned} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} &= \sum_{l_2=0}^{k_2} q^{(l_1-l_2)(k_2-l_2)} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} k_1-l_1 \\ k_2-l_2 \end{bmatrix} \\ &= \sum_{l_2=0}^{k_2} q^{(l_1-l_2)(k_2-l_2)} \sum_{\alpha_2+\beta_2=l_2} q^{(\alpha_1-\alpha_2)(l_2-\alpha_2)} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ &\quad\times \sum_{\gamma_2+\delta_2=k_2-l_2} q^{(\gamma_1-\gamma_2)(k_2-l_2-\gamma_2)} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \end{aligned} \tag{4.39}$$

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to get

$$F_{5_1}(\mu,a,q)=\sum_{0\le k_2\le k_1}\sum_{l_1=0}^{k_1}\sum_{\substack{\alpha_1+\beta_1=l_1\\ \gamma_1+\delta_1=k_1-l_1}}\left(-q^{1/2}\right)^{-2k_1k_2+(k_1-l_1)^2-k_1^2+\alpha_1^2+\gamma_1^2+2\gamma_1 l_1}$$
$$\sum_{l_2=0}^{k_2}\sum_{\substack{\alpha_2+\beta_2=l_2\\ \gamma_2+\delta_2=k_2-l_2}}\left(-q^{1/2}\right)^{2(l_1-l_2)(k_2-l_2)+2(\alpha_1-\alpha_2)(l_2-\alpha_2)+2(\gamma_1-\gamma_2)(k_2-l_2-\gamma_2)}$$
$$\times\frac{\mu^{k_1+2k_2+l_1}a^{\alpha_1+\gamma_1}q^{k_1+k_2+\frac{1}{2}l_1-\frac{3}{2}(\alpha_1+\gamma_1)}}{(q;q)_{\alpha_1-\alpha_2}(q;q)_{\beta_1-\beta_2}(q;q)_{\gamma_1-\gamma_2}(q;q)_{\delta_1-\delta_2}(q;q)_{\alpha_2}(q;q)_{\beta_2}(q;q)_{\gamma_2}(q;q)_{\delta_2}}. \tag{4.40}$$

The change of variables

$$\begin{aligned}
&\alpha_1=d_1+d_5, \quad \beta_1=d_2+d_6, \quad \gamma_1=d_3+d_7, \quad \delta_1=d_4+d_8,\\
&\alpha_2=d_5, \qquad\quad \beta_2=d_6, \qquad\quad \gamma_2=d_7, \qquad\quad \delta_2=d_8\\
&l_1=(d_1+d_5)+(d_2+d_6), \qquad k_1=(d_1+d_5)+(d_2+d_6)+(d_3+d_7)+(d_4+d_8)\\
&l_2=d_5+d_6, \qquad\qquad\qquad\quad k_2=d_5+d_6+d_7+d_8
\end{aligned} \tag{4.41}$$

leads to

$$\begin{aligned}
F_{5_1}(\mu,a,q)&=\left.F_Q(\boldsymbol{x},q)\right|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}},\\
F_Q(\boldsymbol{x},q)&=\sum_{d_1,\ldots,d_8\ge 0}(-q^{1/2})^{\sum_{i,j=1}^{8}C_{ij}d_id_j}\prod_{i=1}^{8}\frac{x_i^{d_i}}{(q;q)_{d_i}},
\end{aligned} \tag{4.42}$$

with $x_i=\mu^{n_i}a^{a_i}q^{l_i}$ given by

$$\begin{aligned}
&x_1=\mu^2a, \qquad x_2=\mu^2q^{3/2}, \qquad x_3=\mu a q^{-1/2}, \qquad x_4=\mu q,\\
&x_5=\mu^4aq, \qquad x_6=\mu^4q^{5/2}, \qquad x_7=\mu^3aq^{1/2}, \qquad x_8=\mu^3q^2,
\end{aligned} \tag{4.43}$$

and

$$C=\begin{bmatrix}
0 & -1 & 0 & -1 & -1 & -1 & 0 & -1\\
-1 & -1 & 0 & -1 & -2 & -2 & 0 & -1\\
0 & 0 & 1 & 0 & -1 & -1 & 0 & 0\\
-1 & -1 & 0 & 0 & -2 & -2 & -1 & -1\\
-1 & -2 & -1 & -2 & -2 & -3 & -2 & -3\\
-1 & -2 & -1 & -2 & -3 & -3 & -2 & -3\\
0 & 0 & 0 & -1 & -2 & -2 & -1 & -2\\
-1 & -1 & 0 & -1 & -3 & -3 & -2 & -2
\end{bmatrix}. \tag{4.44}$$

This is the adjacency matrix corresponding to the cinquefoil knot complement. Performed computations and the matrix itself may seem invloved, but they can be considered as a simple generalisation of the trefoil knot complement case. The most important is the fact that the change of variables (4.41) can be obtained from (4.21). For indices k_1,l_1,α_1,β_1,γ_1,δ_1, we just substitute $d_i\mapsto d_i+d_{i+4}$. In case of k_2,l_2,α_2,β_2,γ_2,δ_2, we start from k_1,l_1,α_1,β_1,γ_1,δ_1

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and shift $d_i \mapsto d_{i+4}$. From the point of view of the quiver adjacency matrix, this corresponds to copying the 4×4 matrix C_{3_1} to fill 8×8 entries:

$$[C_{3_1}] \mapsto \begin{bmatrix} C_{3_1} & C_{3_1} \\ C_{3_1}^T & C_{3_1} \end{bmatrix}. \tag{4.45}$$

We still need to include $q^{-k_1 k_2}$ and the q-binomial $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ which, according to (4.39) and (4.41), contribute to $(-q^{1/2})^{\sum_{i,j} C_{ij} d_i d_j}$ by

$$\begin{aligned} \left(-q^{1/2}\right)^{2(l_1-l_2)(k_2-l_2)+2(\alpha_1-\alpha_2)(l_2-\alpha_2)+2(\gamma_1-\gamma_2)(k_2-l_2-\gamma_2)} &= \left(-q^{1/2}\right)^{2(d_1+d_2)(d_7+d_8)+2d_1d_6+2d_3d_8} \\ \left(-q^{1/2}\right)^{-2k_1k_2} &= \left(-q^{1/2}\right)^{-2(d_1+d_2+\ldots+d_8)(d_5+d_6+d_7+d_8)}. \end{aligned} \tag{4.46}$$

In consequence we have to modify (4.45) to

$$[C_{3_1}] \mapsto \begin{bmatrix} D_0 & R_0 \\ R_0^T & D_1 \end{bmatrix}, \tag{4.47}$$

where

$$D_0 = C_{3_1}, \qquad D_1 = D_0 - \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}, \qquad R_0 = D_0 - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{4.48}$$

We can see that this is consistent with (4.44). The change of variables (4.43) is a direct consequence of (4.23) and (4.41), taking into account that in $F_{5_1}(\mu, a, q)$ we have an extra factor of $\mu^{2k_2} q^{k_2}$.

4.4 General $T^{(2,2p+1)}$ torus knot complements

Now we are ready to move to the general case using recursion. We assume that we know the quiver for $T^{(2,2p+1)}$ torus knot complement and look for the quiver for $T^{(2,2(p+1)+1)} = T^{(2,2p+3)}$. The general formula is given by [43]

$$\begin{aligned} F_{T^{(2,2p+1)}}(\mu, a, q) = \sum_{0 \le k_p \le \ldots \le k_1} & \mu^{2(k_1+\ldots+k_p)-k_1} q^{(k_1+k_2+\ldots+k_p)-\sum_{i=2}^{p} k_{i-1}k_i} \\ & \times \frac{(aq^{-1};q)_{k_1}(\mu;q^{-1})_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}, \end{aligned} \tag{4.49}$$

so when we go from p to $p+1$, the summand in $F_{T^{(2,2p+1)}}$ is multiplied by

$$\mu^{2k_{p+1}} q^{k_{p+1}} (-q^{1/2})^{2k_p k_{p+1}} \begin{bmatrix} k_p \\ k_{p+1} \end{bmatrix}. \tag{4.50}$$

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This means that we copy the rightmost (also downmost and diagonal) 4×4 matrix blocks right (respectively down and further on the diagonal), exactly like we did for $C_{3_1}\mapsto C_{5_1}$, with the same modification as in (4.47)–(4.48). Therefore the quiver for $T^{(2,2p+1)}$ torus knot is given by

$$C=\begin{bmatrix} D_0 & R_0 & R_0 & \dots & R_0 & R_0 \\ R_0^T & D_1 & R_1 & \dots & R_1 & R_1 \\ R_0^T & R_1^T & D_2 & \dots & R_2 & R_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_0^T & R_1^T & R_2^T & \dots & D_{p-2} & R_{p-2} \\ R_0^T & R_1^T & R_2^T & \dots & R_{p-2}^T & D_{p-1} \end{bmatrix}, \tag{4.51}$$

where

$$\begin{aligned} D_n &= \begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} - n\begin{bmatrix} 2&2&2&2\\2&2&2&2\\2&2&2&2\\2&2&2&2 \end{bmatrix} = \begin{bmatrix} -2n & -2n-1 & -2n & -2n-1 \\ -2n-1 & -2n-1 & -2n & -2n-1 \\ -2n & -2n & -2n+1 & -2n \\ -2n-1 & -2n-1 & -2n & -2n \end{bmatrix}, \\ R_n &= D_n - \begin{bmatrix} 1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1 \end{bmatrix} = \begin{bmatrix} -2n-1 & -2n-1 & -2n & -2n-1 \\ -2n-2 & -2n-2 & -2n & -2n-1 \\ -2n-1 & -2n-1 & -2n & -2n \\ -2n-2 & -2n-2 & -2n-1 & -2n-1 \end{bmatrix}. \end{aligned} \tag{4.52}$$

We can see that, similarly to the standard knots-quivers correspondence [1, 2], increasing p does not change any previously determined entries of the matrix, so it makes sense to consider the limit $p\to\infty$ leading to an infinite quiver.

The change of variables $x_i=\mu^{n_i}a^{a_i}q^{l_i}$ for $T^{(2,2p+1)}$ torus knot complement comes directly from the generalisation of (4.23), (4.43) according to (4.50) and is given by

$$\begin{aligned} \sum_i n_i d_i &= 2(d_1+d_2)+1(d_3+d_4) \\ &\quad +4(d_5+d_6)+3(d_7+d_8) \\ &\quad \vdots \\ &\quad +2p(d_{4p-3}+d_{4p-2})+(2p-1)(d_{4p-1}+d_{4p}), \\ \sum_i a_i d_i &= d_1+d_3 \\ &\quad +d_5+d_7 \\ &\quad \vdots \\ &\quad +d_{4p-3}+d_{4p-1}, \\ \sum_i l_i d_i &= 0d_1+\frac{3}{2}d_2-\frac{1}{2}d_3+1d_4 \\ &\quad +1d_5+\frac{5}{2}d_6+\frac{1}{2}d_7+2d_8 \\ &\quad \vdots \\ &\quad +(p-1)d_{4p-3}+\frac{2p+1}{2}d_{4p-2}+\frac{2p-3}{2}d_{4p-1}+pd_{4p}. \end{aligned} \tag{4.53}$$

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We can summarise (4.51)–(4.53) in the concise form of the correspondence between $T^{(2,2p+1)}$ torus knot complements and quivers:

$$\begin{aligned} F_{T^{(2,2p+1)}}(\mu,a,q) &= F_Q(\boldsymbol{x},q)\big|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}} \\ &= \sum_{d_1,\ldots,d_{4p}\geq 0} (-q^{1/2})^{\sum_{i,j=1}^{4p} C_{ij}d_id_j} \prod_{i=1}^{4p} \frac{x_i^{d_i}}{(q;q)_{d_i}}\Bigg|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}}. \end{aligned} \tag{4.54}$$

5 t-deformation

In this section we propose a t-deformation of the results from sections 3–4 using $F_K(\mu,a,q,t)$ given in [43]. These results are based on the t-deformation of GM invariants following the generalisation of HOMFLY-PT polynomials to superpolynomials, which is reflected in equations (2.25) and (2.26).

5.1 Quivers

The connection between GM invariants and superpolynomials [43] suggest the following t-deformed correspondence:

Conjecture 2 *For a given knot complement $M_K = S^3\backslash K$, the GM invariant $F_K(\mu,a,q,t)$ can be written in the form*

$$F_K(\mu,a,q,t) = \sum_{d_1,\ldots,d_m\geq 0} (-q^{1/2})^{\sum_{i,j=1}^{m} C_{ij}d_id_j} \prod_{i=1}^{m} \frac{\mu^{n_id_i}a^{a_id_i}q^{l_id_i}(-t)^{t_id_i}}{(q;q)_{d_i}}, \tag{5.1}$$

where C is a symmetric $m\times m$ matrix and n_i, a_i, l_i, t_i are fixed integers. In consequence, there exist a quiver Q which adjacency matrix is equal to C and motivic generating series

$$F_Q(\boldsymbol{x},q) = \sum_{d_1,\ldots,d_m\geq 0} (-q^{1/2})^{\sum_{i,j=1}^{m} C_{ij}d_id_j} \prod_{i=1}^{m} \frac{x_i^{d_i}}{(q;q)_{d_i}} \tag{5.2}$$

reduces to the GM invariant after the change of variables $x_i = \mu^{n_i}a^{a_i}q^{l_i}(-t)^{t_i}$:

$$F_K(\mu,a,q,t) = F_Q(\boldsymbol{x},q)\big|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}(-t)^{t_i}}\,. \tag{5.3}$$

Let us look at examples, starting from the unknot complement (in unreduced normalisation). Basing on [43], we have

$$F_{0_1}(\mu,a,q,t) = \frac{(\mu q;q)_\infty}{(-\mu a t^3;q)_\infty}. \tag{5.4}$$

We immediately see that the quiver is the same as in (4.3), but the change of variables is given by

$$x_1 = \mu q^{1/2}, \qquad x_2 = -\mu a t^3. \tag{5.5}$$

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For all $T^{(2,2p+1)}$ torus knots (including trefoil and cinquefoil) the situation is similar. The general formula (in reduced normalisation) [43]

$$\begin{aligned} F_{T^{(2,2p+1)}}(\mu,a,q,t) = \sum_{0\leq k_p\leq \ldots \leq k_1} & \mu^{2(k_1+\ldots+k_p)-k_1} q^{(k_1+k_2+\ldots+k_p)-\sum_{i=2}^{p} k_{i-1}k_i} t^{2(k_1+\ldots+k_p)} \\ & \times \frac{(-aq^{-1}t;q)_{k_1}(\mu;q^{-1})_{k_1}}{(q;q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix} \end{aligned} \tag{5.6}$$

leads to the quiver (4.51)–(4.52), the only difference with respect to section 4.4 lies in the change of variables. Now $x_i = \mu^{n_i} a^{a_i} q^{l_i} (-t)^{t_i}$, where

$$\begin{aligned} \sum_i t_i d_i = 3(d_1+d_3) + 2(d_2+d_4) & \\ + 5(d_5+d_7) + 4(d_6+d_8) & \\ \vdots \quad & \\ + (2p+1)(d_{4p-3}+d_{4p-1}) + 2p(d_{4p-2}+d_{4p}). & \end{aligned} \tag{5.7}$$

For the trefoil it leads to the change of variables

$$x_1 = -\mu^2 a t^3, \qquad x_2 = \mu^2 q^{3/2} t^2, \qquad x_3 = -\mu a q^{-1/2} t^3, \qquad x_4 = \mu q t^2. \tag{5.8}$$

Comparing with the quiver adjacency matrix (4.24), we can see that

$$t_i \neq C_{ii}, \tag{5.9}$$

in contrary to the standard knots-quivers correspondence [1, 2].

5.2 BPS states and 3d $\mathcal{N}=2$ effective theories

In section 3.2 we constructed knot complement analogs of LMOV invariants basing on the DT invariants. Knowing the form of the t-deformed change of variables $x_i = \mu^{n_i} a^{a_i} q^{l_i} (-t)^{t_i}$, we can define the t-deformed knot complement analogs of LMOV invariants:

$$N(\mu,a,q,t) = \sum_{r,i,j,k} N_{r,i,j,k} \mu^r a^i q^j (-t)^k = \Omega(\boldsymbol{x},q)\big|_{x_i=\mu^{n_i}a^{a_i}q^{l_i}(-t)^{t_i}}, \tag{5.10}$$

which implies

$$\begin{aligned} F_K(\mu,a,q,t) &= \mathrm{Exp}\left[\frac{N(\mu,a,q,t)}{1-q}\right] = \exp\left[\sum_{n,r,i,j} \frac{N_{r,i,j,k}\mu^{rn}a^{in}q^{jn}(-t)^{kn}}{1-q}\right] \\ &= \prod_{r,i,j,k,l} \left(1-\mu^r a^i q^{j+l}(-t)^k\right)^{-N_{r,i,j,k}} = \prod_{r,i,j,k} (\mu^r a^i q^j (-t)^k;q)_\infty^{-N_{r,i,j,k}}. \end{aligned} \tag{5.11}$$

Since DT invariants are integer numbers, so are $N_{r,i,j,k}$.

In case of the unknot complement, the GM invariant given in terms of infinite q-Pochhammers

$$F_{0_1}(\mu,a,q,t) = \frac{(\mu q;q)_\infty}{(\mu a(-t)^3;q)_\infty} \tag{5.12}$$

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immediately leads to

$$N_{1,0,1,0} = -1, \qquad N_{1,1,0,3} = 1. \tag{5.13}$$

From the point of view of 3d $\mathcal{N}=2$ effective theory $T[M_K]$, the t-deformation of the semiclassical limit (3.8) is given by

$$F_K(\mu,a,q,t) \underset{\hbar\to 0}{\to} \int \prod_i \frac{dz_i}{z_i} \exp\left[\frac{1}{\hbar}\widetilde{\mathcal{W}}_{T[M_K]}(\mu,a,t,z_i) + \mathcal{O}(\hbar^0)\right] \tag{5.14}$$

and can be interpreted as the introduction of the global R-symmetry $\mathrm{U}(1)_F$ with fugacity $(-t)$, associated to rotations in the directions normal to M5-brane inside $\mathbb{R}^4$ (for more details see [12, 14, 16, 18]). Then the moduli space of vacua of $T[M_K]$, obtained by the extremalisation of $\widetilde{\mathcal{W}}_{T[M_K]}$ with respect to z_i and the introduction of variable λ dual to μ, is described by the graph of the super-A-polynomial [16]:

$$\frac{\partial \widetilde{\mathcal{W}}^{\text{eff}}_{T[M_K]}(\mu,a,t)}{\partial \log\mu} = \log\lambda \qquad \Leftrightarrow \qquad A(\mu,\lambda,a,t) = 0. \tag{5.15}$$

$A(\mu,\lambda,a,q,t)$ is the classical limit of the quantum super-A-polynomial, which annihilates superpolynomials (see [14, 16, 18, 48]).

6 Future directions

We conclude with a brief description of interesting directions for future research:

- It would be desirable to understand the relation between quivers found in this paper and in [1, 2]. We could see that for the unknot and the unknot complement quivers are the same, but the quiver corresponding to the trefoil has 3 nodes, whereas the one associated to the trefoil complement has 4 nodes. The core of this issue probably lies in the relation between HOMFLY-PT polynomials and GM invariants. It seems that in general it should be some transformation analogous to the Fourier transform, which for some cases (including $T^{(2,2p+1)}$ torus knots) reduces to the substitution $\mu = q^r$. Investigating the relation between the standard and new correspondence could help in finding quivers for other knot complements. Probably it would also provide a new insight to the duality web of associated 3d $\mathcal{N}=2$ theories studied in [12, 13].

- Since GPPV invariants exhibit peculiar modularity properties and are related to logarithmic conformal field theories, it would be interesting to perform a study of these aspects for GM invariants in the context of the correspondence with quivers.

- GM invariants still lack a proper mathematical definition and a proof that they are ineed topological invariants of knot complements. These goals seem ambitious, but achieving them would potentially allow to properly state and prove the conjectures proposed in this work.

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- One can look for suggestions and constraints on GM invariants coming from the existence of a corresponding quiver. Such approach applied to the case of the standard knots-quivers correspondence enabled finding the formulas for HOMFLY-PT polynomials $P_{K,r}(a,q)$ for 6_2 and 6_3 knots [2].

- The form and meaning of the t-deformation proposed in [43] and applied here are still uncertain. Better understanding of this aspect is crucial for the categorification of 3-manifold invariants, which was the main motivation of introducing GPPV and GM invariants.

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References

[1] P. Kucharski, M. Reineke, M. Stošić and P. Sułkowski, *BPS states, knots and quivers*, *Phys. Rev. D* **96** (2017) 121902 [arXiv:1707.02991] [INSPIRE].

[2] P. Kucharski, M. Reineke, M. Stošić and P. Sułkowski, *Knots-quivers correspondence*, *Adv. Theor. Math. Phys.* **23** (2019) 1849 [arXiv:1707.04017] [INSPIRE].

[3] S. Gukov and C. Manolescu, *A two-variable series for knot complements*, arXiv:1904.06057 [INSPIRE].

[4] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, *Nucl. Phys. B* **577** (2000) 419 [hep-th/9912123] [INSPIRE].

[5] J.M.F. Labastida and M. Mariño, *Polynomial invariants for torus knots and topological strings*, *Commun. Math. Phys.* **217** (2001) 423 [hep-th/0004196] [INSPIRE].

[6] J.M.F. Labastida, M. Mariño and C. Vafa, *Knots, links and branes at large N*, *JHEP* **11** (2000) 007 [hep-th/0010102] [INSPIRE].

[7] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435 [INSPIRE].

[8] M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, *Commun. Num. Theor. Phys.* **5** (2011) 231 [arXiv:1006.2706] [INSPIRE].

[9] A.I. Efimov, *Cohomological Hall algebra of a symmetric quiver*, *Compos. Math.* **148** (2012) 1133 [arXiv:1103.2736].

[10] M. Stošić and P. Wedrich, *Rational links and DT invariants of quivers*, arXiv:1711.03333 [INSPIRE].

[11] M. Stošić and P. Wedrich, *Tangle addition and the knots-quivers correspondence*, arXiv:2004.10837 [INSPIRE].

[12] T. Ekholm, P. Kucharski and P. Longhi, *Physics and geometry of knots-quivers correspondence*, arXiv:1811.03110 [INSPIRE].

[13] T. Ekholm, P. Kucharski and P. Longhi, *Multi-cover skeins, quivers, and* 3d $\mathcal{N} = 2$ *dualities*, *JHEP* **02** (2020) 018 [arXiv:1910.06193] [INSPIRE].

[14] H. Awata, S. Gukov, P. Sułkowski and H. Fuji, *Volume Conjecture: Refined and Categorified*, *Adv. Theor. Math. Phys.* **16** (2012) 1669 [arXiv:1203.2182] [INSPIRE].

[15] M. Aganagic and C. Vafa, *Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots*, arXiv:1204.4709 [INSPIRE].

[16] H. Fuji, S. Gukov and P. Sułkowski, *Super-A-polynomial for knots and BPS states*, *Nucl. Phys. B* **867** (2013) 506 [arXiv:1205.1515] [INSPIRE].

[17] S. Garoufalidis, A.D. Lauda and T.T.Q. Lê, *The colored HOMFLYPT function is q-holonomic*, *Duke Math. J.* **167** (2018) 397 [arXiv:1604.08502] [INSPIRE].

[18] H. Fuji, S. Gukov, M. Stošić and P. Sułkowski, 3*d analogs of Argyres-Douglas theories and knot homologies*, *JHEP* **01** (2013) 175 [arXiv:1209.1416] [INSPIRE].

[19] M. Panfil, M. Stošić and P. Sułkowski, *Donaldson-Thomas invariants, torus knots, and lattice paths*, *Phys. Rev. D* **98** (2018) 026022 [arXiv:1802.04573] [INSPIRE].

[20] M. Panfil and P. Sułkowski, *Topological strings, strips and quivers*, *JHEP* **01** (2019) 124 [arXiv:1811.03556] [INSPIRE].

[21] H. Larraguivel, D. Noshchenko, M. Panfil and P. Sułkowski, *Nahm sums, quiver A-polynomials and topological recursion*, *JHEP* **07** (2020) 151 [arXiv:2005.01776] [INSPIRE].

[22] S. Gukov, P. Putrov and C. Vafa, *Fivebranes and 3-manifold homology*, *JHEP* **07** (2017) 071 [arXiv:1602.05302] [INSPIRE].

[23] S. Gukov, D. Pei, P. Putrov and C. Vafa, *BPS spectra and 3-manifold invariants*, *J. Knot Theor. Ramifications* **29** (2020) 2040003 [arXiv:1701.06567] [INSPIRE].

[24] S. Gukov, M. Mariño and P. Putrov, *Resurgence in complex Chern-Simons theory*, arXiv:1605.07615 [INSPIRE].

[25] M.C.N. Cheng, S. Chun, F. Ferrari, S. Gukov and S.M. Harrison, 3*d Modularity*, *JHEP* **10** (2019) 010 [arXiv:1809.10148] [INSPIRE].

[26] P. Kucharski, $\hat{Z}$ *invariants at rational* τ, *JHEP* **09** (2019) 092 [arXiv:1906.09768] [INSPIRE].

[27] H.-J. Chung, *BPS Invariants for 3-Manifolds at Rational Level K*, arXiv:1906.12344 [INSPIRE].

[28] T. Dimofte, D. Gaiotto and S. Gukov, *Gauge Theories Labelled by Three-Manifolds*, *Commun. Math. Phys.* **325** (2014) 367 [arXiv:1108.4389] [INSPIRE].

[29] S. Cecotti, C. Cordova and C. Vafa, *Braids, Walls, and Mirrors*, arXiv:1110.2115 [INSPIRE].

[30] T. Dimofte, D. Gaiotto and S. Gukov, *3-Manifolds and* 3d *Indices*, *Adv. Theor. Math. Phys.* **17** (2013) 975 [arXiv:1112.5179] [INSPIRE].

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[31] A. Gadde, S. Gukov and P. Putrov, *Walls, Lines, and Spectral Dualities in* 3d *Gauge Theories*, *JHEP* **05** (2014) 047 [arXiv:1302.0015] [INSPIRE].

[32] S. Chun, S. Gukov, S. Park and N. Sopenko, 3*d*-3*d correspondence for mapping tori*, arXiv:1911.08456 [INSPIRE].

[33] H.-J. Chung, *Index for a Model of* 3d-3d *Correspondence for Plumbed 3-Manifolds*, arXiv:1912.13486 [INSPIRE].

[34] K. Bringmann, K. Mahlburg and A. Milas, *Quantum modular forms and plumbing graphs of 3-manifolds*, arXiv:1810.05612 [INSPIRE].

[35] K. Bringmann, K. Mahlburg and A. Milas, *Higher depth quantum modular forms and plumbed 3-manifolds*, arXiv:1906.10722.

[36] M.C.N. Cheng, F. Ferrari and G. Sgroi, *Three-Manifold Quantum Invariants and Mock Theta Functions*, *Phil. Trans. Roy. Soc. Lond.* **378** (2019) 20180439 [arXiv:1912.07997] [INSPIRE].

[37] P. Melvin and H. Morton, *The coloured Jones function*, *Commun. Math. Phys.* **169** (1995) 501.

[38] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, *Invent. Math.* **125** (1996) 103.

[39] L. Rozansky, *A Contribution to the trivial connection to Jones polynomial and Witten's invariant of* 3-*D manifolds. 1.*, *Commun. Math. Phys.* **175** (1996) 275 [hep-th/9401061] [INSPIRE].

[40] L. Rozansky, *The universal R-matrix, Burau representation, and the Melvin-Morton expansion of the colored Jones polynomial*, *Adv. Math.* **134** (1998) 1.

[41] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*, *Geom. Topol. Monographs* **7** (2004) 291 [math/0306230].

[42] S. Gukov, *Three-dimensional quantum gravity, Chern-Simons theory, and the A polynomial*, *Commun. Math. Phys.* **255** (2005) 577 [hep-th/0306165] [INSPIRE].

[43] T. Ekholm, A. Gruen, S. Gukov, P. Kucharski, S. Park and P. Sułkowski, $\widehat{Z}$ *at large N: from curve counts to quantum modularity*, arXiv:2005.13349 [INSPIRE].

[44] L. Diogo and T. Ekholm, *Augmentations, annuli, and Alexander polynomials*, arXiv:2005.09733 [INSPIRE].

[45] S. Park, *Higher rank* $\hat{Z}$ *and* F_K, *SIGMA* **16** (2020) 044 [arXiv:1909.13002] [INSPIRE].

[46] S. Park, *Large color R-matrix for knot complements and strange identities*, arXiv:2004.02087 [INSPIRE].

[47] S. Gukov, P.-S. Hsin, H. Nakajima, S. Park, D. Pei and N. Sopenko, *Rozansky-Witten geometry of Coulomb branches and logarithmic knot invariants*, arXiv:2005.05347 [INSPIRE].

[48] N.M. Dunfield, S. Gukov and J. Rasmussen, *The Superpolynomial for knot homologies*, math/0505662 [INSPIRE].

[49] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, *Bull. Am. Math. Soc.* **12** (1985) 239 [INSPIRE].

[50] J. Przytycki and P. Traczyk, *Invariants of links of Conway type*, *Kobe J. Math.* **4** (1987) 115.

JHEP09(2020)075

[51] E. Witten, *Quantum Field Theory and the Jones Polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [INSPIRE].

[52] S. Garoufalidis, P. Kucharski and P. Sułkowski, *Knots, BPS states, and algebraic curves*, *Commun. Math. Phys.* **346** (2016) 75 [arXiv:1504.06327] [INSPIRE].

[53] P. Kucharski and P. Sułkowski, *BPS counting for knots and combinatorics on words*, *JHEP* **11** (2016) 120 [arXiv:1608.06600] [INSPIRE].

[54] S. Meinhardt and M. Reineke, *Donaldson-Thomas invariants versus intersection cohomology of quiver moduli*, arXiv:1411.4062.

[55] H. Franzen and M. Reineke, *Semi-stable Chow-Hall algebras of quivers and quantized Donaldson-Thomas invariants*, *Alg. Numb. Theor.* **12** (2018) 1001 [arXiv:1512.03748].

[56] T. Dimofte, S. Gukov and L. Hollands, *Vortex Counting and Lagrangian 3-manifolds*, *Lett. Math. Phys.* **98** (2011) 225 [arXiv:1006.0977] [INSPIRE].

[57] Y. Terashima and M. Yamazaki, *Semiclassical Analysis of the 3d/3d Relation*, *Phys. Rev. D* **88** (2013) 026011 [arXiv:1106.3066] [INSPIRE].

[58] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *AMS/IP Stud. Adv. Math.* **23** (2001) 45 [hep-th/9811131] [INSPIRE].

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