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# S-duality, entropy function and transport in $AdS_4/CMT_3$

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Abstract: In this paper we consider Abelian vector plus scalar holographic gravity models for 2+1 dimensional condensed matter transport, and the effect of S-duality on them. We find the transport coefficients from the electric and heat currents via usual membrane paradigm-type calculations, and the effect of S-duality on them. We study the same system also by using the entropy function formalism in the extremal case, and the formalism of holographic Stokes equations, in the case of one-dimensional lattices. We study a few generalizations that appear when considering a supergravity-inspired model, and apply the entropy function method for them.

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## 1 Introduction

AdS/CFT methods have been successfully used in order to calculate transport in condensed matter models, though the particular functional behaviours usually are either different than, or more general than ones obtained in real materials, and one must phenomenologically (ad-hoc) fix parameters and/or functions to obtain a fit. This so-called "AdS/CMT" method is therefore viewed best as a phenomenological one, and must therefore be considered within the most general holographic model available. One is led to consider a system of gravity plus Abelian vector field, plus a scalar that defines the kinetic functions appearing in the Lagrangian.

Transport in such systems has been considered in many papers, but here we will be mostly interested in the methods used in [1–7]. The question we want to ask is, what is the effect of S-duality on this bulk holographic theory on the transport coefficients for the holographic dual field theory? The S-duality should correspond to particle-vortex duality in the boundary [8, 9]. We will not consider the effect of quantum gravitational corrections to the bulk gravity action (those have been addressed in [9]). Since we are after the effect of S-duality, we will consider a vector action that involves both $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$. Transport will be calculated using three different methods, a standard membrane paradigm type method at the horizon for nonextremal black holes, the entropy function formalism for extremal black holes (considered in conjunction with a $T \to 0$ limit of the previous formalism), and the formalism of (fluid) Stokes equations in the case of one-dimensional lattices. The last formalism is also considered in the $T \to 0$ limit and then generalized, in order to take advantage of a supergravity-inspired model for which we can apply the same entropy function formalism. In all of these 3 formalisms, we consider the effect of S-duality of the model on the transport coefficients.

The paper is organized as follows. In section 2 we define the model, the behaviour at the black hole horizon, and we add magnetization currents in the presence of external magnetic fields, studying the resulting thermodynamics. In section 3 we calculate electric and thermal transport in this model, calculating the resulting transport coefficients, and study the effect of S-duality on them. In section 4 we use the entropy function formalism, for extremal black holes, to calculate the transport coefficients, in the corresponding limit of the formulas from section 3, as a function of only the charges of the dual black hole. We also explore a subtlety of S-duality in this limit. In section 5 we consider the formalism of Stokes equations to calculate the transport coefficients, and apply it to one-dimensional lattices. S-duality in this case is also explored. In section 6, we apply the results of section 5 to a supergravity-inspired model, by generalizing the formulas for transport coefficients and using the entropy function formalism. In section 7 we conclude.

## 2 AdS/CMT model and black hole horizon data

### 2.1 Model and black hole horizon

Following the logic from [1], we consider 3+1 dimensional gravity coupled to an Abelian vector field $A_\mu$, with both a Maxwell and a "theta" (topological) term, and kinetic functions $Z(\phi), W(\phi)$ defined by a scalar "dilaton" $\phi$, which has some potential $V(\phi)$. For more generality, in order to break translational invariance in one or two spatial directions, we can consider also two more scalar "axions" $\chi_1, \chi_2$ that have VEV linear in the coordinates

$x, y$ and kinetic function $\Phi(\phi)$. The action is therefore

$$
\begin{aligned}
S =& \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_N}\left(R - \frac{1}{2}[(\partial\phi)^2 + \Phi(\phi)\left((\partial\chi_1)^2 + (\partial\chi_2)^2\right)] - V(\phi)\right)\right.\\
&\left.-\frac{Z(\phi)}{4g_4^2}F_{\mu\nu}^2 - W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \qquad (2.1)
\end{aligned}
$$

where we note the addition of the topological term with coefficient function $W(\phi)$ as compared to [1], in order to be able to study S-duality consistently.

Here the field strength $F_{\mu\nu}$ and the dual field strength $\tilde{F}_{\mu\nu}$ are defined as

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\frac{\tilde{\epsilon}^{\mu\nu\rho\sigma}}{\sqrt{-g}}F_{\rho\sigma}\ , \qquad (2.2)
$$

while the linear axion background solution is

$$
\chi_1 = k_1 x, \quad \chi_2 = k_2 y. \qquad (2.3)
$$

We are interested in models with a holographic dual, so the solutions we want to use must be asymptotically AdS, meaning that the scalar potential must have an AdS solution, so

$$
V(0) = -\frac{6}{L^2}, \quad V'(0) = 0. \qquad (2.4)
$$

The equations of motion for the gravity and the gauge field are

$$
R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}V(\phi) + \frac{(16\pi G_N)}{4g_4^2}Z(\phi)\left(2F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \qquad (2.5)
$$

$$
\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\left(\frac{Z(\phi)}{g_4^2}F^{\mu\nu} + 4W(\phi)\tilde{F}^{\mu\nu}\right)\right] = 0. \qquad (2.6)
$$

We have not written the equation of motion for the scalar dilaton $\phi$ (not for the linear dilatons $\chi_1, \chi_2$), but we assume that it has solutions that asymptotically satisfy the condition (2.4).

For the *isotropic case* (with $\chi_1 = \chi_2 = 0$), the background metric plus gauge field solutions we consider are of the type

$$
\begin{aligned}
ds^2 &= -U\,dt^2 + U^{-1}dr^2 + e^{2V}(dx^2 + dy^2)\\
A &= a(r)dt - Bydx, \qquad (2.7)
\end{aligned}
$$

where $U = U(r), V = V(r)$ (note that $V(r)$ is a factor in the metric and $V(\phi)$ is the scalar potential).

The solutions of interest must have a temperature $T$, since the dual field theory, whose transport we want to calculate, must have the same. That means that we are interested in black hole solutions that asymptote to AdS space, and have event horizons at $r = r_H$. Near it, the background fields are expanded as

$$
\begin{aligned}
U(r) &\simeq U(r_H) + (r - r_H)U'(r_H) + \mathcal{O}((r - r_H)^2) = 4\pi T(r - r_+) + \ldots,\\
a(r) &\simeq a_H(r - r_H) + \ldots,\\
V(r) &\simeq V(r_H) + \ldots,\\
\phi &\simeq \phi_H + \ldots, \qquad (2.8)
\end{aligned}
$$

where we assume $U(r_H) = 0$ for the existence of the event horizon and $U'(r_H) \neq 0$ for a non-extremal solution.

The near-horizon metric for the *non-extremal* black hole then becomes (in the extremal case $U'(r_H) = 0$ also, and we need to go to the next order)

$$ds^2 \simeq -(r - r_H)U'(r_H)dt^2 + \frac{1}{(r - r_H)U'(r_H)}dr^2 + e^{2V(r_H)}(dx^2 + dy^2) , \tag{2.9}$$

which is of the type of two-dimensional Rindler spacetime times $\mathbb{R}^2$. The surface gravity is $\kappa = \pm U'(r_H)/2$, the corresponding temperature (in units where $\hbar = k_B = 1$) being

$$T = \frac{\kappa}{2\pi} = \frac{U'(r_H)}{4\pi}. \tag{2.10}$$

With the change of coordinates $r - r_H = U'(r_H)z^2/4$, the Rindler space part of the metric is

$$ds^2 = -(\kappa z)^2 dt^2 + dz^2. \tag{2.11}$$

The near-horizon solution admits 3 scaling symmetries,

$$t \to \lambda t, \;\; \kappa \to \lambda^{-1}\kappa, \tag{2.12}$$

$$t \to \chi^{-1}t, \;\; (r - r_H) \to \chi(r - r_H), \;\; U'(r_H) \to \chi U'(r_H), \tag{2.13}$$

$$e^{V(r_H)} \to \xi e^{2V(r_H)}, \;\; x \to \xi^{-1}x, \;\; y \to \xi^{-1}y. \tag{2.14}$$

### 2.2 Magnetizations and thermodynamics

In the next section we will study electric and thermal (heat) transport, but it is interesting to consider it in the presence of a magnetic field, for generality of the treatment. In this case however, it is known that there is an extra magnetic contribution to the electric and heat currents $\vec{J}$ and $\vec{Q}$, depending on the magnetization density $M$ and energy magnetization density $M_E$, and being of the Hall (off-diagonal) type,

$$\begin{aligned} J_i^{(\text{mag})} &= \frac{M}{T}\epsilon_{ij}\nabla_j T \\ Q_i^{(\text{mag})} &= M\epsilon_{ij}E_j + \frac{2(M_E - \mu M)}{T}\epsilon_{ij}\nabla_j T. \end{aligned} \tag{2.15}$$

Here both $M$ and $M_E$ are defined for the boundary 2+1 dimensional field theory as responses of the theory to a source that changes the fields, and $M_Q = M_E - \mu M$ is called heat magnetization density. For a source $A_x^{(0)} = -By$, giving a magnetic field $B$ in 2+1 dimensions, the magnetization density is (minus) the variation of the (density of the) Euclidean action with respect to $B$,

$$M = -\frac{1}{Vol}\frac{\partial S_E}{\partial B} , \tag{2.16}$$

whereas the energy magnetization density is the same thing if we apply a change in the (Minkowski) metric of the field theory, with source $\delta g_{tx}^{(0)} = -B_1 y$, and differentiate with respect to $B_1$,

$$M_E = -\frac{1}{Vol}\frac{\partial S_E}{\partial B_1}\bigg|_{B_1=0} . \tag{2.17}$$



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Here the Euclidean action in the bulk is

$$S_E = \int d^4x\sqrt{g}\left[\frac{1}{16\pi G_N}\left(R+\frac{1}{2}(\partial\phi)^2+V(\phi)\right)+\frac{Z(\phi)}{4g_4^2}F^2-W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{2.18}$$

The effect of this source on the boundary is to introduce a $\delta g_{tx}^{(0)} = -U(r)B_1 y$ in the bulk, and by consistency of the equations of motion, we need also to add to $A$ a term $(a(r)-\mu)B_1 y dx$, where $\mu$ is the boundary chemical potential, obtaining a modified background solution of ($\chi_1 = k_1 x, \chi_2 = k_2 y, \phi = \phi(r)$ and)

$$\begin{aligned} A_t &= a(r),\;\; A_x = -By+(a(r)-\mu)B_1 y, \\ ds^2 &= -U(r)(dt+B_1 y dx)^2+\frac{dr^2}{U(r)}+e^{2V(r)}(dx^2+dy^2). \end{aligned} \tag{2.19}$$

The inverse metric is then (in $t, r, x, y$ space)

$$g^{\mu\nu} = \begin{bmatrix} B_1^2 e^{-2V}y^2-\frac{1}{U} & 0 & -B_1 e^{-2V}y & 0 \\ 0 & U & 0 & 0 \\ -B_1 e^{-2V}y & 0 & e^{-2V} & 0 \\ 0 & 0 & 0 & e^{-2V} \end{bmatrix} \tag{2.20}$$

After some algebra, we obtain the Maxwell field Euclidean action in the bulk, on this ansatz, as

$$\begin{aligned} S_E^{\text{Maxwell}} &= \int d^4x\Bigg[\frac{Z(\phi)e^{2V}}{4g_4^2}\left(2(a')^2-2e^{-4V}[-B+(a(r)-\mu)B_1]^2\right) \\ &\quad -4W(\phi)a'(r)(-B+(a(r)-\mu)B_1)\Bigg]. \end{aligned} \tag{2.21}$$

We then obtain the magnetization density, energy magnetization density, and heat magnetization density as

$$M = -\frac{1}{V}\frac{\partial S_E}{\partial B} = \int_{r_H}^{\infty} dr\left(\frac{e^{-2V}Z(\phi)B}{g_4^2}-4W(\phi)a'(r)\right) \tag{2.22}$$

$$M_E = \int_{r_H}^{\infty} dr\left(\frac{e^{-2V}Z(\phi)B}{g_4^2}-4W(\phi)a'(r)\right)(\mu-a(r)) \tag{2.23}$$

$$M_Q = M_E-\mu M = -\int_{r_H}^{\infty} dr\left(\frac{e^{-2V}Z(\phi)B}{g_4^2}-4W(\phi)a'(r)\right)a(r). \tag{2.24}$$

## 3 Transport and S-duality

In this section we calculate electric and heat transport for the background solutions from the previous section, in order to study the effect of S-duality on it.

We add perturbations and electrical and thermal gradient sources to the background solution of the previous section, with the same notation as in [1, 5], in the presence of a magnetic field $B$, but at $B_1 = 0$. The electric field perturbation is sourced by a boundary electric field $E$ and thermal gradient $\frac{1}{T}\nabla_i T$ of

$$E_i = E\delta_{ix}\ , \quad \frac{1}{T}\nabla_i T = \xi\delta_{ix}. \tag{3.1}$$

This results in a extra gauge field term in the bulk of $(-E+\xi a(r))tdx$ and an extra metric term of $\delta g_{tx}^{(0)} = -\xi tU$, so adding relevant perturbations we obtain the perturbed ansatz (the diagonal metric and $A_t$ are unperturbed)

$$\begin{aligned}
A_t &= a(r)\\
A_x &= -By + (-E+\xi a(r))t + \delta A_x(r)\\
A_y &= \delta A_y(r)\\
g_{tx} &= -\xi tU + e^{2V}\delta h_{tx}(r)\\
g_{ty} &= e^{2V}\delta h_{ty}(r)\\
g_{rx} &= e^{2V}\delta h_{rx}(r)\\
g_{ry} &= e^{2V}\delta h_{ry}(r)\\
\chi_1 &= kx + \delta\chi_1(r)\\
\chi_2 &= ky + \delta\chi_2(r).
\end{aligned} \tag{3.2}$$

Note that the logic is that the sources $E, B, \xi$ are small, and they in turn generate the perturbations $\delta h_{\mu\nu}$, solved to linear order from the Einstein's equations, as a function of the sources (linear response theory).

Putting an explicit $\epsilon$ in the perturbation matrix (for Mathematica computation reasons), the metric and its inverse to order $\epsilon$, in matrix form (for a space $t, r, x, y$), and the field strength components, are

$$\begin{aligned}
g &= \begin{pmatrix} -U & 0 & e^{2V}\delta h_{tx}\epsilon - tU\epsilon\xi & e^{2V}\delta h_{ty}\epsilon \\ 0 & \frac{1}{U} & e^{2V}\delta h_{rx}\epsilon & e^{2V}\delta h_{ry}\epsilon \\ e^{2V}\delta h_{tx}\epsilon - tU\epsilon\xi & e^{2V}\delta h_{rx}\epsilon & e^{2V} & 0 \\ e^{2V}\delta h_{ty}\epsilon & e^{2V}\delta h_{ry}\epsilon & 0 & e^{2V} \end{pmatrix},\\
g^{-1} &= \begin{pmatrix} -\frac{1}{U} & 0 & \epsilon\left(\frac{\delta h_{tx}}{U} - e^{-2V}t\xi\right) & \frac{\delta h_{ty}\epsilon}{U} \\ 0 & U & -U\delta h_{rx}\epsilon & -U\delta h_{ry}\epsilon \\ \epsilon\left(\frac{\delta h_{tx}}{U} - e^{-2V}t\xi\right) & -U\delta h_{rx}\epsilon & e^{-2V} & 0 \\ \frac{\delta h_{ty}\epsilon}{U} & -U\delta h_{ry}\epsilon & 0 & e^{-2V} \end{pmatrix}
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
F_{rt} &= a',\\
F_{tx} &= \epsilon(-E+\xi a),\\
F_{xy} &= B,\\
F_{rx} &= \epsilon\xi a' t + \epsilon\delta A_x',\\
F_{ry} &= \epsilon\delta A_y'.
\end{aligned} \tag{3.4}$$

The gauge field equations, $x$ and $y$ components, are

$$0 = \partial_t \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{tx} + 4\sqrt{-g}W(\phi)\tilde{F}^{tx} \right) + \partial_r \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{rx} + 4\sqrt{-g}W(\phi)\tilde{F}^{rx} \right)$$
$$+\partial_y \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{yx} + 4\sqrt{-g}W(\phi)\tilde{F}^{yx} \right) \tag{3.5}$$
$$0 = \partial_t \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{ty} + 4\sqrt{-g}W(\phi)\tilde{F}^{ty} \right) + \partial_r \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{ry} + 4\sqrt{-g}W(\phi)\tilde{F}^{ry} \right)$$
$$+\partial_x \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{xy} + 4\sqrt{-g}W(\phi)\tilde{F}^{xy} \right) , \tag{3.6}$$

and become on the ansatz to leading order

$$0 = -\partial_t \left[ \frac{1}{g_4^2} \left( a' Z e^{2V} \delta h_{rx} + \frac{\xi a Z}{U} + \frac{ZB}{U} \delta h_{ty} - \frac{EZ}{U} \right) + 4W \delta A_y' \right]$$
$$= -\partial_r \left( \frac{\sqrt{-g}Z(\phi)}{g_4^2} F^{rx} + 4\sqrt{-g}W(\phi)\tilde{F}^{rx} \right) , \tag{3.7}$$
$$\partial_t \left[ -\frac{1}{g_4^2} \left( a' e^{2V} Z \delta h_{ry} - \frac{BZ}{U} \delta h_{tx} + ZB e^{-2V} t\xi \right) + 4W(\xi a' t + \delta A_x') \right]$$
$$= -\frac{Z}{g_4^2} \xi e^{-2V} B + 4\xi W a'$$
$$= \partial_r \left( \frac{Z}{g_4^2} \sqrt{-g} F^{yr} + 4\sqrt{-g} W \tilde{F}^{yr} \right) . \tag{3.8}$$

### 3.1 Electric current, conductivity and thermoelectric coefficients

The calculation of the transport coefficients of the dual field theory at the horizon of the black hole relies on the membrane paradigm idea, first presented in the calculation of [10], that the quantities appearing in the currents are independent of the radial position $r$, so instead of calculating them at the boundary at $r \to \infty$, like the AdS/CFT prescription dictates, we can calculate them at the horizon. But if it is the case that the currents do depend on $r$, like in [1], we must redefine them, and find quantities that *can* be calculated at the horizon, being $r$ independent.

The standard (and total) current, defined according to [10] (see also [11]), would be

$$j^{i(\text{tot})} = \frac{\delta S}{\delta \partial_r A_i} = \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{ir} + 4\sqrt{-g} W(\phi) \tilde{F}^{ir} , \tag{3.9}$$

where $S$ is the full bulk action. But we note that, because of (3.8), the $y$ component of the gauge field equation is not $r$-independent, so cannot be calculated at the horizon.

We must calculate instead the modified currents (or fluxes) defined as

$$\mathcal{J}^x = \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{xr} + 4\sqrt{-g} W(\phi) \tilde{F}^{xr},$$
$$\mathcal{J}^y = \frac{Z(\phi)}{g_4^2} \sqrt{-g} F^{yr} + 4\sqrt{-g} W(\phi) \tilde{F}^{yr} - \xi M(r) , \tag{3.10}$$

which are now independent of $r$, since $M(r)$ is a position-dependent magnetization density given by (2.22), only integrated up to $r$ only instead of all the way to $\infty$, so that $\partial_r$ on it gives the bracket in (2.22) as the extra term in (3.8).

Explicitly, we obtain the fluxes

$$\begin{aligned}\mathcal{J}^x &= -\epsilon\frac{Z}{g_4^2}a'e^{2V}\delta h_{tx} - \epsilon\frac{Z}{g_4^2}U\delta A'_x - \epsilon\frac{Z}{g_4^2}UB\delta h_{ry}\\ \mathcal{J}^y &= -\frac{Z}{g_4^2}U\delta A'_y - \frac{Z}{g_4^2}e^{2V}a'\delta h_{ty} + \frac{Z}{g_4^2}BU\delta h_{rx} + 4W(-E+\xi a) - \xi M(r)\,,\end{aligned} \tag{3.11}$$

which can then be evaluated at any $r$, including $r_H$ (the horizon).

The important observation is that, while $\partial_r\mathcal{J}^i = 0$, so we can calculate them at the horizon, at infinity $M(r) = M(\infty) = M$ is just the magnetization, so we just subtract the magnetization currents from the total currents, obtaining the usual transport currents, from which we can calculate the conductivity and thermoelectric coefficients,

$$\mathcal{J}^i(r = r_H) = \mathcal{J}^i(r \to \infty) = j^{i(\text{tot})} - \xi^i M = j^i. \tag{3.12}$$

The advantage of being able to calculate at the horizon is that we can impose the conditions of regularity at the horizon (remember that $E_i = E\delta_{ix}$ and $\xi_i = \xi\delta_{ix}$)

$$\begin{aligned}\delta A_i &= -\frac{E_i}{4\pi T}\ln(r - r_H) + \mathcal{O}(r - r_H),\\ \delta\chi_i &= \mathcal{O}((r - r_H)^0),\\ \delta h_{ti} &= U\delta h_{ri} - \frac{\xi_i U}{4\pi e^{2V}T}\ln(r - r_H) + \mathcal{O}(r - r_H),\end{aligned} \tag{3.13}$$

and moreover, since $M(r)$ is an integral from $r_H$ to $r$, it vanishes at the horizon, simplifying the result. Using (3.13), we obtain that the fluxes at the horizon, equaling the transport currents, are

$$\begin{aligned}j^x = \mathcal{J}^x(r_H) &= \frac{Z}{g_4^2}E_x - \frac{Z}{g_4^2}e^{2V}a'\delta h_{tx} - \frac{Z}{g_4^2}B\delta h_{ty}\Big|_{r_H}\,,\\ j^y = \mathcal{J}^y(r_H) &= \frac{Z}{g_4^2}E_y - e^{2V}Za'\delta h_{ty} + \frac{Z}{g_4^2}B\delta h_{tx} - 4W(E+\xi a)\Big|_{r_H}\,.\end{aligned} \tag{3.14}$$

As we see, it remains to solve for $\delta h_{ti}$ using the Einstein's equations, as a function of the external sources $E, B, \xi$ (linear response theory). Since the topological term with $W(\phi)$ doesn't contribute to Einstein's equations (it is independent of the metric), the linearized Einstein's equations are the same as in [1], namely

$$\begin{aligned}U(e^{4V}\delta h'_{tx})' - \left(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2V}k^2\Phi\right)\delta h_{tx} + \frac{2\kappa^2}{g_4^2}ZBUe^{2V}a'\delta h_{ty} &= -\frac{2\kappa^2}{g_4^2}Ze^{2V}a'\delta a'_x,\\ U(e^{4V}\delta h'_{ty})' - \left(\frac{2\kappa^2}{g_4^2}ZB^2 + e^{2V}k^2\Phi\right)\delta h_{ty} - \frac{2\kappa^2}{g_4^2}ZBUe^{2V}a'\delta h_{tx} &= -\frac{2\kappa^2}{g_4^2}Ze^{2V}a'\delta a'_y\\ &\quad + \frac{2\kappa^2}{g_4^2}ZB(-E+\xi a).\end{aligned} \tag{3.15}$$

Using the regularity conditions at the horizon (3.13), we obtain

$$\begin{aligned}\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)\delta h_{tx}-\frac{2\kappa^2}{g_4^2}ZBe^{2V}a'\delta h_{ty}&=-\frac{2\kappa^2}{g_4^2}Ze^{2V}a'E+e^{2V}U'\xi,\\ \left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)\delta h_{ty}-\frac{2\kappa^2}{g_4^2}ZBe^{2V}a'\delta h_{tx}&=\frac{2\kappa^2}{g_4^2}ZBE\,,\end{aligned}\tag{3.16}$$

and after some algebra we solve the $\delta h_{ti}$ graviton fluctuations in terms of the sources $E, B, \xi$ as

$$\begin{aligned}\delta h_{tx}=&\frac{\frac{2\kappa^2}{g_4^2}Ze^{4V}a'k^2\Phi}{\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)^2+\left(\frac{2\kappa^2}{g_4^2}Z\right)^2B^2e^{4V}a'^2}E\\ &+\frac{\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}a'k^2\Phi\right)e^{2V}U'\xi}{\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)^2+\left(\frac{2\kappa^2}{g_4^2}Z\right)^2B^2e^{4V}a'^2},\\ \delta h_{ty}=&\frac{\frac{2\kappa^2}{g_4^2}ZB}{\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)}E-\frac{2\kappa^2}{g_4^2}ZBe^{2V}a'\frac{e^{2V}U'\xi}{\left[\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)^2+\left(\frac{2\kappa^2}{g_4^2}Z\right)^2B^2e^{4V}a'^2\right]}\\ &-\frac{\frac{2\kappa^2}{g_4^2}ZBe^{2V}a'}{\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)\left[\left(\frac{2\kappa^2}{g_4^2}ZB^2+e^{2V}k^2\Phi\right)^2+\left(\frac{2\kappa^2}{g_4^2}Z\right)^2B^2e^{4V}a'^2\right]}\times\\ &\times e^{4V}\frac{2\kappa^2}{g_4^2}Za'k^2\Phi E.\end{aligned}\tag{3.17}$$

We can now replace the fluctuations (3.17) in the currents (3.14), use the fact that $U'(r_H)=4\pi T$ (meaning that $U'(r_H)\xi=4\pi T\xi$) and separate the terms according to the sources $E$ and $\xi$, via

$$j^i=\sigma_{ij}E^j-\alpha_{ij}(\nabla T)_j=\sigma_{ix}E-\alpha_{ix}T\xi.\tag{3.18}$$

From this, we can identify directly the conductivities and thermoelectric coefficients as

$$\begin{aligned}\sigma_{xx}&=\left.\frac{e^{2V}k^2\Phi(2\kappa_4^2g_4^4\rho^2+2\kappa_4^2B^2Z^2+g_4^2Ze^{2V}k^2\Phi)}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2V}k^2\Phi)^2}\right|_{r_H},\\ \sigma_{xy}&=\left.4\kappa_4^2B\rho\frac{\kappa_4^2g_4^4\rho^2+\kappa_4^2B^2Z^2+g_4^2Ze^{2V}k^2\Phi}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2V}k^2\Phi)^2}-4W\right|_{r_H},\end{aligned}\tag{3.19}$$

$$\begin{aligned}\alpha_{xx}&=\left.\frac{2\kappa_4^2g_4^4s\rho e^{2V}k^2\Phi}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2V}k^2\Phi)^2}\right|_{r_H},\\ \alpha_{xy}&=\left.2\kappa_4^2sB\frac{2\kappa_4^2g_4^4\rho^2+2\kappa_4^2B^2Z^2+g_4^2Ze^{2V}k^2\Phi}{4\kappa_4^4g_4^4B^2\rho^2+(2\kappa_4^2B^2Z+g_4^2e^{2V}k^2\Phi)^2}\right|_{r_H},\end{aligned}\tag{3.20}$$

which have been expressed in terms of the boundary charge density $\rho=-Ze^{2V}a'$ and the entropy density (Hawking formula) $s=4\pi e^{2V(r_H)}$.

We note that the only new contribution is from the $W$ term in the Hall conductivity $\sigma_{xy}$.



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### 3.2 Heat current and heat conductivity

We move on to the heat current, considered (also with a topological term) from the point of view of transport coefficients in the presence of magnetization in [6]. The heat current itself was defined in [3].

The (total) heat current is obtained from the energy-momentum tensor by subtracting the electric current,

$$Q^{(\mathrm{tot})i} = T^{(\mathrm{tot})i0} - \mu J^{(\mathrm{tot})i}. \tag{3.21}$$

But in [3, 6] it was noticed that the result one obtains from this equals, at $r \to \infty$, the flux

$$Q^{(\mathrm{tot})i} = \sqrt{-g}G^{ri} , \tag{3.22}$$

where the bulk 2-form $G^{\mu\nu}$ is defined through

$$G^{\mu\nu} = -2\nabla^{[\mu}k^{\nu]} - Zk^{[\mu}F^{\nu]\rho}A_\rho - \frac{1}{2}(\psi - 2\theta)H^{\mu\nu}, \tag{3.23}$$

where

$$H^{\mu\nu} \equiv Z(\phi)F^{\mu\nu} + 4g_4^2 W(\phi)\tilde{F}^{\mu\nu} , \tag{3.24}$$

and where $k^\mu$ is the vector $\partial_t$. However, more generally, an arbitrary vector satisfying $\nabla_\mu k^\mu = 0$ will also satisfy (as can be easily checked) the general property

$$\nabla_\mu(\nabla^{[\mu}k^{\nu]}) = \nabla_\mu(\nabla^{(\mu}k^{\nu)}) - R^\nu{}_\mu k^\mu, \tag{3.25}$$

which is the only one we need. The functions $\psi$ and $\theta$ are defined by the relations

$$\nabla_\rho\psi = (\mathcal{L}_k A)_\rho = k^\mu\partial_\mu A_\rho + A_\mu\partial_\rho k_\mu, \tag{3.26}$$

$$\nabla_\rho\theta = k^\mu F_{\mu\rho} - \frac{1}{2}\xi_\rho k^\mu A_\mu. \tag{3.27}$$

After some involved algebra, we obtain

$$\begin{aligned}\nabla_\mu G^{\mu\nu} &= Vk^\nu - 2\nabla_\mu(\nabla^{(\mu}k^{\nu)}) + \frac{1}{2}ZF^{\nu\mu}s_\mu - \frac{Z}{2}A_\rho(\mathcal{L}_k F)^{\nu\rho} \\ &\quad -2g_4^2(\partial_\mu W)\tilde{F}^{\mu\rho}A_\rho k^\nu - 2g_4^2 W\tilde{F}^{\mu\nu}\nabla_\mu(\psi - 2\theta) ,\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}(\mathcal{L}_k F)^{\nu\rho} &= k^\mu\nabla_\mu F^{\nu\rho} - \nabla_\mu k^\nu F^{\mu\rho} - \nabla_\mu k^\rho F^{\nu\mu} \\ s_\mu &\equiv k^\nu F_{\nu\mu} - \nabla_\mu\theta ,\end{aligned} \tag{3.29}$$

and where we define *on-shell* $V$ by

$$2R^\mu{}_\nu k^\nu = Vk^\mu. \tag{3.30}$$

We also calculate, in the $A_r = 0$ gauge and in the background (no fluctuations), and using the fact that $k^\mu = (\partial_t)^\mu$,

$$\begin{aligned}\int dx^\rho\nabla_\rho(\psi - 2\theta) &= \int dx^\rho k^\mu\partial_\mu A_\rho + \int dx^\rho A_\mu\nabla_\rho k^\mu - 2\int dx^\rho k^\mu F_{\mu\rho} + \int dx^\rho\xi_\rho i_k A \\ &= Ex + 2a.\end{aligned} \tag{3.31}$$

Further,

$$\begin{aligned} G^{ri} &= -\nabla^r k^i + \nabla^i k^r - Z(\Phi) k^{[r} F^{i]\sigma} A_\sigma - \frac{1}{2}(2a(r) + Ex) H^{ri} \\ &= -g^{r\alpha}\Gamma^i_{\alpha t} + g^{i\alpha}\Gamma^r_{\alpha t} - \frac{1}{2}(2a(r) + Ex) H^{ri} \,, \end{aligned} \tag{3.32}$$

so, after some calculations *in the presence of fluctuations*, we find that *at* $r \to \infty$, when $a(r)$ dominates over $Ex$, we have

$$-Q^{(\mathrm{tot})i} = -\sqrt{-g} G^{ri} = U^2 \left(\frac{e^{2V}\delta h_{ti}}{U}\right)' + a(r)\sqrt{-g}(Z(\phi)F^{ri} + 4g_4^2 W(\phi)\tilde{F}^{ri}). \tag{3.33}$$

Note that

$$\begin{aligned} F^{rx} &= \epsilon(a'\delta h_{tx} + Ue^{-2V}\delta A'_x + Ue^{-2V}B\delta h_{ry}) \\ F^{ry} &= \epsilon(a'\delta h_{ty} + Ue^{-2V}\delta A'_y - Ue^{-2V}B\delta h_{rx}) \\ \sqrt{-g}\tilde{F}^{rx} &= 0 \\ \sqrt{-g}\tilde{F}^{ry} &= -\epsilon(-E + \xi a). \end{aligned} \tag{3.34}$$

However, from (3.28), we find that

$$\sqrt{-g}\nabla_\mu G^{\mu i} = \partial_\mu(\sqrt{-g}G^{\mu i}) \neq 0 \,, \tag{3.35}$$

and it equals zero only in the absence of thermal fluctuations (which we are interested in). If it would be true, we would have that the linearized fluxes $\sqrt{-g}G^{ri}$ would be independent of $r$, and could be evaluated at the horizon.

As it is, we obtain from evaluating (3.28) the *modified* conservation laws,

$$\begin{aligned} \partial_r(\sqrt{-g}G^{rx}) &= -\partial_t(\sqrt{-g}G^{tx}) - \partial_y(\sqrt{-g}G^{yx}), \\ \partial_r(\sqrt{-g}G^{ry}) &= -\partial_t(\sqrt{-g}G^{ty}) - \partial_x(\sqrt{-g}G^{xy}) + \sqrt{-g}H^{xy}a(r). \end{aligned} \tag{3.36}$$

Moreover, we calculate

$$\begin{aligned} G^{tx} &= -g^{tt}\Gamma^x_{tt} + g^{xr}\Gamma^t_{rt} - \frac{1}{2}ZF^{xt}A_t - \frac{1}{2}ZF^{xy}A_y - \frac{1}{2}(2a+Ex)(ZF^{tx} + 4g_4^2 WF_{yr}) \\ G^{xy} &= -\frac{1}{2}(2a+Ex)(Ze^{-4V}B - 4g_4^2 e^{-2V}a') = -\frac{1}{2}(2a+Ex)H^{xy} \\ G^{ty} &= -\frac{U'}{U}\delta h_{ry} - \frac{1}{2}Z(e^{-4V}B\xi t + \cdots)a \\ &\quad -\frac{1}{2}Z\left[\left(-e^{-4V}B + \frac{\delta h_{ty}e^{-2V}}{U}(-E+\xi a)\right)(-E+\xi a)t + \ldots\right] \\ &\quad -\frac{1}{2}(2a+Ex)[Z(e^{-4V}B\xi t + \cdots) - 4g_4^2 W(\xi a' t + \delta A'_x)], \end{aligned} \tag{3.37}$$

where "$\cdots$" represents terms that do not depend on the $t$ coordinate, resulting in

$$\begin{aligned} \partial_t(\sqrt{-g}G^{tx}) &= 0 \\ \partial_x(\sqrt{-g}G^{xy}) &= \frac{E}{2}(Ze^{-2V}B - 4g_4^2 Wa') \\ \partial_t(\sqrt{-g}G^{ty}) &= -\frac{1}{2}Ze^{-2V}B\xi a + \frac{1}{2}Ze^{-2V}B(-E+\xi a) \\ &\quad -\frac{1}{2}(2a+Ex)(Ze^{-2V}B\xi - 4g_4^2 Wa'\xi). \end{aligned} \tag{3.38}$$

Note that we consider always the case when $a(x)$ dominates over $Ex$.



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Finally, one obtains

$$
\begin{aligned}
\partial_r(\sqrt{-g}G^{rx}) &= 0, \\
\partial_r(\sqrt{-g}G^{ry}) &= e^{2V}H^{xy}(E-2\xi a(r)) \\
&= -(e^{-2V}Z\phi)B - 4g_4^2W(\phi)a')(E-2\xi a(r)) \, ,
\end{aligned} \tag{3.39}
$$

where we have used $F^{xy} = Ze^{-4V}B, \sqrt{-g}\tilde{F}^{xy} = -a'$ and $\sqrt{-g} = e^{2V}$, which can be easily calculated. This in turn is consistent with a particular example of the more general formula presented in [6],

$$
\partial_r(\sqrt{-g}G^{ir}) = \partial_j(\sqrt{-g}G^{ji}) + 2\sqrt{-g}G^{ij}\xi_j + \sqrt{-g}H^{ij}E_j \, , \tag{3.40}
$$

upon specializing to $\xi_i = \xi\delta_{ix}, E_i = E\delta_{ix}$ and using $G^{yx} = -aH^{yx}$.

Since there is an extra term in the conservation law (3.39), like in the case of the electric current, we can add an extra term to the heat current, obtaining the fluxes (compare with (3.33))

$$
\begin{aligned}
\mathcal{Q}^x &= U^2\left(\frac{e^{2V}\delta h_{tx}}{U}\right)' - a(r)\sqrt{-g}H^{rx}, \\
\mathcal{Q}^y &= U^2\left(\frac{e^{2V}\delta h_{ty}}{U}\right)' - a(r)\sqrt{-g}H^{ry} - M(r)E - 2M_Q(r)\xi,
\end{aligned} \tag{3.41}
$$

where $M(r)$ and $M_Q(r)$ are given by (2.24) and (2.22), only integrated until $r$ instead of infinity. Note that their integrands match the right hand side of the non-conservation in (3.39), so by derivating with respect to $r$ we obtain the needed extra term to cancel the non-conservation, so that

$$
\partial_r\mathcal{Q}^i = 0 \, , \tag{3.42}
$$

as wanted. But by construction $M(r)$ and $M_Q(r)$ (which are integrated from the horizon to $r$) vanish at the horizon. Moreover, at the boundary $r\to\infty$, $M(r)\to M, M_Q(r)\to M_Q$, so the extra term are the magnetization currents, and subtracting them we obtain the pure transport currents,

$$
Q^i = Q^{(\mathrm{tot})i} - ME - 2M_Q\xi = \mathcal{Q}^i(r\to\infty) = \mathcal{Q}^i(r_H) \tag{3.43}
$$

so that at the horizon we calculate the transport currents.

At the horizon, not only $M(r_H) = M_Q(r_H) = 0$, but also $a(r_H) = 0$ and $U(r_H) = 0$ (but $U'(r_H) \neq 0$) by the boundary (regularity) condition there, which means that finally we obtain

$$
Q^i = -\left. U'e^{2V}\delta H_{ti}\right|_{r=r_H} . \tag{3.44}
$$

This is the same formula as in the case without topological term, in [1]. The graviton perturbations in the presence of $E, B, \xi$ sources was already calculated in (3.17), so substituting them in the above, and comparing with the general formula

$$
Q^i = T\alpha_{ij}E_j - \kappa_{ij}\nabla_j T \, , \tag{3.45}
$$

with $\nabla_i T = \xi\delta_{ix}T, E_i = E\delta_{ix}$ and $U'(r_H) = 4\pi T$, we thus extract the coefficients of $TE$ and $\xi T$ as

$$\begin{aligned}
\alpha_{xx} &= \frac{s\rho e^{2V}k^2\Phi}{B^2\rho^2 + (B^2 Z + e^{2V}k^2\Phi)^2}, \\
\alpha_{xy} &= sB\frac{(B^2Z^2 + \rho e^{2V}k^2\Phi + \rho^2)}{B^2\rho^2 + (B^2 Z + e^{2V}k^2\Phi)^2} \\
\kappa_{xx} &= \frac{s^2T(B^2Z + e^{2V}k^2\Phi)}{B^2\rho^2 + (B^2 Z + e^{2V}k^2\Phi)^2} \\
\kappa_{xy} &= \frac{s^2T\rho B}{B^2\rho^2 + (B^2 Z + e^{2V}k^2\Phi)^2}.
\end{aligned} \tag{3.46}$$

The thermoelectric coefficients agree with the results obtained from the electric current in (3.20), as they should, by general transport theory. We have no new contributions from the topological term with $W(\phi)$.

### 3.3 S-duality

The general conductivity formulas (3.19), (3.20), (3.46) contain explicitly a nonzero electric charge $\rho$, and magnetic field $B$, but no nonzero magnetic charge or electric field, so as they are, they do not exhibit manifest S-duality (Maxwell duality in a more general setting). However, we can consider $\rho = 0, B = 0$ in them, and obtain

$$\begin{aligned}
\sigma_{xx} &= Z(r_H) \\
\sigma_{xy} &= -4W(r_H) \\
\alpha_{xx} &= 0 = \alpha_{xy} = \frac{\kappa_{xy}}{T} \\
\frac{\kappa_{xx}}{T} &= \frac{s^2}{e^{2V(r_H)}k^2\Phi(r_H)}.
\end{aligned} \tag{3.47}$$

We see that the isotropic thermal conductivity $\kappa_{xx}$ is singular for $\Phi(r_H) \to 0$, but we keep it finite. In any case, the $\alpha^{ij}$ and $\kappa^{ij}$ coefficients are invariant under changes of the eletric/magnetic variables (S-duality). The other formulas are consistent with previous results at $\rho = B = 0$, where we know the effect of S-duality [9].

Indeed, we can explicitly check that our action (2.1) is invariant under the transformation

$$\begin{aligned}
F_{\mu\nu} &\to Z(\phi)\tilde{F}_{\mu\nu} - \bar{W}(\phi)F_{\mu\nu} \equiv Z(\phi)\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} - \frac{W(\phi)}{4} \\
Z(\phi) &\to -\frac{Z(\phi)}{Z(\phi)^2 + \bar{W}(\phi)^2} \\
\bar{W}(\phi) &\to \frac{\bar{W}(\phi)}{Z(\phi)^2 + \bar{W}(\phi)^2},
\end{aligned} \tag{3.48}$$

where we have defined $\bar{W}(\phi) \equiv W(\phi)/4$.

It was shown in [9] that this transformation comes from a simple duality transformation on the action (going to a master action and then writing a dual action in terms of a previously auxiliary field). Moreover, as we can see, since $\sigma_{xx} = Z(r_H)$ and $\sigma_{xy} = -\bar{W}(r_H)$, this transformation becomes

$$\begin{aligned}\sigma'_{xx} &= \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \\ \sigma'_{xy} &= -\frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}\,,\end{aligned} \tag{3.49}$$

or, by defining the usual complex conductivity $\sigma \equiv \sigma_{xy} + i\sigma_{xx}$, simply the usual S-duality formula acting on complex objects,

$$\sigma' = -\frac{1}{\sigma}. \tag{3.50}$$

This is indeed the effect of particle-vortex duality (standing in for S-duality in 2+1 dimensions) in the dual field theory, as seen for instance in [8, 12].

# 4 Transport via entropy function and S-duality

We next consider an alternative treatment of transport, relevant for *extremal* black holes (unlike the nonextremal case in the previous section) using the entropy function formalism, and generalize the work in [7, 13] to the case with a topological term.

The entropy function formalism was developed by Sen [14, 15], having in mind the application to the attractor mechanism [16, 17]. Within the context of transport, the first application was in [7], whose logic we will follow here.

## 4.1 Entropy function formalism

The entropy function formalism calculates the entropy and other quantities at the horizon of an *extremal* black hole by the extremization of a function called the entropy function. Since as we saw in the previous section often transport properties are determined at the horizon of a black hole in a gravity dual, this formalism will allow us to do the calculations easily.

The specific case we are interested in is the case of an extremal dyonic black hole in four dimensions, which is known to have a near-horizon geometry of the type $AdS_2 \times S^2$, or $AdS_2 \times \mathbb{R}^2$, in the case of a planar horizon. The near-horizon metric in this latter (planar) case is

$$ds^2 = v\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + wd\vec{x}^2, \tag{4.1}$$

where $v$ is the $AdS_2$ radius, $w$ is the $\mathbb{R}^2$ radius. The Ricci scalar for this metric is

$$R = -\frac{2}{v}. \tag{4.2}$$

The attractor mechanism [16, 17] means that the values for the fields at the horizon are independent on the values at infinity, depend only on the electric and magnetic charges of the black hole, and can be found from the extremization of the entropy function. For an application in the AdS/CFT correspondence, see [18]. The constant values taken by the scalar and vector fields at the horizon are denoted by

$$\phi_s = u_s, \;\; F^{(A)}_{rt} = e_A, \;\; F^{(A)}_{\theta\phi} = B_A, \tag{4.3}$$

where $e_A$ and $B_A$ are related to the electric and magnetic charges respectively.

We define the function $f(u_s, v, w, e_A, p_A)$ as the Lagrangian density $\sqrt{-\det g}\mathcal{L}$ evaluated for the near-horizon geometry (4.1) and integrated over the coordinates of the planar horizon [14, 15],

$$f(u_s, v_i, e_A, p_A) = \int dxdy \sqrt{-\det g}\mathcal{L}. \tag{4.4}$$

Then the entropy function is

$$\mathcal{E}(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p}) \equiv 2\pi[e_A Q^A - f(\vec{u},\vec{v},\vec{e},\vec{p})]. \tag{4.5}$$

Its equations of motion,

$$\frac{\partial\mathcal{E}}{\partial u_s} = 0, \;\; \frac{\partial\mathcal{E}}{\partial v} = 0, \;\; \frac{\partial\mathcal{E}}{\partial w} = 0, \;\; \frac{\partial\mathcal{E}}{\partial e_A} = 0 \,, \tag{4.6}$$

are called attractor equations, and fix the horizon data $(u_s, v, w, e_A)$ as a function of the electric and magnetic charges of the black hole, $Q_A, p_A$, thus defining the attractor solution.

At the extremum (for the true values of the horizon data at the horizon), the entropy function equals the entropy of the black hole,

$$S_{BH} = \mathcal{E}(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p}). \tag{4.7}$$

Note that in the case of the $\mathbb{R}^2$ horizon black hole, as $f$ is an integral over the horizon (which has infinite volume, or rather area), we must consider the entropy density instead.

### 4.2 Electrical and heat conductivities

We want to apply the entropy function formalism, for extremal black holes in an asymptotically AdS gravity dual, in order to calculate the transport coefficients, using the formulas (3.20), (3.19), (3.46).

However, as we mentioned, these results from last section were for nonextremal black holes. But we can consider the particular case of extremal black holes by taking the temperature to zero, $T \to 0$. Indeed, for an extremal black hole we have

$$U(r) \approx U(r_H) + (r - r_H)U'(r_H) + \frac{(r-r_H)^2}{2}U''(r_H) + \mathcal{O}(r^3) \,, \tag{4.8}$$

where $U'(r_H) = 4\pi T = 0$. Therefore the near-horizon metric is

$$ds^2 = -\frac{(r-r_H)^2}{2}U''(r_H)dt^2 + \frac{2}{(r-r_H)^2 U''(r_H)}dr^2 + e^{2V(r_H)}(dx^2 + dy^2) \,, \tag{4.9}$$

and by the coordinate redefinition

$$r - r_H = \tilde{\rho}, \;\; t = \frac{2}{U''(r_H)}\tau, \tag{4.10}$$

we obtain the $AdS_2 \times \mathbb{R}^2$ metric

$$ds^2 = \frac{2}{U''(r_H)}\left(-\tilde{\rho}^2 d\tau^2 + \frac{d\tilde{\rho}^2}{\tilde{\rho}^2}\right) + e^{2V(r_H)}(dx^2 + dy^2), \tag{4.11}$$

where therefore

$$v = \frac{2}{U''(r_H)}\,, \quad w = e^{2V(r_H)}. \tag{4.12}$$

We can then apply the formalism from the previous section with $T \to 0$, and then use the entropy function formalism from the previous subsection to calculate the horizon data as a function of the electric and magnetic charges.

Moreover, from the previous section, the ansatz for the field strength to leading order (in the absence of perturbations) was

$$F = a'(r)dr \wedge dt + Bdx \wedge dy. \tag{4.13}$$

Changing to the near-horizon coordinates, we obtain

$$F = \frac{2a'(r_H)}{U''(r_H)} d\tilde{\rho} \wedge d\tau + Bdx \wedge dy. \tag{4.14}$$

Comparing with the ansatz for the entropy function formalism at the horizon, (4.3), we also obtain

$$e = \frac{2a'(r_H)}{U''(r_H)} = va'(r_H). \tag{4.15}$$

In order to use the entropy function formalism, we consider $\Phi(\phi) = 0$ in (2.1), so that we don't have axions, obtaining

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_N}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) - \frac{Z(\phi)}{4g_4^2}F_{\mu\nu}F^{\mu\nu} - W(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{4.16}$$

Using (4.1), (4.2) and (4.3), we compute the Lagrangian in the near-horizon region,

$$\sqrt{-g}\mathcal{L} = \frac{1}{16\pi G_N}(-2w - wvV(u_D)) + \frac{Z(u_D)}{2g_4^2}\left(\frac{w}{v}e^2 - \frac{v}{w}B^2\right) + 4W(u_D)eB\,, \tag{4.17}$$

where $u_D$ is the value of the dilaton field on the horizon.

The entropy function (4.5) is then

$$\mathcal{E} = 2\pi[e_A Q^A - \mathrm{Vol}\mathbb{R}^2\sqrt{-g}\mathcal{L}]. \tag{4.18}$$

The attractor equations (equations of motion of the entropy function) for our system are then

$$\frac{Q}{\mathrm{Vol}\mathbb{R}^2} - \frac{Z(u_D)}{g_4^2}\frac{w}{v}e - 4W(u_D)B = 0, \tag{4.19}$$

$$\frac{Z(u_D)}{2g_4^2}\left(\frac{w}{v^2}e^2 + \frac{B^2}{w}\right) + \frac{w}{16\pi G_N}V(u_D) = 0, \tag{4.20}$$

$$\frac{2}{16\pi G_N} - \frac{Z(u_D)}{2g_4^2}\left(\frac{1}{v}e^2 + \frac{v}{w^2}B^2\right) + \frac{v}{16\pi G_N}V(u_D) = 0, \tag{4.21}$$

$$-\frac{1}{2g_4^2}\frac{\partial Z(u_D)}{\partial u_D}\left(\frac{w}{v}e^2 - \frac{v}{w}B^2\right) - 4\frac{\partial W(u_D)}{\partial u_D}eB + \frac{wv}{16\pi G_N}\frac{\partial V(u_D)}{\partial u_D} = 0. \tag{4.22}$$

Using (4.19) we eliminate $Q$ from (4.18), and obtain

$$\mathcal{E} = 2\pi \mathrm{Vol}\mathbb{R}^2 \left[ \frac{1}{(16\pi G_N)} \left(2w + wvV(u_D)\right) + \frac{Z(u_D)}{2g_4^2} \left( \frac{w}{v} e^2 + \frac{v}{w} B^2 \right) \right]. \tag{4.23}$$

We combine equations (4.20) and (4.21) and obtain

$$V(u_D) = -\frac{1}{v}, \tag{4.24}$$

$$\frac{Z(u_D)}{2g_4^2} \left( \frac{e^2}{v^2} + \frac{B^2}{w^2} \right) = \frac{1}{(16\pi G_N)} \frac{1}{v}\,, \tag{4.25}$$

and replacing this in (4.18), we obtain the entropy (value of the entropy function on the solution of the attractor equations)

$$\mathcal{E} = \frac{4\pi w \mathrm{Vol}\mathbb{R}^2}{16\pi G_N} = \frac{w \mathrm{Vol}\mathbb{R}^2}{4G_N} = \frac{A}{4G_N}. \tag{4.26}$$

This is the expected Hawking formula for the entropy of the black hole, which shows that the attractor mechanism for the entropy function does work in this case as well.

Moving on to the transport, the electric current is defined in the gravity dual as before, as

$$\langle J^\mu \rangle = \frac{\delta S_{\text{on-shell}}}{\delta \partial_r A_\mu}\bigg|_{\text{boundary}} = \sqrt{-g} \left( \frac{Z(\phi)}{g_4^2} F^{\mu\nu} + 4W(\phi) \tilde{F}^{\mu\nu} \right). \tag{4.27}$$

As we saw in the previous section, by subtracting a magnetization term that vanishes at the horizon, we obtain the pure transport current (not the total one), and the resulting flux is $r$-independent, so can be calculated at the horizon. That means that the charge density $J^0 \equiv \rho$ of the dual field theory can be calculated at the horizon, obtaining[1]

$$\rho = \frac{Z(u_D) w a'(r_H)}{g_4^2} + 4W(u_D)B. \tag{4.28}$$

Replacing (4.28) in the attractor equation (4.19), with the identification (4.15), we obtain that the charge density of the dual field theory $\rho$ equals the charge density of the gravity dual black hole in the entropy function formalism,

$$\rho = \tilde{Q} \equiv \frac{Q}{\mathrm{Vol}\mathbb{R}^2}. \tag{4.29}$$

Moreover, the entropy density of the dual field theory equals the entropy density of the black hole, which because of (4.26) becomes

$$s = \frac{4\pi w}{16\pi G_N}. \tag{4.30}$$

[1]Remember that $A_t = a(r)$ vanishes at the horizon due to the regularity conditions, but $a'(r)$ does not.



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Replacing these $\rho, s$, together with $T \to 0, \Phi(\phi) = 0$ in (3.20), (3.19), (3.46), gives the finite results

$$\begin{aligned}
\sigma_{xx} &= 0, \\
\sigma_{xy} &= \frac{\rho}{B} - 4W, \\
\alpha_{xx} &= 0, \\
\alpha_{xy} &= \frac{s}{B}, \\
\frac{\bar{\kappa}_{xx}}{T} &= \frac{s^2 Z}{g_4^2 \left(\rho^2 + \frac{B^2 Z^2}{g_4^4}\right)}, \\
\frac{\bar{\kappa}_{xy}}{T} &= \frac{\rho}{B} \frac{s^2}{\left(\rho^2 + \frac{B^2 Z^2}{g_4^4}\right)},
\end{aligned} \tag{4.31}$$

where we wrote $\bar{\kappa}_{ij}/T$, since this is usually the relevant finite quantity.

While in the above analysis we have considered the case of nonzero $B$, let us comment on the case of zero magnetic field. In order to obtain regular planar black holes with only electric charge (and finite chemical potential) for the Einstein-Maxwell-dilaton theory, the dilaton potential must be non-zero.[2] As it was pointed out on page 13 of ref. [7], the regularity of the solutions in the extremal limit is guaranteed if the attractor equations admit solutions, since they are also solutions to the equations of motion with $AdS_2 \times \mathbb{R}^2$ near-horizon geometry. So, Sen's formalism is applicable for the Einstein-Maxwell-dilaton theory with only electric charge if there is a non-zero dilaton potential, i.e., zero magnetic field and finite chemical potential (consider eqs. 3.7 and 3.10 in [7] relating $\mu \neq 0, B = 0$ with $\rho = \tilde{Q} \neq 0$ and $v$ finite, which by our eq. (4.24) means nonzero potential). Notice that the coupling $\gamma$ inside the dilaton potential (for instance $\gamma_m$ in (4.47) below) must be chosen in order to obtain regular solutions, since it might be possible to find limits when the solutions are non-regular. As it was also pointed out in the same reference, Sen's formalism is not applicable for general theories with Lifshitz symmetry since these don't have an $AdS_2 \times \mathbb{R}^2$, although it might also be possible to obtain finite chemical potential with no magnetic charge in this case.

### 4.3 Examples

Finally, since we have obtained the formulas for the transport coefficients as a function of $\rho/B, s/B$ and $W(u_D)$, it remains to solve the attractor equations in specific cases, so as to write explicit formulas for the transport coefficients as a function only of the charges and the magnetic field $B$.

#### 4.3.1 Constant potential and power law topological term

We consider first the case that the potential is just a constant negative cosmological constant (giving the AdS vacuum at infinity), while the topological term is a power law of the

[2]One can write the attractor equations for zero magnetic charge and non-zero electric charge and find the solutions only in the case when the dilaton potential is nonzero.



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kinetic function $Z(\phi)$,

$$V(\phi) = \frac{-6}{L^2}\,, \quad W(\phi) = \beta Z^n(\phi). \tag{4.32}$$

We manipulate the attractor equations so that we can write $s, \rho, W(u_D)$ in terms of the charges.

Equation (4.24) gives $v$, which now is a constant,

$$v = \frac{L^2}{6}. \tag{4.33}$$

Equation (4.19) gives

$$\tilde{Q} - \frac{Z}{g_4^2}\frac{w}{v}e - 4\beta Z^n B = 0\,, \tag{4.34}$$

which can be solved for $e$ as

$$e = \frac{g_4^2}{Z}\frac{v}{w}(\tilde{Q} - 4\beta Z^n B). \tag{4.35}$$

Using

$$\frac{\partial W}{\partial u_D} = \frac{\partial W}{\partial Z}\frac{\partial Z}{\partial u_D} = \beta n Z^{n-1}\frac{\partial Z}{\partial u_D} \tag{4.36}$$

in (4.22) and (4.21), we obtain

$$\frac{Z}{g_4^2}\left(\frac{e^2}{v^2} + \frac{B^2}{w^2}\right) - \frac{1}{16\pi G_N}\frac{2}{v} = 0, \tag{4.37}$$

$$\frac{Z}{g_4^2}\left(\frac{e^2}{v^2} - \frac{B^2}{w^2}\right) + \frac{8\beta n Z^n e B}{wv} = 0. \tag{4.38}$$

Substituting $e$ from (4.35) in the above equations, and eliminating $w$ from the two, as

$$w^2 = \frac{v}{\alpha}\left[\frac{Z}{g_4^2}B^2 - 4\beta n Z^n \tilde{Q} B \frac{g_4^2}{Z} + (4\beta Z^n B)^2 n \frac{g_4^2}{Z}\right]\,, \tag{4.39}$$

where $\alpha \equiv \frac{1}{16\pi G_N}$, we obtain the polynomial equation for $\tilde{Q}$,

$$\tilde{Q}^2 - \frac{Z^2}{g_4^4}B^2 - 8\beta\tilde{Q}B(1-n)Z^n + (4\beta B)^2(1-2n)Z^{2n} = 0. \tag{4.40}$$

- The $n = 0$ case.

In this case, solving (4.40) gives

$$\frac{Z}{g_4^2} = \pm\left(\frac{\tilde{Q}}{B} - 4\beta\right). \tag{4.41}$$

Substituting back into (4.39) and (4.35), we obtain

$$\begin{aligned} w &= \sqrt{\pm\frac{L^2(16\pi G_N)B}{6}[\tilde{Q} - 4\beta B]} \\ e &= \pm\sqrt{\pm\frac{L^2}{6(16\pi G_N)}\frac{B}{(\tilde{Q} - 4\beta B)}}. \end{aligned} \tag{4.42}$$

Finally now we can put everything back into (4.31) and obtain the nonzero transport coefficients as a function of the charges as

$$
\begin{aligned}
\sigma_{xy} &= \frac{\tilde{Q}}{B} - 4\beta, \\
\alpha_{xy} &= 4\pi\sqrt{\pm\frac{L^2}{6(16\pi G_N)}\left(\frac{\tilde{Q}}{B} - 4\beta\right)}, \\
\frac{\bar{\kappa}_{xx}}{T} &= (4\pi)^2\frac{L^2}{6(16\pi G_N)}\frac{(\tilde{Q}-4\beta B)^2}{\tilde{Q}^2+(\tilde{Q}-4\beta B)^2} \\
\frac{\bar{\kappa}_{xy}}{T} &= \pm(4\pi)^2\frac{L^2}{6(16\pi G_N)}\frac{\tilde{Q}(\tilde{Q}-4\beta B)}{\tilde{Q}^2+(\tilde{Q}-4\beta B)^2}.
\end{aligned} \tag{4.43}
$$

- The $n = 1$ case.

In this case, solving (4.40) gives

$$
\frac{Z}{g_4^2} = \pm\frac{\tilde{Q}}{B}\frac{1}{\sqrt{1+(4\beta g_4^2)^2}}. \tag{4.44}
$$

Substituting back into (4.39) and (4.35), we obtain

$$
\begin{aligned}
w^2 &= \frac{v}{\alpha}\tilde{Q}B4\beta g_4^2\left[\pm\sqrt{1+\frac{1}{(4\beta g_4^2)^2}}-1\right] \\
e &= \sqrt{v\alpha\tilde{Q}B(\pm\sqrt{1+(4\beta g_4^2)^2}-4\beta g_4^2)}.
\end{aligned} \tag{4.45}
$$

Putting everything back into (4.31), we obtain the nonzero transport coefficients as a function of the charges as

$$
\begin{aligned}
\sigma_{xy} &= \frac{\tilde{Q}}{B}\left(1 \mp \frac{4\beta g_4^2}{\sqrt{1+(4\beta g_4^2)^2}}\right), \\
\alpha_{xy} &= 4\pi\sqrt{\frac{L^2}{6(16\pi G_N)}\frac{\tilde{Q}}{B}\left(\pm\sqrt{(4\beta g_4^2)^2+1}-4\beta g_4^2\right)} \\
\frac{\kappa_{xx}}{T} &= (4\pi)^2\frac{L^2}{6(16\pi G_N)}\left(\pm\sqrt{1+(4\beta g_4^2)^2}-4\beta g_4^2\right)\frac{\sqrt{1+(4\beta g_4^2)^2}}{2+(4\beta g_4^2)^2} \\
\frac{\kappa_{xy}}{T} &= (4\pi)^2\frac{L^2}{6(16\pi G_N)}\left(\pm\sqrt{(4\beta g^2)^2+1}-4\beta g_4^2\right)\frac{1+(4\beta g^2)^2}{2+(4\beta g^2)^2}.
\end{aligned} \tag{4.46}
$$

### 4.3.2 Power law potential and power law topological term

Next we want to consider the more general case when the potential is polynomial, specifically

$$
V(u_D) = \sum_m \gamma_m Z^m. \tag{4.47}
$$

Now we still have

$$
v = -\frac{1}{V(u_D)}\,, \tag{4.48}
$$

because of (4.24), just that the right-hand side is not a constant anymore. Further, (4.19) is unchanged, so we can still solve for $e$ in the same way, obtaining again (4.35).

However, now from (4.22) and (4.21), we obtain

$$\frac{2\alpha}{v} - \frac{\tilde{Z}}{2}\left(\frac{e^2}{v^2} + \frac{B^2}{w^2}\right) + \alpha \sum \gamma_m Z^m = 0, \tag{4.49}$$

$$-\frac{1}{2{g^2}_4}\left(\frac{e^2}{v^2} - \frac{B^2}{w^2}\right)\frac{\partial Z}{\partial u_D} - 4\beta n Z^{n-1}\frac{eB}{wv}\frac{\partial Z}{\partial u_D} + \alpha \sum m\gamma_m Z^{m-1}\frac{\partial Z}{\partial u_D} = 0. \tag{4.50}$$

Now, if $\frac{\partial Z}{\partial u} \neq 0$, substituting $e$ from (4.35) in the above equations, and eliminating $w$ from the two, we obtain a new polynomial equation for $\tilde{Q}$,

$$(m-2n+1)Z^{2n} - 2(m-n+1)\left(\frac{\tilde{Q}}{4\beta B}\right)Z^n + (m-1)\frac{Z^2}{(4\beta g^2)^2} + (m+1)\left(\frac{\tilde{Q}}{4\beta B}\right)^2 = 0. \tag{4.51}$$

Moreover, (4.49) can be used to solve for $w$, if we substitute in it $e$ from (4.35) and $v$ from (4.48).

- The $n = 0$ case.

In this case, solving (4.51) leads to

$$\frac{Z}{g_4^2} = \pm\sqrt{-\frac{m+1}{m-1}}\left(\frac{\tilde{Q}}{B} - 4\beta\right). \tag{4.52}$$

- The $n = 1$ case.

In this case, (4.51) becomes

$$(m-1)\left[1 + \frac{1}{(4\beta g_4^2)^2}\right]Z^2 - 2m\frac{\tilde{Q}}{4\beta B}Z + (m+1)\frac{\tilde{Q}^2}{(4\beta B)^2} = 0. \tag{4.53}$$

For small perturbations, $4\beta g_4^2 \gg 1$, its solution behaves like

$$Z \sim \frac{\tilde{Q}}{4\beta B}, \tag{4.54}$$

but otherwise the full solution is unenlightening.

In principle we could proceed as before, and solve for $w$ and replace everything in the transport coefficients, but the calculations are difficult (we obtain higher order algebraic equations) and the solutions unenlightening.

### 4.4 S-duality

In this case, we have a different limit of the conductivity formulas with respect to the case at section 3, since now we have first $\Phi \to 0, T \to 0$, and then nonzero $\rho, B, s$ (the opposite of section 3). As mentioned there, we cannot check S-duality explicitly on this background, since we have $\rho \neq 0, B \neq 0$, but $\rho_m = 0 = E$. Moreover (and related) we have black holes with $Q \neq 0, B \neq 0$, but $P = 0, E = 0$. We can however take the limit (notice the order of limits though, we first took $\Phi \to 0$, and then took $\rho \to 0$, unlike in section 3) $\rho \to 0, s \to 0$ and obtain

$$\sigma_{xx} = 0\,, \quad \sigma_{xy} = -4W(r_H) = -\bar{W}(r_H)\,, \quad \alpha_{xx} = 0 = \alpha_{xy} = \kappa_{xy} = \kappa_{xx}. \tag{4.55}$$

Then we obtain a subset of the S-duality of section 3, namely

$$\bar{W} \to \frac{1}{\bar{W}} \Rightarrow \sigma_{xy} \to -\frac{1}{\sigma_{xy}}\,, \tag{4.56}$$

namely what we obtain by restricting to $\sigma_{xx} = 0$.

Notice however that we still have $Z(r_H) \neq 0$, and that is due to the order of limits we took (the limits are non-commutative).

## 5 Transport from Stokes equations and S-duality

Starting with [4], and developed in [2, 6], the transport coefficients $(\sigma, \alpha, \bar{\alpha}, \kappa)_{ij}$ for electric and thermal transport were also obtained from a formalism of perturbations of black hole solutions that leads to generalized Stokes equations. In the limit when hydrodynamics is valid, it was shown in [19] that the formalism turns into the fluid/gravity correspondence formalism [20].

Here we will apply the formulas of [6] to some one-dimensional lattices and take a relevant $T \to 0$ limit, with the goal of, in the next section, make some generalizations for that, and use the entropy function formalism for a supergravity-inspired model.

### 5.1 Stokes equations from black hole horizons

We consider the action (2.1) at $\Phi(\phi) = 0$, i.e., the Einstein-Maxwell-dilaton action (4.16), which has a topological term for the gauge field.

We consider electrically charged black holes solutions in 3+1 dimensions, with a metric and gauge field

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^i dx^j + 2g_{tr}dtdr + 2g_{ti}dtdx^i + 2g_{ri}drdx^i, \\ A &= A_t dt + A_r dr + A_i dx^i. \end{aligned} \tag{5.1}$$

At infinity, the solution should go to $AdS_4$ with sources, so

$$\begin{aligned} ds^2 &\to r^{-2}dr^2 + r^2[g_{tt}^{(\infty)}dt^2 + g_{ij}^{(\infty)}dx^i dx^j + 2g_{ti}^{(\infty)}dtdx^i], \\ A &\to A_t^{(\infty)}dt + A_i^{(\infty)}dx^i, \\ \phi &\to r^{\Delta-3}\phi^{(\infty)}\,, \end{aligned} \tag{5.2}$$

where $A_t^{(\infty)} = \mu(x)$ is the spatially-dependent chemical potential (source for particle number in the dual CFT), $g_{tt}^{(\infty)} = \tilde{G}(x)$ and $g_{ij}^{(\infty)} = \tilde{g}_{ij}(x)$ define the source for the energy-momentum tensor of the dual CFT, and $\phi^{(\infty)} = \tilde{\phi}(x)$ is a source for the dual scalar operator in the CFT.



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The solution should have a horizon at $r = r_H$, and near it, we expect the expansion

$$\begin{aligned}
g_{tt}(r,x) &= -U(r)(G^{(0)}(x)+\ldots)\\
g_{rr}(r,x) &= U^{-1}(r)(G^{(0)}(x)+\ldots)\\
g_{ti}(r,x) &= U(r)(g^{(0)}_{tr}(x)+\ldots)\\
g_{ti}(r,x) &= U(r)(G^{(0)}(x)\chi^{(0)}_i(x)+\ldots)\\
A_t(r,x) &= U(r)\left(\frac{G^{(0)}(x)}{4\pi T}A^{(0)}_t(x)+\ldots.\right)\\
g_{ij}(r,x) &= h^{(0)}_{ij}(x)+\ldots\\
g_{ir}(r,x) &= g^{(0)}_{ir}(x)+\ldots\\
A_i(r,x) &= A^{(0)}_i(x)+\ldots\\
A_r(r,x) &= A^{(0)}_r(x)+\ldots\\
\phi(r,x) &= \phi^{(0)}(x)+\ldots\ ,
\end{aligned}\tag{5.3}$$

where the dots refer to higher orders in $r - r_H$ and, as before, $U(r) = 4\pi T(r - r_H) + \ldots$, which means that the fields proportional to $U$ vanish at the horizon. The most relevant horizon data are then $T, h^{(0)}_{ij}, A^{(0)}_t, \chi^{(0)}_i$ and $\phi^{(0)}$.

The metric, gauge field and scalar perturbation that introduces sources for the electric and heat currents is

$$\begin{aligned}
\delta(ds^2) &= \delta g_{\mu\nu}dx^\mu dx^\nu + 2tg_{tt}\xi_i dt dx^i + t(g_{ti}\xi_j + g_{tj}\xi_i)dx^i dx^j + 2tg_{tr}\xi_i dr dx^i\\
\delta A &= \delta a_\mu dx^\mu - tE_i dx^i + tA_t\xi_i dx^i, \qquad \delta\phi\ ,
\end{aligned}\tag{5.4}$$

where as before we have $E_i(x)dx^i$ electric source and $\xi_i(x)dx^i$ thermal gradient, but are considered periodic, and closed as one-forms, $dE = 0 = d\xi$.

Regularity at the horizon $r_H$ gives the conditions

$$\begin{aligned}
\delta g_{tt} &= U(r)(\delta g^{(0)}_{tt}(x)+\mathcal{O}(r-r_H)), & \delta g_{rr} &= \frac{1}{U(r)}(\delta g^{(0)}_{rr}(x)+\mathcal{O}(r-r_H)),\\
\delta g_{ij} &= \delta g^{(0)}_{ij}(x)+\frac{2\ln(r-r_H)}{4\pi T}g_{t(i}\xi_{j)}+\mathcal{O}(r-r_H), & \delta g_{tr} &= \delta g^{(0)}_{tr}(x)+\mathcal{O}(r-r_H),\\
\delta g_{ti} &= \delta g^{(0)}_{ti}(x)+g_{tt}\xi_i\frac{\ln(r-r_H)}{4\pi T}+\mathcal{O}(r-r_H),\\
\delta g_{ri} &= \frac{1}{U}(\delta g^{(0)}_{ri}(x)+\frac{\ln(r-r_H)}{4\pi T}g_{tr}\xi_i+\mathcal{O}(r-r_H)),\\
\delta a_t &= \delta a^{(0)}_t(x)+\mathcal{O}(r-r_H)\ , & \delta a_r &= U^{-1}(\delta a^{(0)}_t(x)+\mathcal{O}(r-r_H)\\
\delta a_i &= \frac{\ln(r-r_H)}{4\pi T}(-E_i+A_t\xi_i)+\delta a^{(0)}_i(x)+\mathcal{O}(r-r_H)\ ,\\
\delta\phi &= \delta\phi^{(0)}(x)+\mathcal{O}(r-r_H).
\end{aligned}\tag{5.5}$$

As we already saw, we can define fluxes that are $r$-independent, by subtracting magnetization terms to the total currents, and then at the boundary these are just the transport currents, but they can also be calculated at the horizon, where the extra terms vanish:

$$\begin{aligned}\mathcal{J}^i &= J^{(\text{tot})i} - M^{ij}_{(b)}\xi_j\\ \mathcal{Q}^i &= Q^{(\text{tot})i} - M^{ij}_{(b)}E_j - 2M^{ij}_{Q(b)}\xi_j \ ,\end{aligned} \tag{5.6}$$

where $(b)$ means for the background (no fluctuations) and

$$M^{ij}(r) = \int_{r_H}^{r} dr\sqrt{-g}H^{ij} \ , \quad M^{ij}_{Q} = \int_{r_H}^{r} dr\sqrt{-g}G^{ij}. \tag{5.7}$$

The equality of the transport and horizon currents, via the radially independent fluxes, is written as

$$J^i = \mathcal{J}^i = J^i_{(0)} \ , \quad Q^i = \mathcal{Q}^i = Q^i_{(0)} \ , \tag{5.8}$$

where the (0) index signifies horizon value.

Then [6] obtains Stokes equations for a charged "fluid" (is a real fluid only in the hydrodynamics limit, as we said) for the variables $(v_i, p, w)$, standing in for velocity of the fluid, pressure, and (electric) scalar potential, respectively, defined as

$$\begin{aligned}v_i &\equiv -\delta g^{(0)}_{ti},\\ p &\equiv -\frac{4\pi T}{G^{(0)}}\left(\delta g^{(0)}_{rt} - h^{ij}_{(0)}g^{(0)}_{ir}\delta g^{(0)}_{tj}\right) - h^{ij}_{(0)}\frac{\partial_i G^{(0)}}{G^{(0)}}\delta g^{(0)}_{tj},\\ w &\equiv \delta a^{(0)}_t.\end{aligned} \tag{5.9}$$

Here $h^{ij}_{(0)}$ is the inverse metric for $h^{(0)}_{ij}$.

The resulting (generalized) Stokes equations are

$$\begin{aligned}&-2\nabla^j\nabla_{(i}v_{j)} + v^j[\nabla_j\phi^{(0)}\nabla_i\phi^{(0)} - 4\pi T d\chi^{(0)}_{ji}] - F^{(0)}_{ij}\frac{J^i_{(0)}}{\sqrt{h^{(0)}}}\\ &\qquad = \frac{\rho_H}{\sqrt{h^{(0)}}}(E_i + \nabla_i w) + 4\pi T\xi_i - \nabla_i p.\\ &\nabla_i v^i = 0, \qquad \partial_i J^i_{(0)} = 0,\end{aligned} \tag{5.10}$$

where the local charge density at the horizon (the horizon data for the zeroth component of the electric current) is

$$\rho_H \equiv J^t_{(0)} = \sqrt{h^{(0)}}\left(Z^{(0)}A^{(0)}_t - \frac{1}{2}W^{(0)}\epsilon^{ij}F^{(0)}_{ij}\right) \ , \tag{5.11}$$

we can define a magnetic field at the horizon by

$$B_H \equiv \sqrt{h^{(0)}}\frac{1}{2}\epsilon^{ij}F^{(0)}_{ij} \ , \tag{5.12}$$

$W^{(0)} = W(\phi^{(0)})$ is the horizon data for the coefficient of the topological term, and the electric and heat currents at the horizon are

$$\begin{aligned}J^i_{(0)} &= \rho_H v^i + \sqrt{h^{(0)}}\left(Z^{(0)}h^{ij}_{(0)} - W^{(0)}\epsilon^{ij}\right)\left(E_j + \nabla_j w + F^{(0)}_{jk}v^j\right)\\ Q^i_{(0)} &= 4\pi T\sqrt{h^{(0)}}v^i.\end{aligned} \tag{5.13}$$

For a particular case, one can next calculate these currents, and as before, identify the coefficients of $T\xi$ and $TE$ as the transport coefficients.



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### 5.2 Results for one-dimensional lattices

Here we mostly follow [6].

The relevant case we are interested in is of one-dimensional lattices, where the only nontrivial dependence is on a single coordinate $x$, and the fields are independent of the others. Then, in particular for the spatial metric in boundary directions at the horizon (horizon data) we consider

$$h_{ij}^{(0)}dx^i dx^j = g_{ij}^{(0)}dx^i dx^j = \gamma(x)dx^2 + \lambda(x)dy^2. \tag{5.14}$$

Then one of the Stokes equations, the incompressibilty condition $\nabla_i v^i = 0$ becomes (for a single nonvanishing component $v^x$, $0 = \nabla_x v^x = \frac{1}{\sqrt{-h}}\partial_x(\sqrt{-h}v^x)$, and denoting the constant by $v_0$, we solve it by

$$v^x = (\gamma g_{d-1})^{-1/2}v_0. \tag{5.15}$$

Moreover, we consider also

$$F_{xy}^{(0)} = B_H(x)\ , \quad 4\pi T\chi_y(x) = \chi(x)\ , \quad \chi_x = 0\ , \quad \phi^{(0)} = \phi^{(0)}(x)\ ; \quad A_t^{(0)} = A_t^{(0)}(x)\ , \tag{5.16}$$

and all the horizon data depending on $x$ are periodic with period $L$. We can define also the average over a period, $\int \equiv (1/L)\int_0^L dx$, and then the zero modes

$$B = \int B_H, \qquad \rho = \int \rho_H, \qquad s = \int s_H. \tag{5.17}$$

Note that the entropy density of the horizon is (by the Hawking formula)

$$s_H = 4\pi\sqrt{\gamma\lambda}. \tag{5.18}$$

Moreover, separate the zero modes of $B_H$ and $\rho_H$, and write the remainder as $\partial_x$ of something, defining

$$B_H = B + \partial_x \hat{A}_y\ , \quad \rho_H = \rho + \partial_x C. \tag{5.19}$$

We also define $x$-dependent averages $\int^x$ as the average with $L$ replaced by $x$ in the upper limit of integration. Then consider

$$w_1(x) = \rho\left(\frac{1}{B}\int^x B_H - \frac{1}{\rho}\int^x \rho_H\right), \quad w_2(x) = Ts\left(\frac{1}{B}\int^x B_H - \frac{1}{s}\int^x s_H\right)\ , \tag{5.20}$$

and then construct the periodic functions

$$u_i = \int^x \frac{\gamma^{1/2}\Sigma_i}{\lambda^{3/2}} - \frac{\int \frac{\gamma^{1/2}\Sigma_i}{\lambda^{3/2}}}{\int \frac{\gamma^{1/2}}{\lambda^{3/2}}}\int^x \frac{\gamma^{1/2}}{\lambda^{3/2}}\ , \tag{5.21}$$

where $\Sigma_i$ stands for the set of periodic functions $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5) = (\chi, w_1, w_2, \hat{A}_y, C)$.

Finally, define the matrix with constant components

$$\mathcal{U}_{ij} = \int \frac{\lambda^{3/2}}{\gamma^{1/2}}\partial_x u_i \partial_x u_j. \tag{5.22}$$

For the transport coefficients, it turns out that one needs to define also the constant

$$X = \int \frac{(\partial_x \lambda)^s}{\lambda^{5/2}\gamma^{1/2}} + \int \frac{(\partial_x \phi^{(0)})^2}{(\gamma\lambda)^{1/2}} + \int \frac{(\rho_H + B_H W^{(0)})^2}{\lambda Z^{(0)}(\lambda\gamma)^{1/2}} + \int \frac{B_H^2 Z^{(0)}}{\lambda(\lambda\gamma)^{1/2}} + \mathcal{U}_{11}. \tag{5.23}$$

Then one solves the Stokes equations for the velocities $v^i$ and currents $J^i_{(0)}$ as a function of the souces $E_i, \xi_i$, and extracts the transport coefficients.

### 5.2.1 Constant $B_H$, $\gamma(x) = \lambda(x)$ and $T \to 0$ limit

The case that we will mostly be interested in is of $B_H(x) = B$ =constant and $\lambda(x) = \gamma(x)$. The last condition can be thought of as using residual diffeomorphism invariance to fix $\lambda = \gamma$.

Then we obtain first

$$u_i = \int^x \frac{\Sigma_i}{\lambda} - \frac{\int \frac{\Sigma_i}{\lambda}}{\int \frac{1}{\lambda}} \int^x \frac{1}{\lambda}, \tag{5.24}$$

and then

$$\mathcal{U}_{ij} = \int \partial_x u_i \Sigma_j = \int \frac{\Sigma_i \Sigma_j}{\lambda} - \frac{\int \frac{\Sigma_i}{\lambda} \int \frac{\Sigma_j}{\lambda}}{\int \frac{1}{\lambda}}. \tag{5.25}$$

Next, we have $s_H = 4\pi\lambda$, and then

$$\begin{aligned} w_1(x) &= \rho x - \int^x \rho_H, \\ w_2(x) &= 4\pi T \left( x \int \lambda - \int^x \lambda \right) \\ X &= \int \frac{(\partial_x \lambda)^2}{\lambda^3} + \int \frac{(\partial_x \phi^{(0)})^2}{\lambda} + \int Z^{(0)} A_t^{(0)2} + \int \frac{Z^{(0)} B^2}{\lambda^2} + \int \frac{\chi^2}{\lambda} - \frac{(\int \frac{\chi}{\lambda})^2}{\int \frac{1}{\lambda}}. \end{aligned} \tag{5.26}$$

With the above formulas, putting $\gamma = \lambda$ and $B_H(x) = B$ in the more general formulas obtained in [6], we find for the electric conductivities

$$\begin{aligned} \sigma^{xx} &= 0 \\ \sigma^{yy} &= \mathcal{U}_{22} + \int Z^{(0)} + \int \frac{(\frac{\rho}{B} + W^{(0)})^2}{Z^{(0)}} \\ &\quad - \frac{1}{X} \left( \mathcal{U}_{12} - \int \left( \frac{\rho}{B} + W^{(0)} \right) A_t^{(0)} - \int \frac{B Z^{(0)}}{\lambda} \right)^2. \\ \sigma^{xy} &= -\sigma^{yx} = \frac{\rho}{B}\,, \end{aligned} \tag{5.27}$$

for the thermoelectric conductivities

$$
\begin{aligned}
\alpha^{xx} &= \bar{\alpha}^{xx} = 0, \\
\alpha^{yy} &= \bar{\alpha}^{yy} = \frac{\mathcal{U}_{23}}{T} + \frac{s}{B}\int \frac{(\frac{\rho}{B} + W^{(0)})}{Z^{(0)}} \\
&\quad - \frac{1}{X}\left(\mathcal{U}_{12} - \int \left(\frac{\rho}{B} + W^{(0)}\right) A_t^{(0)} - \int \frac{BZ^{(0)}}{\lambda}\right)\left(\frac{\mathcal{U}_{13}}{T} - \frac{s}{B}\int A_t^{(0)}\right), \\
\alpha^{xy} &= \bar{\alpha}^{yx} = \frac{s}{B}, \\
\alpha^{yx} &= \bar{\alpha}^{xy} = \frac{4\pi}{X}\left(\mathcal{U}_{12} - \int \left(\frac{\rho}{B} + W^{(0)}\right) A_t^{(0)} - \int \frac{BZ^{(0)}}{\lambda}\right),
\end{aligned} \tag{5.28}
$$

and for the thermal conductivities

$$
\begin{aligned}
\kappa^{xx} &= \frac{16\pi^2 T}{X}, \\
\kappa^{yy} &= \frac{\mathcal{U}_{33}}{T} + \frac{s^2 T}{B^2}\int \frac{1}{Z^{(0)}} - \frac{T}{X}\left(\frac{\mathcal{U}_{13}}{T} - \frac{s}{B}\int A_t^{(0)}\right)^2, \\
\kappa^{xy} &= \bar{\kappa}^{yx} = -\frac{4\pi T}{X}\left(\frac{\mathcal{U}_{13}}{T} - \frac{s}{B}\int A_t^{(0)}\right).
\end{aligned} \tag{5.29}
$$

Note that in our case we have

$$
\frac{\rho}{B} + W^{(0)} = \frac{\lambda Z^{(0)}}{B} A_t^{(0)}. \tag{5.30}
$$

Finally, for application to the extremal case (which will be done in the next section), we want to take the limit $T \to 0$, and also (see previous sections), we need to consider $\chi = 0$, which means that $\mathcal{U}_{1i} = 0$. Also note that, because of (5.26), $w_2/T$ remains finite as $T \to 0$, so then so does $\mathcal{U}_{23}/T$ and $\mathcal{U}_{33}/T^2$.

We obtain for the nonzero electric conductivities

$$
\begin{aligned}
\sigma^{yy} &= \mathcal{U}_{22} + \int Z^{(0)} + \int \lambda^2 Z^{(0)} A_t^{(0)2} - \frac{1}{X}\left[\int \frac{\lambda Z^{(0)}}{B}\left(A_t^{(0)2} + \frac{B^2}{\lambda^2}\right)\right]^2 \\
\sigma^{xy} &= \frac{\int \lambda Z^{(0)} A_t^{(0)} - \int W^{(0)} B_H}{B} = \frac{\rho}{B},
\end{aligned} \tag{5.31}
$$

for the nonzero thermoelectric conductivities

$$
\begin{aligned}
\alpha^{yx} &= -\frac{4\pi}{X}\int \frac{\lambda Z^{(0)}}{B}(A_t^{(0)2} + \frac{B^2}{\lambda^2}) \\
\alpha^{xy} &= \frac{s}{B} \\
\alpha^{yy} &= \frac{\mathcal{U}_{23}}{T} + \frac{s}{B^2}\left(\int \lambda A_t^{(0)} - \frac{1}{X}\int A_t^{(0)} \int \lambda Z^{(0)}\left(A_t^{(0)2} + \frac{B^2}{\lambda^2}\right)\right),
\end{aligned} \tag{5.32}
$$

and for the nonzero and finite thermal conductivities $\frac{\kappa^{ij}}{T}$,

$$\begin{aligned}
\frac{\kappa^{yy}}{T} &= \frac{\mathcal{U}_{33}}{T^2} + \frac{s^2}{B^2}\int \frac{1}{Z^{(0)}} - \frac{1}{X}\frac{s^2}{B^2}\left(\int A_t^{(0)}\right)^2 \\
\frac{\kappa^{xy}}{T} &= \frac{4\pi}{X}\frac{s}{B}\int A_t^{(0)} \\
\frac{\kappa^{xx}}{T} &= \frac{16\pi^2}{X}.
\end{aligned} \tag{5.33}$$

Here $X$ is (for $\lambda = e^{-w}$)

$$X = \int [e^{-w(x)}((\partial_x w)^2 + (\partial_x \phi)^2) + Z^{(0)}(A_t^{(0)2} + e^{-2w(x)}B^2)]. \tag{5.34}$$

Also, the finite thermal conductivity at zero electric current (obtained by putting $J^i = 0$, and thus relating the electric field with the thermal gradient, and substituting it in the heat current) $\kappa^{ij}_{J^i=0} = \kappa^{ij} - T\alpha^{il}(\sigma^{-1})_{lm}\alpha^{mj}$, is

$$\begin{aligned}
\frac{\kappa^{xx}_{J=0}}{T} &= \frac{1}{T}\left(\kappa^{xx} - T\alpha^{xy}(\sigma^{-1})_{yx}\alpha^{xy}\right) \\
&= \frac{(4\pi)^2}{X}\left[1 - \left(\int \lambda\right)^2 \frac{X}{\rho B}\right] \\
\frac{\kappa^{xy}_{J=0}}{T} &= \frac{1}{T}\left(\kappa^{xy} - T\alpha^{xy}(\sigma^{-1})_{yx}\alpha^{xy}\right) \\
&= \frac{(4\pi)^2}{X}\left[\rho\int A_t^{(0)} - X\int \lambda\right]\frac{\int \lambda}{B\rho}.
\end{aligned} \tag{5.35}$$

### 5.3 S-duality

The generalized Stokes equations are invariant under an S-duality transformation of the horizon data [2, 6]. Indeed, consider the transformation

$$\begin{aligned}
B_H &\to \rho_H & \rho_H &\to -B_H \\
Z^{(0)} &\to \frac{Z^{(0)}}{Z^2_{(0)} + W^2_{(0)}}, & W^{(0)} &\to -\frac{W^{(0)}}{Z^2_{(0)} + W^2_{(0)}} \\
(E_i + \nabla_i w) &\to -\frac{1}{\sqrt{h^{(0)}}}\epsilon_{ij}J^j_{(0)}, & J^i_{(0)} &\to -\sqrt{h^{(0)}}\epsilon^{ij}(E_j + \nabla_j w).
\end{aligned} \tag{5.36}$$

Then, it is easy to check that the Stokes equations (5.10) are left invariant. The transformation on $(Z^{(0)}, W^{(0)})$ is understood as a transformation that must be performed on the right-hand side of the definition of $J^i_{(0)}$ in (5.13), together with the transformation of the other horizon data, namely $(B_H, \rho_H, (E_i + \nabla_i w))$, and then by again replacing $J^i_{(0)}$ from (5.13) in the result, to finally obtain the transformation of $J^i_{(0)}$.

Defining the horizon data and its inverse S-dual,

$$D_H = (\rho_H, B_H, Z^{(0)}, W^{(0)}) \to D'_H = \left(B_H, -\rho_H, \frac{Z^{(0)}}{Z^2_{(0)} + W^2_{(0)}}, -\frac{W^{(0)}}{Z^2_{(0)} + W^2_{(0)}}\right), \tag{5.37}$$

then the action on the electric and thermal conductivities is (here we define $\epsilon^{xy} = +1$)

$$\begin{aligned}
\sigma^{ij}(D'_H) &= -\epsilon^{ik}\sigma^{-1}_{kl}\epsilon^{lj} \\
\alpha^{ij}(D'_H) &= -\epsilon^{ik}\sigma^{-1}_{kl}(D_H)\alpha^{lj}(D_H) \\
\bar{\alpha}^{ij}(D'_H) &= -\bar{\alpha}^{ik}(D_H)\sigma^{-1}_{kl}(D_H)\epsilon^{lj} \\
\kappa^{ij}(D'_H) &= \kappa^{ij}_{J=0}(D_H) \;,
\end{aligned} \tag{5.38}$$

where as usual the heat conductivity at zero electrical current is $\kappa^{ij}_{J=0} = \kappa^{ij} - T\bar{\alpha}^{ik}\sigma^{-1}_{kl}\alpha^{lj}$.

But if $D_H$ is a solution for horizon data, $D'_H$ is not necessarily also a solution. Only if the bulk theory is S-duality invariant, specifically under

$$\begin{aligned}
\phi &\to -\phi \\
Z(\phi) &\to \frac{Z(\phi)}{Z^2(\phi)+W^2(\phi)} \\
W(\phi) &\to -\frac{W(\phi)}{Z^2(\phi)+W^2(\phi)} \\
F_{\mu\nu} &\to Z(\phi)\tilde{F}_{\mu\nu} - W(\phi)F_{\mu\nu} \;,
\end{aligned} \tag{5.39}$$

which we can check that reduces on the horizon data to (5.36), is $D'_H$ also a solution, and then the transformation (5.38) of the transport coefficients is indeed a symemtry of the dual field theory.

Our action (2.1) certainly falls within that category, since as we saw in section 3, the S-duality (5.39) is an invariance of the action. This matches with the analysis of S-duality in section 3. We will consider more such bulk theories, inspired from ones arising from supergravity, in the next section.

# 6 Supergravity-inspired model and generalizations of transport relations for entropy function formalism

We now consider, as an example, a supergravity-inspired model that contains several scalar fields and a potential for them that is polynomial in the field.

Consider the action for $U(1)^4$ gauge fields $A^I_\mu$ coupled to scalars $X_I$ and gravity,

$$\begin{aligned}
S = \int d^4x\sqrt{-g}\Bigg[&\frac{1}{16\pi G_N}\left(R - \frac{1}{32}\left(3\sum_{I=1}^{4}(\partial_\mu\lambda_I)^2 - 2\sum_{I<J}\partial_\mu\lambda_I\partial^\mu\lambda_J\right) - V(X)\right) \\
&-\frac{1}{4g_4^2}\sum_{I=1}^{4}Z_I(X)(F^I_{\mu\nu})^2 - \sum_{I=1}^{4}W_I(X)F^I_{\mu\nu}\tilde{F}^{\mu\nu I}\Bigg] \;,
\end{aligned} \tag{6.1}$$

where $I = 1,2,3,4$ labels the scalars $X_I$, subject to the constraint

$$X_1X_2X_3X_4 = 1 \;, \tag{6.2}$$

the $\lambda_I$ are redefinitions of $X_I$ via

$$\frac{X_I}{\sqrt{8}} = e^{-\frac{\lambda_I}{2}} \;, \tag{6.3}$$



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the field strengths of the abelian vectors are as usual $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$, and the potential for the scalar fields is

$$V(X) = -\frac{g_4^2}{4}\sum_{I<J}\frac{1}{X_I X_J}. \tag{6.4}$$

This is a generalization of the $U(1)^4$ gauged supergravity model, obtained by dimensional reduction of 11 dimensional supergravity on $S^7$ and truncation to the Cartan sector in [21], and which has been considered in the entropy function formalism in [22]. To restrict to that model, we put $W_I = 0$ and $Z_I = X_I^2$. The generalization considered here is consistent with the rest of the paper, having arbitrary $Z(\phi), W(\phi)$, only now generalized to a sum over $I = 1, 2, 3, 4$. To completely generalize, we would consider an arbitrary potential $V(X)$, but instead we want to keep the features of the supergravity truncation. For the same reason, we also keep the constraint $X_1X_2X_3X_4 = 1$. Note that taking $g \to 0$ leads to the vanishing of the scalar potential, so that is another situation that can be analyzed.

### 6.1 Entropy function formalism and solution in terms of charges

We follow the same method for the entropy function with the attractor mechanism considered in section 4. The near-horizon geometry of an extremal planar black hole solution of this model will again be $AdS_2 \times \mathbb{R}^2$, using the same general ansatz (2.7) for the solution as in the rest of the paper. Note that because we consider the planar horizon case (with $\mathbb{R}^2$ factor) instead of the spherical horizon case (with $S^2$ factor) as in [22], the entropy function and attractor equations will differ from that paper, not only by the topological term, but also by the absence of the $2/v_2$ term coming from the Ricci scalar of the horizon factor. In this section we will use the notation of [22] and denote $v$ by $v_1$, and $w$ by $v_2$, also since we reserve $w$ for use in one-dimensional lattices. The horizon data for the abelian vector fields and the scalars is written as

$$X_I = u_I, \qquad F^I_{rt} = e^I, \qquad F^I_{xy} = p^I, \tag{6.5}$$

and similarly as before, this leads to the entropy function

$$\mathcal{E} = 2\pi\left\{\sum_{I=1}^4 e_I q^I - v_1 v_2\left[\frac{1}{16\pi G_N}\left(-\frac{2}{v_1} - V(X)\right) + \sum_{I=1}^4 \frac{Z_I}{2g^2}\left(\frac{e_I^2}{v_1^2} - \frac{p_I^2}{v_2^2}\right) + 4\sum_{I=1}^4\frac{W_I e_I p^I}{v_1 v_2}\right]\right\}. \tag{6.6}$$

The attractor equations derived from it are

$$\begin{aligned}
\frac{\partial \mathcal{E}_B}{\partial e_I} &= 2\pi\left[q^I - v_1 v_2\left(\sum_I \frac{Z_I}{g_4^2}\frac{e_I^2}{v_1^2}\right) - 4\sum_I W_I p^I\right] = 0 \qquad (6.7)\\
-\frac{\partial \mathcal{E}_B}{\partial v_1} &= 2\pi\left[\left(+\frac{1}{2g_4^2}\sum_I Z_I\left(-\frac{v_2}{v_1^2}e_I^2 - \frac{p_I^2}{v_2}\right) - \frac{v_2 V}{16\pi G_N}\right)\right] = 0\\
-\frac{\partial \mathcal{E}_B}{\partial v_2} &= 2\pi\left[\frac{-2}{16\pi G_N} + \frac{1}{2g_4^2}\sum_I Z_I\left(\frac{e_I^2}{v_1} + \frac{v_1 p_I^2}{v_2^2}\right) - \frac{v_1 V}{16\pi G_N}\right] = 0\\
-\frac{\partial \mathcal{E}_B}{\partial u_I} &= 2\pi\left[v_1 v_2\left(\frac{1}{2g_4^2}\sum_J \frac{\partial Z_J}{\partial u_I}\left(\frac{e_J^2}{v_1^2} - \frac{p_J^2}{v_2^2}\right) - \frac{1}{16\pi G_N}\frac{\partial V}{\partial u_I}\right) + 4\sum_J\frac{\partial W_J}{\partial u_I}e_J p^J\right] = 0.
\end{aligned}$$

The first equation in (6.7) can be solved for $e_I$ in terms of the charges and other parameters, as

$$e_I = g^2 \frac{v_1}{v_2} \frac{1}{Z_I}(q_I - 4W_I p^I). \tag{6.8}$$

Substituting this in the second and third equation in (6.7), and adding and subtracting the result, we obtain

$$-\frac{1}{16\pi G_N}\left(\frac{2}{v_1}\right) + \sum_I \frac{Z_I}{g_4^2}\frac{p^{I2}}{v_2^2} + g_4^2 \frac{1}{v_2^2}\sum_I \frac{(q_I - 4W_I p^I)^2}{Z_I} = 0, \tag{6.9}$$

$$\left(2V + \frac{2}{v_1}\right)\frac{1}{16\pi G_N} = 0 \Rightarrow V(u_I) = -\frac{1}{v_1}. \tag{6.10}$$

These give the possibility to write 2 of the 3 horizon data, $v_1, v_2, V(u_I)$, as a function of the third, and the charges $(q_I, p^I)$, and $W_I(u)$.

Finally, one should be able to solve the last of the equations in (6.7), for polynomial $Z_I = \sum_m c_m u_I^m$ and $W_I = \sum_n d_n u_I^n$, to obtain $u_I$ as a function of the same data, reducing to dependence on the charges. However, before that, we would have to remember that we have the constraint $X_1 X_2 X_3 X_4 = 1$, which means that

$$u_1 u_2 u_3 u_4 = 1\ , \tag{6.11}$$

and the potential depends only on 3 of them (the independent ones), while the fourth is found from the above constraint. For instance, if $u_4$ is taken to be dependent, and solved for, we have

$$V(u_1, u_2, u_3) = -\frac{g_4^2}{4}\left[u_1 u_2 + u_2 u_3 + u_3 u_1 + \frac{1}{u_1 u_2} + \frac{1}{u_2 u_3} + \frac{1}{u_3 u_1}\right]. \tag{6.12}$$

Alternatively, we could consider the same theory *without* the constraint, so $V(u_1, u_2, u_3, u_4)$. In that case, we would have

$$\sum_I u_I \frac{\partial V(u_D)}{\partial u_I} = -2V(u_D)\ , \tag{6.13}$$

and, as an example, substituting in (6.7) a pure power law case, $Z_I = u_I^m, W_I = W_0 u_I^n$, after some manipulations we would obtain

$$(4p^I)^2(m - 2n - 2)u_I^{2n} - 8q_I p^I(m - n - 2)u_I^n - (m+2)\frac{p_I^2}{g_4^4}u_I^{2m} \\ +(m-2)q_I^2 = 0. \tag{6.14}$$

This would allow us to solve for $u_I$, in terms of the charges and either $v_2$, or $V(u_D)$ (obtainable from the previous equations, relating $V(u_D), v_1, v_2$). For example, for $m = 2, n = 2$, we would obtain

$$-u_I^4\left(1 + \frac{1}{(4g_4^2 W_0)^2}\right) + \frac{q_I}{4p_I W_0}u_I^2 = 0 \Rightarrow u_I^2 = \frac{q_I}{p_I}\frac{4W_0 g_4^4}{1 + (4W_0 g_4^2)^2}. \tag{6.15}$$

This can be then substituted into $V(u_D)$, resulting in

$$V = -\frac{g_4^2}{4}\sum_{I<J}\frac{1}{u_I u_J} = -\frac{g_4^2}{4}\frac{1+(4W_0 g_4^2)^2}{4W_0 g_4^4}\sum_{I<J}\sqrt{\frac{p_I p_J}{\tilde{q}_I\tilde{q}_J}}\,, \tag{6.16}$$

and then from the attractor equations (6.10), the second fixes $v_1$,

$$v_1 = -\frac{1}{V}\,, \tag{6.17}$$

and replacing in the first we fix $v_2$,

$$v_2^2 = \frac{2}{\alpha g_4^2}\frac{1}{1+(4W_0 g_4^2)^2}\frac{\sum_I^4 \tilde{q}_I p_I}{\sum_{I<J}\sqrt{\frac{p_I p_J}{\tilde{q}_I\tilde{q}_J}}}\,, \tag{6.18}$$

finally fixing all horizon data in terms of the charges. Then the entropy density at the horizon (minimum of the entropy function) would be (Hawking fromula)

$$s = \frac{4\pi v_2}{16\pi G_N} \equiv 4\pi\alpha v_2 = \frac{4\pi}{g_4}\sqrt{\frac{2\alpha}{1+(4W_0 g_4^2)^2}}\sqrt{\frac{\sum_I^4 \tilde{q}_I p_I}{\sum_{I<J}\sqrt{\frac{p_I p_J}{\tilde{q}_I\tilde{q}_J}}}}. \tag{6.19}$$

### 6.2 Transport formulas for this generalization

To use the transport formulas from the previous section, we need to generalize them to this case. But since the only such generalization is the fact that we have several scalars $X_I$ and gauge fields $A_\mu^I$, the only thing we need to be careful about is where to put the sums over $I$.

The horizon data is

$$\begin{aligned}
\rho_{H,I} &\equiv J^t_{(0)I} = \sqrt{h^{(0)}}\left(Z_I^{(0)}A_t^{I(0)} - \frac{1}{2}W_I^{(0)}\epsilon^{ij}F_{ij}^{I(0)}\right)\\
B_{H,I} &\equiv \sqrt{h^{(0)}}\frac{1}{2}\epsilon^{ij}F_{ij}^{I(0)}\\
J_I^{i(0)} &= \rho_{H,I}v^i + \sqrt{h^{(0)}}\left(Z_I^{(0)}h_{(0)}^{ij} - W_I^{(0)}\epsilon^{ij}\right)\left(E_j^I + \nabla_j w^I + F_{jk}^{I(0)}v^j\right)\,,
\end{aligned} \tag{6.20}$$

and we can define the sums over $I$ (total value)

$$\rho_H = \sum_I \rho_{H,I}\,, \quad B_H = \sum_I B_{H,I}\,, \quad J^{i(0)} = \sum_I J_I^{i(0)}. \tag{6.21}$$

and, in the case of one-dimensional lattices that we will be interested in, the averages

$$B_I = \int B_{H,I}\,, \quad B = \int B_H\,, \quad \rho_I = \int \rho_{H,I}\,, \quad \rho = \int \rho_H. \tag{6.22}$$

Then we have a multiply-charged (pseudo-)fluid with variables $(v_i, p, w_I)$, standing in for velocity and pressure of the fluid and electric scalar potentials defined by

$$\begin{aligned}
v_i &\equiv -\delta g_{ti}^{(0)},\\
p &\equiv -\frac{4\pi T}{G^{(0)}}\left(\delta g_{rt}^{(0)} - h_{(0)}^{ij}g_{ir}^{(0)}\delta g_{tj}^{(0)}\right) - h_{(0)}^{ij}\frac{\partial_i G^{(0)}}{G^{(0)}}\delta g_{tj}^{(0)},\\
w_I &\equiv \delta a_t^{I(0)}.
\end{aligned} \tag{6.23}$$

The resulting Stokes equations are

$$
\begin{aligned}
&-2\nabla^j\nabla_{(i}v_{j)} + v^j[\nabla_j\phi^{(0)}\nabla_i\phi^{(0)} - 4\pi T d\chi^{(0)}_{ji}] - \sum_I F^{I(0)}_{ij}\frac{J^{iI}_{(0)}}{\sqrt{h^{(0)}}} \\
&\quad = \sum_I \frac{\rho_{H,I}}{\sqrt{h^{(0)}}}(E^I_i + \nabla_i w_I) + 4\pi T\xi_i - \nabla_i p. \\
&\nabla_i v^i = 0, \qquad \partial_i J^{iI}_{(0)} = 0,
\end{aligned}
\tag{6.24}
$$

Next, we consider one-dimensional lattices. As we described, the case we are most interested in is of $\chi = 0$ and $\hat{A}_y = C = 0$, and moreover, since we want to apply to extremal black holes, of $T \to 0$. That means that among the $\Sigma_i$ we consider nonzero only $\Sigma_2 = w_1$ and $\Sigma_3 = w_2$, which have now to be generalized to $\Sigma_{2I} = w^I_1$ and $\Sigma_3 = w_2$, defined as

$$
w^I_1(x) = \rho\left(\frac{1}{B_I}\int^x B_{H,I} - \frac{1}{\rho, I}\int^x \rho_{H,I}\right), \quad w_2(x) = Ts\left(\frac{1}{B}\int^x B_H - \frac{1}{s}\int^x s_H\right). \tag{6.25}
$$

That means that the nonzero components of the $\mathcal{U}_{ij}$ matrix are $\mathcal{U}_{2I2I}, \mathcal{U}_{2I3}, \mathcal{U}_{33}$. Moreover, as before, the finite values as $T \to 0$ are $\mathcal{U}_{2I2I}, \mathcal{U}_{2I3}/T, \mathcal{U}_{33}/T^2$.

We can next follow the steps outlined in appendix D of [2] in order to solve the Stokes equations for $J^i_{(0)}, v^i$ as a function of the sources $E_i, \xi_i$, and find first $v^x = v_0/\sqrt{-h}$ as before, then $v^y$ as a linear function of $v_0$ (involving a sum over $I$), then $J^{(0)x}_I, J^{y(0)}_I$ as a linear function of $v_0$; and finally $v_0$ is obtained as a sum over $I$.

We can consider $E^i_I = E^i$ (equal electric fields for the all the four gauge fields), and define conductivities by $J^i_I = \sigma^{ij}_I E_j + T\alpha^{ij}_I \xi$, in which case we obtain the the electric conductivities

$$
\begin{aligned}
\sigma^{xx}_I &= 0 \\
\sigma^{yy}_I &= \mathcal{U}_{2I2I} + \int Z^{(0)}_I + \int \frac{\left(\frac{\rho_I}{B_I} + W^{(0)}_I\right)^2}{Z^{(0)}_I} \\
&\quad -\frac{1}{X}\left(\int\left(\frac{\rho_I}{B_I} + W^{(0)}_I\right)^2 \frac{B_I}{\lambda Z^{(0)}_I} + \int \frac{B_I Z^{(0)}_I}{\lambda}\right) \times \\
&\quad \times \sum_J \left(\int\left(\frac{\rho_J}{B_J} + W^{(0)}_J\right)^2 \frac{B_J}{\lambda Z^{(0)}_J} + \int \frac{B_J Z^{(0)}_J}{\lambda}\right) \\
\sigma^{xy}_I &= -\sigma^{yx}_I = \frac{\rho_I}{B_I},
\end{aligned}
\tag{6.26}
$$

for the thermoelectric conductivities

$$
\begin{aligned}
\alpha^{xx}_I &= \bar{\alpha}^{xx}_I = 0, \\
\alpha^{yy}_I &= \bar{\alpha}^{yy}_I = \frac{\mathcal{U}_{2I3}}{T} + \frac{s}{B_I}\int \frac{\left(\frac{\rho_I}{B_I} + W^{(0)}_I\right)}{Z^{(0)}} \\
&\quad -\frac{1}{X}\left(\int\left(\frac{\rho_I}{B_I} + W^{(0)}_I\right)^2 \frac{B_I}{\lambda Z^{(0)}_I} + \int \frac{B_I Z^{(0)}_I}{\lambda}\right)\sum_J\left(\frac{s}{B_J}\int \frac{B_J}{\lambda Z^{(0)}_J}\left(\frac{\rho_J}{B_J} + W^{(0)}_J\right)\right),
\end{aligned}
$$

$$\begin{aligned}
\alpha_I^{xy} &= \bar{\alpha}_I^{yx} = \frac{s}{B_I}, \\
\alpha_I^{yx} &= \bar{\alpha}_I^{xy} = -\frac{4\pi}{X}\left(\int\left(\frac{\rho_I}{B_I}+W_I^{(0)}\right)^2\frac{B_I}{\lambda Z_I^{(0)}}+\int\frac{B_I Z_I^{(0)}}{\lambda}\right),
\end{aligned} \tag{6.27}$$

and for the thermal conductivities

$$\begin{aligned}
\frac{\kappa^{xx}}{T} &= \frac{16\pi^2}{X}, \\
\frac{\kappa^{yy}}{T} &= \frac{\mathcal{U}_{33}}{T^2}+\sum_I\frac{s^2}{B_I^2}\int\frac{1}{Z_I^{(0)}}+\frac{1}{X}\left(\sum_I\frac{s}{B_I}\int\left(\frac{\rho_I}{B_I}+W_I^{(0)}\right)\frac{B_I}{\lambda Z_I^{(0)}}\right)^2, \\
\frac{\kappa^{xy}}{T} &= \frac{\bar{\kappa}^{yx}}{T} = \frac{4\pi}{X}\sum_I\frac{s}{B_I}\int\left(\frac{\rho_I}{B_I}+W_I^{(0)}\right)\frac{B_I}{\lambda Z_I^{(0)}}.
\end{aligned} \tag{6.28}$$

If we consider the total conductivities $\sigma^{ij}$ and $\alpha^{ij}$, we have an additional sum over $I$ in the respective formulas. On the other hand, if we consider only a single nonzero $E_I$ (the previous case), all the formulas have no sums at all, and only $I$ indices.

We should note that we have the choice of whether one of the currents $J_I$, or their sum, refers to the electric charge current, since in AdS/CMT one takes a phenomenological approach, so *any* gauge current in the bulk could a priori stand for it, either one of the $U(1)^4$ ones, or the diagonal one (the sum of the currents).

Finally, in order to be able to use the results from the previous subsection, we compare the one-dimensional lattice case with the set-up for the extremal black hole with $AdS_2\times\mathbb{R}^2$ horizon. First, since the $(x,y)$ space corresponds to $\mathbb{R}^2$, we have that

$$\lambda = v_2. \tag{6.29}$$

That also implies that $\sqrt{h^{(0)}}=\lambda=v_2$. Second, we have the constant magnetic field at the horizon

$$B_I = B_{H,I} = \frac{1}{2}\sqrt{h^{(0)}}\epsilon_{ij}F_{ij}^I = v^2 p^I. \tag{6.30}$$

Finally, the electric field is (in the gauge $A_r=0$)

$$G_{rt}^I = \partial_r A_t^I = e^I \Rightarrow A_t^I = e^I(r-r_H)\,, \tag{6.31}$$

to be compared with the general formula (for $G^{(0)}=1$) near the horizon,

$$A_t^I = (r-r_H)(A_t^{(0)}+\ldots) \Rightarrow A_t^{(0)} = e^I\,, \tag{6.32}$$

which finally gives

$$\rho_I = \rho_{H,I} = \sqrt{h^{(0)}}Z_I^{(0)}A_t^{I(0)} - W_I^{(0)}B_{H,I} = v_2\left(Z_I^{(0)}e^I - W_I^{(0)}p^I\right). \tag{6.33}$$

With $v_2, e^I$ written in the previous subsection in terms of the charges $q_I, p^I$, this completes calculating the transport coefficients in terms of the charges of the dual black holes.

# 7 Conclusions

In this paper we have considered electric and thermal transport, in the presence of magnetic fields and electric charges and a topological term with coefficient $W$, and the effect of S-duality in such theories. We have also found that we can use the entropy function formalism and the attractor mechanism to give results for the transport coefficients as a function of the charges of the black hole in the gravity dual.

We have found that the only modification of the transport coefficients from previously found formulas is an extra term $-4W(r_H)$ in $\sigma_{xy}$, which however means that S-duality acts on the transport coefficients consistently with results at $\rho = B = 0$. The entropy function formalism was extended to this case, obtaining, in conjunction with the general formulas, explicit formulas depending on the charges of the dual black hole. S-duality still acts naturally on the transport coefficients, but an order of limits is important now.

The formalism of Stokes equations for determination of the transport coefficients, especially as it applies to one-dimensional lattices, was also considered, and was applied for the case of extremal black holes relevant for the entropy function formalism. S-duality is defined now more generally. A supergravity-inspired model, obtained by extending the $U(1)^4$ Cartan subgroup of $\mathcal{N} = 8, d = 4$ gauged supergravity in order to make it consistent with the rest of the paper, was also considered. The attractor mechanism, used in conjunction with generalized formulas for transport from Stokes equations, which we obtained, allowed us to write the transport coefficients of this generalized model in terms of the charges of the dual black hole.

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