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Non-chiral 2d CFT with integer energy levels

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Abstract: The partition function of 2d conformal field theory is a modular invariant function. It is known that the partition function of a holomorphic CFT whose central charge is a multiple of 24 is a polynomial in the Klein function. In this paper, by using the medium temperature expansion we show that every modular invariant partition function can be mapped to a holomorphic partition function whose structure can be determined similarly. We use this map to study partition function of CFTs with half-integer left and right conformal weights. We show that the corresponding left and right central charges are necessarily multiples of 4. Furthermore, the degree of degeneracy of high-energy levels can be uniquely determined in terms of the degeneracy in the low energy states.

Keywords: Conformal Field Theory, AdS-CFT Correspondence

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Contents

1 Introduction

An important question in conformal field theory (CFT) is to what extent a theory can be identified in terms of its constraints and symmetries. The bootstrap hypothesis [1–3] is based on the crossing symmetry. Recently in 4d CFT the crossing symmetry has been used to obtain an upper bound on the weights of the fields that appear in the operator product expansion of scalar operators [4]–[8] and a lower bound on the stress tensor central charge [9, 10]. Similarly an upper bound on the scaling dimension of the first scalar operator appearing in the OPE of two quasi-primary scalar operators has been obtained in two dimensions [4].

In two dimensions, the infinite dimensional group of the conformal symmetry makes the bootstrap project more efficient. Furthermore, the partition function of a 2d CFT should be invariant under modular transformations. The modular group $\mathbf{PSL}(2,\mathbb{Z})$ is the disconnected diffeomorphism group of the torus

$$(\tau,\bar\tau)\to(\tau',\bar\tau')=\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar\tau+b}{c\bar\tau+d}\right),\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathbf{PSL}(2,\mathbb{Z}),\tag{1.1}$$

where $\tau = \tau_1 + i\tau_2$ is the complex structure taking value in the upper half plane ($\tau_2 \geq 0$) and $\bar{\tau} = \tau_1 - i\tau_2$. The generators of the modular group are

$$T: \ (\tau, \bar{\tau}) \to (\tau + 1, \bar{\tau} + 1), \qquad\qquad S: \ (\tau, \bar{\tau}) \to \left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right). \tag{1.2}$$

Invariance under T-transformation (henceforth T-invariance) constrains the spin of states and the difference between the left and right central charges of the conformal field theory. S-invariance constrains the density of states and the spectrum of the theory.

In [11] the S-invariance of partition function has been used to estimate the density of states in the saddle-point approximation for a unitary CFT. It is seen that the density of states at conformal dimension h grows exponentially with the square root of h [12]. The Cardy formula is a key ingredient in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence; it reproduces the Bekenstein-Hawking entropy of the BTZ-black holes [13].[1] S-invariance has been also used to compute the 'logarithmic correction' [12] and 'the beyond the logarithmic corrections' to the Cardy formula [15]. In [16] it is shown that in theories with sparse light spectrum and large central charge, the Cardy formula also works for energies greater than the central charge.

Recently, the modular invariance of partition function has been used in order to obtain an upper bound on conformal dimensions of the primary fields. In [17] for holomorphically factorizable models whose left and right central charges are multiples of 24, an upper bound on the lowest primary fields has been obtained,

$$\Delta \leq \min\left(\frac{c_L}{24} + 1, \frac{c_R}{24} + 1\right). \tag{1.3}$$

This upper bound is saturated in extremal CFTs [18, 19]. Extremal CFT's are promising holographic duals to the pure gravity with negative cosmological constant [17, 20]. The vacuum state corresponds to the AdS space, and the primary fields above the vacuum correspond to the BTZ black hole. Modular invariance is enough to determine the partition function of an extremal CFT. For $c = 24$ an extremal CFT is known and its uniqueness has been conjectured [21, 22]. The holomorphic and anti-holomorphic parts of the partition function are modular functions. A modular function can be written in terms of a polynomial in the Klein function J [23]. While for the other values of the central charge the partition functions are known it is not clear whether such CFTs exist [24].

In general, for CFT's in which there is no chiral algebra beyond the Virasoro algebra and $c \gg 1$, the following upper bound on the lowest primary operator has been obtained [25]–[28]

$$\Delta \leq \frac{c_{\text{tot}}}{12} + \mathcal{O}(1), \qquad\qquad c_{\text{tot}} := c_L + c_R. \tag{1.4}$$

In fact for asymptotically large central charge this inequality is valid for Δ_n with $n \leq e^{\frac{\pi c}{12}}$ [29, 30]. The upper bound (1.4) can be computed by using the medium temperature

[1] The asymptotic symmetry group of an asymptotically AdS_3 spacetime with radius ℓ in Planck units is given by two copies of the Virasoro algebra whose central charge $\sim \ell$ [14]. The validity of the semi-classical gravity requires that $\ell \gg 1$.

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expansion. This method uses the S-invariance of the partition function at the self-dual point $\tau = -\bar{\tau} = i$ [25]. Considering a small neighborhood of $\tau = -\bar{\tau} = i$

$$\tau = i\, e^{s}, \qquad \bar{\tau} = -i\, e^{s}, \tag{1.5}$$

in the limit $s \to 0$, one obtains an infinite set of constraints on the partition function:

$$\left(\tau \frac{\partial}{\partial \tau}\right)^{N_R} \left(\bar{\tau} \frac{\partial}{\partial \bar{\tau}}\right)^{N_L} Z(\tau, \bar{\tau})\Big|_{\tau=-\bar{\tau}=i} = 0 \qquad \text{for} \quad N_L + N_R = \text{odd}. \tag{1.6}$$

Combining the constraints that can be obtained by different selections of (N_L, N_R) leads to certain universal constraints on the spectrum [25, 29–32].

In this work we use the medium temperature expansion method in a different manner. We note that eq. (1.6) indicates that for any (smooth) odd function $f(x,y) = -f(-x,-y)$

$$f\left(\tau \frac{\partial}{\partial \tau}, \bar{\tau} \frac{\partial}{\partial \bar{\tau}}\right) Z(\tau, \bar{\tau})\Big|_{\tau=-\bar{\tau}=i} = 0. \tag{1.7}$$

This observation leads to an interesting result: *corresponding to every S-invariant non-chiral partition function $Z(\tau, \bar{\tau})$, there exist an S-invariant chiral function $\mathcal{Z}(\tau)$.* The map $Z(\tau, \bar{\tau}) \to \mathcal{Z}(\tau)$ can be interpreted as the *chiralization* of the partition function. The chiral function corresponding to the non-chiral partition function can be easily obtain by inserting $\bar{\tau} = -\tau$ in $Z(\tau, \bar{\tau})$. That is,

$$\mathcal{Z}(\tau) := Z(\tau, -\tau). \tag{1.8}$$

This equation implies that $\mathcal{Z}(\tau)$ can be obtained by analytic continuation of the 'canonical' partition function

$$Z_{\text{canonical}}(\beta) := Z(\tau, \bar{\tau})|_{\tau=-\bar{\tau}=\frac{i\beta}{2\pi}}. \tag{1.9}$$

to the complex β-plane. The behavior of the chiral function $\mathcal{Z}(\tau)$ under T transformation depends on the spectrum of the main theory.

Focusing on a special class of CFTs whose primary operators have half integer scaling dimensions (henceforth HI-CFT), we show that the corresponding chiral partition function is an eigen-function of T whose eigen-value is $e^{\frac{-i\pi c_{\text{tot}}}{12}}$.

$$T : \mathcal{Z}(\tau) \to e^{\frac{-i\pi c_{\text{tot}}}{12}} \mathcal{Z}(\tau). \tag{1.10}$$

Since $\mathcal{Z}(\tau)$ is by construction S-invariant, the identity $(ST)^3 = 1$ implies that $c_{\text{tot}} \in 8\,\mathbb{Z}$. Thus, in such theories c_L and c_R are inevitably multiples of 4. We show that the corresponding chiral partition function $\mathcal{Z}(\tau)$ can be determined in terms of $1 + [\frac{k}{3}]$ positive integers.

$$\mathcal{Z}(\tau) = J^{k/3} \sum_{r=0}^{[\frac{k}{3}]} n_r J^{-r}, \qquad n_r \in \mathbb{N}. \tag{1.11}$$

Since the degree of degeneracy of levels in $Z(\tau, \bar{\tau})$ and $\mathcal{Z}(\tau)$ are equivalent (as can be inferred from eq. (1.9)), eq. (1.11) implies that the degree of degeneracy of high-energy levels in $Z(\tau, \bar{\tau})$ can be uniquely determined in terms of the degeneracy in the low energy states.

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The organization of the paper is as follows. In sections 2 we review the effect of two constraints on the partition function. One of them is the T-invariance and the other one is the simple fact that partition function should be real-valued. In section 3 we study the S-invariance of partition function and use the medium temperature expansion to obtain the chiralization map. Sections 4 and 5 are devoted to the HI-CFT's. We study the chiral partition function $\mathcal{Z}(\tau)$ in section 4, and identify a subclass of HI-CFT partition functions in terms of free-fermions in section 5. Some technical details are relegated to the appendices. Our main results are summarized in section 6.

2 Constraints on the spin values

Consider a two dimensional unitary CFT on a circle of length 2π. The partition function of the theory at temperature $\frac{1}{\beta}$ and chemical potential μ_c is as follows

$$Z(\beta, \mu_c) := \operatorname{Tr} e^{-\beta H + i\mu P} = \sum_{\Delta, j} \rho(\Delta, j) e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} e^{i\mu\left(j - \frac{c_{\text{dif}}}{24}\right)}, \tag{2.1}$$

in which $\mu := \mu_c \beta$, H is the Hamiltonian and P is the momentum on the compact spatial direction. The eigenvalues of H and P are $\Delta - \frac{c_{\text{tot}}}{24}$ and $j - \frac{c_{\text{dif}}}{24}$ respectively [33].

$$c_{\text{tot}} := c_L + c_R, \qquad\qquad c_{\text{dif}} := c_L - c_R, \tag{2.2}$$

where c_L and c_R are the left and right central charges. This partition function can be interpreted as a CFT partition function on a torus whose complex structure is given by

$$\tau := \frac{\mu + i\beta}{2\pi}, \qquad\qquad \bar{\tau} := \frac{\mu - i\beta}{2\pi}. \tag{2.3}$$

In this picture, the conformal weights are given by

$$h := \frac{1}{2}(\Delta + j), \qquad\qquad \bar{h} := \frac{1}{2}(\Delta - j), \tag{2.4}$$

and the partition function can be written as follows.[2]

$$Z(\tau, \bar{\tau}) = q^{\frac{-c_L}{24}} \bar{q}^{\frac{-c_R}{24}} \sum_{h, \bar{h}=0} \rho(h, \bar{h})\, q^h \bar{q}^{\bar{h}}, \tag{2.5}$$

in which

$$q := e^{2i\pi\tau}, \qquad\qquad \bar{q} := e^{-2i\pi\bar{\tau}}. \tag{2.6}$$

Henceforward we assume the following:

- The partition function is invariant under modular transformation;
- The spectrum contains the identity operator $h = \bar{h} = 0$;

[2] In a unitary CFT $h \geq 0$ and $\bar{h} \geq 0$. Therefore, $-\Delta \leq j \leq \Delta$.

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- The density of states $\rho(h, \bar{h})$ are positive integer numbers;
- The partition function is real.

In the following, we show that T-invariance indicates that the spin $j \in \mathbb{Z}$ and $c_{\text{dif}} \in 24\mathbb{Z}$. Furthermore we show that since the partition function is real-valued, at each energy level, the number of states with spin j and $-j + \frac{c_{\text{dif}}}{12}$ are equivalent.

2.1 T-invariance of partition function

Under T transformation

$$\mu \to \mu + 2\pi, \qquad \beta \to \beta. \tag{2.7}$$

Therefore, T-invariance of the partition function requires that

$$\sum_{\Delta,j} \rho(\Delta, j) e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} e^{i\mu\left(j - \frac{c_{\text{dif}}}{24}\right)} = \sum_{\Delta,j} \rho(\Delta, j) e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} e^{i(\mu+2\pi)\left(j - \frac{c_{\text{dif}}}{24}\right)}. \tag{2.8}$$

For $\mu = 0$ eq. (2.8) gives

$$\sum_{\Delta,j} \rho(\Delta, j) e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} \left[1 - \cos 2\pi \left(j - \frac{c_{\text{dif}}}{24}\right)\right] = 0, \tag{2.9}$$

$$\sum_{\Delta,j} \rho(\Delta, j) e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} \sin 2\pi \left(j - \frac{c_{\text{dif}}}{24}\right) = 0. \tag{2.10}$$

The summands in (2.9) are non-negative. Consequently $j - \frac{c_{\text{dif}}}{24}$ is necessarily an integer. The vacuum state ($j = 0$) enforces that $c_{\text{dif}} \in 24\mathbb{Z}$. Therefore, $j \in \mathbb{Z}$. From eq. (2.8) one verifies that these conditions are also sufficient.

2.2 Partition function is real-valued

The imaginary part of the partition function (2.1) is zero.

$$\sum_{\Delta} \sum_{j \in \mathcal{J}_\Delta} \rho(\Delta, j) e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} \sin \left[\mu \left(j - \frac{c_{\text{dif}}}{24}\right)\right] = 0, \tag{2.11}$$

where $\mathcal{J}_\Delta \subset [-\Delta, \Delta]$ denotes the set of spins of states with energy Δ. From the T-invariance we already know that $j - \frac{c_{\text{dif}}}{24} \in \mathbb{Z}$. Using the orthogonality of $\sin\left[\left(j - \frac{c_{\text{dif}}}{24}\right)\mu\right]$ (as a function of μ) in eq. (2.11) one obtains

$$\sum_{\Delta} \left[\rho(\Delta, j) - \rho\left(\Delta, -j + \frac{c_{\text{dif}}}{12}\right)\right] e^{-\beta\left(\Delta - \frac{c_{\text{tot}}}{24}\right)} = 0. \tag{2.12}$$

Assuming the ordering $\Delta_1 < \Delta_2 < \cdots$, eq. (2.12) reads

$$\rho(\Delta_1, j) - \rho\left(\Delta_1, -j + \frac{c_{\text{dif}}}{12}\right) + \sum_{\Delta=\Delta_2} \left[\rho(\Delta, j) - \rho\left(\Delta, -j + \frac{c_{\text{dif}}}{12}\right)\right] e^{-\beta(\Delta - \Delta_1)} = 0. \tag{2.13}$$

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By considering the $\beta \to \infty$ limit one verifies that

$$\rho(\Delta_1, j) = \rho\bigg(\Delta_1, -j + \frac{c_{\rm dif}}{12}\bigg). \tag{2.14}$$

Using eq. (2.14) in eq. (2.13), the same argument implies that $\rho(\Delta_2, j) = \rho\left(\Delta_2, -j + \frac{c_{\rm dif}}{12}\right)$. Iteration gives,

$$\rho(\Delta_m, j) = \rho\bigg(\Delta_m, -j + \frac{c_{\rm dif}}{12}\bigg). \tag{2.15}$$

Since $j - \frac{c_{\rm dif}}{24}$ is the momentum eigen-value, we conclude that

Corollary 2.1. *A 2d CFT whose partition function is real-valued and T-invariant is parity even.*

3 Invariance of partition function under S-transformation

S-invariance of the partition function,

$$Z(\tau, \bar\tau) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar\tau}\right), \tag{3.1}$$

implies that [25],[3]

$$\left(\tau\frac{\partial}{\partial\tau}\right)^{N_L}\left(\bar\tau\frac{\partial}{\partial\bar\tau}\right)^{N_R} Z(\tau,\bar\tau) = (-1)^{N_L+N_R}\left(\omega\frac{\partial}{\partial\omega}\right)^{N_L}\left(\bar\omega\frac{\partial}{\partial\bar\omega}\right)^{N_R} Z\left(\omega,\bar\omega\right), \tag{3.2}$$

in which

$$\omega := -\frac{1}{\tau}, \qquad\qquad \bar\omega := -\frac{1}{\bar\tau}. \tag{3.3}$$

At the self dual point $\tau = \omega = i$, and $\bar\tau = \bar\omega = -i$, this condition reads

$$\hat D_L^{N_L}\hat D_R^{N_R} Z(\tau,\bar\tau)\bigg|_{\tau=+i,\bar\tau=-i} = 0 \quad \text{for} \quad N_L + N_R = \text{odd}, \tag{3.4}$$

where $\hat D_L = \tau\frac{\partial}{\partial\tau}$ and $\hat D_R = \bar\tau\frac{\partial}{\partial\bar\tau}$ are respectively the left and the right dilatation operators. For a holomorphic test function $\mathcal{F}(\tau)$

$$e^{x\hat D}\mathcal{F}(\tau) = \mathcal{F}(e^x\tau), \qquad\qquad x \in \mathbb{C}. \tag{3.5}$$

Eq. (3.4) implies that for any (smooth) odd function $f(-x_L, -x_R) = -f(x_L, x_R)$

$$f(\hat D_L, \hat D_R) Z(\tau,\bar\tau)\bigg|_{\tau=i,\bar\tau=-i} = 0. \tag{3.6}$$

Using $f_1(\hat D_L, \hat D_R) = \sinh(x_L\hat D_L)\cosh(x_R\hat D_R)$ and $f_2(\hat D_L, \hat D_R) = \cosh(x_L\hat D_L)\sinh(x_R\hat D_R)$ and for $x_L, x_R \in \mathbb{C}$ one verifies that

$$Z(u_L, u_R) + Z\left(u_L, \frac{-1}{u_R}\right) - Z\left(\frac{-1}{u_L}, u_R\right) - Z\left(\frac{-1}{u_L}, \frac{-1}{u_R}\right) = 0, \tag{3.7}$$

$$Z(u_L, u_R) - Z\left(u_L, \frac{-1}{u_R}\right) + Z\left(\frac{-1}{u_L}, u_R\right) - Z\left(\frac{-1}{u_L}, \frac{-1}{u_R}\right) = 0, \tag{3.8}$$

[3]We assume that the partition function is a smooth function of β and μ i.e. there are no phase transitions.

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where $u_L = ie^{x_L}$ and $u_R = -ie^{ix_R}$.[4] An immediate result of the identities (3.7) and (3.8) is

Corollary 3.1. *Every S-invariant partition function $Z(\tau, \bar{\tau})$ is extended S-invariant, i.e.*

$$Z(u_L, u_R) = Z\left(\frac{-1}{u_L}, \frac{-1}{u_R}\right), \tag{3.10}$$

where u_L and u_R are two independent $\mathbb{C}$ parameters taking value in the upper half-plane and in the lower half-plane respectively.

3.1 Chiralization of the partition function

Consider the case $u_L = -u_R = \tau$ and define

$$\mathcal{Z}(\tau) := Z(\tau, -\tau). \tag{3.11}$$

Eq. (2.5) (for $q = \bar{q}$) gives

$$\mathcal{Z}(\tau) = q^{-\frac{c_{\text{tot}}}{24}} \sum_{\Delta=0} \hat{\rho}(\Delta) q^{\Delta}, \tag{3.12}$$

where[5]

$$\hat{\rho}(\Delta) := \sum_{j \in \mathcal{J}_\Delta} \rho(\Delta, j). \tag{3.13}$$

From (3.10) we learn that the function $\mathcal{Z}(\tau)$ is invariant under S-transformation. In summary,

Corollary 3.2. *Corresponding to every S-invariant partition function $Z(\tau, \bar{\tau})$, there is a an S-invariant chiral function $\mathcal{Z}(\tau) := Z(\tau, -\tau)$.*

We call the map

$$ch : Z(\tau, \bar{\tau}) \to \mathcal{Z}(\tau), \tag{3.14}$$

the chiralization map and $\mathcal{Z}(\tau)$ the ch-image of $Z(\tau, \bar{\tau})$. Table 1 shows some example of the known partition function and the corresponding ch-images.

[4] Since the growth of $\rho(h, \bar{h})$ in eq. (2.5) is controlled by the Cardy formula, the partition function

$$Z(u_L, u_R) = e^{-\frac{\pi i u_L c_L}{12}} e^{\frac{\pi i u_R c_R}{24}} \sum_{h, \bar{h}=0} \rho(h, \bar{h}) e^{2\pi i u_L h} e^{-2\pi i u_R \bar{h}}, \tag{3.9}$$

is convergent if the imaginary parts of u_L and u_R are positive and negative respectively. We assume that $Z(u_L, u_R)$ gives a biholomorphic analytic continuation of $Z(\beta, \mu)$ to complex u_L in the upper half-plane and u_R in the lower half-plane.

[5] $\mathcal{Z}(\tau)$ corresponds to the analytic continuation of the canonical partition function $Z_{\text{canonical}}(\beta)$ defined in eq. (1.9) to the complex β-plane. It is known that $Z_{\text{canonical}}(\beta)$ is a real analytic function [34], thus $\mathcal{Z}(\tau)$ is well-defined. The S-invariance of $\mathcal{Z}(\tau)$ is also indicated by the S-invariance of $Z_{\text{canonical}}(\beta)$.

Model	$Z(\tau,\bar{\tau})$	*ch*-image
Ising model	$\frac{1}{2}\left(\left\|\frac{\theta_2}{\eta}\right\|+\left\|\frac{\theta_3}{\eta}\right\|+\left\|\frac{\theta_4}{\eta}\right\|\right)$	$\frac{1}{2}\left(\frac{\theta_2}{\eta}+\frac{\theta_3}{\eta}+\frac{\theta_4}{\eta}\right)$
Free boson	$\frac{1}{\sqrt{\tau_i}}\frac{1}{\|\eta(\tau)\|^2}$	$\frac{1}{\sqrt{-i\tau}}\frac{1}{(\eta(\tau))^2}$
Free boson on a circle of $r=1$	$\frac{1}{2}\left(\left\|\frac{\theta_2}{\eta}\right\|^2+\left\|\frac{\theta_3}{\eta}\right\|^2+\left\|\frac{\theta_4}{\eta}\right\|^2\right)$	$\frac{1}{2}\left[\left(\frac{\theta_2}{\eta}\right)^2+\left(\frac{\theta_3}{\eta}\right)^2+\left(\frac{\theta_4}{\eta}\right)^2\right]$

Table 1. Examples of non-chiral partition functions and the corresponding *ch*-images.

4 CFT's with half-integer conformal weights

In this section we investigate a family of CFT's in which $\Delta \in \mathbb{Z}$. Since $j \in \mathbb{Z}$, the corresponding conformal weights are half-integers. Hence we call such a CFT an *HI-CFT*. In the following $Z(\tau,\bar{\tau})$ and $\mathcal{Z}(\tau)$ denote the partition function of an HI-CFT and the corresponding *ch*-image respectively.

From eq. (3.12) one verifies that

$$T:\ \mathcal{Z}(\tau) \to e^{-i\pi\frac{c_{\rm tot}}{12}}\mathcal{Z}(\tau). \tag{4.1}$$

Since $\mathcal{Z}(\tau)$ is S-invariant, using the identity $(TS)^3 = 1$ one obtains

$$e^{-2\pi i\frac{c_{\rm tot}}{8}} = 1. \tag{4.2}$$

Consequently,

$$c_{\rm tot} \in 8\,\mathbb{N}. \tag{4.3}$$

From the T-invariance of $Z(\tau,\bar{\tau})$ we have learned that $c_{\rm dif} \in 24\,\mathbb{Z}$. Therefore,

Corollary 4.1. *For an HI-CFT*

$$c_L \in 4\,\mathbb{N}, \qquad\qquad c_R \in 4\,\mathbb{N}. \tag{4.4}$$

Now we are ready to obtain the basis for $\mathcal{Z}(\tau)$. Let's start with $c_L, c_R \in 12\mathbb{N}$. In that case $c_{\rm tot} \in 24\mathbb{Z}$ and $\mathcal{Z}(\tau)$ defined by eq. (3.12) is a well-defined modular invariant meromorphic function in the upper half plane. Therefore it can be given as a polynomial in the Klein function J [23],

$$\mathcal{Z} = \sum_{r=-\frac{c_{\rm tot}}{24}}^{0} a_r J^{-r}. \tag{4.5}$$

The Klein function can be written in terms of the Jacobi Theta functions $\theta_i(\tau)$ $(i=2,3,4)$ and the Dedekind function $\eta(\tau)$.

$$J = \mathrm{j}^3 \tag{4.6}$$

$$= q^{-1} + 744 + 196884\,q + \cdots\,, \tag{4.7}$$

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where

$$\begin{aligned}
\mathfrak{j}(\tau) &:= \frac{1}{2}\left[\left(\frac{\theta_2(\tau)}{\eta(\tau)}\right)^8 + \left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^8 + \left(\frac{\theta_4(\tau)}{\eta(\tau)}\right)^8\right] \\
&= q^{\frac{-1}{3}}\left(1 + 248\, q + \cdots\right). \qquad (4.8)
\end{aligned}$$

In the following we show that for $c_{\text{tot}} \in 8\mathbb{N}$, $\mathcal{Z}(\tau)$ can be written in terms of a polynomial in $\mathfrak{j}$.

Lemma 4.2. *Let $f^{(r)}\left(\{a^{(r)}\},\tau\right)$ be an S-invariant function with Fourier expansion*

$$f^{(r)}\left(\{a^{(r)}\},\tau\right) = q^{\frac{-p}{3}}\left[\sum_{n=-r}^{0} a_n^{(r)} q^n + \sum_{n=1}^{\infty} a_n^{(r)} q^n\right], \qquad p \in \{0,1,2\}. \qquad (4.9)$$

in the upper half τ-plane. Then

a. *$f^{(r)}\left(\{a^{(r)}\},\tau\right)$ is T^3-invariant.*

b. *It is a polynomial in $\mathfrak{j}$.*

Proof. T^3-invariance is obvious. Eq. (4.7) and eq. (4.8) imply that there exist $\{a^{(r-1)}\}$ such that

$$q^{\frac{-p}{3}} \sum_{n=-r}^{\infty} a_n^{(r)} q^n = a_{-r}^{(r)}\, \mathfrak{j}^p\, J^r + q^{\frac{-p}{3}} \sum_{n=-r+1}^{\infty} a_n^{(r-1)} q^n. \qquad (4.10)$$

Therefore,

$$f^{(r)}\left(\{a^{(r)}\},\tau\right) = a_r^{(r)}\, \mathfrak{j}^p\, J^r + f^{(r-1)}\left(\{a^{(r-1)}\},\tau\right). \qquad (4.11)$$

The order of the poles of $f^{(r)}\left(\{a^{(r)}\},\tau\right)$ and $f^{(r-1)}\left(\{a^{(r-1)}\},\tau\right)$ are r and $r-1$ respectively. The S-invariance of $f^{(r)}\left(\{a^{(r)}\},\tau\right)$, $\mathfrak{j}$ and J imply that $f^{(r-1)}\left(\{a^{(r-1)}\},\tau\right)$ is also S-invariant. By iteration one obtains

$$f^{(r)}\left(\{a^{(r)}\},\tau\right) = \mathfrak{j}^p\left[a_{-r}^{(r)} J^r + a_{-(r-1)}^{(r-1)} J^{r-1} + \cdots + a_0^{(0)}\right] + f^{(-1)}\left(\{a^{(-1)}\},\tau\right), \qquad (4.12)$$

where

$$f^{(-1)}\left(\{a^{(-1)}\},\tau\right) = q^{1-\frac{p}{3}} \sum_{m\geq 1} a_m^{(-1)} q^{m-1}. \qquad (4.13)$$

The function $\left[f^{(-1)}\left(\{a^{(-1)}\},\tau\right)\right]^3$ is modular invariant. It has no pole in the upper half plane and is zero at $\tau = i\infty$. Thus it is zero in the upper half plane. □

Corollary 4.3. *The ch-image of the HI-CFT partition function with total central charge $c_{\text{tot}} = 8k$, has an expansion in terms of $\mathfrak{j}$ as follows*

$$\mathcal{Z}(\tau) = \mathfrak{j}^k \sum_{r=0}^{[k/3]} n_r J^{-r}, \qquad n_r \in \mathbb{N}. \qquad (4.14)$$

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central charge	*ch*-Image of partition function
$k=1$	$\mathrm{j} = q^{\frac{-1}{3}}(1+248q+4124q^2+34752q^3+\cdots)$
$k=2$	$\mathrm{j}^2 = q^{\frac{-2}{3}}(1+496q+69752q^2+2115008q^3+\cdots)$
$k=3$	$J+n_1 = q^{-1}\left[1+(744+n_1)q+196884q^2+21493760q^3+\cdots\right]$

Table 2. The *ch*-image of the HI-CFT partition function with $c_{\text{tot}} \in \{8,16,24\}$.

The degeneracy of the vacuum state is given by n_0. In the following we assume that $n_0 = 1$. Eq. (4.14) shows that $\mathcal{Z}(\tau)$, and consequently the number of states with energy Δ i.e. $\hat{\rho}(\Delta)$ can be uniquely determined if c_{tot} and the integers n_r, or equivalently, the low-energy (i.e. $\Delta \leq \left[\frac{k}{3}\right]$) density of states are given.[6]

Finally, consider an HI-CFT whose *ch*-image $\mathcal{Z}(\tau)$ is extremal, i.e. $\mathcal{Z}(\tau) = q^{-k/3}\left[1+\mathcal{O}(q)\right]$. In that case, the coefficients n_r can be uniquely determined in terms of the central charge. Furthermore, the scaling dimension of the first primary field after identity is $\Delta_1 = \frac{c_{\text{tot}}}{24}+1$, which is in agreement with the upper bound given in eq. (1.4).

4.1 AdS/CFT correspondence

It is known that the Cardy formula reproduces the Bekenstein-Hawking entropy at $\Delta \gg 1$. In [17] it has been observed that for $k \in 3\,\mathbb{N}$,[7] the number of primary fields is given by the Cardy formula

$$\hat{\rho}(\Delta) \cong e^{\mathcal{S}(k,\Delta)}, \qquad \mathcal{S}(k,\Delta) := 4\pi\sqrt{\frac{k\left(\Delta-\frac{k}{3}\right)}{3}}. \tag{4.15}$$

Therefore it is natural to assume that the primary fields correspond to the micro-states of the BTZ black hole.

In table 2 the Fourier expansion of the *ch*-image is given for $c_{\text{tot}} = 8, 16, 24$. The coefficients of the expansions determine the density of state $\hat{\rho}(\Delta)$ which equals the number of states with energy Δ and spin $j \in [-\Delta, \Delta]$. For $k = 1,2,3$ the first high energy state (i.e. $\Delta = 1 + \left[\frac{k}{3}\right]$) has weight $\Delta = 1,1,2$ respectively. It is an interesting observation that the corresponding number of states can be estimated by the Cardy formula.

5 A basis for HI-CFT partition function

In the previous section we have observed that the *ch*-image of the HI-CFT partition function is a polynomial in j. Motivated by the fact that j is the *ch*-image of $\frac{1}{2}\sum_{i=1}^{3}\left|\frac{\theta_i}{\eta}\right|^8$ in this section we study a class of HI-CFT's whose partition functions can be given as a polynomial in $\sqrt{\frac{\theta_i}{\eta}}$ and $\sqrt{\frac{\bar{\theta}_i}{\eta}}$.

[6]The existence and the uniqueness of such CFT's is an open problem.

[7]In our conventions, $c_{\text{tot}} = 8k$ while in [17] $c_{\text{tot}} = 24k$.

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The functions $\sqrt{\frac{\theta_i}{\eta}}$ have the following Fourier expansion.

$$\sqrt{\frac{\vartheta_2}{\eta}} = q^{\frac{-1}{48}+\frac{1}{16}} \sum_{n=0}^{\infty} C_n^{(\frac{1}{16})} q^n, \tag{5.1}$$

$$\sqrt{\frac{\vartheta_3}{\eta}} = q^{\frac{-1}{48}} \left(\sum_{n=0}^{\infty} C_n^{(0)} q^n + \sum_{n=1}^{\infty} C_n^{(\frac{1}{2})} q^{n+\frac{1}{2}} \right), \tag{5.2}$$

$$\sqrt{\frac{\vartheta_4}{\eta}} = q^{\frac{-1}{48}} \left(\sum_{n=0}^{\infty} C_n^{(0)} q^n - \sum_{n=1}^{\infty} C_n^{(\frac{1}{2})} q^{n+\frac{1}{2}} \right), \tag{5.3}$$

where

$$C_n^{(i)} \in \mathbb{N}, \qquad i = 0, \frac{1}{16}, \frac{1}{2}. \tag{5.4}$$

The S-transformation of the Dedekind function η and the Theta functions are as follows.

$$\theta_2 \to (-i\tau)^{1/2}\theta_4, \qquad \theta_4 \to (-i\tau)^{1/2}\theta_2, \tag{5.5}$$

$$\theta_3 \to (-i\tau)^{1/2}\theta_3, \qquad \eta \to (-i\tau)^{1/2}\eta. \tag{5.6}$$

T-transformation of these functions is given by,

$$\theta_2 \to e^{i\pi/4}\theta_2, \qquad \theta_3 \leftrightarrow \theta_4, \qquad \eta \to e^{i\pi/12}\eta. \tag{5.7}$$

Since an HI-CFT only contains primary fields with half integer scaling dimension, from eqs. (5.1)–(5.7) one infers that the corresponding partition function is a polynomial in x, y and z defined as follows.

$$x := \left(\sqrt{\frac{\vartheta_2}{\eta}}\right)^8 = q^{\frac{-1}{6}+\frac{1}{2}} C(q), \tag{5.8}$$

$$y := \left(\sqrt{\frac{\vartheta_3}{\eta}}\right)^8 = q^{\frac{-1}{6}} \left[A(q) + q^{\frac{1}{2}} B(q)\right], \tag{5.9}$$

$$z := \left(\sqrt{\frac{\vartheta_4}{\eta}}\right)^8 = q^{\frac{-1}{6}} \left[A(q) - q^{\frac{1}{2}} B(q)\right], \tag{5.10}$$

where $A(q)$, $B(q)$ and $C(q)$ are polynomials in q with positive integer coefficients. The functions x, y and z are not independent. They are related through the standard relations between the Theta functions and Dedekind function.

$$x - y + z = 0, \tag{5.11}$$

$$xyz = 16. \tag{5.12}$$

By using eq. (5.12) and the transformation rules

$$S: \; y \to y, \qquad x \leftrightarrow z, \tag{5.13}$$

$$T: \; x \to e^{2i\pi/3}x, \qquad y \to e^{-i\pi/3}z, \qquad z \to e^{-i\pi/3}y, \tag{5.14}$$

one can show that the most general modular covariant combination of x, y, z can be written as follows.

$$\begin{aligned} R_{a,b,c,d} = &\, x^c\bar{x}^a\left(y^d\bar{z}^b + \alpha z^d\bar{y}^b\right) \\ &+ y^c\bar{y}^a\left(\beta z^d\bar{x}^b + \tilde{\beta}x^d\bar{z}^b\right) \\ &+ z^c\bar{z}^a\left(\gamma x^d\bar{y}^b + \tilde{\gamma}y^d\bar{x}^b\right), \end{aligned} \tag{5.15}$$

where a, b, c and d are some positive integer number and $\alpha, \beta, \tilde{\beta}, \gamma$ and $\tilde{\gamma}$ are complex number. By covariance we mean that

$$S:\ R_{a,b,c,d} \to e^{i\sigma}R_{a,b,c,d}, \tag{5.16}$$

$$T:\ R_{a,b,c,d} \to e^{i\delta}R_{a,b,c,d}, \tag{5.17}$$

where σ and δ are real numbers. Eq. (5.16) implies that

$$\tilde{\gamma} = e^{i\sigma}, \qquad \tilde{\gamma}^2 = e^{2i\sigma} = 1, \qquad \gamma = e^{i\sigma}\alpha, \qquad \tilde{\beta} = e^{i\sigma}\beta. \tag{5.18}$$

Using (5.17) and (5.18) one obtains,

$$e^{i\delta} = (-1)^{a+c}e^{\frac{i\pi}{3}(a+b-c-d)}\alpha, \qquad \beta = (-1)^{a+c+d}, \tag{5.19}$$

$$\tilde{\gamma} = (-1)^{b+d}\alpha, \qquad \alpha^2 = 1. \tag{5.20}$$

Consequently,

$$\begin{aligned} R^{\pm}_{a,b,c,d} = &\, x^c\bar{x}^a\left((-y)^d\bar{z}^b \pm z^d(-\bar{y})^b\right) \\ &+ (-y)^c(-\bar{y})^a\left(z^d\bar{x}^b \pm x^d\bar{z}^b\right) \\ &+ z^c\bar{z}^a\left(x^d(-\bar{y})^b \pm (-y)^d\bar{x}^b\right), \end{aligned} \tag{5.21}$$

where we have dropped an overall phase $(-1)^d$. $R^-_{a,b,c,d}$ and $R^+_{a,b,c,d}$ are respectively odd and even under S-transformation.

$$SR^{\pm}_{a,b,c,d} = \pm R^{\pm}_{a,b,c,d}. \tag{5.22}$$

S-invariance of the partition function implies that $Z(\tau, \bar{\tau})$ should be an even function in $R^-_{a,b,c,d}$. In appendix A we show that $R^-_{a,b,c,d}R^-_{a',b',c',d'}$ is a linear combination of $R^+_{a,b,c,d}$. Hence, we concentrate on polynomials in $R^+_{a,b,c,d}$ and drop the $+$ sign for simplicity.

Noting that

$$\mathrm{j} = \frac{1}{2}(x^2 + y^2 + z^2), \tag{5.23}$$

one can use eq. (5.11) to show that

$$\mathrm{j} = x^2 + yz = z^2 + xy = y^2 - xz. \tag{5.24}$$

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These identities together with eq. (5.12) result in the following recurrence relations.

$$
\begin{aligned}
R_{a+2,b,c,d} &= \bar{\mathfrak{j}} R_{a,b,c,d} - 16 R_{a-1,b,c,d}, && (5.25)\\
R_{a,b+2,c,d} &= \bar{\mathfrak{j}} R_{a,b,c,d} - 16 R_{a,b-1,c,d}, && (5.26)\\
R_{a,b,c+2,d} &= \mathfrak{j} R_{a,b,c,d} - 16 R_{a,b,c-1,d}, && (5.27)\\
R_{a,b,c,d+2} &= \mathfrak{j} R_{a,b,c,d} - 16 R_{a,b,c,d-1}. && (5.28)
\end{aligned}
$$

Eqs. (5.25)–(5.28) show that every $R_{a,b,c,d}$ is a polynomial in $\mathfrak{j}$, $\bar{\mathfrak{j}}$,

$$\mathfrak{h} := \frac{1}{2}\left(|x|^2 + |y|^2 + |z|^2\right), \tag{5.29}$$

and

$$\mathfrak{k} := x^2\bar{x} - y^2\bar{y} + z^2\bar{z}. \tag{5.30}$$

To show it, we first consider the chiral function

$$R_{c,d} := R_{0,0,c,d} = x^c\left((-y)^d + z^d\right) + (-y)^c(z^d + x^d) + z^c(x^d + (-y)^d\right). \tag{5.31}$$

Noting that

$$
\begin{aligned}
R_{0,0} &= 6, & R_{1,0} &= 0, & R_{2,0} &= 4\mathfrak{j}, && (5.32)\\
R_{0,1} &= 0, & R_{1,1} &= -2\mathfrak{j}, & R_{2,1} &= 48, && (5.33)\\
R_{0,2} &= 4\mathfrak{j}, & R_{1,2} &= 48, & R_{2,2} &= 2\mathfrak{j}^2, && (5.34)
\end{aligned}
$$

one verifies that $\mathfrak{j}$ is the single generator of $R_{c,d}$.

In appendix B we show that

$$
\begin{aligned}
R_{a,b,0,1} &= -R_{b,a,1,0} - R_{a,b,1,0}, && (5.35)\\
R_{a,b,1,1} &= -\mathfrak{j} R_{a,b,0,0} + R_{b,a,2,0}, && (5.36)\\
R_{a,b,0,2} &= 2\mathfrak{j} R_{a,b,0,0} - R_{a,b,2,0} - R_{b,a,2,0}, && (5.37)\\
R_{a,b,2,2} &= \mathfrak{j}^2 R_{a,b,0,0} - \mathfrak{j} R_{b,a,2,0} - 16 R_{b,a,1,0}, && (5.38)\\
R_{a,b,1,2} &= -\mathfrak{j} R_{a,b,0,1} + R_{b,a,2,1}. && (5.39)
\end{aligned}
$$

Thus all of $R_{a,b,c,d}$ can be obtained in terms of $R_{a,b,c',d'}$ where $\{c', d'\} \in \{(1,0),\ (2,0),\ (2,1)\}$. Using the identity

$$\bar{R}_{a,b,c,d} = R_{c,d,a,b}, \tag{5.40}$$

one can also determine $R_{2,0,1,0}$, $R_{2,1,1,0}$, and $R_{2,1,2,0}$ in terms of $R_{1,0,2,0}$, $R_{1,0,2,1}$, and $R_{2,0,2,1}$ respectively. Using eq. (5.36) one can determine $R_{1,1,1,0}$, $R_{1,1,2,0}$ and $R_{1,1,2,1}$. Similarly, eq. (5.38) can be used to compute $R_{2,2,1,0}$, $R_{2,2,2,0}$ and $R_{2,2,2,1}$. Therefore all that we need

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to compute $R_{a,b,c,d}$ are the following functions.

$$\begin{aligned}
R_{1,0,1,0} &= -2R_{0,1,1,0} = 4\mathfrak{h}, & R_{0,1,2,0} &= \bar{R}_{0,2,1,0} = -\frac{1}{2}R_{1,0,2,0} = -\mathfrak{k}, \\
R_{1,2,1,0} &= -\bar{R}_{1,0,2,1} = 2\bar{\mathfrak{j}}\,\mathfrak{h}, & R_{0,1,2,1} &= 0, \\
R_{2,0,2,0} &= \frac{8h^2}{3} + \frac{4|\mathfrak{j}|^2}{3}, & R_{0,2,2,0} &= -\frac{4\mathfrak{h}^2}{3} + \frac{10|\mathfrak{j}|^2}{3}, \\
R_{2,0,2,1} &= -\mathfrak{j}\bar{\mathfrak{k}} + 32\mathfrak{j}, & R_{0,2,2,1} &= 32\bar{\mathfrak{j}}, \\
R_{2,1,2,1} &= \frac{8}{3}|\mathfrak{j}|^2\,\mathfrak{h} - \frac{8\mathfrak{h}^3}{3} + 6(16)^2.
\end{aligned} \tag{5.41}$$

In summery every $R_{a,b,c,d}$ is a polynomial in $\mathfrak{h}, \mathfrak{k}$ and $\bar{\mathfrak{k}}$ as follows

$$\mathfrak{R} = g_0 + g_1\mathfrak{h} + g_2\mathfrak{h}^2 + g_3\mathfrak{h}^3 + g_4\mathfrak{k} + g_5\bar{\mathfrak{k}}, \tag{5.42}$$

where $g_i = g_i(\mathfrak{j}, \bar{\mathfrak{j}})$ are polynomials in $\mathfrak{j}$ and $\bar{\mathfrak{j}}$.

Since the HI-CFT partition function is modular invariant, we investigate the invariance of $\mathfrak{R}$ under T transformation. $\mathfrak{h}$ is modular invariant. $\mathfrak{j}$ and $\mathfrak{k}$ are eigen-functions of T with eigen-values $e^{\frac{-2\pi i}{3}}$ and $e^{\frac{2\pi i}{3}}$ respectively. In order to determine $g_i, (i = 0\cdots5)$, we write them as follows

$$g_i = F_i^{(0)} + \left(F_i^{(1)}\mathfrak{j} + F_i^{(2)}\mathfrak{j}^2 + \text{h.c.}\right), \qquad i = 0,1,2,3, \tag{5.43}$$

$$g_4 = F_4^{(0)} + F_4^{(1)}\mathfrak{j} + F_4^{(2)}\mathfrak{j}^2 + G_4^{(1)}\bar{\mathfrak{j}} + G_4^{(2)}\bar{\mathfrak{j}}^2, \tag{5.44}$$

$$g_5 = \bar{g}_4. \tag{5.45}$$

where $F_i^{(a)}$ and $G_4^{(a)}$ are polynomials in $|\mathfrak{j}|^2$, J and $\bar{J}$. In writing eqs. (5.43)–(5.45) we have noted that $\mathfrak{R}$ as a partition function should be real-valued. Using eqs. (5.43)–(5.45) in eq. (5.42) and the identity $T\mathfrak{R} + T^2\mathfrak{R} = 2\mathfrak{R}$ one obtains

$$\left[\sum_{i=0}^{3}\left(\mathfrak{j}F_i^{(1)} + \mathfrak{j}^2F_i^{(2)}\right)\mathfrak{h}^i + \left(F_4^{(0)} + F_4^{(2)}\mathfrak{j}^2 + G_4^{(1)}\bar{\mathfrak{j}}\right)\mathfrak{k}\right] + \text{c.c.} = 0. \tag{5.46}$$

Therefore, every HI-CFT partition function can be written as

$$Z(\tau,\bar{\tau}) = \sum_{i=0}^{3} F_i^{(0)}\mathfrak{h}^i + \left[\left(F_4^{(1)}\mathfrak{j} + G_4^{(2)}\bar{\mathfrak{j}}^2\right)\mathfrak{k} + \text{c.c.}\right]. \tag{5.47}$$

Noting that the *ch*-image of $\mathfrak{h}$ is $\mathfrak{j}$ and the *ch*-image of $\mathfrak{k}$ equals -48,[8] one easily verifies that the *ch*-image of $Z(\tau,\bar{\tau})$ is a function of $\mathfrak{j}$ in agreement with corollary 4.3.

[8]The *ch*-image of $\mathfrak{k}$ is $\frac{1}{2}R_{0,0,3,0}$ which can be easily computed by using eq. (5.27).

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5.1 Examples of HI-CFT Partition function

In this section we study HI-CFT's with $c_{\text{tot}} = 8, 16$.[9]

- $c_{\text{tot}} = 8$. In this case there is only one partition function

$$Z(\tau,\bar\tau) = \mathfrak{h} = \frac{1}{2}\frac{{\vartheta_2}^4(\tau){\bar\vartheta_2}^4(\bar\tau) + {\vartheta_3}^4(\tau){\bar\vartheta_3}^4(\bar\tau) + {\vartheta_4}^4(\tau){\bar\vartheta_4}^4(\bar\tau)}{\eta^4(\tau)\bar\eta^4(\bar\tau)}, \tag{5.48}$$

which corresponds to 8 right-handed and 8 left-handed fermions. The corresponding *ch*-image is

$$\mathcal{Z}(\tau) = \frac{1}{2}\frac{{\vartheta_2}^8 + {\vartheta_3}^8 + {\vartheta_4}^8}{\eta^8(\tau)} = \mathfrak{j}. \tag{5.49}$$

- $c_{\text{tot}} = 16$. In this case the partition function is not unique.

$$Z(\tau,\bar\tau) = \frac{1}{a+b}\left(a\mathfrak{h}^2 + b\,|\mathfrak{j}|^2\right). \tag{5.50}$$

The *ch*-image of $Z(\tau,\bar\tau)$ is $\mathfrak{j}^2$ (independent of a and b). The factor $\frac{1}{a+b}$ indicates that there is a single vacuum state. The coefficients a an b should be determined in such a way that the density of states are positive integers. By inspecting the first few terms in the Fourier expansion of $Z(\tau,\bar\tau)$, one can obtain the following necessary condition.

$$\frac{384a}{a+b} = n, \qquad \frac{56a + 248b}{a+b} = m, \tag{5.51}$$

in which m and n are nonnegative integers. This gives

$$m, n \in 8\mathbb{Z} \qquad\qquad 2m' + n' = 62, \tag{5.52}$$

where $m' := \frac{m}{8}$ and $n' := \frac{n}{8}$. Using eq. (5.52) in eq. (5.50) one obtains

$$Z(\tau,\bar\tau) = \frac{1}{24}\left[(31 - m')\mathfrak{h}^2 + (m' - 7)\,|\mathfrak{j}|^2\right]. \tag{5.53}$$

For $7 \leq m' \leq 31$ the energy densities are obviously positive integers. We have not been able to exclude the partition functions corresponding to $0 \leq m' \leq 6$. Therefore, we are optimistic that there should be 32 different HI-CFT's with $c_{\text{tot}} = 16$.

6 Summary

In this work we have studied modular invariant partition functions of unitary CFT's whose conformal weights are half-integers, hence HI-CFT's. By using the medium temperature expansion we have obtained a chiralization map which maps every S-invariant non-chiral partition function to an S-invariant chiral partition function. We have used the chiralization map to show that the left and right central charges of an HI-CFT are multiples of 4. Furthermore, we have shown that the partition function after chiralization can be written

[9] $c_{\text{dif}} \in 24\mathbb{Z}$ implies that the corresponding left and right central charges are $c_L = c_R = 4$ and $c_L = c_R = 8$ respectively.

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as a polynomial in $j = J^{1/3}$, where J is the Klein function. In this way we have realized that the degree of degeneracy of the high energy levels $\Delta > \left[\frac{c_L+c_R}{24}\right]$ can be uniquely determined in terms $1 + \left[\frac{c_L+c_R}{24}\right]$ integers corresponding to the degeneracy in the low energy states.

We have identified a class of HI-CFT's whose partition functions can be given in terms of the Jacobi Theta function θ_i and the Dedekind function η. In eq. (5.47) we have given the most general form of such partition functions.

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A S-invariant combinations of $R_{a,b,c,d}$

The multiplication rule for $R^-_{a,b,c,d}$ can be obtained as follows.

$$
\begin{aligned}
R^-_{a,b,c,d}R^-_{a',b',c',d'} = {} & R^+_{a+a',b+b',c+c',d+d'} \\
& -\Big[x^{c+c'}(-y)^{d'}z^d\bar{x}^{a+a'}(-\bar{y})^b\bar{z}^{b'} + x^{c+c'}(-y)^d z^{d'}\bar{x}^{a+a'}(-\bar{y})^{b'}\bar{z}^b \\
& \quad +x^{d'}(-y)^{c+c'}z^d\bar{x}^b(-\bar{y})^{a+a'}\bar{z}^{b'} + x^d(-y)^{c+c'}z^{d'}\bar{x}^{b'}(-\bar{y})^{a+a'}\bar{z}^b \\
& \quad +x^d(-y)^{d'}z^{c+c'}\bar{x}^{b'}(-\bar{y})^b\bar{z}^{a+a'} + x^{d'}(-y)^d z^{c+c'}\bar{x}^b(-\bar{y})^{b'}\bar{z}^{a+a'}\Big] \\
& +\Big[x^c(-y)^{c'+d}z^{d'}\bar{x}^{a+b'}(-\bar{y})^{a'}\bar{z}^b + x^c(-y)^{d'}z^{c'+d}\bar{x}^{a+b'}(-\bar{y})^b\bar{z}^{a'} \\
& \quad +x^{d'}(-y)^c z^{c'+d}\bar{x}^b(-\bar{y})^{a+b'}\bar{z}^{a'} + x^{d'}(-y)^{d+c'}z^c\bar{x}^b(-\bar{y})^{a'}\bar{z}^{a+b'} \\
& \quad +x^{c'+d}(-y)^c z^{d'}\bar{x}^{a'}(-\bar{y})^{a+b'}\bar{z}^b + x^{c'+d}(-y)^{d'}z^c\bar{x}^{a'}(-\bar{y})^b\bar{z}^{a+b'}\Big] \\
& +\Big[x^{c'}(-y)^{c+d'}z^d\bar{x}^{a'+b}(-\bar{y})^a\bar{z}^{b'} + x^{c'}(-y)^d z^{c+d'}\bar{x}^{a'+b}(-\bar{y})^{b'}\bar{z}^a \\
& \quad +x^d(-y)^{c'}z^{c+d'}\bar{x}^{b'}(-\bar{y})^{a'+b}\bar{z}^a + x^d(-y)^{d'+c}z^{c'}\bar{x}^{b'}(-\bar{y})^a\bar{z}^{a'+b} \\
& \quad +x^{c+d'}(-y)^{c'}z^{d'}\bar{x}^a(-\bar{y})^{a'+b}\bar{z}^{b'} + x^{c+d'}(-y)^d z^{c'}\bar{x}^a(-\bar{y})^{b'}\bar{z}^{a'+b}\Big] \\
& -\Big[x^{c+d'}(-y)^{c'+d}\bar{x}^a(-\bar{y})^{a'}\bar{z}^{b+b'} + x^{c+d'}z^{c'+d}\bar{x}^a(-\bar{y})^{b+b'}\bar{z}^{a'} \\
& \quad +(-y)^{c+d'}z^{c'+d}\bar{x}^{b+b'}(-\bar{y})^a\bar{z}^{a'} + (-y)^{d+c'}z^{c+d'}\bar{x}^{b+b'}(-\bar{y})^{a'}\bar{z}^a \\
& \quad +x^{c'+d}(-y)^{c+d'}\bar{x}^{a'}(-\bar{y})^a\bar{z}^{b+b'} + x^{c'+d}z^{c'+d}\bar{x}^{a'}(-\bar{y})^{b+b'}\bar{z}^a\Big] \\
& -\Big[x^c(-y)^{c'}z^{d+d'}\bar{x}^{a+b'}(-\bar{y})^{a'+b} + x^c(-y)^{d+d'}z^{c'}\bar{x}^{a+b'}\bar{z}^{a'+b} \\
& \quad +x^{d+d'}(-y)^c z^{c'}(-\bar{y})^{a+b'}\bar{z}^{a'+b} + x^{d+d'}(-y)^{c'}z^c(-\bar{y})^{a'+b}\bar{z}^{a+b} \\
& \quad +x^{c'}(-y)^c z^{d+d'}\bar{x}^{a'+b}(-\bar{y})^{a+b'} + x^{c'}(-y)^{d+d'}z^c\bar{x}^{a'+b}\bar{z}^{a+b'}\Big].
\end{aligned}
\tag{A.1}
$$

We have separated the above terms in 5 combinations. We show that each combination is an R^+. It is clear that these terms have the following structure.

$$
\begin{aligned}
I(a,b,c,d,a',b',c',d') = {} & x^a(-y)^b z^c\bar{x}^{a'}(-\bar{y})^{b'}\bar{z}^{c'} + x^a(-y)^c z^b\bar{x}^{a'}(-\bar{y})^{c'}\bar{z}^{b'} \\
& +x^b(-y)^a z^c\bar{x}^{b'}(-\bar{y})^{a'}\bar{z}^{c'} + x^c(-y)^a z^b\bar{x}^{c'}(-\bar{y})^{b'}\bar{z}^{a'} \\
& +x^b(-y)^c z^a\bar{x}^{b'}(-\bar{y})^{c'}\bar{z}^{a'} + x^c(-y)^b z^a\bar{x}^{c'}(-\bar{y})^{b'}\bar{z}^{a'}.
\end{aligned}
$$

where

$$I(a,b,c,a',b',c') = I(b,a,c,b',a',c') = I(c,b,a,c',b',a'). \tag{A.2}$$

In order to proceed we need to classify different orderings of a, b, c and (a', b', c'). In general, there are nine of them as follows

$$\begin{array}{lll}
1. & a \le b,c & a' \le b',c', \\
2. & b \le a,c & b' \le a',c', \\
3. & c \le a,b & c' \le a',b', \\
4. & a \le b,c & b' \le a',c', \\
5. & b \le a,c & a' \le b',c', \\
6. & c \le a,b & a' \le b',c', \\
7. & a \le b,c & c' \le a',b', \\
8. & b \le a,c & c' \le a',b', \\
9. & c \le a,b & b' \le a',c',
\end{array} \tag{A.3}$$

We first consider cases 1, 4 and 7.

$$1.\quad I(a,b,c,a',b',c') = 16^{a+a'}(-1)^{b+c}\sum_{k=0}^{c-a}\frac{(c-a)!}{k!(c-a-k)!}R^{+}_{b'-a',c'-a',b-a+k,c-a-k}, \tag{A.4}$$

$$4.\quad I(a,b,c,a',b',c') = (-16)^{a+b'}R^{+}_{c'-b',a'-b',c-a,b-a}, \tag{A.5}$$

$$7.\quad I(a,b,c,a',b',c') = (-16)^{a+c'}R^{+}_{b'-c',a'-c',b-a,c-a}. \tag{A.6}$$

By using eq. (A.2) and eq. (A.4) and switching $(a \leftrightarrow b, a' \leftrightarrow b')$, and $(a \leftrightarrow c, a' \leftrightarrow c')$, one can resolve the cases 2 and 3. Similarly, the cases 5 and 6 and the cases 8 and 9 can be obtained from the cases 4 and 7 respectively.

B Basis for $R_{a,b,c,d}$

In this appendix we prove eqs. (5.35)–(5.39).

By definition,

$$\begin{aligned}
R_{a,b,0,1} &= \bar{x}^a\left((-y)\bar{z}^b + z(-\bar{y})^b\right) + (-\bar{y})^a(x\bar{z}^b + z\bar{x}^b) \\
&\quad + \bar{z}^a\left((-y)\bar{x}^b + x(-\bar{y})^b\right).
\end{aligned} \tag{B.1}$$

Thus, the identity (5.11) gives eq. (5.35). Similarly,

$$\begin{aligned}
R_{a,b,1,1} &= x\bar{x}^a\left((-y)\bar{z}^b + z(-\bar{y})^b\right) + (-y)(-\bar{y})^a(x\bar{z}^b + z\bar{x}^b) \\
&\quad + z\bar{z}^a\left((-y)\bar{x}^b + x(-\bar{y})^b\right).
\end{aligned} \tag{B.2}$$

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Therefore eq. (5.36) is a result of the identity (5.24). Eq. (5.37) can be verified by using eq. (5.23) in

$$\begin{aligned} R_{a,b,0,2} &= \bar{x}^a \left(y^2\bar{z}^b + z^2(-\bar{y})^b\right) + (-\bar{y})^a(x^2\bar{z}^b + z^2\bar{x}^b) \\ &\quad + \bar{z}^a \left(y^2\bar{x}^b + x^2(-\bar{y})^b\right), \end{aligned} \tag{B.3}$$

eq. (5.24) and eq. (5.12) give

$$z^4 = z^2(\mathrm{j} - xy) = \mathrm{j}z^2 - 16z. \tag{B.4}$$

Therefore,

$$x^2y^2 = \mathrm{j}^2 - \mathrm{j}z^2 - 16z. \tag{B.5}$$

Similarly,

$$x^2z^2 = \mathrm{j}^2 - \mathrm{j}y^2 + 16y, \tag{B.6}$$

$$y^2z^2 = \mathrm{j}^2 - \mathrm{j}x^2 - 16x. \tag{B.7}$$

Using eqs. (B.5)–(B.7) in

$$\begin{aligned} R_{a,b,2,2} &= x^2\bar{x}^a \left(y^2\bar{z}^b + z^2(-\bar{y})^b\right) + y^2(-\bar{y})^a(x^2\bar{z}^b + z^2\bar{x}^b) \\ &\quad + z^2\bar{z}^a \left(y^2\bar{x}^b + x^2(-\bar{y})^b\right), \end{aligned} \tag{B.8}$$

one obtains eq. (5.38). Finally, using the identity

$$x\bar{x}^a \left[y^2\bar{z}^b + z^2(-\bar{y})^b\right] = \bar{x}^a \left[(-y)\left(-\mathrm{j} + z^2\right)\bar{z}^b + \left(-\mathrm{j} + y^2\right) z(-\bar{y})^b\right], \tag{B.9}$$

in

$$\begin{aligned} R_{a,b,1,2} &= x\bar{x}^a \left(y^2\bar{z}^b + z^2(-\bar{y})^b\right) + (-y)(-\bar{y})^a(x^2\bar{z}^b + z^2\bar{x}^b) \\ &\quad + z\bar{z}^a \left(y^2\bar{x}^b + x^2(-\bar{y})^b\right), \end{aligned} \tag{B.10}$$

one obtains eq. (5.39).

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