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# Dimensional reduction for conformal blocks

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Abstract: We consider the dimensional reduction of a CFT, breaking multiplets of the $d$-dimensional conformal group $\mathrm{SO}(d+1,1)$ up into multiplets of $\mathrm{SO}(d,1)$. This leads to an expansion of $d$-dimensional conformal blocks in terms of blocks in $d-1$ dimensions. In particular, we obtain a formula for $3d$ conformal blocks as an infinite sum over ${}_2F_1$ hypergeometric functions with closed-form coefficients.

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## Contents



# 1 Introduction

Conformal Field Theories (CFTs) in $d \geq 2$ spacetime dimensions are being intensively studied via the bootstrap program, revived in ref. [1]. The conformal bootstrap has led to a deeper analytic understanding of (super)conformal field theories [2–8] but also to precise numerical predictions for critical exponents, see e.g. [9, 10]. A pedagogical treatment of the subject is given in [11, 12].

A crucial role in this program is played by *conformal blocks*, special functions that are determined completely by conformal symmetry. They were first described in the 1970s [13–16] but have received much attention in recent years after breakthrough results by Dolan and Osborn [17–19]. Currently, simple expressions for conformal blocks are only known in even spacetime dimension $d$. Recent work has led to systematic methods for computing these blocks in any $d$ [20–23]. Moreover, much is now known about conformal blocks that appear in four-point functions with spinning operators [24–33] and about superconformal blocks, see e.g. [34].

In this note we develop a new representation of conformal blocks in $d$ dimensions. This representation arises from the "dimensional reduction" of a CFT, i.e. the restriction of the conformal group $\mathrm{SO}(d+1,1)$ to a subgroup $\mathrm{SO}(d,1)$ that preserves a hyperplane of codimension one. Although this is similar in spirit to a Kaluza-Klein reduction, we are not actually truncating the theory: rather, we simply organize all states in the Hilbert space of the CFT in representations of $\mathrm{SO}(d,1)$ instead of the full conformal group. In particular, a $d$-dimensional conformal block will decompose into infinitely many $(d-1)$-dimensional conformal blocks with computable coefficients. As a corollary, this strategy provides an explicit formula for $3d$ and $5d$ conformal blocks in terms of ${}_2F_1$ hypergeometric functions.

This paper is organized as follows. Section 2 reviews basic facts about conformal blocks and develops the promised dimensional reduction. In section 2.3, we compare our expansion in $d-1$ dimensional blocks to an expansion in $2d$ blocks. Finally section 3 discusses several



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directions for future work. Appendix A is a consistency check of the formalism developed in this note, applying it to the four-point function of the free scalar field.

## 2 Dimensional reduction

Let's start by recalling the definition of conformal blocks. For concreteness, consider a scalar operator $\phi$ of scaling dimension $\Delta_\phi$ in a unitary $d$-dimensional CFT. Conformal invariance requires that its four-point function is of the following form:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}_\phi(u,v)}{|x_1-x_2|^{2\Delta_\phi}|x_3-x_4|^{2\Delta_\phi}} \tag{2.1}$$

where the function $\mathcal{G}_\phi(u,v)$ depends only on two conformally invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \qquad\qquad x_{ij} := x_i - x_j\,. \tag{2.2}$$

The four-point function (2.1) can be computed using the operator product expansion (OPE):

$$\phi(x)\phi(0) = \frac{1}{|x|^{2\Delta_\phi}} + \sum_{\mathcal{O}=[\Delta,\ell]} \frac{\lambda_{\mathcal{O}}}{|x|^{2\Delta_\phi-\Delta}}\, C_\Delta^{(\ell)}(x,\partial)^{\mu_1\cdots\mu_\ell}\mathcal{O}_{\mu_1\cdots\mu_\ell}(0)\,. \tag{2.3}$$

Here the sum runs over all primary operators $\mathcal{O}_{\mu_1\ldots\mu_\ell}(x)$ of even spin $\ell$ in the theory; with $\Delta$ we denote their scaling dimension, and the OPE coefficient $\lambda_{\mathcal{O}}$ is the constant of proportionality appearing in the three-point function $\langle\phi\phi\mathcal{O}\rangle$. The differential operator $C_\Delta^{(\ell)}(x,\partial)^{\mu_1\cdots\mu_\ell}$ depends only on the quantum numbers $\Delta$ and $\ell$. In passing, we note that unitary puts a lower bound on the possible values that $\Delta$ can have:

$$\Delta \;\geq\; \begin{cases} (d-2)/2 & \ell = 0 \\ \ell + d - 2 & \ell \geq 1 \end{cases}\,. \tag{2.4}$$

By applying the OPE twice to the four-point function (2.1), one can show that $\mathcal{G}_\phi(u,v)$ can be written as follows:

$$\mathcal{G}_\phi(u,v) = 1 + \sum_{\mathcal{O}=[\Delta,\ell]} (\lambda_{\mathcal{O}})^2\, G_\Delta^{(\ell)}(u,v;d)\,. \tag{2.5}$$

The functions $G_\Delta^{(\ell)}(u,v;d)$ are conformal blocks, hence eq. (2.5) is known as a conformal block (CB) decomposition. As the notation indicates, the blocks only depend on the quantum numbers $\Delta$ and $\ell$ and the spacetime dimension $d$. In practice, they can be computed by solving a second-order PDE [18] while imposing the following asymptotic behaviour:

$$G_\Delta^{(\ell)}(u,v;d) \underset{u\to 0,\, v\to 1}{\sim} c_\ell^{(d)}\, u^{\Delta/2}\, \widehat{C}_\ell^{(\nu)}\!\left(\frac{1-v}{2\sqrt{u}}\right), \tag{2.6}$$

where $\widehat{C}_j^{(\nu)}$ is a rescaled Gegenbauer polynomial with parameter $\nu := (d-2)/2$:

$$\widehat{C}_j^{(\nu)}(\xi) := \frac{j!}{(2\nu)_j}\mathrm{Geg}_j^{(\nu)}(\xi), \qquad (x)_n := \Gamma(x+n)/\Gamma(x)\,. \tag{2.7}$$

By construction, these functions obey $\widehat{C}_j^{(\nu)}(1) = 1$ and have a finite limit as $d \to 2$, contrary to the normal Gegenbauer polynomials. A natural choice for the normalization coefficients $c_\ell^{(d)}$ is [24]

$$c_\ell^{(d)} = \frac{(-1)^\ell (2\nu)_\ell}{2^\ell (\nu)_\ell} \tag{2.8}$$

although we will leave $c_\ell^{(d)}$ arbitrary in the rest of this paper.

In even spacetime dimensions, simple expressions for the conformal blocks exist [17–19]. These are easiest to state in the Dolan-Osborn coordinates $z, \bar{z}$, defined through $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$. On the Euclidean section, $z$ is a complex coordinate and $\bar{z} = z^*$ its conjugate. After defining

$$k_{2\beta}(x) := x^\beta \, {}_2F_1(\beta, \beta; 2\beta; x) \tag{2.9}$$

the $2d$ and $4d$ conformal blocks are:

$$G_\Delta^{(\ell)}(z, \bar{z}; 2) = \frac{c_\ell^{(2)}}{2} \left[ k_{\Delta+\ell}(z) k_{\Delta-\ell}(\bar{z}) \; + \; (z \leftrightarrow \bar{z}) \right] \tag{2.10a}$$

$$G_\Delta^{(\ell)}(z, \bar{z}; 4) = \frac{c_\ell^{(4)}}{\ell+1} \frac{z\bar{z}}{z - \bar{z}} \left[ k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) \; - \; (z \leftrightarrow \bar{z}) \right]. \tag{2.10b}$$

No similar formulas in odd $d$ are known, although some simplifications occur when specializing to the "diagonal" line $z = \bar{z}$ [21, 35].

The conformal block $G_\Delta^{(\ell)}$ has a representation-theoretical meaning: it is the contribution of a conformal multiplet of dimension $\Delta$ and spin $\ell$ to the four-point function (2.1), containing a primary operator $\mathcal{O}_{\mu_1 \cdots \mu_\ell}(x)$ and all of its derivatives. Such a multiplet can be described in a concrete fashion through the state-operator correspondence. The multiplet of $\mathcal{O}$ is built on top of the primary state $|\mathcal{O}\rangle_{\mu_1 \cdots \mu_\ell} := \lim_{x \to 0} \mathcal{O}_{\mu_1 \ldots \mu_\ell}(x)|0\rangle$, where $|0\rangle$ is the CFT vacuum. All other states in the multiplet are obtained by acting on $|\mathcal{O}\rangle$ with $P_\mu$, the generator of translations of the conformal algebra. A complete basis[1] of these descendants is spanned by the following states:

$$(P^2)^k P_{\mu_1} \cdots P_{\mu_m} P^{\nu_1} \cdots P^{\nu_p} |\mathcal{O}\rangle_{\nu_1 \cdots \nu_p \, \mu_{m+1} \cdots \mu_{m+r}} \, , \qquad r = \ell - p \, , \quad 0 \le p \le \ell \, . \tag{2.11}$$

It is understood that the $\mu$ indices must be symmetrized and made traceless. The state shown in (2.11) then has scaling dimension $\Delta + 2k + m + p$ and spin $\ell + m - p$. It follows that a descendant of level $n$ — that is to say, with dimension $\Delta + n$ — can have the following spins:

$$j = \ell + n, \; \ell + n - 2, \; \ldots, \; \max(\ell - n, \; \ell - n \bmod 2). \tag{2.12}$$

For a suitable choice of coordinates, there is one-to-one correspondence between a descendant of level $n$ and spin $j$ and a term in the conformal block $G_\Delta^{(\ell)}$. To make this concrete, we pass to the following coordinates:

$$s := |z| = \sqrt{z\bar{z}} \, , \quad \xi := \cos(\arg z) = \frac{z + \bar{z}}{2\sqrt{z\bar{z}}} \, . \tag{2.13}$$

[1] We are ignoring descendants that transform in mixed or antisymmetric representations of the Lorentz group, since such descendants do not contribute to a scalar four-point function.

In the $(s,\xi)$ coordinates, the contribution of a level-$n$ spin-$j$ descendant to the conformal block can be shown [20] to be proportional to $\mathbf{P}^{(d)}_{\Delta+n,j}(s,\xi)$, where

$$\mathbf{P}^{(d)}_{E,j}(s,\xi) := s^E\, \widehat{C}^{(\nu)}_j(\xi)\,. \tag{2.14}$$

This is consistent with the fact that Gegenbauer polynomials are $d$-dimensional spherical harmonics. Consequently, conformal blocks admit an expansion of the form

$$G^{(\ell)}_\Delta(s,\xi;d) = \sum_{n=0}^{\infty}\sum_j a^{(d)}_{n,j}(\Delta,\ell)\, \mathbf{P}^{(d)}_{\Delta+n,j}(s,\xi) \tag{2.15}$$

with $j$ again restricted to the range (2.12). The coefficients $a^{(d)}_{n,j}$ are fixed by conformal invariance, and are known in closed form as ${}_4F_3$ hypergeometrics evaluated at unity [18].

As advertised, we will break the conformal group down to a subgroup of conformal transformations in $d-1$ dimensions, and we want to analyze the consequences of this dimensional reduction for conformal blocks. Let us first consider a toy example of what will happen, namely the restriction of the rotation group SO($d$) to SO($d$–1). If we think of SO($d$) as the isometry group of the sphere $S^{d-1}$, this means that we take the subgroup of rotations that leave the equator invariant. Under this restriction, the spin-$\ell$ representation of SO($d$), denoted as $[\ell]_d$, breaks up into SO($d$–1) irreps as follows:

$$[\ell]_d = [0]_{d-1} + [1]_{d-1} + \ldots + [\ell]_{d-1}\,. \tag{2.16}$$

The branching rule (2.16) can be understood by realizing $[\ell]_d$ as a traceless symmetric tensor of rank $\ell$. For instance, the first $d-1$ components of a vector $v_\mu \in \mathbb{R}^d$ form a vector representation of SO($d$–1), whereas the last component $v_d$ transforms as a SO($d$–1) scalar.

Since spherical harmonics form a representation of SO($d$), the branching rule (2.16) applies in particular to the (rescaled) Gegenbauer polynomials. Concretely, the spin-$\ell$ Gegenbauer polynomial $\widehat{C}^{(\nu)}_\ell$ can be written in the following form:

$$\widehat{C}^{(\nu)}_\ell(\xi) = \sum_{j=0}^{\ell} Z^j_\ell\, \widehat{C}^{(\nu-1/2)}_j(\xi) \tag{2.17}$$

since the $\widehat{C}^{(\nu-1/2)}_j$ are Gegenbauer polynomials in $d-1$ dimensions. As a matter of fact, only spins $j=\ell,\ \ell-2,\ldots,\ \ell$ mod 2 appear in the r.h.s. of eq. (2.17), owing to the selection rule

$$\widehat{C}^{(\nu)}_j(-\xi) = (-1)^j\, \widehat{C}^{(\nu)}_j(\xi)\,. \tag{2.18}$$

The coefficients $Z^j_\ell$ in eq. (2.17) can be computed using explicit expressions for the Gegenbauer polynomials [36] together with their orthogonality. This yields

$$Z^j_\ell = \frac{(1/2)_p\, \ell!}{p!\, j!}\, \frac{(\nu)_{j+p}(2\nu-1)_j}{(\nu-1/2)_{j+p+1}(2\nu)_\ell}(j+\nu-1/2)\,, \qquad p \equiv (\ell-j)/2\,. \tag{2.19}$$



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It will prove useful later in this work to have a bound on the coefficients $Z_\ell^j$. It is easy to see that all $Z_\ell^j$ are positive, provided that $d \geq 2$.[2] Moreover, the normalization condition $\widehat{C}_\ell^{(\nu)}(1) = 1$ implies that for fixed $\ell$ we have

$$\sum_j Z_\ell^j = 1\,. \tag{2.20}$$

We conclude that $0 \leq Z_\ell^j \leq 1$ for all $\ell, j$.

Having considered the restriction $\mathrm{SO}(d) \to \mathrm{SO}(d{-}1)$, we now turn our attention to the conformal group $\mathrm{SO}(d+1,1)$. We will restrict the full group to a subgroup $\mathrm{SO}(d,1)$ that preserves the hyperplane $x_d = 0$. Doing so, a primary $d$-dimensional representation breaks up into infinitely many primary $(d{-}1)$-dimensional representations. The argument is the following. Recall that a state is a primary of $\mathrm{SO}(d+1,1)$ if and only if it is annihilated by all $d$ generators of special conformal transformations, which we denote here by $K_\mu$. Therefore any state that is annihilated by $K_1, \ldots, K_{d-1}$ but not by $K_d$ is a descendant of $\mathrm{SO}(d+1,1)$ but a primary of $\mathrm{SO}(d,1)$. Among all descendants shown in eq. (2.11), the following states fit that description:

$$|\mathcal{O}; j, m\rangle_{\alpha_1\cdots\alpha_j} = (P_d)^m |\mathcal{O}\rangle_{\alpha_1\cdots\alpha_j\, d\cdots d}\,, \qquad 0 \leq j \leq \ell\,,\ m = 0, 1, 2, \ldots\,. \tag{2.21}$$

The state $|\mathcal{O}; j, m\rangle$ has $\mathrm{SO}(d-1)$ spin $j$ and scaling dimension $\Delta + m$. We arrive at the following branching rule: any $\mathrm{SO}(d+1,1)$ multiplet of dimension $\Delta$ and spin $\ell$ splits up into infinitely many $\mathrm{SO}(d,1)$ multiplets of spin $0 \leq j \leq \ell$ and dimension $\Delta + m$ with $m \geq 0$.[3]

Consequently, a conformal block $G_\Delta^{(\ell)}(u, v; d)$ can be written as an infinite sum over the conformal blocks $G_{\Delta+m}^{(j)}(u, v; d-1)$ with $0 \leq j \leq \ell$ and $m \geq 0$. There are however some selection rules that apply, as was the case for the Gegenbauer polynomials. We will derive these in the $\rho$ kinematics of [39], passing to the $(r, \eta)$ coordinates defined as

$$u = \left(\frac{4r}{1 + 2r\eta + r^2}\right)^2, \quad v = \left(\frac{1 - 2r\eta + r^2}{1 + 2r\eta + r^2}\right)^2. \tag{2.22}$$

In the $(r, \eta)$ coordinates, conformal blocks have an expansion where only descendants of *even* level appear [20]:

$$G_\Delta^{(\ell)}(r, \eta; d) = \sum_{n=0}^{\infty} \sum_j b_{n,j}^{(d)}(\Delta, \ell)\, \mathbf{P}_{\Delta+2n,j}^{(d)}(r, \eta) \tag{2.23}$$

with $j$ restricted to the range (2.12). The $\mathrm{SO}(d{+}1,1) \to \mathrm{SO}(d,1)$ branching rule described above must apply to any coordinate set, in particular to the $(r, \eta)$ coordinates. By consistency with eq. (2.23), it follows that only $\mathrm{SO}(d,1)$ primaries of even level can appear

[2] This is a special case of the fact that Gegenbauer polynomials in $D$ dimensions can be written as a sum over Gegenbauer polynomials in $d < D$ dimensions with positive coefficients, see [37] and references therein. We thank A. Zhiboedov for pointing this out.

[3] Such branching rules can also be derived or checked by decomposing the characters of $\mathrm{SO}(d+1,1)$ into $\mathrm{SO}(d,1)$ characters, see e.g. [38]. This approach may be useful when dealing with more complicated representations.



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in the decomposition of $G_\Delta^{(\ell)}(u,v;d)$. Likewise, only spins $j=\ell,\,\ell-2,\ldots,\;\ell \bmod 2$ can contribute. In conclusion, there exists a decomposition of $G_\Delta^{(\ell)}$ of the following form:

$$G_\Delta^{(\ell)}(u,v;d)=\sum_{n=0}^{\infty}\sum_j \mathcal{A}_{n,j}(\Delta,\ell)\,G_{\Delta+2n}^{(j)}(u,v;d-1) \tag{2.24}$$

with the sum running over

$$j=\ell\,,\,\ell-2\,,\ldots,\,\ell \bmod 2\,. \tag{2.25}$$

The coefficients $\mathcal{A}_{n,j}(\Delta,\ell)$ are fixed by conformal invariance. In the following section, we will explain one method to compute them.

### 2.1 Recursion relation for coefficients

In this section, we will compute the coefficients $\mathcal{A}_{n,j}(\Delta,\ell)$ appearing in eq. (2.24). Our discussion will rely heavily on the representation (2.15) of conformal blocks in the $(s,\xi)$ coordinates. In particular, we will use the fact that the coefficients $a_{n,j}^{(d)}(\Delta,\ell)$ obey a three-term recursion relation:

$$\begin{aligned}\Big[\mathrm{C}^{(d)}(\Delta+n,j)-\mathrm{C}^{(d)}(\Delta,\ell)\Big]\,&a_{n,j}^{(d)}(\Delta,\ell)\\ &=\beta_{j-1}^{(d)}(\Delta+n-1)\,a_{n-1,j-1}^{(d)}(\Delta,\ell)\;+\;\gamma_{j+1}^{(d)}(\Delta+n-1)\,a_{n-1,j+1}^{(d)}(\Delta,\ell)\,.\end{aligned} \tag{2.26}$$

Here $\mathrm{C}^{(d)}(\Delta,\ell):=\Delta(\Delta-d)+\ell(\ell+d-2)$ is the eigenvalue of the quadratic conformal Casimir and

$$\beta_j^{(d)}(x):=\frac{(x+j)^2(j+2\nu)}{2(j+\nu)}\,,\quad \gamma_j^{(d)}(x):=\frac{(x-j-2\nu)^2 j}{2(j+\nu)}\,. \tag{2.27}$$

We can use this recurrence to compute the coefficients $a_{n,j}^{(d)}(\Delta,\ell)$ to arbitrary order in $n$, starting from the initial condition

$$a_{0,j}^{(d)}(\Delta,\ell)=c_\ell^{(d)}\,\delta_{j,\ell} \tag{2.28}$$

that is imposed by eq. (2.6). A comprehensive discussion of this recursion relation is given in [20].

We will compute the $\mathcal{A}_{n,j}(\Delta,\ell)$ coefficients by formulating a second recursion relation. As a starting point, remark that the conformal block $G_\Delta^{(\ell)}(s,\xi;d)$ admits an expansion in terms of the functions $\mathbf{P}_{\Delta+n,j}^{(d-1)}(s,\xi)$. This expansion takes the following form:

$$G_\Delta^{(\ell)}(s,\xi;d)=\sum_{n=0}^{\infty}\sum_j Y_{n,j}^{(\ell)}\,\mathbf{P}_{\Delta+n,j}^{(d-1)}(s,\xi)\,,\qquad j\in\{\ell+n,\,\ell+n-2,\ldots,\,\ell \bmod 2\} \tag{2.29}$$

for some coefficients $Y_{n,j}^{(\ell)}$ that we will determine. On the one hand, eq. (2.29) can be obtained by applying the Gegenbauer identity (2.19) to the $(s,\xi)$ representation of eq. (2.15). We can therefore express the coefficients $Y_{n,j}^{(\ell)}$ as follows:

$$Y_{n,j}^{(\ell)}=\sum_{p=0}^{p_*} Z_{j+2p}^{j}\,a_{n,j+2p}^{(d)}(\Delta,\ell),\qquad p_*=(\ell+n-j)/2\,. \tag{2.30}$$

On the other hand, we can first apply the dimensional reduction formula (2.24) to the conformal block $G_\Delta^{(\ell)}(u,v;d)$. Next, we expand every $(d-1)$-dimensional block on the r.h.s. in terms of the $(s,\xi)$ representation. Doing so leads to a different expression for the $Y_{n,j}^{(\ell)}$, namely

$$Y_{n,j}^{(\ell)} = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{\ell} \mathcal{A}_{m,k}(\Delta,\ell)\, a_{n-2m,j}^{(d-1)}(\Delta+2m,k)\,. \tag{2.31}$$

The sum over $k$ is also restricted to $j+2m-n \le k \le j+n-2m$ and $k \equiv \ell \bmod 2$.

Now fix $j \in \{\ell, \ell-2, \dots, \ell \bmod 2\}$ and $n \ge 0$. Requiring that the two expressions for $Y_{2n,j}^{(\ell)}$ agree, we obtain the following identity:

$$c_j^{(d-1)} \mathcal{A}_{n,j}(\Delta,\ell) = \sum_{q=0}^{q_*} Z_{j+2q}^{j}\, a_{2n,j+2q}^{(d)}(\Delta,\ell) - \sum_{m=0}^{n-1} \sum_{k} \mathcal{A}_{m,k}(\Delta,\ell)\, a_{2n-2m,j}^{(d-1)}(\Delta+2m,k) \tag{2.32}$$

where $q_* = (\ell+2n-j)/2$ and $k$ is restricted to

$$\max(0, j+2m-2n) \le k \le \min(\ell, j+2n-2m), \quad k \equiv \ell \bmod 2\,. \tag{2.33}$$

Notice that the r.h.s. of (2.32) only involves coefficients $\mathcal{A}_{m,k}(\Delta,\ell)$ with $m<n$. Moreover, the coefficients $a_{2n,j+2q}^{(d)}(\Delta,\ell)$ and $a_{2n-2m,j}^{(d-1)}(\Delta+2m,k)$ can be computed by means of the recursion relation (2.26). We can therefore use eq. (2.32) to compute the coefficients $\mathcal{A}_{n,j}(\Delta,\ell)$ recursively, up to arbitrary $n$, starting from $n=0$. To be precise, Eq, (2.32) must be understood as a set of $\lfloor \ell/2 \rfloor$ coupled recursion relations, one for every allowed value of $j$. Finally, we remark that the above recursion relation is inhomogeneous, which means that the "initial condition" $\mathcal{A}_{0,j}(\Delta,\ell)$ is not arbitrary. Concretely, setting $n=0$ in eq. (2.32) yields

$$\mathcal{A}_{0,j}(\Delta,\ell) = Z_\ell^j \frac{c_\ell^{(d)}}{c_j^{(d-1)}} \tag{2.34}$$

which is consistent with the asymptotics imposed by eq. (2.6).

Although the recursion relation (2.32) looks complicated, its solution can be written down in closed form:

$$\begin{aligned}\mathcal{A}_{n,j}(\Delta,\ell) = Z_\ell^j \frac{c_\ell^{(d)}}{c_j^{(d-1)}} &\frac{\big((\Delta+j)/2\big)_n \big((\tau+\ell-j+1)/2\big)_n}{\big((\Delta+j-1)/2\big)_n \big((\tau+\ell-j)/2\big)_n} \\ \times\ &\frac{(1/2)_n}{16^n\, n!} \frac{(\Delta-1)_{2n} \big((\Delta+\ell)/2\big)_n (\tau/2)_n}{(\Delta-\nu)_n (\Delta-\nu-1/2+n)_n \big((\Delta+\ell+1)/2\big)_n \big((\tau+1)/2\big)_n}\,,\end{aligned} \tag{2.35}$$

writing $\tau := \Delta-(\ell+d-2)$ for the conformal twist. While we don't have a rigorous proof of this formula, we have checked that it satisfies (2.32) for $\ell, n \le 20$ and we conjecture that it holds in general. Clearly eq. (2.35) can be checked in other ways, e.g. using expansions of conformal blocks in the $z$ or $\rho$ coordinates and in the diagonal limit.

In passing, we notice that for the scalar ($\ell=0$) block, only terms with $j=0$ are allowed in (2.24), and the formula for the coefficients simplifies:

$$\mathcal{A}_{n,0}(\Delta,0) = \frac{c_0^{(d)}}{c_0^{(d-1)}} \frac{(1/2)_n}{4^n\, n!} \frac{(\Delta/2)_n^3}{(\Delta-\nu)_n (\Delta-\nu-1/2+n)_n \big((\Delta+1)/2\big)_n}\,. \tag{2.36}$$

A similar simplication occurs for $\ell=1$.

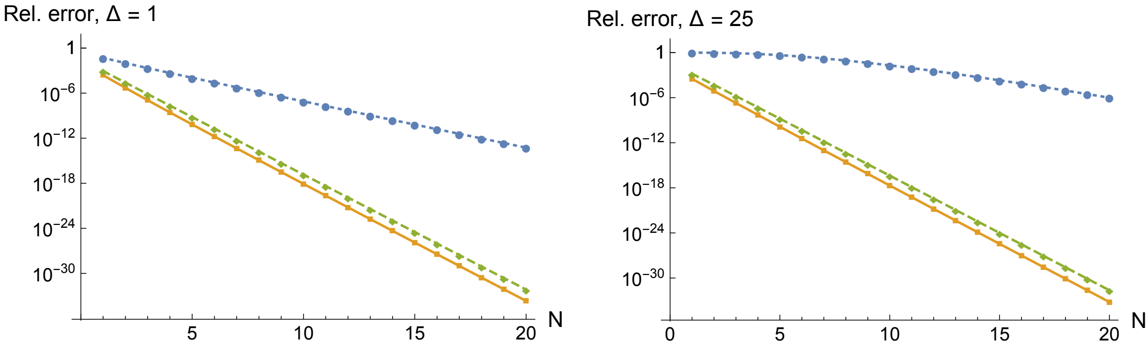


**Figure 1**. Comparison of different conformal block expansions. Horizontal axis: the order of truncation $N$, vertical axis: relative error in the numerical value of the conformal block — notice the logarithmic scale. Solid orange: dimensional reduction with $n \leq N$ terms; dashed green: $\rho$-series with $n \leq N$ terms; dotted blue: $z$-series with $n \leq 2N$ terms. The points are joined by lines to guide the eye. The left plot shows the $3d$ scalar block at the point $u = v = 1/4$ with $\Delta = 1$, the right plot corresponds to $\Delta = 25$.

### 2.2 Convergence

Equations (2.24) and (2.35) are the main result of this note. At this stage, we want to point out two important properties of the coefficients $\mathcal{A}_{n,j}$. For convenience, we will set $c_\ell^{(d)} \equiv 1$ in what follows. First, we note that all $\mathcal{A}_{n,j}(\Delta, \ell)$ are positive, provided that $\Delta$ satisfies the unitarity bound (2.4) and $d \geq 2$. Second, we remark that $\mathcal{A}_{n,j}$ decays exponentially fast with $n$. To prove this, let's consider the coefficient $\mathcal{A}_{n,j}(\Delta, \ell)$ as a function of $\Delta$, keeping $\ell, j$ and $n$ fixed. We notice that $\mathcal{A}_{n,j}(\Delta, \ell)$ is a rational function of $\Delta$ of the form $p(\Delta)/q(\Delta)$, where $p$ and $q$ are polynomials of equal degree. Furthermore $p$ and $q$ completely factorize over the reals, with all zeroes at values of $\Delta$ at or below the unitarity bound. This means that *above* the unitarity bound, $\mathcal{A}_{n,j}(\Delta, \ell)$ is a slowly varying function of $\Delta$, and it is well approximated by its value in the limit $\Delta \to \infty$:

$$\mathcal{A}_{n,j}(\Delta, \ell) \underset{\Delta \to \infty}{\sim} Z_\ell^j \, \frac{(1/2)_n}{16^n \, n!} \left[ 1 + \mathrm{O}\left( \frac{1}{\Delta} \right) \right]. \tag{2.37}$$

As promised, the coefficient $\mathcal{A}_{n,j}(\Delta, \ell)$ decreases exponentially with $n$, as $\sim n^{-1/2} \, 16^{-n}$. Remarkably, this exponential behaviour holds not only asymptotically, but already starts at $n = 1$.

So far, we have encountered three different expansions for $d$-dimensional conformal blocks: the "$z$-series" from eq. (2.15), the "$\rho$-series" from (2.23) and the expansion in terms of lower-dimensional blocks (2.24). In figure 1 we compare their convergence rates numerically, by truncating these expansions at finite order $N$ and evaluating them at the crossing symmetric point $u = v = 1/4$. The results corroborate that the truncation error of (2.24) decreases exponentially with $N$.

For completeness, we can verify that the exponential decay with $n$ also holds for $\Delta$ close to the unitarity bound. For spinning operators ($\ell \geq 1$), the limit $\tau \to 0$ is continuous, meaning that there are no important corrections to (2.37), and the exponential decay at large $n$ persists. This is confirmed by an explicit expression for $\mathcal{A}_{n,j}$ at $\tau = 0$ shown in



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appendix A. As is well known, the scalar ($\ell = 0$) block diverges at the unitarity bound $\Delta = \nu$, where a level-two descendant becomes null. Using a conformal representation theory argument [22, 30, 40, 41], we have

$$G_\Delta^{(0)}(u,v;d) \underset{\Delta\to\nu}{\sim} \frac{1}{\Delta-\nu}\frac{\nu^3}{16(\nu+1)}G^{(0)}_{d/2+1}(u,v;d) + \dots \tag{2.38}$$

omitting terms that are regular as $\Delta \to \nu$. Hence near the unitarity bound, $G_\Delta^{(0)}$ is dominated by a conformal block with $\Delta = d/2+1$, which itself is well above the unitarity bound. Therefore the estimate (2.37) applies, and we are done. The same conclusion can be reached by expanding eq. (2.36) around $\Delta = \nu$.

### 2.3 Comparison to $2d$ expansion

The dimensional reduction discussed in this paper has a counterpart on the lightcone, i.e. the Minkowski section of a CFT, where $z$ and $\bar{z}$ are independent, real variables. Lightcone kinematics turn out to be particularly simple: in the limit $z \to 0$ at fixed $\bar{z}$ the conformal blocks become effectively two-dimensional, up to an unimportant prefactor:

$$G_\Delta^{(\ell)}(z,\bar{z};d) \underset{z\to 0}{\sim} c_\ell^{(d)}\frac{(\nu)_\ell}{(2\nu)_\ell}\,z^{(\Delta-\ell)/2}\,k_{\Delta+\ell}(\bar{z})\,. \tag{2.39}$$

The study of CFT crossing equations in this limit has led to many analytic bootstrap results, initiated in [2, 3] with follow-up work in refs. [42–55].

It may be interesting to systematically compute corrections to the leading-order behaviour (2.39). There is a group-theoretical approach to this problem, first discussed in appendix A of ref. [46] (see also [56]). We will briefly review their argument here. The idea is to restrict $\mathrm{SO}(d,2)$ — the conformal group in Minkowski signature — to $\mathrm{SO}(2,2)$, the group of conformal transformations acting on the $(z,\bar{z})$ plane. On the level of its Lie algebra, the latter splits into two copies of $\mathfrak{sl}(2)$, spanned by three chiral generators $L_0, L_{\pm1}$ and three anti-chiral generators $\bar{L}_0, \bar{L}_{\pm1}$. Any $\mathrm{SO}(2,2)$ primary state is therefore labeled by two numbers $h, \bar{h}$, the eigenvalues of $L_0$ resp. $\bar{L}_0$; such a state lifts to a local operator with scaling dimension $h+\bar{h}$ and spin $|h-\bar{h}|$.

Under this restriction, any $d$-dimensional conformal multiplet breaks up into infinitely many "lightcone primaries". As with the dimensional reduction discussed in this paper, this implies that any $d$-dimensional conformal block can be decomposed into $2d$ blocks. Concretely, we have:

$$G_\Delta^{(\ell)}(z,\bar{z};d) = \sum_{h,\bar{h}} P_{h,\bar{h}}(\Delta,\ell;d)\,G_{h+\bar{h}}^{(|h-\bar{h}|)}(z,\bar{z};2) \tag{2.40}$$

for some coefficients $P_{h,\bar{h}}(\Delta,\ell;d)$ fixed by conformal symmetry. Every term in the r.h.s. corresponds to a different lightcone primary with quantum numbers $(h,\bar{h})$. In the limit $z\to 0$, the sum (2.40) is dominated by a single block with $h = (\Delta-\ell)/2$ and $\bar{h} = (\Delta+\ell)/2$, all other terms being suppressed by powers of $z$.

In order to compute corrections to (2.39) it is therefore sufficient to determine the coefficients $P_{h,\bar{h}}(\Delta,\ell;d)$. This has not been done so far, to our knowledge. We remark



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however that the expression (2.35) for the coefficients $\mathcal{A}_{n,j}(\Delta,\ell)$ is sufficient to determine the $P_{h,\bar{h}}(\Delta,\ell;d)$ for all integer $d$. For $d=3$ this is obvious: after relabeling $h,\bar{h}$ in the r.h.s. of (2.40) in terms of scaling dimensions and spins, the coefficients $P_{h,\bar{h}}$ are identical to the coefficients $\mathcal{A}_{n,j}$ with $d\to 3$. The generalization to $d>3$ is straightforward: in order to determine the coefficients $P_{h,\bar{h}}(\Delta,\ell;d>3)$ one has to "dimensionally reduce" $d-2$ times.

## 3 Discussion

This note has presented a new method to compute conformal blocks in $d$-dimensional CFTs, by relating them to conformal blocks of CFTs in $d-1$ dimensions. In particular, eqs. (2.24) and (2.35) together form an explicit formula for blocks in odd $d$: for $d=3$ (resp. $d=5$) our method leads to an expression in terms of $2d$ (resp. $4d$) blocks shown in eq. (2.10), which in turn are given by ${}_2F_1$ hypergeometric functions. Moreover, the expansion in lower-dimensional blocks converges exponentially fast, which may prove to be useful for numerical applications.

Currently only two closed-form expressions are known for conformal blocks in odd $d$: the $z$-series expansion (2.15) and a formula that uses Mellin-Barnes integrals [19, 57–59]. The latter involves so-called Mack polynomials that don't admit very compact expressions. The coefficients $\mathcal{A}_{n,j}$ from eq. (2.35) may therefore be easier to deal with in practice. In particular, they may be useful for the analytic bootstrap [60, 61] in three dimensions, since the two-dimensional crossing kernel is known in closed form [62].

There are a few obvious ways to extend the results presented in this note. First, it is possible write down a similar expansion for conformal blocks with non-zero external dimensions. The resulting expressions are somewhat more complicated, as the selection rule described below (2.23) does not apply. Second, it is possible to dimensionally reduce more complicated representations of the Lorentz group. A starting point for this would be the "seed" conformal blocks in three and four dimensions [29, 31]. An even further generalization consists of dimensionally reducing superconformal multiplets and the resulting superconformal blocks. We leave all of these issues for future work.

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## A Free field theory

A consistency check of the results obtained in this paper is furnished by free scalar field CFT in $d$ dimensions. We recall that the four-point function of the free scalar $\phi$ is

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle=\frac{\mathcal{G}_{\text{free}}(u,v)}{|x_{12}|^{d-2}|x_{34}|^{d-2}}\,,\qquad \mathcal{G}_{\text{free}}(u,v;d)=1+u^{\nu}+(u/v)^{\nu}\,. \quad \text{(A.1)}$$

The above four-point function has a well-known CB decomposition, namely

$$\mathcal{G}_{\text{free}}(u,v;d)=1+2\sum_{\ell\text{ even}}f_\ell\,G^{(\ell)}_{\ell+d-2}(u,v;d)\,,\qquad f_{2p}=\frac{1}{c^{(d)}_{2p}}\frac{(\nu)_p(2\nu)_{2p}}{4^p\,(2p)!\,(\nu-1/2+p)_p}\,. \quad \text{(A.2)}$$

At the same time, we can decompose $\mathcal{G}_{\text{free}}(u,v)$ in terms of $d-1$ dimensional conformal blocks. From the $d-1$ dimensional point of view, the four-point function (A.2) belongs to a free field theory with a non-local action, known as a generalized free field [63]. The CB decomposition has the following form:

$$\mathcal{G}_{\text{free}}(u,v;d) = 1 + 2\sum_{\ell \text{ even}}\sum_{n=0}^{\infty} g_{\ell,n}\, G^{(\ell)}_{\ell+d-2+2n}(u,v;d-1)\,. \tag{A.3}$$

The coefficients $g_{\ell,n}$ appearing here are given by [64]

$$g_{\ell,n} = \frac{c_\ell^{(d)}}{c_\ell^{(d-1)}}\, f_\ell\, Z_\ell^\ell \times \begin{cases} 1 & n=0 \\ 2\lambda_n(\ell) & n\geq 1 \end{cases}, \tag{A.4}$$

where we introduce the notation

$$\lambda_n(\ell) := \frac{(1/2)_n}{16^n\, n!}\frac{(\ell+2\nu-1)_{2n}(\ell+\nu)_n}{(\ell+\nu-1/2)_{2n}(\ell+\nu+1/2)_n}\,. \tag{A.5}$$

We want to verify that the CB decompositions (A.2) and (A.3) are consistent with the dimensional reduction formula (2.35). Notice that in (A.2) only operators with twist $\tau=0$ appear. In the zero-twist limit, the coefficients $\mathcal{A}_{n,j}$ simplify:

$$\mathcal{A}_{n,j}(\Delta,\ell) \underset{\tau\to 0}{\sim} Z_\ell^j\, \frac{c_\ell^{(d)}}{c_j^{(d-1)}} \times \begin{cases} \lambda_n(\ell) & j=\ell \\ \delta_{n,0} & j<\ell \end{cases}. \tag{A.6}$$

Hence applying (2.35) to eq. (A.2) gives the following CB decomposition in $d-1$ dimensions:

$$\mathcal{G}_{\text{free}}(u,v;d) = 1 + 2\sum_{\ell \text{ even}}\sum_{n=0}^{\infty} h_{\ell,n}\, G^{(\ell)}_{d-2+\ell+2n}(u,v;d-1) \tag{A.7}$$

with coefficients

$$h_{\ell,0} = \frac{c_\ell^{(d)}}{c_\ell^{(d-1)}}\, Z_\ell^\ell\, f_\ell\,, \qquad h_{\ell,n\geq 1} = \frac{c_\ell^{(d)}}{c_\ell^{(d-1)}}\, Z_\ell^\ell\, f_\ell\, \lambda_n(\ell) + \frac{c_{\ell+2n}^{(d)}}{c_\ell^{(d-1)}}\, Z_{\ell+2n}^\ell\, f_{\ell+2n}\,. \tag{A.8}$$

Consistency with (A.3) requires that $g_{\ell,n} = h_{\ell,n}$ for all $\ell, n$. For $n=0$ this is obvious, and for $n\geq 1$ this follows from the identity

$$c_{\ell+2n}^{(d)}\, Z_{\ell+2n}^\ell\, f_{\ell+2n} = c_\ell^{(d)}\, Z_\ell^\ell\, f_\ell\, \lambda_n(\ell)\,. \tag{A.9}$$

Similar consistency checks could be performed for more complicated four-point functions in free field or generalized free field CFTs.

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